4. Permutations

We'll now talk more about permutations. For deeper treatments, see [Bóna: Combinatorics of permutations] and [Stanley: Enumerative Combinatorics, vol. 1, ch. 1].

Recall: a *permutation* of a set X is a bijection from X to X.

4.1. Definitions

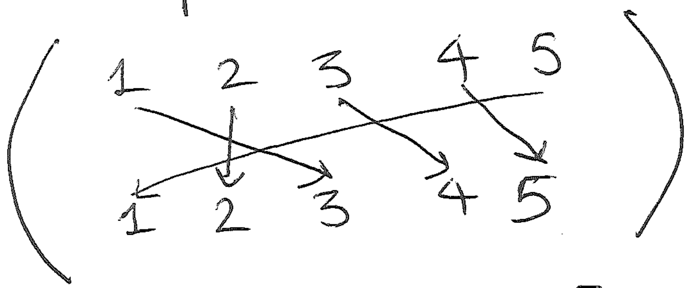
Def. Given $n \in \mathbb{N}$. Let S_n be the set of all permutations of $[n]$.
This set S_n is called the *n-th symmetric group*.

It is closed under composition (i.e., for any $\alpha \in S_n$ and $\beta \in S_n$, we have $\alpha \circ \beta \in S_n$), and under inverses (i.e., for any $\sigma \in S_n$, we have $\sigma^{-1} \in S_n$), and contains $\mathrm{id}_{[n]}$.

Def. Let $n \in \mathbb{N}$ and $\sigma \in S_n$. We introduce 2 notations for σ:

(a) The *one-line notation* of σ is the n-tuple
$[\sigma(1), \sigma(2), \ldots, \sigma(n)]$. (The use of ~~sq~~ square brackets is a convention.)

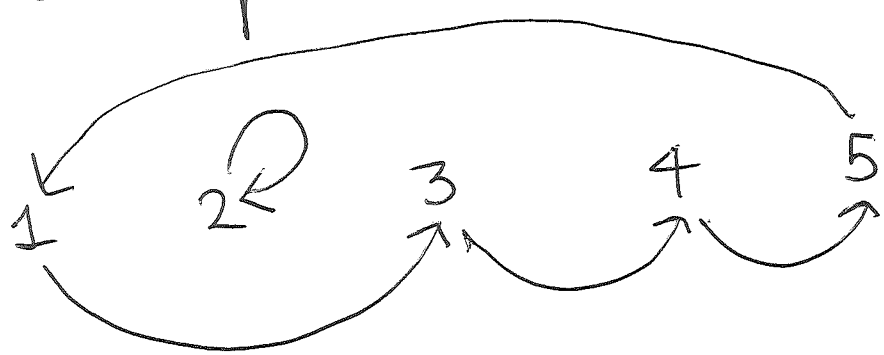
E.g., the permutation



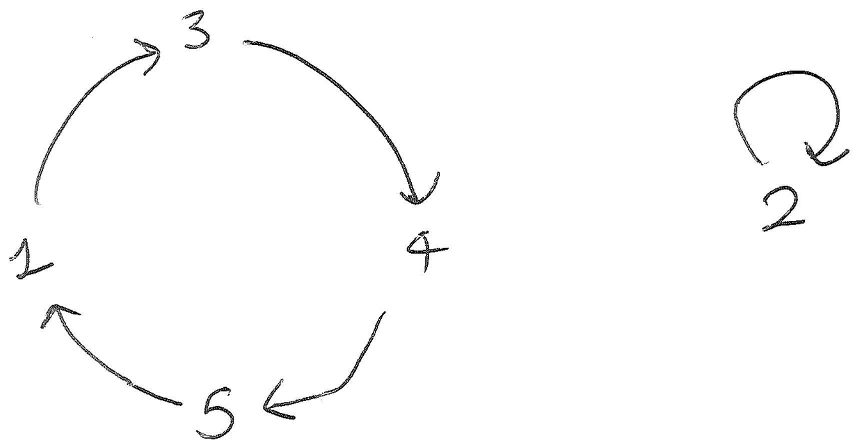
$\in S_5$

has one-line notation $[3, 2, 4, 5, 1]$, or, short, 32451.

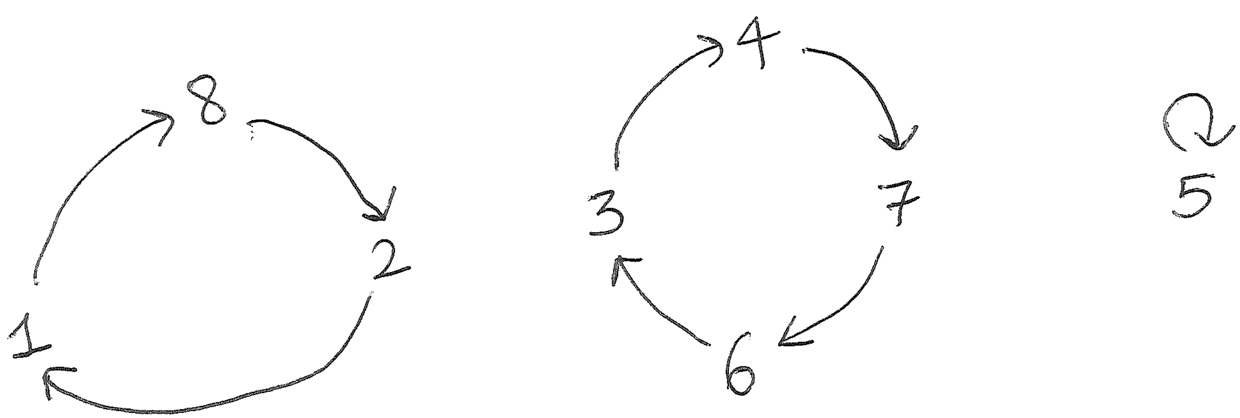
(b) The cycle digraph of σ is defined (informally) as follows:
For each $i \in [n]$, draw a point ("node") labelled i.
For each $i \in [n]$, draw an arrow ("arc") from the node labelled i to the node labelled $\sigma(i)$.
~~E.g., the~~ The result is called the cycle digraph of σ.
E.g., the above permutation has cycle digraph



or



E.g., the permutation in S_8 whose one-line notation is $[8, 1, 4, 7, 5, 3, 6, 2]$ has cycle digraph



<u>Prop. 4.1.</u> Let $n \in \mathbb{N}$. Then, the map

$$S_n \to \{n\text{-tuples of distinct elements of } [n]\},$$
$$\sigma \mapsto (\text{the one-line notation of } \sigma) = [\sigma(1), \sigma(2), \ldots, \sigma(n)]$$

is a bijection.

<u>Proof.</u> Permutations of $[n]$ are the same as injective maps $[n] \to [n]$ (by the Pigeonhole Principle for Injections). $\square$

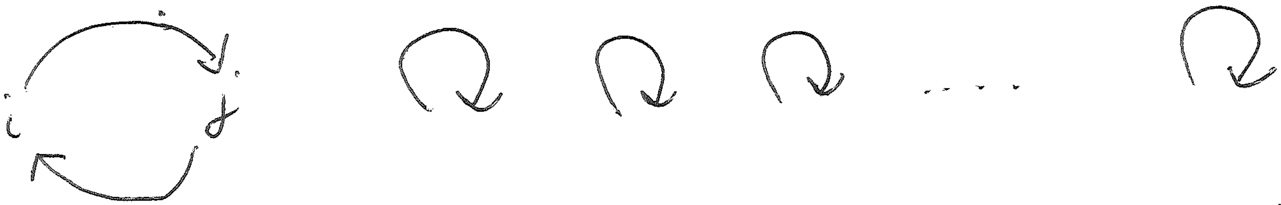
4.2. ~~Inversions & lengths~~ Transpositions & cycles

<u>Def.</u> (a) Let i and j be two elements of a set X. Assume $i \neq j$. Then, the <u>transposition $t_{i,j}$</u> is the permutation of X that sends i to j, j to i, and leaves everything else in its place.

If $X = [n]$ for some $n \in \mathbb{N}$, and if $i < j$, then the one-line notation of $t_{i,j}$ is

$$[1, 2, \ldots, i-1, j, i+1, i+2, \ldots, j-1, i, j+1, j+2, \ldots, n].$$

The cycle digraph of $t_{i,j}$ is



(b) Let $n \in \mathbb{N}$, and $i \in [n-1]$. Then, the *simple transposition* s_i is defined by $s_i = t_{i,i+1} \in S_n$.

Convention: If α and β are two permutations of a set X, we write $\alpha\beta$ for $\alpha \circ \beta$. Also, $\alpha^i := \underbrace{\alpha \circ \alpha \circ \cdots \circ \alpha}_{i \text{ times } \alpha}$.

Prop. 4.2. Let $n \in \mathbb{N}$. (Recall: $s_i^2 = s_i \circ s_i$.)

(a) We have $s_i^2 = \mathrm{id}$ $\forall i \in [n-1]$.

(b) We have $s_i s_j = s_j s_i$ $\forall i, j \in [n-1]$ with $|i-j| > 1$.

(c) We have $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} = t_{i,i+2}$ $\forall i \in [n-2]$.

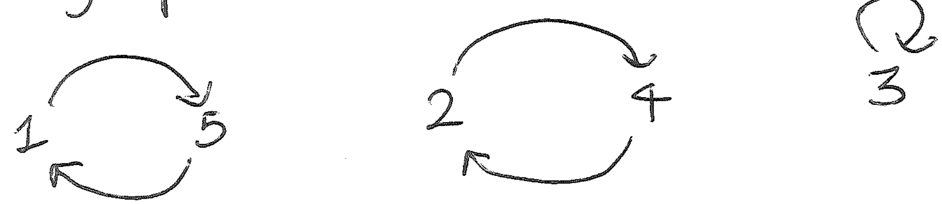
("the braid relation for permutations")

Proof. Straightforward verification that both sides send each $k \in [n]$ to the same image. $\square$

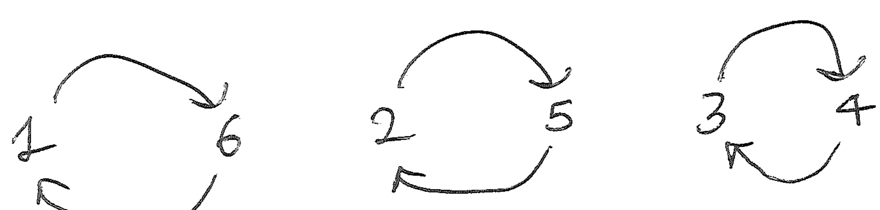
Def. Let $n \in \mathbb{N}$. Let w_0 be the permutation in S_n that sends each i to $n+1-i$.

In other words, it "reflects" numbers across the middle of n.
It is the unique strictly decreasing permutation of $[n]$.

Examples: If $n=5$, then $w_0 = [5,4,3,2,1]$ in one-line notation, with cycle digraph

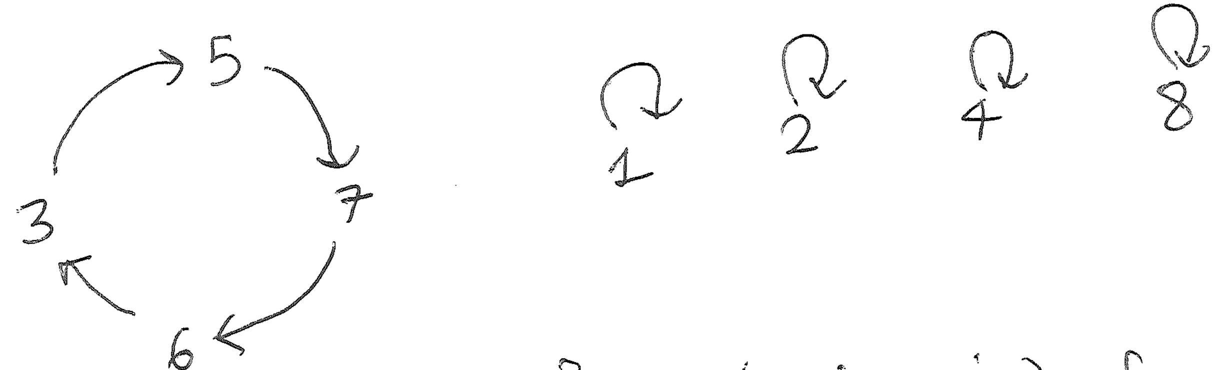


If $n=6$, then $w_0 = [6,5,4,3,2,1]$ in one-line notation, with cycle digraph



Def. Let X be a set. Let $i_1, i_2, \ldots, i_k$ be k distinct elements of X. Then, $\operatorname{cyc}_{i_1, i_2, \ldots, i_k}$ means the permutation of X that sends $i_1 \mapsto i_2$, $i_2 \mapsto i_3$, $i_3 \mapsto i_4$, ..., $i_{k-1} \mapsto i_k$, $i_k \mapsto i_1$, and leaves all other elements of X unchanged.
This is called a <u>k-cycle</u>.

Example: The cycle digraph of $\operatorname{cyc}_{3,5,7,6} \in S_8$



Remark: People often write $(i_1, i_2, \ldots, i_k)$ for $\operatorname{cyc}_{i_1, i_2, \ldots, i_k}$.

Remark: A permutation α is called an <u>involution</u> if $\alpha^2 = \mathrm{id}$. Both $t_{i,j}$ and w_0 are involutions, but k-cycles with $k > 2$ are not.

Prop. 4.3. Let $n \in \mathbb{N}$.

(a) For any k distinct elements $i_1, i_2, \ldots, i_k$ of $[n]$, we have

$$\operatorname{cyc}_{i_1, i_2, \ldots, i_k} = t_{i_1, i_2} t_{i_2, i_3} \cdots t_{i_{k-1}, i_k}.$$

(b) For any $i \in [n]$ and $k \in \mathbb{N}$ such that $i + k - 1 \leq n$, we have $\operatorname{cyc}_{i, i+1, \ldots, i+k-1} = s_i s_{i+1} \cdots s_{i+k-2}$.

(c) For any $i \in [n]$, we have $\operatorname{cyc}_i = \operatorname{id}$.

(d) For any distinct $i, j \in [n]$, we have $\operatorname{cyc}_{i,j} = t_{i,j}$.

(e) For any k distinct elements $i_1, i_2, \ldots, i_k$ of $[n]$, we have

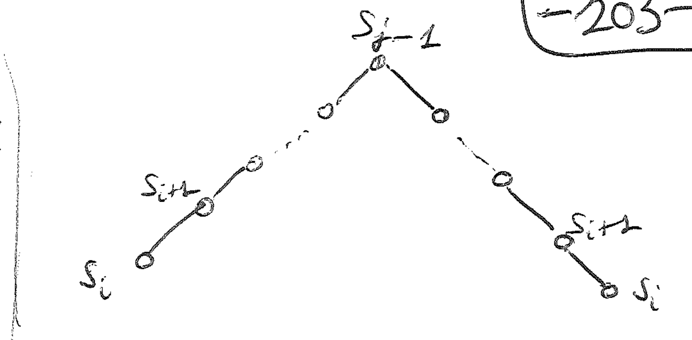
$$\operatorname{cyc}_{i_1, i_2, \ldots, i_k} = \operatorname{cyc}_{i_k, i_1, i_2, \ldots, i_{k-1}}.$$

(f) For any k distinct elements $i_1, i_2, \ldots, i_k$ of $[n]$ and any $\sigma \in S_n$, we have $\cancel{\sigma \operatorname{cyc}_{i_1, i_2, \ldots, i_k} \sigma^{-1} = \operatorname{cyc}_{\sigma(i_1), \sigma(i_2), \ldots, \sigma(i_k)}}$

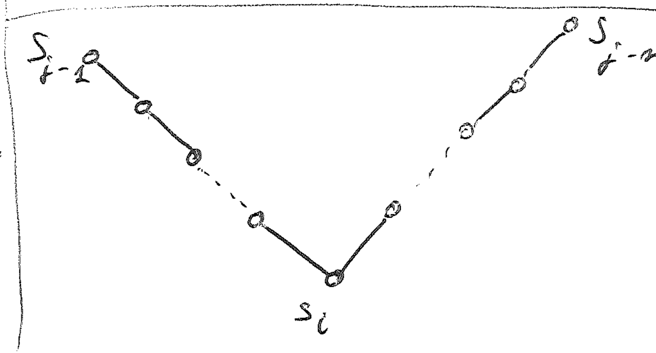
$$\sigma \operatorname{cyc}_{i_1, i_2, \ldots, i_k} \sigma^{-1} = \operatorname{cyc}_{\sigma(i_1), \sigma(i_2), \ldots, \sigma(i_k)}.$$

(g) If $1 \leq i < j \leq n$, then

$$t_{i,j} = s_i \, s_{i+1} \cdots s_{j-1} \cdots s_{i+1} s_i$$



$$= s_{j-1} s_{j-2} \cdots s_i \cdots s_{j-2} s_{j-1}.$$



Both products have $2(j-i)-1$ factors.

(h) $$w_0 = \mathrm{cyc}_{1,2,\ldots,n} \, \mathrm{cyc}_{1,2,\ldots,n-1} \cdots \mathrm{cyc}_1$$
$$= (s_1 s_2 \cdots s_{n-1})(s_1 s_2 \cdots s_{n-2}) \cdots (s_1 s_2) s_1$$
$$= \mathrm{cyc}_1 \, \mathrm{cyc}_{2,1} \, \mathrm{cyc}_{3,2,1} \cdots \mathrm{cyc}_{n,n-1,\ldots,2,1}$$
$$= s_1 (s_2 s_1)(s_3 s_2 s_1) \cdots (s_{n-1} s_{n-2} \cdots s_1).$$

<u>Proof.</u> (a) Applying $t_{i_1,i_2} t_{i_2,i_3} \cdots t_{i_{k-1},i_k}$ to i_j for $j \in [k-1]$, we get

$$i_j \xmapsto{t_{i_{k-1},i_k}} i_j \xmapsto{t_{i_{k-2},i_{k-1}}} i_j \longmapsto \cdots \mapsto i_j \xmapsto{t_{i_j,i_{j+1}}} i_{j+1} \xmapsto{t_{i_{j-1},i_j}} i_{j+1}$$

$$\longmapsto \cdots \longmapsto i_{j+1} \xmapsto{t_{i_2,i_3}} i_{j+1} \xmapsto{t_{i_1,i_2}} i_{j+1}.$$

Thus, $\forall j \in [k-1]$, we have

(21) $$(t_{i_1,i_2} t_{i_2,i_3} \cdots t_{i_{k-1},i_k})(i_j) = i_{j+1} = cyc_{i_1,i_2,\ldots,i_k}(i_j).$$

Applying $t_{i_1,i_2} t_{i_2,i_3} \cdots t_{i_{k-1},i_k}$ to i_k, we get

$$i_k \xmapsto{t_{i_{k-1},i_k}} i_{k-1} \xmapsto{t_{i_{k-2},i_{k-1}}} i_{k-2} \longmapsto \cdots \longmapsto i_2 \xmapsto{t_{i_1,i_2}} i_1.$$

So

(22) $$(t_{i_1,i_2} t_{i_2,i_3} \cdots t_{i_{k-1},i_k})(i_k) = i_1 = cyc_{i_1,i_2,\ldots,i_k}(i_k).$$

For any $x \notin \{i_1, i_2, \ldots, i_k\}$, we have

(23) $$(t_{i_1,i_2} t_{i_2,i_3} \cdots t_{i_{k-1},i_k})(x) = x = cyc_{i_1,i_2,\ldots,i_k}(x).$$

Combining (21), (22) and (23), we get

$$(t_{i_1,i_2} t_{i_2,i_3} \cdots t_{i_{k-1},i_k})(x) = cyc_{i_1,i_2,\dots,i_k}(x) \qquad \forall x \in [n].$$

Hence, $t_{i_1,i_2} t_{i_2,i_3} \cdots t_{i_{k-1},i_k} = cyc_{i_1,i_2,\dots,i_k}$.

(b) Apply (a) to $i_1 = i$, $i_2 = i+1$, $i_3 = i+2$, ..., $i_k = i+k-1$.

(c), (d), (e): trivial.

(f): Easy (see [detnotes, Exercise 5.16 (a)]).

(g)

$$\underbrace{s_i s_{i+1} \cdots s_{j-1}} s_{j-2} \cdots s_{i+1} s_i$$

$$\overset{(b)}{=} cyc_{i,i+1,\dots,j}$$

$$\overset{(e)}{=} cyc_{j,i,i+1,\dots,j-1}$$

$$\overset{(a)}{=} t_{j,i} t_{i,i+1} t_{i+1,i+2} \cdots t_{j-2,j-1}$$

$$\overset{(d)}{=} t_{j,i} s_i s_{i+1} \cdots s_{j-2}$$

$$= t_{j,i} s_i s_{i+1} \cdots \underbrace{s_{j-2} s_{j-2}}_{= \mathrm{id}} \cdots s_{i+1} s_i$$

$$= t_{j,i} s_i s_{i+1} \cdots \underbrace{s_{j-3} s_{j-3}}_{= \mathrm{id}} \cdots s_{i+1} s_i$$

$$= \cdots\cdots = t_{j,i} = t_{i,j}.$$

This proves the 1st equality. The 2nd is similar.

(h) For each $k \in \{0, 1, \ldots, n\}$, define the permutation

$$\varsigma_k := \mathrm{cyc}_{1,2,\ldots,k} \, \mathrm{cyc}_{1,2,\ldots,k-1} \cdots \mathrm{cyc}_1 .$$

Thus,

$$\varsigma_1 = \mathrm{cyc}_1 = \mathrm{id};$$

$$\varsigma_n = \mathrm{cyc}_{1,2,\ldots,n} \, \mathrm{cyc}_{1,2,\ldots,n-1} \cdots \mathrm{cyc}_1 ;$$

(24) $$\varsigma_k = \mathrm{cyc}_{1,2,\ldots,k} \, \varsigma_{k-1} \quad \text{for each } k \in [n].$$

Now, we claim:

<u>Claim 1:</u> For each $k \in \{0, 1, \ldots, n\}$, the permutation ς_k sends $1, 2, \ldots, k$ to $k, k-1, \ldots, 1$ but leaves

$k+1, k+2, \ldots, n$ in their places.

(For $k=n$, this says that $\varsigma_n = w_0$.)

Proof of Claim 1: Induction on k:

Base case ($k=0$):

$\varsigma_0 =$ (empty product of permutations) $=$ id.

Step ($k-1 \to k$): Let $k \in [n]$. Assume (as IH) that ς_{k-1} sends $1, 2, \ldots, k-1$ to $k-1, k-2, \ldots, 1$ but leaves $k, k+1, \ldots, n$ in their places.

We must prove that ς_k sends $1, 2, \ldots, k$ to $k, k-1, \ldots, 1$ but leaves $k+1, k+2, \ldots, n$ in their places.

In other words, we must prove

(25) $\varsigma_k(i) = k+1-i \quad \forall\, i \in [k]$, and

(26) $\varsigma_k(i) = i \quad \forall\, i > k$.

Both of these equalities follow straightfly from (24) and from the IH. (See Math 4707 Spring '18 for details) ■

Claim 1 (applied to $k=n$) proves the 1st equality of (h).

The 2nd equality is a consequence of (b).

The 3rd & 4th equalities can be proven similarly, or can be derived from the previous ones as follows:

Recall that $(\alpha_1 \alpha_2 \cdots \alpha_k)^{-1} = \alpha_k^{-1} \alpha_{k-1}^{-1} \cdots \alpha_1^{-1}$ $\forall$ bijections $\alpha_1, \alpha_2, \ldots, \alpha_k$. Now,

$$w_0 = (s_1 s_2 \cdots s_{n-1})(s_1 s_2 \cdots s_{n-2}) \cdots (s_1 s_2) s_1$$

$$\Longrightarrow \quad w_0^{-1} = ((s_1 s_2 \cdots s_{n-1})(s_1 s_2 \cdots s_{n-2}) \cdots (s_1 s_2) s_1)^{-1}$$

$$= s_1^{-1} (s_2^{-1} s_1^{-1})(s_3^{-1} s_2^{-1} s_1^{-1}) \cdots (s_{n-1}^{-1} s_{n-2}^{-1} \cdots s_1^{-1})$$

$$= s_1 (s_2 s_1)(s_3 s_2 s_1) \cdots (s_{n-1} s_{n-2} \cdots s_1)$$

(since $s_i^{-1} = s_i$ $\forall i$)

(since w_0 is an involution) $w_0^{-1} = w_0$

$$= \mathrm{cyc}_1 \, \mathrm{cyc}_{2,1} \, \mathrm{cyc}_{3,2,1} \cdots \mathrm{cyc}_{n,n-1,\ldots,2,1}.$$ □

4.3. Inversions & lengths

Def. Let $n \in \mathbb{N}$ and $\sigma \in S_n$.

(a) An *inversion* of σ is a pair (i,j) of elements of $[n]$ such that $i < j$ and $\sigma(i) > \sigma(j)$.

(b) The *length* of σ is the # of inversions of σ. It is called $\ell(\sigma)$. \ell

(aka *Coxeter length*)

Example: Let $\pi = [3, 1, 4, 2] \in S_4$.

The inversions of π are $(1,2)$ (since $1 < 2$ and $\underset{\overset{\parallel}{3}}{\pi(1)} > \underset{\overset{\parallel}{1}}{\pi(2)}$),

$(1,4)$ (since $1 < 4$ and $\pi(1) > \pi(4)$),

and $(3,4)$.

So the length of π is 3.

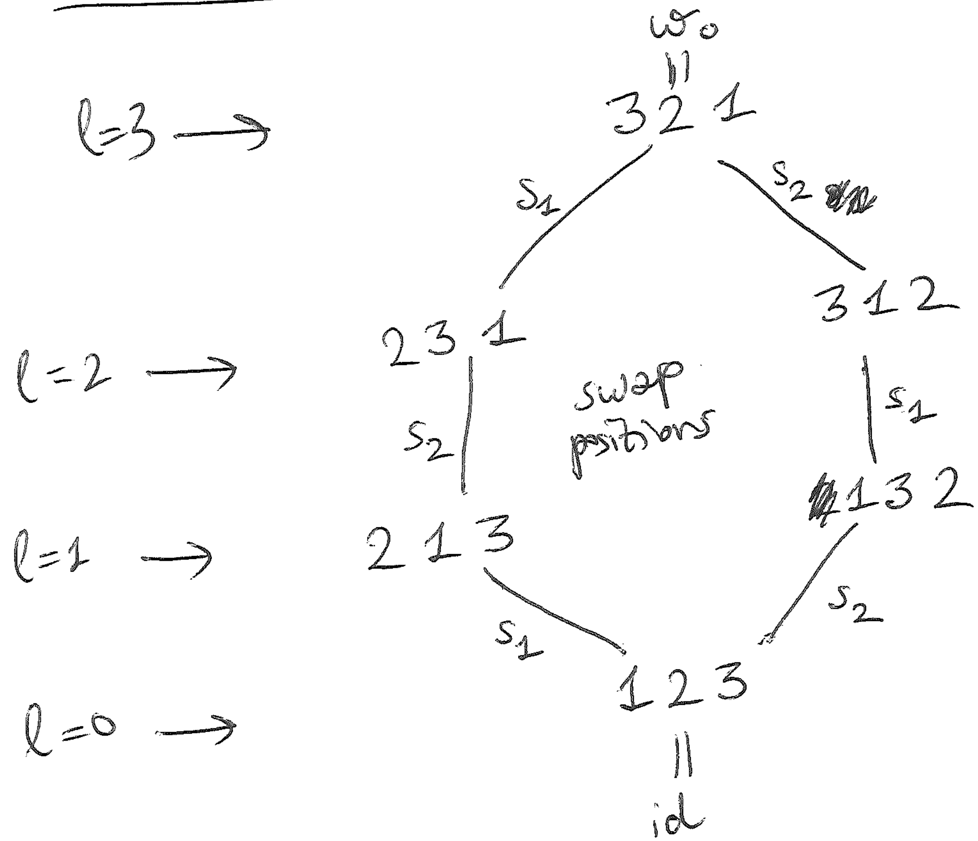
Remark: If $\sigma \in S_n$, then $0 \leq \ell(\sigma) \leq \binom{n}{2}$.

The only $\sigma \in S_n$ with $\ell(\sigma) = 0$ is id.

The only $\sigma \in S_n$ with $\ell(\sigma) = \binom{n}{2}$ is w_0.

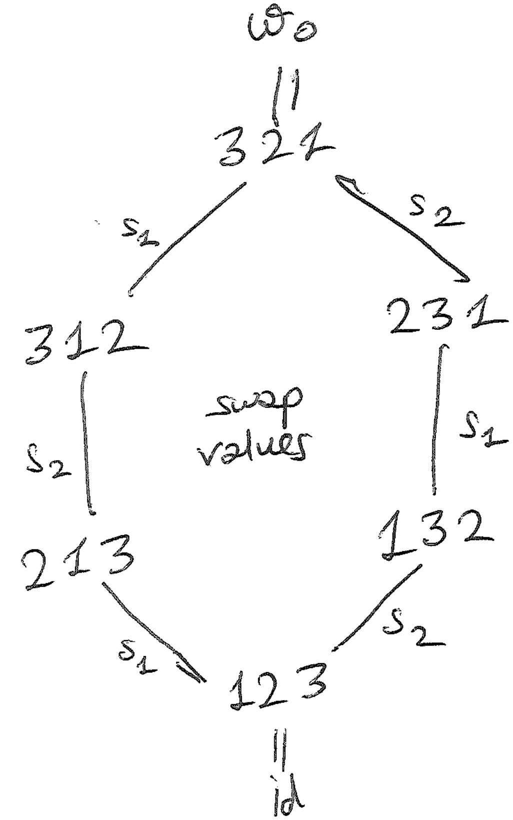
In between, there are many:

S_3 (permutations in one-line notation):



Here, we draw an edge $\alpha \,\overset{s_i}{\text{———}}\, \beta$ if $\alpha = \beta s_i$

(or, equivalently, $\beta = \alpha s_i$)



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