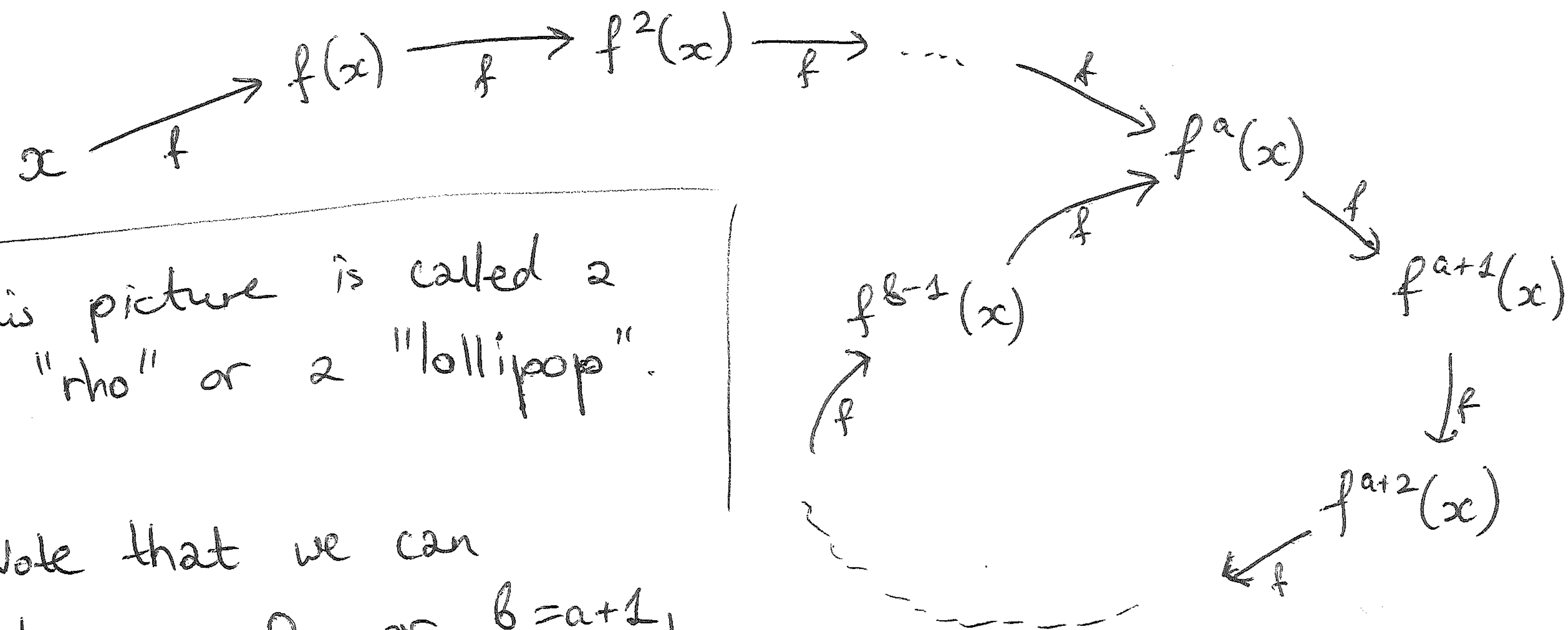
Rmk. Let $n \in \mathbb{N}$. Let S be an n-element set. Let $f : S \to S$ be a map. Let $x \in S$. Then, the $n+1$ elements $f^0(x), f^1(x), \ldots, f^n(x)$ of the n-element set S cannot all be distinct (by Pigeonhole). Thus, $\exists\ a, b \in \{0, 1, \ldots, n\}$ such that $f^a(x) = f^b(x)$ and $a < b$. Pick such a pair (a, b) with minimum b. Then, $f^0(x), f^1(x), \ldots, f^{b-1}(x)$ are distinct (else, b would not be minimum). Thus, we get the following picture:



This picture is called a "rho" or a "lollipop".

(Note that we can have $a = 0$ or $b = a+1$, in which case the picture is just a cycle, or the cycle is a loop.)

5.2. Two counting problems

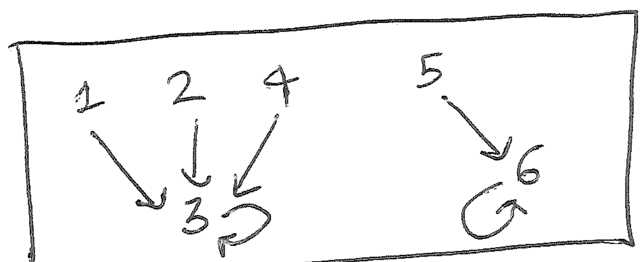
Def. Let S be a set. A map $f : S \to S$ is called idempotent if it satisfies $f^2 = f$.

Note that this is equivalent to requiring ($f^k = f \ \forall k \geq 1$).

Exercise. Let $n \in \mathbb{N}$. Let S be an n-element set. How many idempotent maps $f : S \to S$ exist?

Answer: $\sum_{k=0}^{n} \binom{n}{k} k^{n-k}$. (Fall 2017 Math 4990 hw #6 exercise 1 (b).)

[Example: $S = [6]$ and f has cycle digraph



.]

[Proof idea: f being idempotent means that $f(i) = i \ \forall i \in f(S)$.

Thus, we can construct each idempotent $f : S \to S$ as follows:

- Decide on $k := |f(S)|$. (Note $k \in \{0, 1, \ldots, n\}$)
- Choose $f(S)$. (There are $\binom{n}{k}$ choices.)

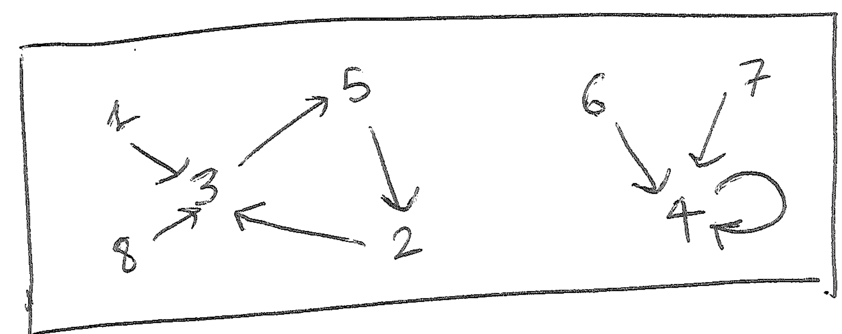
- Set $f(i) = i \quad \forall i \in f(S)$.
- Choose $f(i)$ among the k elements of $f(S)$ for each $i \in S \setminus f(S)$. (There are k^{n-k} choices, since we have k choices for each of the $n-k$ different i's.)

]

<u>Def.</u> Let S be a set. A map $f : S \to S$ is called <u>image-injective</u> if $\forall a, b \in S$, we have: (if $f^2(a) = f^2(b)$, then $f(a) = f(b)$).
(Equivalently, this means $f|_{f(S)}$ is injective.)

<u>Exercise.</u> Let $n \in \mathbb{N}$. Let S be an n-element set. How many image-injective maps $f : S \to S$ exist?

<u>Example:</u> $S = [8]$ and



Idea: f is image-injective

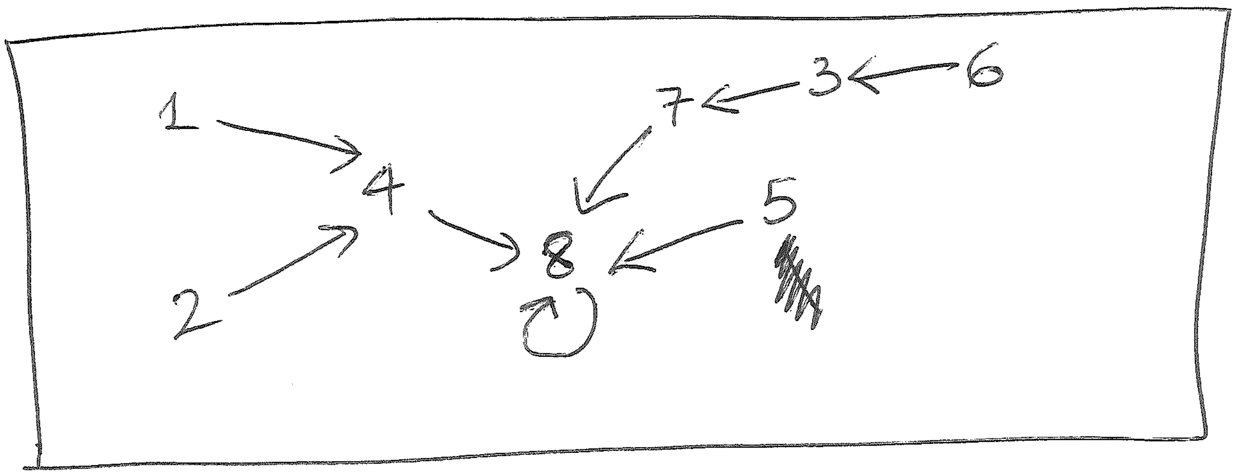
$\Longleftrightarrow \forall x \in S$, the element $f(x)$ is recurrent.

(Details: see Spring 2018 Math 4707 hw #3 exercise 4.)

5.3. Maps f with $|f^n(S)| = 1$

How does a map $f : S \to S$ with $|f^n(S)| = 1$ look like?

Example: $S = [8]$, and f has cycle digraph



This f has $f^n(S) = \{8\}$.

Thm. 5.2 (Cayley's, in map form). Let $n \geq 1$, and let $S = [n]$. Then, the # of maps $f : S \to S$ with $f^n(S) = \{n\}$ is n^{n-2}.

Example: For $n=3$, there are $3^{3-2} = 3$ such maps:

$1 \to 2 \to 3 \circlearrowleft$	$2 \to 1 \to 3 \circlearrowleft$	$1 \to 3 \circlearrowleft$, $2 \to 3$

Remark. Let $n \geq 1$, and let $S = [n]$. Let $f : S \to S$ be a map with $f^n(S) = \{n\}$. Then,

$$f(n) = n. \tag{41}$$

This is because $n \in \{n\} = f^n(S)$, thus $f(n) \in f(f^n(S)) = f^{n+1}(S) \subseteq f^n(S) = \{n\}$, so $f(n) = n$.

Thm. 5.2 is commonly stated in the language of trees.

Quick summary:

Def. A simple graph is a pair (V, E) of

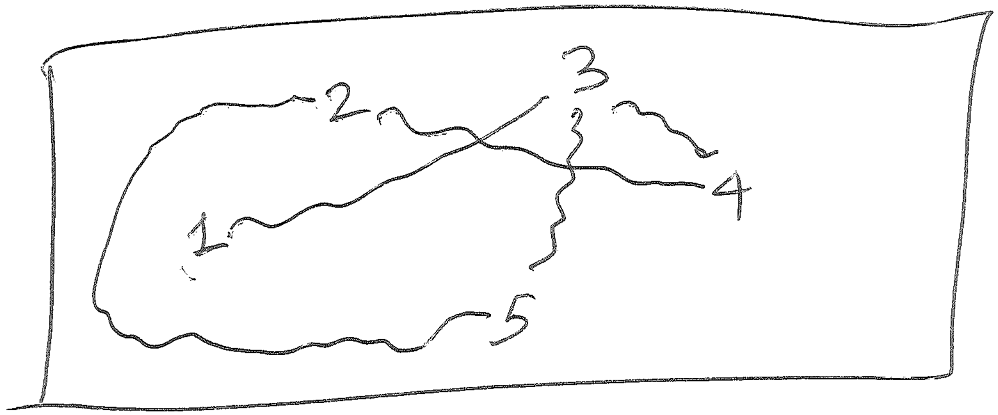
- a finite set V (called its vertex set), and
- a ~~subset~~ set E of 2-element subsets of V (called its edge set).

It can be drawn as follows:

- For each $v \in V$, draw a node labelled v somewhere in the plane.
- For each $\{v, w\} \in E$, draw a curve joining the node labelled v with the node labelled w.

Example: The simple graph $([5], \{\{1,3\}, \{2,4\}, \{3,4\}, \{3,5\}, \{2,5\}\})$

can be drawn as



Def. Given a simple graph ~~(V,E)~~ $G=(V,E)$.

(a) If $v\in V$ and $w \in V$, then a walk from v to w in G means a $(k+1)$-tuple $(v_0, v_1, \dots, v_k) \in V^{k+1}$ (for some $k \geq 0$) such that $v_0 = v$ and $v_k = w$ and $\{v_{i-1}, v_i\} \in E$ $\forall i \in [k]$.

[Example: In the above graph, $(1,3,5,2)$ and $(1,3,4,2)$ and $(1,3,5,3,5,2)$ and many other tuples are walks from 1 to 2.]

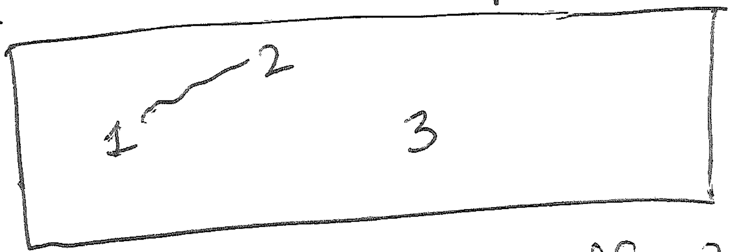
(b) A cycle in G means a $(k+1)$-tuple $(v_0, v_1, \dots, v_k) \in V^{k+1}$ such that $k \geq 3$, and the vertices $v_0, v_1, \dots, v_{k-1}$ are distinct, and $v_k = v_0$, and $\{v_{i-1}, v_i\} \in E$ $\forall i \in [k]$.

[Example: In the above graph, $(2,4,3,5,2)$ is a cycle.]

(c) The graph G is <u>connected</u> if $V \neq \emptyset$ and $\forall\, v, w \in V$ $\exists$ walk from v to w in G.

[<u>Example:</u> The above graph is connected.

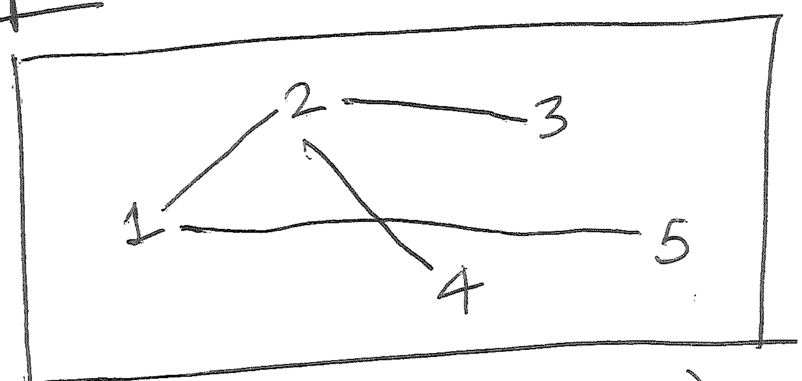
But



is not.]

(d) The graph G is a <u>tree</u> if it is connected & has no cycles.

[<u>Example:</u> The above 2 graphs are not trees, but



is a tree.]

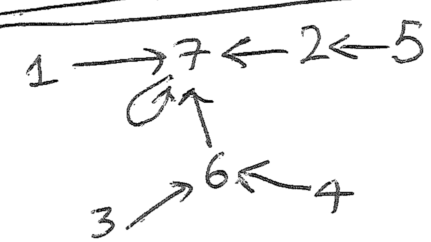
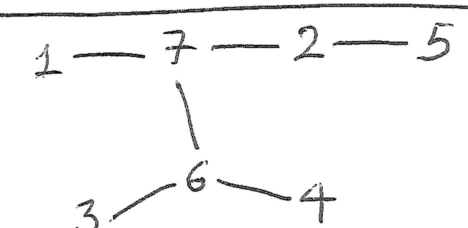
<u>Theorem 5.3 (Cayley, in tree form).</u> Let $n \geq 1$, and let $S = [n]$. Then, the # of trees with vertex set S is n^{n-2}.

Proof of Thm. 5.3 using Thm. 5.2. This follows from Thm. 5.2, once we have shown that the map

$$A : \{\text{maps } f : S \to S \text{ with } f^n(S) = \{n\}\} \to \{\text{trees with vertex set } S\},$$
$$f \mapsto (S, \{\{i, f(i)\} \mid i \in [n-1]\})$$

is well-defined & bijective.

[Example: (for $n = 7$)

f	$A(f)$
	

.]

(Idea: A removes the arrowheads from the cycle digraph of f and the "degenerate" edge $\{n, n\}$.)

This is shown using some basic graph theory.
(E.g., the inverse map A^{-1} sends a tree G to the map $f : S \to S$ sending n to n and each $i \in [n-1]$ to the neighbor of i that is closest to n in G.) $\square$

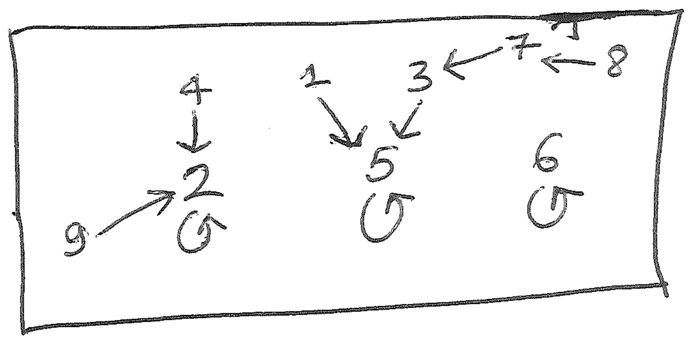
Proof of Thm. 5.2. (Fourth proof in Ch. 33 of [Aigner/Ziegler, "Proofs from the Book", 6th edition 2018])

For each $T \subseteq S$, we set

$$\Phi(T) := \{ f : S \to S \mid f^n(S) \subseteq T \text{ and } T \subseteq \operatorname{Fix} f \},$$

where $\operatorname{Fix} f := \{ x \in S \mid f(x) = x \}$.

[Example: If $n = 9$ and $T = \{2, 5, 6\}$, then the map f with cycle digraph



is in $\Phi(T)$.]

Note that n & S are fixed. Thus, $|\Phi(T)|$ depends only on $|T|$ (since we can relabel elements).

Thus, there are numbers $g_0, g_1, \ldots, g_n$ such that

(42) $\quad g_i = |\Phi(T)|$ for any i-element subset T of S.

Note: $g_0 = 0$ (since $n \geq 1 \Rightarrow S \neq \emptyset$, so $f^n(S) \neq \emptyset$ $\forall f : S \to S$, and thus $f^n(S) \subseteq \emptyset$ is impossible).

Also,

(43) $g_n = 1$,

since (42) (for $T = S$) yields $g_n = |\Phi(S)| = |\{id\}| = 1$.

We want to prove: $g_1 = n^{n-2}$.

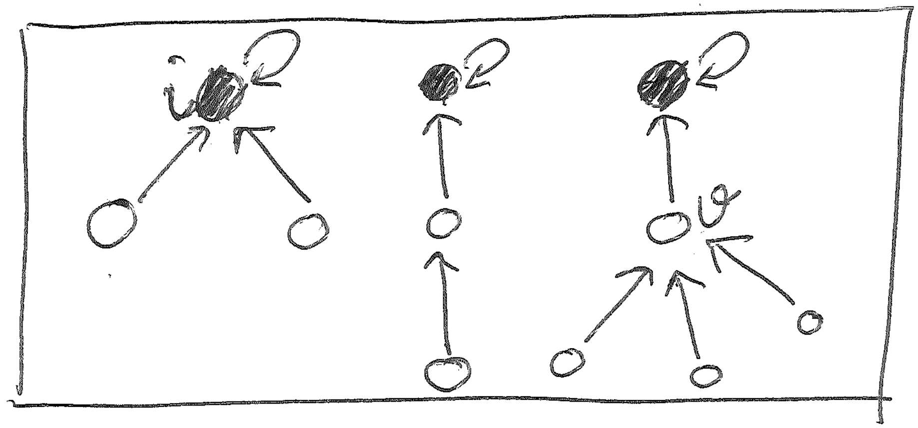
Idea: express g_k through g_{k+1}, then proceed recursively.

So fix $k \in \{0, 1, \ldots, n-1\}$. Fix a $(k+1)$-element subset T of S. Let

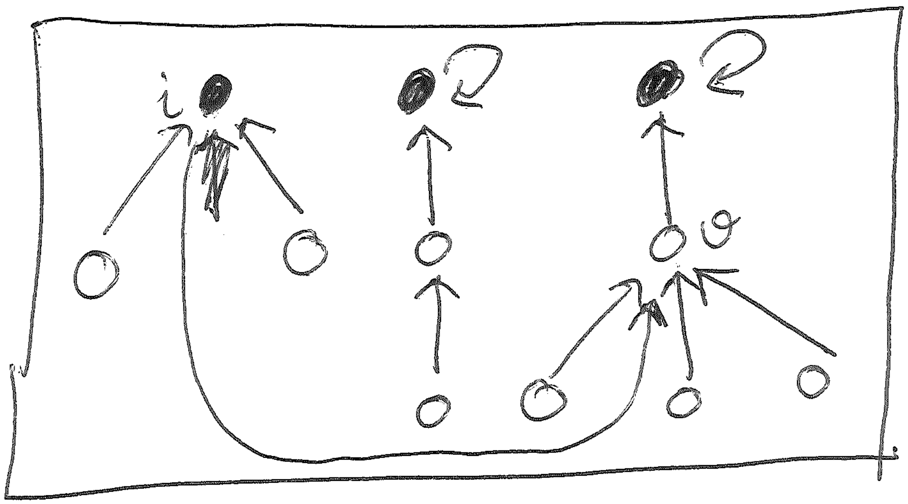
$$A := \{(f, v, i) \mid f \in \Phi(T) \text{ and } v \in S \text{ and } i \in T \setminus \{f^n(v)\}\}$$

and $B := \{(\overline{f}, i) \mid i \in T \text{ and } \overline{f} \in \Phi(T \setminus \{i\})\}$.

[Example: ($T = \{\text{black circles}\}$)



$\in A$,



$\in B$.]

Note that (42) yields

(44) $$|A| = g_{k+1}\, n\, k$$

(because we have g_{k+1} choices for f, then n choices for v, then k choices for i)

and

(45) $$|B| = (k+1)\, g_k$$ (because we have $k+1$ choices for i & then g_k choices for $\overline{f}$).

Now, define two bijections $\alpha : B \to A$ and $\beta : A \to B$ as follows:

- If $(\overline{f}, i) \in B$, then $\alpha(\overline{f}, i) = (f, v, i) \in A$, where $v := \overline{f}(i)$ and $f : S \to S$ is defined by

$$f(x) = \begin{cases} \overline{f}(x), & \text{if } x \neq i; \\ i, & \text{if } x = i \end{cases} \qquad \forall x \in S.$$

- If $(f, v, i) \in A$, then $\beta(f, v, i) = (\overline{f}, i) \in B$, where $\overline{f} : S \to S$ is defined by

$$\overline{f}(x) = \begin{cases} f(x), & \text{if } x \neq i \\ v, & \text{if } x = i \end{cases} \qquad \forall x \in S.$$

It is not hard to check that α & β are well-defined (the well-definedness of β relies on $f^n(v) \neq i$) and bijective. $\Longrightarrow$ $|A| = |B|$.

By (44) and (45), this rewrites as $g_{k+1} nk = (k+1) g_k$.

Hence,
$$g_k = n \cdot \frac{k}{k+1} g_{k+1}.$$

Now, this holds for <u>every</u> $k \in \{0, 1, \ldots, n-1\}$.

Hence, each $k \in \{0, 1, \ldots, n\}$ satisfies
$$g_k = n \cdot \frac{k}{k+1} g_{k+1} = n \cdot \frac{k}{k+1} \cdot n \cdot \frac{k+1}{k+2} g_{k+2}$$
$$= n \cdot \frac{k}{k+1} \cdot n \cdot \frac{k+1}{k+2} \cdot n \cdot \frac{k+2}{k+3} g_{k+3} = \cdots\cdots$$
$$= n \cdot \frac{k}{\cancel{k+1}} \cdot n \cdot \frac{\cancel{k+1}}{\cancel{k+2}} \cdot n \cdot \frac{\cancel{k+2}}{\cancel{k+3}} \cdot \ldots \cdot n \cdot \frac{\cancel{n-1}}{n} \cdot \underbrace{g_n}_{\overset{(43)}{=} 1}$$
$$= n^{n-k} \cdot \frac{k}{n} = k n^{n-k-1}.$$

Applied to $k=1$, this yields $g_1 = 1 n^{n-1-1} = n^{n-2}$.

Now, (# of maps $f : S \to S$ such that $f^n(S) = \{n\}$)

$= |\Phi(\{n\})| = g_1$ (by (42), since $|\{n\}| = 1$)

$= n^{n-2}$. $\square$

(For different proofs, see [Aigner/Ziegler], but also [Galvin].)

Theorem 5.4 (the Matrix-Tree Theorem, in map form).

Let $n \geq 1$. Let $S = [n]$.

For any distinct $i, j \in [n]$, let $a_{i,j}$ be a number (or an indeterminate).

For each $i \in [n]$, let $b_i = a_{i,1} + a_{i,2} + \cdots + \widehat{a_{i,i}} + \cdots + a_{i,n}$ (where the hat over the "$a_{i,i}$" means "don't include $a_{i,i}$").

Let L be the $(n-1)\times(n-1)$-matrix whose (i,j)-th entry is

$$\begin{cases} b_i, & \text{if } i=j; \\ -a_{i,j}, & \text{if } i\neq j \end{cases} \qquad \forall i,j\in[n-1].$$

Then,

$$\sum_{\substack{f:S\to S;\\ f^n(S)=\{n\}}} \prod_{i\in[n-1]} a_{i,f(i)} = \det L.$$

For a proof, see [Zeilberger, A Combinatorial approach to Matrix Algebra].

You can derive Thm. 5.2 from Thm. 5.4, using

$$\det\begin{pmatrix} n-1 & -1 & -1 & \cdots & -1 \\ -1 & n-1 & -1 & \cdots & -1 \\ -1 & -1 & n-1 & & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & n-1 \end{pmatrix} = n^{n-2}.$$