Math 5705 Oct 15, 2018

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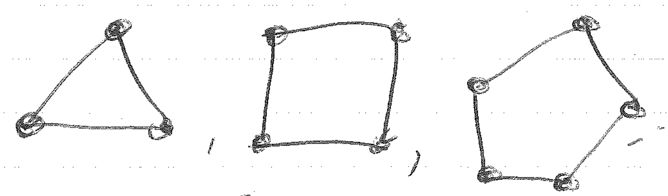
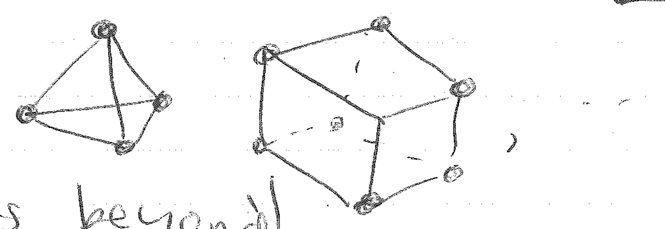
A taste of Ehrhart theory

Let's look at polytopes from a combinatorial point of view. A (convex) polytope is the convex hull of finitely many points in Euclidean space: $P = \text{conv Hull}(v_1, \ldots, v_m)$, $v_1, \ldots, v_m \in \mathbb{R}^n$. If this list $v_1, \ldots, v_m$ is irredundant ($P \neq \text{conv Hull}(v_1, \ldots, \hat{v}_i, \ldots, v_m)$ for any i) then the v_i are the vertices of P.

Examples:

2-dim'l polygons: 

3-dim'l polytopes: 

And in dimensions beyond!

What does it mean to study polytopes combinatorially? We could e.g. enumerate faces (vertices, edges, ...) of polytopes. This is a huge area.

Instead, I'll talk about counting lattice points of polytopes: $\#(P \cap \mathbb{Z}^n)$.

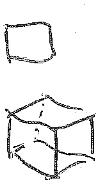
This gets us to "Ehrhart theory"

Great reference: "Computing the Continuous Discretely" by Beck + Robins

Def: The *k^{th} dilate* of P is $kP := \{kx : x \in P\}$.
The *Ehrhart function* of P is
$L_P(k) := \#(kP \cap \mathbb{Z}^n)$, i.e., the number of lattice pts in the k^{th} dilate of P.

Examples:

1) $P = \square_n \subseteq \mathbb{R}^n$, the standard n-dim'l hypercube:



$$\square_n := \{(x_1, \dots, x_n) \in \mathbb{R}^n : 0 \le x_i \le 1 \ \forall i = 1, \dots, n\}.$$

What is $L_{\square^n}(k)$?

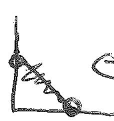
$$(k\square_n \cap \mathbb{Z}^n) = \{(x_1, \dots, x_n) \in \mathbb{Z}^n : 0 \le x_i \le k\}$$

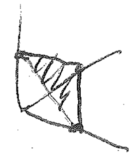
n independent choices from $\{0, \dots, k\}$ ← $k+1$ elements

$$\Rightarrow L_{\square^n}(k) = (k+1)^n$$

2) $P = \Delta^n \subseteq \mathbb{R}^{n+1}$, the standard n-dim'l simplex

$$\Delta^n := \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : 0 \le x_i \ \forall i, \ \sum_{i=1}^{n+1} x_i = 1\}$$

E.g.,  ← $\Delta^1 \subseteq \mathbb{R}^2$

 ← $\Delta^2 \subseteq \mathbb{R}^3$

What is $L_{\Delta^n}(k)$?

$$(k\Delta^n \cap \mathbb{Z}^n) = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : 0 \le x_i \ \forall i, \ \sum x_i = k\}$$

<u>Claim</u>: $L_{\Delta^n}(k) = \binom{n+k}{n} \left(= \binom{n+k}{k}\right)$

<u>Pf</u>: "Stars and bars"

$$\underbrace{**}_{x_1} \,|\, \underbrace{\ }_{x_2} \,|\, \underbrace{*}_{x_3} \,|\, \underbrace{*}_{x_4} \,|\, \underbrace{**}_{x_5}$$

k $*$'s and n $|$'s

We say P is a <u>lattice polytope</u> if $v_1, \dots, v_m \in \mathbb{Z}^n$.

<u>Thm</u> (Ehrhart's theorem) (this is important!)

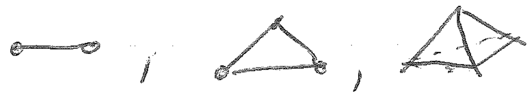
Let P be a lattice polytope. Then $L_P(k)$ is a polynomial in k of degree $= \dim(P)$.

<u>E.g.</u> $L_{\Delta^n}(k) = \binom{k+n}{n} = \dfrac{(k+n)(k+n-1)\cdots(k+1)}{n!}$

<u>Sketch of proof</u>:

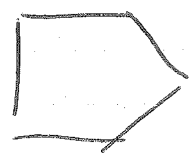
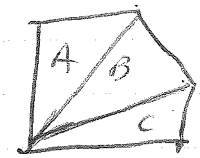
<u>Part 1</u>: Any polytope P has a triangulation.

A <u>simplex</u> is the convex hull of affinely independent pts:



etc.

A <u>triangulation</u> of P is a decomposition of P into simplices, whose vertices are vertices of P, such that the intersection of any two of these simplices is again a simplex (and a "common face of both".)

E.g. $P =$  , triangulation of P = 

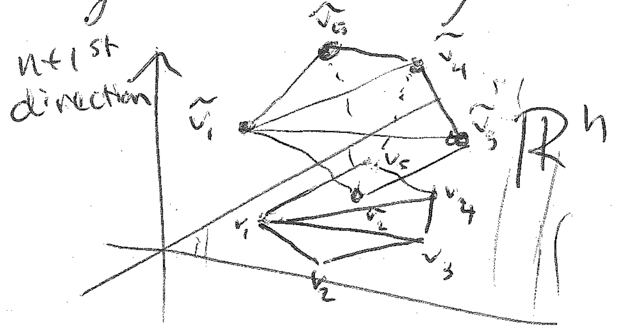
w/ triangles A, B, C

How to construct a triangulation?

a) Lift $v_1, \ldots, v_m \in \mathbb{R}^n$ to $\tilde{v}_1, \ldots, \tilde{v}_m \in \mathbb{R}^{n+1}$ where the last coordinates are <u>generic</u> numbers

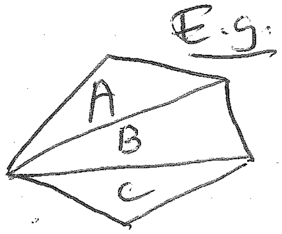
b) $\tilde{P} = \text{conv Hull}(\tilde{v}_1, \ldots, \tilde{v}_m)$ will be a <u>simplicial polytope</u> (all faces = simplices)

c) If we project the "upper cover" of $\tilde{P}$ (the "part of $\tilde{P}$ we can see from above") onto P, we get a triangulation.



> Tricky bonus question: Are all triangulations obtained in this way?

<u>Step 2</u> Use inclusion-exclusion on the triangulation



<u>E.g.</u> Pentagon P is triangulated into triangles A, B, C. How can we count lattice pts in P?

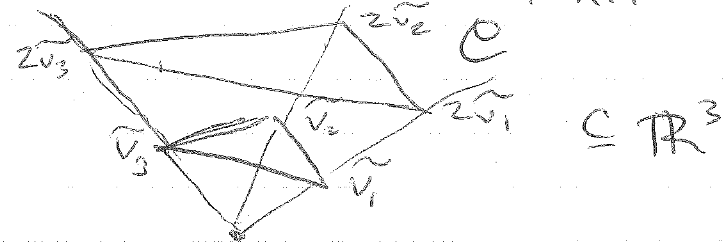
$$\#(P \cap \mathbb{Z}^n) = \#(A \cap \mathbb{Z}^n) + \#(B \cap \mathbb{Z}^n) + \#(C \cap \mathbb{Z}^n)$$
$$- \#((A \cap B) \cap \mathbb{Z}^n) - \#((A \cap C) \cap \mathbb{Z}^n) - \#((B \cap C) \cap \mathbb{Z}^n)$$
$$+ \#((A \cap B \cap C) \cap \mathbb{Z}^n)$$

But $A, B, C, A\cap B, A \cap C, B \cap C$, and $A \cap B \cap C$ are all simplices! So if the Ehrhart poly's exist for those, they exist for P.

Step 3 The case where P is a lattice simplex

Now we can assume P is a lattice simplex. Again lift $v_1, \dots, v_m \in \mathbb{R}^n$ to $\tilde{v}_1, \dots, \tilde{v}_m \in \mathbb{R}^{n+1}$, but now with the last coordinate $x_{n+1} = 1$.

E.g. $P =$



Let $\mathcal{C} := \{a_1\tilde{v}_1 + a_2\tilde{v}_2 + \cdots + a_m\tilde{v}_m;\ a_i \in \mathbb{R}_{\geq 0}\}$ be the cone generated by $\tilde{v}_1, \dots, \tilde{v}_m$.

Observe: $\mathcal{C} \cap \{x_{n+1} = k\} \simeq kP$.

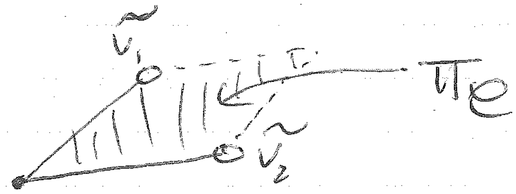
We want to understand the generating function of all lattice pts in $\mathcal{C}$:

$$\sum_{v=(a_1,\dots,a_{n+1}) \in \mathcal{C}\cap\mathbb{Z}^{n+1}} \underbrace{z_1^{a_1} z_2^{a_2} \cdots z_{n+1}^{a_{n+1}}}_{\text{shorthand } \vec{z}^{\,v}} =: F_{\mathcal{C}}(z_1, \dots, z_{n+1})$$

To do this we introduce the fundamental half-open parallelopiped of $\mathcal{C}$

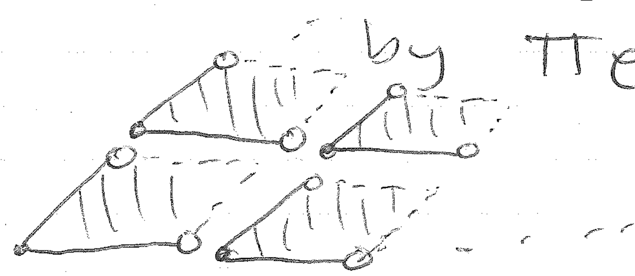
$$\Pi_{\mathcal{C}} := \{a_1\tilde{v}_1 + \cdots + a_m\tilde{v}_m : 0 \leq a_i < 1 \ \forall i = 1, \dots, m\}.$$

E.g.



Claim: $\mathcal{C}$ is tiled by $\Pi_{\mathcal{C}}$:

Pf by picture:



That $\Pi_{\mathcal{C}}$ tiles $\mathcal{C}$ implies:

$$F_{\mathcal{C}}(z_1, \dots, z_{n+1}) = \left(1 + \vec{z}^{\,\tilde{v}_1} + \vec{z}^{\,2\tilde{v}_1} + \cdots\right)\left(1 + \vec{z}^{\,\tilde{v}_2} + \vec{z}^{\,2\tilde{v}_2} + \cdots\right)$$

$$\cdots \left(1 + \vec{z}^{\,\tilde{v}_m} + \vec{z}^{\,2\tilde{v}_m} + \cdots\right) \cdot \sum_{v \in (\Pi_{\mathcal{C}} \cap \mathbb{Z}^n)} \vec{z}^{\,v}$$

$$= \left(\prod_{i=1}^{m} \frac{1}{(1 - \vec{z}^{\,\tilde{v}_i})} \right) \cdot \sum_{v \in (\Pi_{\mathcal{C}} \cap \mathbb{Z}^n)} \vec{z}^{\,v}$$

"choose which tile your point is in" "choose a point in that tile"

<u>Next</u>, note that if we substitute $z_1 = z_2 = \cdots = z_n = 1$ and $z_{n+1} = z$, we get

$$F_{\mathcal{C}}(0, 0, \dots, 0, z) = \sum_{k=0}^{\infty} L_P(k) \cdot z^k$$

So, (*)

$$\sum_{k=0}^{\infty} L_P(k) \cdot z^k = \frac{h_P^*(z)}{(1-z)^m},$$

where $h_P^*(z) = h_0^* + h_1^* z + \cdots + h_{m-1}^* z^{m-1}$ is a polynomial in z of degree $< m$.

In fact, (*) is enough to conclude $L_P(k)$ is a polynomial, as we will now explain!

Let $\ell < m$, then

$$\frac{z^\ell}{(1-z)^m} = z^\ell \cdot \underbrace{(1 + z + z^2 + \cdots)(1 + z + z^2 + \cdots) \cdots (1 + z + z^2 + \cdots)}_{m \text{ copies}}$$

$$= \sum_{k=0}^{\infty} \#\{(a_1,\dots,a_m) \in \mathbb{Z}_{\geq 0}^m : a_1+\dots+a_m = k-\ell\} \cdot z^k$$

we saw this when we looked at $L_{\Delta^n}(k)$!

$$= \sum_{k=0}^{\infty} \binom{m-1+k-\ell}{m-1} \cdot z^k$$

$$= \sum_{k=0}^{\infty} \frac{(k+(m-\ell-1))(k+(m-\ell-2))\cdots(k+(-\ell+1))}{(m-1)!} \cdot z^k$$

polynomial in k ✓

$\Rightarrow L_P(k)$ is a polynomial in k, as claimed.

The degree of $L_P(k)$ is $\max\{\ell \in \mathbb{Z}_{\geq 0} : \lim_{k\to\infty} \frac{L_P(k)}{k^\ell} \neq 0\}$, and it is easy to see geometrically that this $= \dim P$. ∎

What is known about Ehrhart polynomials?

Write $L_P(k) = a_0 + a_1 k + \cdots + a_d k^d$.

Then $a_0 = 1$, $a_d =$ "relative volume" of P,

and $a_{d-1} = \frac{1}{2} \sum_{F \text{ facet of } P}$ "relative volume" of F.

Compare to <u>Pick's theorem</u>:

For P a lattice polygon ($=$ 2-dim'l),

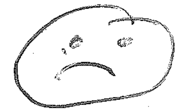
$\text{area}(P) = i + \frac{b}{2} - 1$, where

i = # "interior" lattice pts of P,
b = # "boundary" lattice pts of P.

What about other coefficients?

Example (Reeve tetrahedron) $P = \text{Conv Hull}\{(0,0,0), (1,0,0), (0,1,0), (1,1,r)\}$

Then $L_P(k) = \frac{r}{6}k^3 + k^2 + (2-\frac{r}{6})k + 1$.

negative for $r \geq 13$: 

In general, no known formula for the other coefficients of the Ehrhart poly.

But... if we write:

$$\sum_{k=0}^{\infty} L_P(k) \cdot z^k = \frac{h_0^* + h_1^* z + \cdots + h_d^* z^d}{(1-z)^{d+1}},$$

then it is known that $h_0^*, \ldots, h_d^*$ are nonnegative integers (Stanley, 1980).

The proof I sketched above starts to show why.

In general, understanding Ehrhart polynomials (and "h^*"-polynomials) of lattice polytopes, either in general or for specific families of polytopes, remains an active area of research in geometric combinatorics.