Remark: Let p be a prime number.

Then, ~~[illegible]~~ $p \mid \left[{p \atop k} \right]$ for all $k \in \{2, 3, \ldots, p-1\}$.

(Cf. remark after Prop. 3.12.)

Proof outline. Similar to the Remark after Prop. 3.12:

For any $\sigma \in S_n$, we define the shift of σ by replacing $1, 2, \ldots, n-1, n$ by $2, 3, \ldots, n, 1$ in the DCD of σ.

Thus, if $n = 7$ and $\sigma = cyc_{3,4,7} \circ cyc_{2,6} \circ cyc_1 \circ cyc_5$,

then $\text{shift}(\sigma) = cyc_{4,5,1} \circ cyc_{3,7} \circ cyc_2 \circ cyc_6$.

Again, define equivalence relation $\overset{\text{shift}}{\sim}$.

~~Again, its class~~ If σ has k cycles, then so does $\text{shift}(\sigma)$.

~~The shift-equivalen~~ Now, let $n = p$ be a prime, and $k \in \{2, 3, \ldots, p-1\}$. The $\overset{\text{shift}}{\sim}$-equivalence classes of permut.s $\sigma \in S_p$ with k cycles have size p. (Again, this follows from algebra, once you know that $\sigma \neq \text{shift}(\sigma)$ $\forall \sigma \in S_p$

with k cycles. The latter fact can be proven by contradiction: If we had $\sigma = \operatorname{shift}(\sigma)$, then $\forall i \in [n]$, where $n=p$, the number $i^+ := \begin{cases} i+1, & \text{if } i<n; \\ 1, & \text{if } i=n \end{cases}$ would lie in a σ-cycle of the same length as the one i lies in $\Longrightarrow$ all the numbers $1, 2, \ldots, n$ lie in cycles of the same length $\Longrightarrow$ all cycles have the same length $\Longrightarrow$ all cycles have length 1 or p (since $n=p$ is prime) $\Longrightarrow$ there are p or 1 cycles; but this contradicts $k \in \{2, 3, \ldots, p\}$.) $\Longrightarrow p \mid \left[{p \atop k}\right]$. $\square$

<u>A corollary:</u> The two polynomials $x^{\underline{p}} = x(x-1)\cdots(x-p+1)$ and $x^p - x$ are congruent modulo p (where p is prime), in the sense that corresponding coefficients are congruent (by Prop. 4.15 (b)).

Note that their values at $x \in \mathbb{Z}$ are always $\equiv 0 \bmod p$, but they are not $\equiv 0 \bmod p$ as polynomials.

4.7. Eulerian numbers

Def. Let $n, k \in \mathbb{N}$. Then, $\left\langle {n \atop k} \right\rangle$ denotes the # of $\sigma \in S_n$ having exactly k descents.

(Recall: a descent of a $\sigma \in S_n$ is an $i \in [n-1]$ such that $\sigma(i) > \sigma(i+1)$.)

These $\left\langle {n \atop k} \right\rangle$ are called Eulerian numbers.

Prop. 4.16. $\left\langle {n \atop k} \right\rangle = \left\langle {n \atop n-1-k} \right\rangle$.

Proof. HW #4 exercise 3. $\square$

Prop. 4.17. (a) $\left\langle {n \atop 0} \right\rangle = 1 \quad \forall n \in \mathbb{N}$.

(b) $\left\langle {n \atop k} \right\rangle = 0 \quad \forall k \geq n$.

(c) $\left\langle {0 \atop k} \right\rangle = [k=0] \quad \forall k \in \mathbb{N}$.

(d) $\left\langle {n \atop k} \right\rangle = (k+1) \left\langle {n-1 \atop k} \right\rangle + (n-k) \left\langle {n-1 \atop k-1} \right\rangle$

$\forall$ positive integers n, k.

Proof. (a)–(c) are easy. For (d), see MT #2. $\square$

<u>Thm. 4.18.</u> Let $n \in \mathbb{N}$ and $k \in \mathbb{N}$. ~~Then~~ Assume $n > 0$. Then,

$$\left\langle {n \atop k} \right\rangle = \sum_{i=0}^{k+1} (-1)^i \binom{n+1}{i} (k+1-i)^n.$$

<u>Proof (Stanley/Thomas, 2003):</u> A <u>k-barpe</u> (short for: k-barred permutation) shall mean an n-tuple containing each of the numbers $1, 2, \ldots, n$ exactly once (i.e., the one-line notation of some $\sigma \in S_n$), with (altogether) k <u>bars</u> placed between some of its entries (or at the very start, or at the very end), subdividing the n-tuple into $k+1$ (possibly empty) <u>compartments</u>, with the property that the entries in each compartment are ~~...~~ in increasing order.

Examples (for $n = 8$):

- a 5-barpe: $5 \mid 1\ 3 \mid 8 \mid\mid 2\ 4\ 6 \mid 7$
- a 5-barpe: $5 \mid 1\ 3\ \ 8 \mid 2\ 4 \mid\mid\mid 6\ \ 7$
- a 5-barpe: $\mid\mid 5 \mid 1\ 3\ 8 \mid 2\ 4\ \ 6\ 7 \mid$

- not a 5-barpe: $\underbrace{5\ 1}_{\text{not increasing}} | 3\ \underbrace{8\ \ 2}_{\text{not increasing}} || 4 | 6 | 7$

Note:
- We omit commas & parentheses.
- Several bars can be placed between 2 consecutive entries (or at start, or at end).
- We have

(36) $$(\#\text{ of } k\text{-barpes}) = (k+1)^n,$$

since a k-barpe can be constructed by deciding, for each $i \in [n]$, which of the $k+1$ compartments we place i in.

If B is a k-barpe, then:
- a <u>wall</u> of B means a bar of B that is not immediately followed by another bar (e.g., $5|13|8||246|7$). not a wall
- a <u>useless wall</u> of B means a wall of B such that removing that wall leaves a $(k-1)$-barpe (e.g., $5|13|8||246|7$). useless walls

Then, $\left\langle {n \atop k} \right\rangle = (\#$ of k-barpes with no useless walls)

(since k-barpes with no useless walls are in bijection with permutations $\sigma \in S_n$ having k descents, because each non-useless wall marks a descent).

Compute the RHS using Thm. 2.23 (PIE).

Let $U = \{\text{all } k\text{-barpes}\}$ (note: n is fixed).

For each $i \in [n+1]$, let A_i be the set of all k-barpes which have a useless wall between the $(i-1)$-st & i-th entries (if $i = 1$, this means at front; if $i = n+1$, this means at end). Then,

$$\left\langle {n \atop k} \right\rangle = (\#\text{ of } k\text{-barpes with no useless walls})$$
$$= \left| U \setminus \bigcup_{i=1}^{n+1} A_i \right|$$

Theorem 2.23(b) (applied to $n+1$ instead of n)

$$\sum_{I \subseteq [n+1]} (-1)^{|I|} \underbrace{\left| \bigcap_{i \in I} A_i \right|}_{\substack{= |\{k\text{-barpes } B \text{ such that } \forall i \in I, \text{ the } k\text{-barpe } B \text{ has a useless wall between its } (i-1)\text{-st } \& \text{ } i\text{-th entries}\}| \\ = |\{(k-|I|)\text{-barpes}\}| \\ = (\# \text{ of } (k-|I|)\text{-barpes}) \\ \overset{(36)}{=} \begin{cases} (k-|I|+1)^n, & \text{if } |I| \leq k \\ 0, & \text{if } |I| > k \end{cases}}}$$

by bijection: just remove the useless walls at those positions

$$= \underbrace{\sum_{I \subseteq [n+1]}}_{=\sum_{i=0}^{n+1} \sum_{\substack{I \subseteq [n+1];\\ |I|=i}}} (-1)^{|I|} \begin{cases} (k-|I|+1)^n, & \text{if } |I| \leq k \\ 0, & \text{if } |I| > k \end{cases}$$

$$= \sum_{i=0}^{n+1} \underbrace{\sum_{\substack{I \subseteq [n+1];\\ |I|=i}} (-1)^i \begin{cases} (k-i+1)^n, & \text{if } i \leq k \\ 0, & \text{if } i > k \end{cases}}_{= \binom{n+1}{i} (-1)^i \begin{cases} (k-i+1)^n, & \text{if } i \leq k \\ 0, & \text{if } i > k \end{cases}}$$

$$= \sum_{i=0}^{n+1} \binom{n+1}{i} (-1)^i \begin{cases} (k-i+1)^n, & \text{if } i \leq k \\ 0, & \text{if } i > k \end{cases}$$

$$\overset{\circ}{=} \sum_{i=0}^{k} \binom{n+1}{i} (-1)^i (k-i+1)^n \overset{\circ}{=} \sum_{i=0}^{k+1} \binom{n+1}{i} (-1)^i (k-i+1)^n .$$

$\square$

4.8. ~~[illegible]~~ How many fixed points does an average $\sigma \in S_n$ have?

Exercise: Let $n \in \mathbb{N}$.

For any $f : [n] \to [n]$, let $\operatorname{Fix} f := \{ i \in [n] \mid f(i) = i \}$ be the set of fixed points of f.

Find $\displaystyle\sum_{w \in S_n} |\operatorname{Fix} w|$.

Answer: $n!$.

Proof:

$$\sum_{w \in S_n} \underbrace{|\operatorname{Fix} w|}_{= \sum_{i \in [n]} [i \in \operatorname{Fix} w]} = \sum_{w \in S_n} \sum_{i \in [n]} [i \in \operatorname{Fix} w]$$

$$= \sum_{i \in [n]} \underbrace{\sum_{w \in S_n} [i \in \operatorname{Fix} w]}_{\substack{= (\# \text{ of } w \in S_n \mid i \in \operatorname{Fix} w) \\ \overset{\text{obvious bijection}}{=} (\# \text{ of permutations of } [n] \setminus \{i\}) \\ = (n-1)!}} = \sum_{i \in [n]} (n-1)! = n \cdot (n-1)! = n!.$$

<u>Exercise:</u> Let $n \in \mathbb{N}$.

(a) For each $k \in \{0, 1, \dots, n\}$, prove that $\sum_{w \in S_n} \binom{|\mathrm{Fix}\, w|}{k} = \frac{n!}{k!}$.

(b) Find $\sum_{\substack{w \in S_n; \\ w \text{ is even}}} |\mathrm{Fix}\, w|$.

4.9. The Lehmer code

<u>Def.</u> Let $n \in \mathbb{N}$.

(a) If $i \in [n]$ and $\sigma \in S_n$, then $\ell_i(\sigma) := (\#\text{ of } j > i \text{ such that } \sigma(i) < \sigma(j))$.

(b) If $\sigma \in S_n$, then the n-tuple $(\ell_1(\sigma), \ell_2(\sigma), \dots, \ell_n(\sigma))$ is called the <u>Lehmer code</u> of σ.

<u>Def.</u> Let $m \in \mathbb{Z}$. Then, $[m]_0 = \{0, 1, \dots, m\}$.

<u>Def.</u> Let $(a_1, a_2, \dots, a_n)$ and $(b_1, b_2, \dots, b_n)$ be two n-tuples of integers. We say that $(a_1, a_2, \dots, a_n) \underset{\text{lex}}{<} (b_1, b_2, \dots, b_n)$ if & only if $\exists\, k \in [n]$ such that $a_k < b_k$, but $a_i = b_i \ \forall i < k$.

<u>Examples:</u> $(3,4,6) <_{\text{lex}} (3,5,2)$.

$(4,1,2,5) <_{\text{lex}} (4,1,3,0)$.

$(1,1,0,1) <_{\text{lex}} (2,1,0,0)$.

<u>Thm. 4.19.</u> Let $n \in \mathbb{N}$.

(a) For each $\sigma \in S_n$, we have $\ell(\sigma) = \ell_1(\sigma) + \ell_2(\sigma) + \cdots + \ell_n(\sigma)$.

(Note: $\ell_n(\sigma) = 0$.)

(b) If $\sigma \in S_n$ and $\tau \in S_n$ satisfy

$$(\sigma(1), \sigma(2), \ldots, \sigma(n)) <_{\text{lex}} (\tau(1), \tau(2), \ldots, \tau(n)),$$

then

$$(\ell_1(\sigma), \ell_2(\sigma), \ldots, \ell_n(\sigma)) <_{\text{lex}} (\ell_1(\tau), \ell_2(\tau), \ldots, \ell_n(\tau)).$$

(c) The map

$$L : S_n \to [n-1]_0 \times [n-2]_0 \times \cdots \times [n-n]_0$$
$$\sigma \mapsto (\ell_1(\sigma), \ell_2(\sigma), \ldots, \ell_n(\sigma))$$

is well-defined & bijective.

Proof. [detnotes, §5.8] or UMN Fall 2017 Math 4990 HW #7 exercise 5. □

Proof of Proposition 4.9.

$$\sum_{w \in S_n} x^{\ell(w)} = \sum_{\sigma \in S_n} x^{\ell(\sigma)} \underset{}{\overset{\text{Thm. 4.19(a)}}{=\!=}} \sum_{\sigma \in S_n} x^{\ell_1(\sigma) + \ell_2(\sigma) + \dots + \ell_n(\sigma)}$$

$$\overset{\text{Thm. 4.19(c)}}{=\!=} \sum_{(i_1, i_2, \dots, i_n) \in [n-1]_0 \times [n-2]_0 \times \dots \times [n-n]_0} \underbrace{x^{i_1 + i_2 + \dots + i_n}}_{= x^{i_1} x^{i_2} \dots x^{i_n}}$$

$$= \left(\sum_{i_1 \in [n-1]_0} x^{i_1} \right) \left(\sum_{i_2 \in [n-2]_0} x^{i_2} \right) \cdots \left(\sum_{i_n \in [n-n]_0} x^{i_n} \right)$$

$$= \left(1 + x + x^2 + \dots + x^{n-1}\right)\left(1 + x + x^2 + \dots + x^{n-2}\right) \cdots (1)$$

$$= \prod_{i=0}^{n-1} \left(1 + x + x^2 + \dots + x^{i}\right) = \prod_{i=1}^{n-1} \left(1 + x + x^2 + \dots + x^{i}\right). \qquad \square$$

Thm. 4.20. Let $n \in \mathbb{N}$. Let $\sigma \in S_n$.

For each $i \in [n]$, let $a_i = \operatorname{cyc}_{i', i'-1, \ldots, i} = s_{i'-1} s_{i'-2} \cdots s_i$,

where $i' = i + \ell_i(\sigma)$.

Then, $\sigma = a_1 a_2 \cdots a_n$. (Note: $a_n = \mathrm{id}$.)

Proof. MT#2 ? $\square$

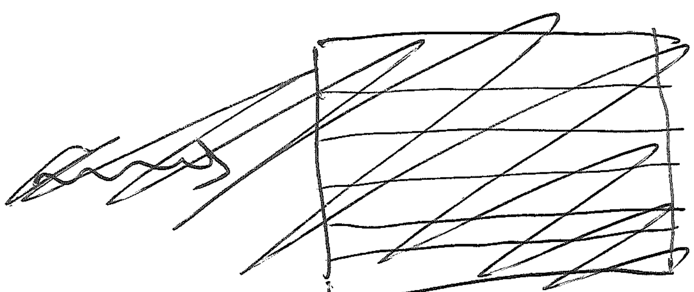
Note that Thm. 4.20 yields a new, explicit proof of Thm. 4.6.

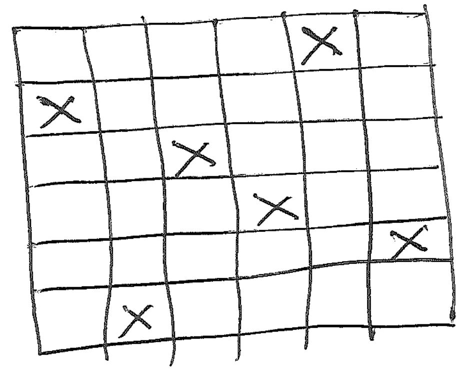
Remark. Let $n \in \mathbb{N}$ and $\sigma \in S_n$. Here is a visual way to think of the Lehmer code:

Draw an (empty) $n \times n$-matrix.

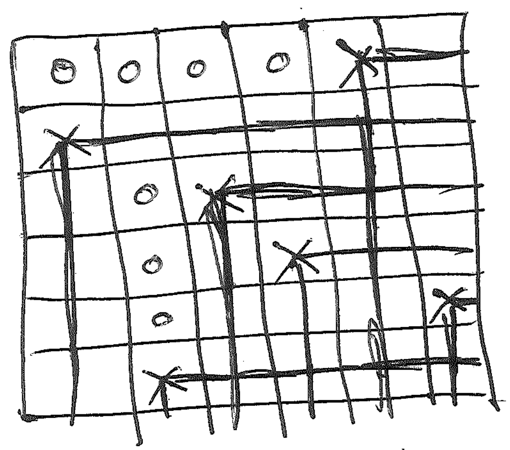
For each $i \in [n]$, put an X into its cell $(i, \sigma(i))$.

Running example: $n = 6$ and $\sigma = [5, 1, 3, 4, 6, 2]$ (in 1-line notation)



$\rightsquigarrow$ 

From each X, draw a vertical line downwards & a horizontal line eastwards:



Draw a o into each cell that does not fall on any line.

This is called the <u>Rothe diagram</u> of σ.

Explicitly: a cell (i,j) has a $\circ$ in it

$\Longleftrightarrow$ $\sigma(i) > j$ & $\sigma^{-1}(j) > i$.

In other words, a cell $(i, \sigma(j))$ has a $\circ$ in it

$\Longleftrightarrow$ $\sigma(i) > \sigma(j)$ & $j > i$

$\Longleftrightarrow$ (i,j) is an inversion of σ.

Thus, $\ell(\sigma) = (\#$ of inversions of $\sigma) = (\#$ of $\circ$'s$)$.

Furthermore, $\ell_i(\sigma) = (\#$ of $\circ$'s in row $i)$ $\forall i \in [n]$.

Finally, let us label the $\circ$'s as follows:

For each $i \in [n]$, label the $\circ$'s in row i from right to left by $s_i, s_{i+1}, s_{i+2}, \ldots, s_{i'-1}$ where $i' = i + \ell_i(\sigma)$.

Then, read the matrix row by row, starting with the top row, from left to right. $\Longrightarrow$ This gets you a product of simples that equals σ (by Thm. 4.20).

4.10. Permutation patterns (a glimpse)

Def. Let $m, n \in \mathbb{N}$. Let $\tau \in S_m$ and $\sigma \in S_n$.

A τ-pattern in σ means a subsequence of $(\sigma(1), \ldots, \sigma(n))$ whose entries come in the same order as $\tau(1), \ldots, \tau(m)$.

(Rigorously: it means an m-tuple $(i_1 < i_2 < \ldots < i_m)$ of numbers in $[n]$ such that $\forall x, y \in [m]$, we have

$$(\sigma(i_x) < \sigma(i_y)) \iff \tau(x) < \tau(y)).$$

We say σ is τ-avoiding if $\nexists$ τ-pattern in σ.

Example: • A $\underbrace{21}_{=[2,1]}$-pattern in a $\sigma \in S_n$ is an inversion of σ.

Thus, the only 21-avoiding $\sigma \in S_n$ is id.

• A 12-pattern in a $\sigma \in S_n$ is a non-inversion (= a pair (i, j) with $1 \leq i < j \leq n$ and $\sigma(i) < \sigma(j)$) of σ.

- A 123-pattern in a $\sigma \in S_n$ is a triple $(i<j<k)$ with $\sigma(i) < \sigma(j) < \sigma(k)$.

- A 231-pattern in a $\sigma \in S_n$ is a triple $(i<j<k)$ with $\sigma(k) < \sigma(i) < \sigma(j)$.

- Is $[2, 1, 5, 3, 4]$
  - 123-avoiding? 2 1 5 3 4, so no.
  - 132-avoiding? 2 1 5 3 4, so no.
  - 231-avoiding? yes!
  - 321-avoiding? 2 1 5 3 4, so yes.

Thm. 4.21. Let $n \in \mathbb{N}$. Then,

(# of 123-avoiding perms $\sigma \in S_n$)

= (# of 132 – —— // ——)

= (# of 213 – —— // ——)

= (# of 231 – —— // ——)

= (# of 312 – —— // ——)

=

$= (\#\text{ of } 321- \text{———} // \text{————})$

$$= \frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n-1}$$

(a so-called Catalan number).

(For more about patterns, see:

- [Kitaev: Patterns in Permutations and Words],
- [Bóna: Combinatorics of Permutations].)