Def. Let $n \in \mathbb{N}$. A permutation $\sigma \in S_n$ is said to be <u>even</u> if $(-1)^\sigma = 1$ (i.e., if $\ell(\sigma)$ is even), and <u>odd</u> if $(-1)^\sigma = -1$ (i.e., if $\ell(\sigma)$ is odd).

Cor. 4.11. Let $n \geq 2$. Then,

$$(\# \text{ of even } \sigma \in S_n) = (\# \text{ of odd } \sigma \in S_n) = n!/2.$$

Proof. The map

$$\{\text{even } \sigma \in S_n\} \longrightarrow \{\text{odd } \sigma \in S_n\}, \quad \sigma \longmapsto \sigma \circ s_1$$

is a bijection. $\square$

Example: The 15-game

1	2	3	4
5	6	7	8
9	10	11	12
13	15	14	

$\longrightarrow$ via swaps of the empty cell with a neighboring square ("slides")

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

Claim: This is impossible.

Proof sketch:

(a) Consider the 3×3-analogue

$$\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5&6\\\hline 8&7&\\\hline\end{array} \xrightarrow{\text{slides}} \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5&6\\\hline 7&8&\\\hline\end{array}$$

Claim: This is also impossible.

Proof: For each position P, let σ_P be the permutation of $[8]$ whose one-line notation is what you get if you read P row by row left-to-right.

$$P = \begin{array}{|c|c|c|}\hline a&b&c\\\hline d&e&\\\hline f&g&h\\\hline\end{array} \rightsquigarrow \sigma_P = [a,b,c,d,e,f,g,h].$$

Now, a slide changes σ_P either not at all or by multiplying it with a $\underbrace{\text{3-cycle}}_{=\text{cyc}_{p,q,r} \text{ for some } p,q,r}$ $(\sigma_P \rightsquigarrow \sigma_P \circ \text{cyc}_{*,*,*})$.

$\Rightarrow$ The sign of σ_p never changes (since 3-cycles have sign 1).

Thus, the 3×3-problem is unsolvable (since the initial ~~pos~~ position & the target position have σ_p's of different sign).

(b) Now to the 4×4-version.

The sign of σ_p is no longer invariant.

Instead,

$$(-1)^{\sigma_p} \cdot (-1)^{\text{which row the empty square lies in}}$$

is invariant. $\square$

Which positions can be reached from one another?

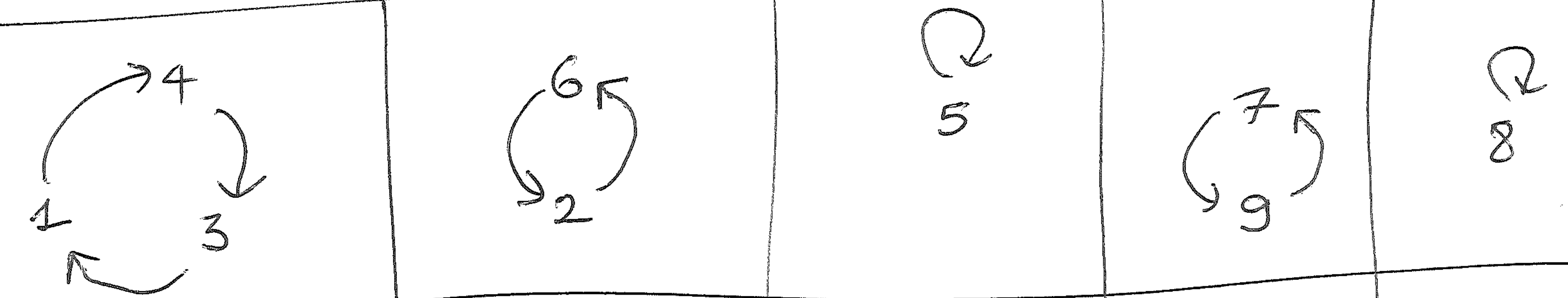
See Wikipedia.

Rmk. For $n \in \mathbb{N}$, we denote the set of all even permutations $\sigma \in S_n$ as the alternating group A_n.

4.5. Cycle decomposition

Example: Let $\sigma = [4, 6, 1, 3, 5, 2, 9, 8, 7] \in S_9$.

Its cycle digraph is



$\mathrm{cyc}_{1,4,3}$	$\mathrm{cyc}_{2,6}$	cyc_{5}	$\mathrm{cyc}_{7,9}$	cyc_{8}

$$\Longrightarrow \quad \sigma = \mathrm{cyc}_{1,4,3} \circ \mathrm{cyc}_{2,6} \circ \mathrm{cyc}_{5} \circ \mathrm{cyc}_{7,9} \circ \mathrm{cyc}_{8}$$

(since the LHS and the RHS act on each $h \in [9]$ in the same way).

Similarly, each $\pi \in S_n$ can be written as a composition of cycles $\mathrm{cyc}_{i_1, i_2, \ldots, i_k}$ with each element of $[n]$ appearing in exactly 1 of these cycles.

This representation of π is unique up to

- swapping the cycles;
- rotating each cycle ($\mathrm{cyc}_{i_1, i_2, \ldots, i_k} = \mathrm{cyc}_{i_2, i_3, \ldots, i_k, i_1}$
$= \mathrm{cyc}_{i_3, i_4, \ldots, i_k, i_1, i_2} = \ldots$).

__Thm. 4.12.__ Let σ be a permutation of a finite set X.

(a) There is a list

$$\begin{aligned}
((&a_{1,1}, a_{1,2}, \ldots, a_{1,n_1}), \\
(&a_{2,1}, a_{2,2}, \ldots, a_{2,n_2}), \\
&\ldots \\
(&a_{k,1}, a_{k,2}, \ldots, a_{k,n_k}))
\end{aligned}$$

of lists of elements of X, such that:

- each element of X appears exactly once in
$a_{1,1}, a_{1,2}, \ldots, a_{k,n_k}$;

- $\sigma = \mathrm{cyc}_{a_{1,1}, a_{1,2}, \dots, a_{1,n_1}} \circ \mathrm{cyc}_{a_{2,1}, a_{2,2}, \dots, a_{2,n_2}} \circ \dots \circ \mathrm{cyc}_{a_{k,1}, a_{k,2}, \dots, a_{k,n_k}}$.

Such a list is called a <u>disjoint cycle decomposition (DCD)</u> of σ.

(b) Any two DCDs of σ can be obtained from each other by swapping sublists & rotating each sublist.

(c) If we additionally require

- $a_{1,1} > a_{2,1} > \dots > a_{k,1}$;
- $a_{i,1} \leqslant a_{i,p}$ $\forall i \in [k]$ $\forall p \in [n_i]$,

then this list is unique.

<u>Proof outline.</u> Follow the example above.

<u>Existence:</u> Fall 2017 Math 4990 hw #7 exercise ~~7~~ (e) + (d).

<u>Uniqueness:</u> Any DCD of σ must have the property that

whatever sublist contains $i \in [n]$ must also contain $\sigma(i)$ as its next entry (or first entry, if i was its last entry). This uniquely determines the sublists (up to swapping & rotating).

For an alternative proof, see [Goodman, "Algebra: Abstract & Concrete", Theorem 1.5.3 & its proof]. $\square$

<u>Def.</u> Let X be a set. Let σ be a permutation of X. The <u>cycles</u> of σ are the sublists in the DCD of σ. (They are defined only up to rotation.)

<u>Prop. 4.13.</u> Let $n \in \mathbb{N}$. Let $\sigma \in S_n$ have exactly k $\underbrace{\text{cycles}}_{\text{this includes 1-cycles}}$. Then,

$$(-1)^\sigma = (-1)^{n-k}.$$

Proof. Let notations be as in Thm. 4.12.

Then $\sigma = (\text{an } n_1\text{-cycle}) \circ (\text{an } n_2\text{-cycle}) \circ \cdots \circ (\text{an } n_k\text{-cycle})$.

Thus, $(-1)^{\sigma} = (-1)^{(\text{an } n_1\text{-cycle})} (-1)^{(\text{an } n_2\text{-cycle})} \cdots (-1)^{(\text{an } n_k\text{-cycle})}$

(by Thm. 4.10 (f))

$= (-1)^{n_1 - 1} (-1)^{n_2 - 1} \cdots (-1)^{n_k - 1}$

(since Thm. 4.10 (c) yields $(-1)^{(\text{a } p\text{-cycle})} = (-1)^{p-1}$)

$= (-1)^{(n_1 - 1) + (n_2 - 1) + \cdots + (n_k - 1)} = (-1)^{n_1 + n_2 + \cdots + n_k - k}$

$= (-1)^{n-k}$,

since $n_1 + n_2 + \cdots + n_k = n$ (since $a_{1,1}, a_{1,2}, \ldots, a_{k, n_k}$ contain each $i \in [n]$ exactly once). $\square$

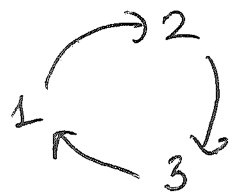
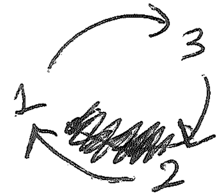
Exercise. Let n be a positive integer.

How many n-cycles exist in S_n?

Examples:

For $n=2$, the only 2-cycle in S_2 is $cyc_{1,2} = s_1$.

For $n=3$, the only 3-cycles in S_3 are

$cyc_{1,2,3}$	$cyc_{1,3,2}$
	

There is a bijection

$$\{\text{permutations of } \{2,3,\ldots,n\}\} \longrightarrow \{n\text{-cycles in } S_n\},$$

$$\sigma \longmapsto cyc_{1,\sigma(2),\sigma(3),\ldots,\sigma(n)}.$$

$\Longrightarrow$ The # of n-cycles in S_n is $(n-1)!$. $\square$

Exercise. More generally: Fix $n \in \mathbb{N}$, and fix n nonnegative integers $m_1, m_2, \ldots, m_n$ with $1m_1 + 2m_2 + \cdots + nm_n = n$.

How many permutations $\sigma \in S_n$ have ~~exactly~~

- exactly m_1 1-cycles,
- exactly m_2 2-cycles,

  ⋮

- exactly m_n n-cycles ?

$\Big\}$ in their DCD

(Example: The σ in the first example of §4.5 has
- exactly 2 1-cycles;
- exactly 2 2-cycles;
- exactly 1 3-cycle;
- exactly 0 4-cycles;
- exactly 0 5-cycles.)

Answer:
$$\frac{n!}{m_1!\, m_2! \cdots m_n!\; 1^{m_1} 2^{m_2} \cdots n^{m_n}}.$$

Proof idea: Construct such a σ by choosing a DCD for σ

in which the m_1 1-cycles come first, the m_2 2-cycles come after them, & so on.

There are $n!$ ways to do this, because $1m_1 + 2m_2 + \dots + nm_n = n$.

But we are getting every σ exactly $m_1! m_2! \cdots m_n! 1^{m_1} 2^{m_2} \cdots n^{m_n}$ many times, since our cycles can be swapped and rotated without changing the σ (see Thm. 4.12 (b)).

<u>Corollary:</u> $m_1! m_2! \cdots m_n! \, 1^{m_1} 2^{m_2} \cdots n^{m_n} \mid n!$ & $\forall m_1, m_2, \dots, m_n \in \mathbb{N}$ such that $1m_1 + 2m_2 + \dots + nm_n = n$.

<u>Corollary:</u>

$$\sum_{\substack{(m_1, m_2, \dots, m_n) \in \mathbb{N}^n; \\ 1m_1 + 2m_2 + \dots + nm_n = n}} \frac{1}{m_1! m_2! \cdots m_n! \, 1^{m_1} 2^{m_2} \cdots n^{m_n}} = 1.$$

4.6. Stirling numbers of the 1st kind

<u>Def.</u> Let $n, k \in \mathbb{N}$. Then, $\begin{bmatrix} n \\ k \end{bmatrix}$ denotes the # of $\sigma \in S_n$ having exactly k cycles (in their DCD).

These numbers $\left[{n \atop k}\right]$ are called the unsigned Stirling numbers of the 1st kind.

(The signed ones are $(-1)^{n-k}\left[{n \atop k}\right]$.)

Prop. 4.14. (a) $\left[{n \atop 0}\right] = [n=0]$ $\forall n \in \mathbb{N}$.

(b) $\left[{n \atop n}\right] = 1$ $\forall n \in \mathbb{N}$.

(c) $\left[{n \atop k}\right] = 0$ $\forall k > n$.

(d) $\left[{n \atop 1}\right] = (n-1)!$ $\forall n > 0$.

(e) $\left[{n \atop k}\right] = (n-1)\left[{n-1 \atop k}\right] + \left[{n-1 \atop k-1}\right]$ $\forall n > 0,\ k > 0$.

Proof. (a), (b), (c): easy.

(d) This says: (# of n-cycles in S_n) $= (n-1)!$. Proven above.

(e) Let us say that a permutation $\sigma \in S_n$ having k cycles is

- <u>red</u> if $\sigma(n) \neq n$;
- <u>green</u> if $\sigma(n) = n$.

There is a bijection

$$\{\text{green } \sigma \in S_n\} \to \{\tau \in S_{n-1} \text{ having } k-1 \text{ cycles}\},$$

$$\sigma \mapsto \underbrace{\sigma|_{[n-1]}^{[n-1]}}.$$

(This means the map $[n-1] \to [n-1]$ obtained by restricting σ.)

Thus,

$$(\# \text{ of green } \sigma \in S_n) = \left[{n-1 \atop k-1}\right].$$

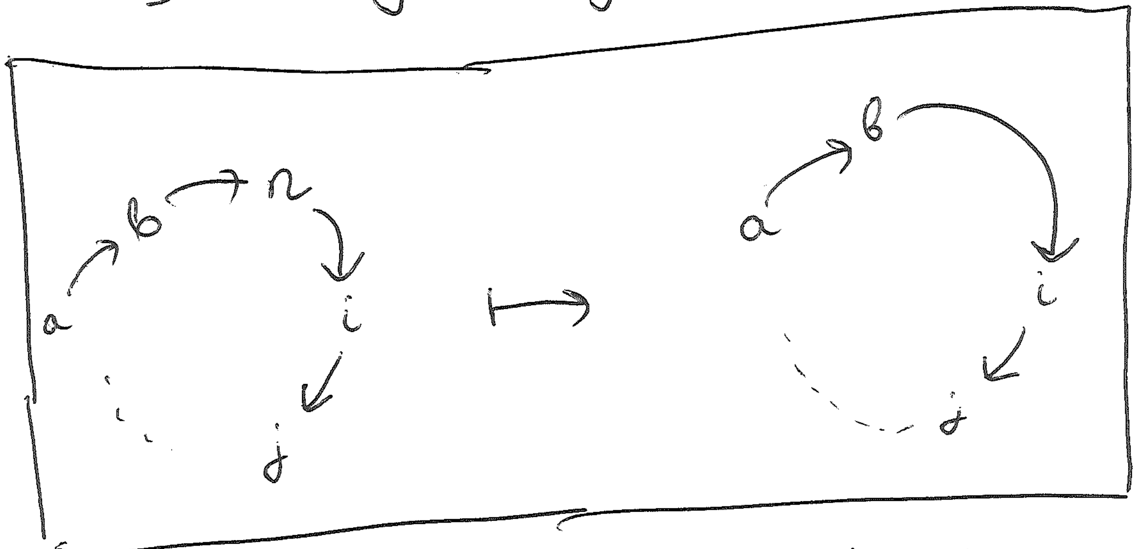
For each $i \in [n-1]$, there is a bijection

$$\{\text{red } \sigma \in S_n \mid \sigma(n) = i\} \to \{\tau \in S_{n-1} \text{ having } k \text{ cycles}\},$$

$$\sigma \mapsto (t_{i,n} \circ \sigma)|_{[n-1]}^{[n-1]}.$$

(What this bijection does is removing n from its cycle

2nd closing the cycle again:



The inverse map inserts n back into the cycle that contains i, right before i.)

Thus, for each $i \in [n-1]$, we have

$$(\text{\# of red } \sigma \in S_n \text{ satisfying } \sigma(n) = i)$$
$$= (\text{\# of } \tau \in S_{n-1} \text{ having } k \text{ cycles}) = \left[{n-1 \atop k} \right].$$

Summing this equation up over all $i \in [n-1]$, we find

$$(\text{\# of red } \sigma \in S_n) = (n-1) \left[{n-1 \atop k} \right].$$

Each $\sigma \in S_n$ with k cycles is either red or green (but not both). Hence,

$$\left[{n \atop k}\right] = \underbrace{(\#\text{ of red } \sigma \in S_n)}_{=(n-1)\left[{n-1 \atop k}\right]} + \underbrace{(\#\text{ of green } \sigma \in S_n)}_{=\left[{n-1 \atop k-1}\right]}$$

$$= (n-1)\left[{n-1 \atop k}\right] + \left[{n-1 \atop k-1}\right].$$

$\square$

Prop. 4.15. Let $m \in \mathbb{N}$ and $x \in \mathbb{Q}$.

(a) We have $x^m = \sum_{i=0}^{m} \left\{{m \atop i}\right\} x^{\underline{i}}$.

(b) We have $x^{\underline{m}} = \sum_{i=0}^{m} (-1)^{m-i} \left[{m \atop i}\right] x^i$.

Proof. By the polynomial identity trick, it suffices to prove these for $x \in \mathbb{N}$. So assume $x \in \mathbb{N}$.

(a) Theorem 2.15 (applied to $k = x$) yields

$$x^m = \sum_{i=0}^{m} \operatorname{sur}(m,i) \underbrace{\binom{x}{i}}_{=x^{\underline{i}}/i!} = \sum_{i=0}^{m} \underbrace{\frac{\operatorname{sur}(m,i)}{i!}}_{=\left\{{m \atop i}\right\}} x^{\underline{i}}$$

$$= \sum_{i=0}^{m} \left\{ {m \atop i} \right\} x^{\underline{i}}.$$

(b) 1st proof: Induction on m, using Prop. 4.14 (e) and $x^{\underline{m+1}} = (x-m)\, x^{\underline{m}}$. LTTR.

2nd proof (sketch): (This is from [Galvin, §36]).

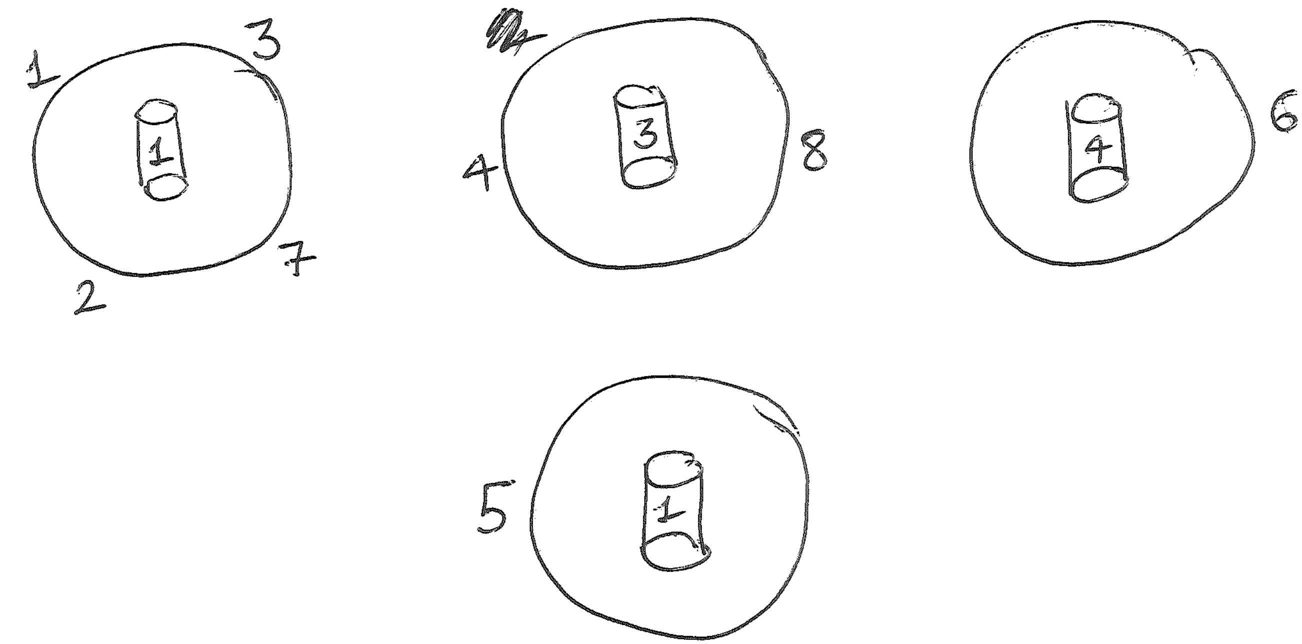
We shall show

(35) $$y(y+1)\cdots(y+m-1) = \sum_{i=0}^{m} \left[{m \atop i} \right] y^i, \quad \text{for all } y \in \mathbb{N}.$$

This will yield (35) for all $y \in \mathbb{R}$. Then, by setting $y = -x$, we will get the claim of (b).

Consider m people and a bar that has y brands of beer. We want to seat the people at several round tables, and put a pitcher of beer on each (occupied) table. (But the tables are indistinguishable, and there are $\geq m$ tables.) How many $\underbrace{\text{seatings}}_{\text{incl. choices of beer}}$ exist?

Example: $m = 8$, $y = 4$



- There are $\sum_{i=0}^{m} \left[{m \atop i}\right] y^i$ many seatings, since a seating is just a permutation of the m people + a choice of beer for each cycle of the permutation.
- There are $y(y+1)\cdots(y+m-1)$ many seatings, since
  - $*$ person 1 enters & chooses a beer ($\rightsquigarrow y$ options);
  - $*$ person 2 enters & chooses either a beer (and ~~[illegible]~~ to sit alone) or

to sit by person 1
($\rightsquigarrow$ $y+1$ options);

* person 3 enters & chooses either a beer
(and to sit alone) or
which of persons 1 and 2 to
sit by ($\rightsquigarrow$ $y+2$ options);

⋮

* person k enters & chooses either a beer
(and to sit alone), or
to sit on the right of one of persons
$1, 2, \ldots, k-1$ ($\rightsquigarrow$ $y+k-1$ options);

⋱

$\Longrightarrow$ (35) is proven. $\square$

Remark. For linear algebraists: Fix $n \in \mathbb{N}$. Consider the
$\mathbb{Q}$-vector space $\mathbb{Q}[x]_{\leq n}$ of all polynomials of degree
$\leq n$ in x over $\mathbb{Q}$.

It is known that $(1, x, x^2, \dots, x^n) = (x^0, x^1, x^2, \dots, x^n)$ is a basis of this vector space.

Another basis is $(x^{\underline{0}}, x^{\underline{1}}, x^{\underline{2}}, \dots, x^{\underline{n}})$.

The change-of-bases matrices for these 2 bases are

$$\left(\left\{ {i \atop j} \right\} \right)_{\substack{0 \le i \le n, \\ 0 \le j \le n}}, \qquad \left((-1)^{i-j} \left[{i \atop j} \right] \right)_{\substack{0 \le i \le n, \\ 0 \le j \le n}}.$$

Thus, these 2 matrices are mutually inverse.

$\Rightarrow$ For each $i, j \in \mathbb{N}$, we have

$$\sum_{k} \left\{ {i \atop k} \right\} (-1)^{k-j} \left[{k \atop j} \right] = [i = j]$$

and

$$\sum_{k}^{i} (-1)^{i-k} \left[{i \atop k} \right] \left\{ {k \atop j} \right\} = [i = j].$$