Prop. 3.12. Let $n \in \mathbb{N}$ and $k \in \mathbb{N}$.

(a) We have $\left\{ {n \atop 0} \right\} = [n=0]$.

(b) We have $\left\{ {0 \atop k} \right\} = [k=0]$.

(c) $\left\{ {n \atop k} \right\} = 0$ if $k > n$.

(d) $\left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} + k \left\{ {n-1 \atop k} \right\}$ if $n > 0$ & $k > 0$.

(e) $\left\{ {n \atop k} \right\} = \sum_{j=0}^{n-1} \binom{n}{j} \left\{ {j \atop k-1} \right\} / k$.

(f) $\left\{ {n \atop k} \right\} = \frac{1}{k!} \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} i^n$.

Pf. Follows from Prop. 2.10, Prop. 2.11, Prop. 2.12, Thm. 2.14. $\square$

Remark: Let p be a prime number.

We know: $p \mid \binom{p}{k}$ $\forall k \in \{1, 2, \ldots, p-1\}$. (This is Thm. 1.24.)

But also $p \mid \left\{ {p \atop k} \right\}$ $\forall k \in \{2, 3, \ldots, p-1\}$. Why?

Let us reprove $p \mid \binom{p}{k}$.

Let $\mathcal{P}_k([n])$ be the set of all k-element subsets of $[n]$.
We define the <u>shift</u> of some $X \in \mathcal{P}_k([n])$ to be the subset obtained from X by replacing $1, 2, \ldots, n-1, n$
by $2, 3, \ldots, n, 1$.

(For example, if $n=6$, then the shift of $\{2,3,5\}$ is $\{3,4,6\}$,
and — // — $\{2,3,6\}$ is $\{3,4,1\}$.)

Now, let $n=p$ be prime and $k \in \{1, \ldots, p-1\}$.
Define an equivalence relation $\underset{\text{shift}}{\sim}$ on $\mathcal{P}_k([n])$ by letting $X \underset{\text{shift}}{\sim} Y$ if & only if Y can be obtained from X by shifts. E.g. for $n=6$, we have

$$\{2,3,5\} \underset{\text{shift}}{\sim} \{3,4,6\} \underset{\text{shift}}{\sim} \{4,5,1\} \underset{\text{shift}}{\sim} \{5,6,2\} \underset{\text{shift}}{\sim} \{6,1,3\} \underset{\text{shift}}{\sim} \{1,2,4\} \underset{\text{shift}}{\sim} \{2,3,5\},$$

Any $\underset{\text{shift}}{\sim}$-equivalence class has $\leq n$ elements, since shifting a subset n times brings it back home.

In general, it can have $< n$ elements:

E.g., if $n=6$ & $k=3$, then $\{3,6\} \underset{\text{shift}}{\sim} \{4,1\} \underset{\text{shift}}{\sim} \{5,2\} \underset{\text{shift}}{\sim} \{6,3\}$

so the class only has 3 elements.

But if $n=p$ prime & $k \in \{1,2,\ldots,p-1\}$, it will always have exactly $n=p$ elements (easy algebra: need to check that the shift of $S \in \mathcal{P}_k([n])$ is not S itself).

Thus, the relation $\underset{\text{shift}}{\sim}$ subdivides the set $\mathcal{P}_k([n])$ into $\underset{\text{shift}}{\sim}$-equivalence classes, and each of these classes has size p.

Thus, $|\mathcal{P}_k([n])| = (\#\text{ of } \underset{\text{shift}}{\sim}\text{-equiv. classes}) \cdot p$.

Hence, $p \mid |\mathcal{P}_k([n])| = \binom{n}{k} = \binom{p}{k}$.

The proof of $p \mid {p \brace k}$ (for $k \in \{2,3,\ldots,p-1\}$) is similar.

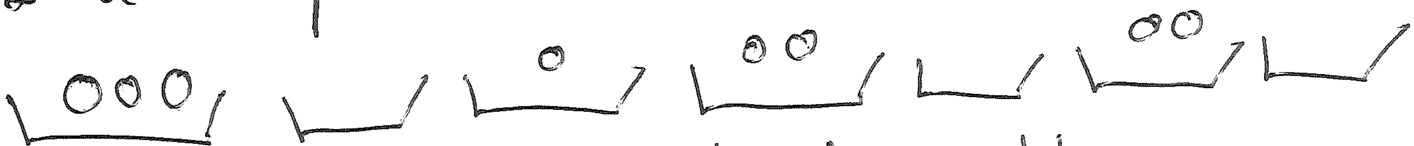
<u>Remark.</u> Let $n \in \mathbb{N}$. The <u>Bell number</u> $B(n)$ is the # of <u>all</u> set partitions of $[n]$. Thus, $B(n) = {n \brace 0} + {n \brace 1} + \cdots + {n \brace n}$.

It satisfies $B(n+1) = \sum_{i=0}^{n} \binom{n}{i} B(i)$. (Exercise.)

See Spring 2018 Math 4707 for more.

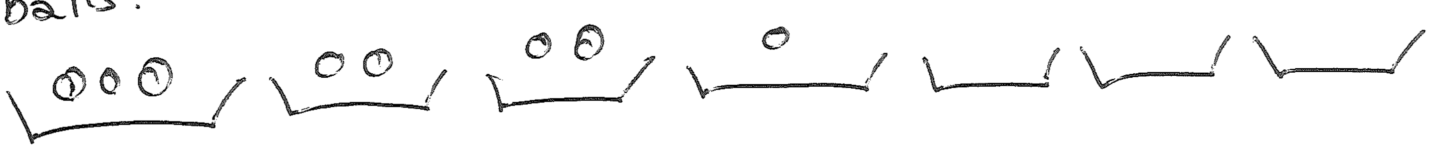
3.6. $\mathcal{U} \to \mathcal{U}$ and integer partitions

Idea: a $\mathcal{U} \to \mathcal{U}$ placement looks like this:



but the boxes, too, are interchangeable.

Thus, we can always order the boxes by decreasing number of balls:



You can encode this $U \to U$ placement by a sequence of numbers, telling how many balls each box gets:

$$(3, 2, 2, 1, 0, 0, 0).$$

The decreasing order makes the sequence unique.

<u>Def.</u> A <u>partition</u> of an integer n is a weakly decreasing list $(a_1, a_2, \dots, a_k)$ of positive integers whose sum is n (that is, $a_1 \geq a_2 \geq \dots \geq a_k > 0$ and $a_1 + a_2 + \dots + a_k = n$).

The integers $a_1, a_2, \dots, a_k$ are called the <u>parts</u> of the partition.

<u>Example:</u> The partitions of 5 are

(5), $(4,1)$, $(3,2)$, $(3,1,1)$, $(2,2,1)$, $(2,1,1,1)$, $(1,1,1,1,1)$.

<u>Rmk:</u> A partition is the same as a weakly decreasing composition.

<u>Def.</u> Let $n \in \mathbb{Z}$ and $k \in \mathbb{N}$. Then, $p_k(n)$ means the # of partitions of n into k parts (= having k parts).

<u>Examples:</u> $p_0(5) = 0$, $p_1(5) = 1$, $p_2(5) = 2$, $p_3(5) = 2$, $p_4(5) = 1$, $p_5(5) = 1$, $p_k(5) = 0 \quad \forall k > 5$.

<u>Prop. 3.13.</u> (a) $p_k(n) = 0 \quad \forall n < 0$.

(b) $p_k(n) = 0$ if $k > n$.

(c) $p_0(n) = [n = 0]$.

(d) $p_1(n) = [n > 0]$.

(e) $p_k(n) = p_k(n-k) + p_{k-1}(n-1) \quad \forall n \in \mathbb{Z}$ & $k \geq 1$.

(f) $p_2(n) = \lfloor n/2 \rfloor \quad \forall n \geq 0$.

<u>Proof.</u> (a) A sum of positive integers is never negative.

(b) A partition into k parts has sum $\geq \underbrace{1 + 1 + \cdots + 1}_{k \text{ ones}} = k$.

(c) The only partition into 0 parts is $()$.

(d) The only partition of n into 1 part is (n), which only exists if $n>0$.

(e) Classify the partitions of n into k parts into 2 types:
<u>Type 1:</u> partitions that have 1 as a part;
<u>Type 2:</u> partitions that don't.

There is a bijection
$$\{\text{Type-1 partitions of } n \text{ into } k \text{ parts}\} \to \{\text{partitions of } n-1 \text{ into } k-1 \text{ parts}\},$$
$$(\lambda_1, \lambda_2, \ldots, \lambda_{k-1}, 1) \longmapsto (\lambda_1, \lambda_2, \ldots, \lambda_{k-1})$$
(because any Type-1 partition must end with a 1).
Thus, ($\#$ of Type-1 partitions) $= p_{k-1}(n-1)$.
Also, there is a bijection
$$\{\text{Type-2 partitions of } n \text{ into } k \text{ parts}\} \to \{\text{partitions of } n-k \text{ into } k \text{ parts}\},$$
$$(\lambda_1, \lambda_2, \ldots, \lambda_k) \longmapsto (\lambda_1 - 1, \lambda_2 - 1, \ldots, \lambda_k - 1).$$

Thus, (# of Type-2 partitions) $= p_k(n-k)$.

Adding these equalities together, we get the claim of (e).

(f) The partitions of n into 2 parts are

$(n-1, 1)$, $(n-2, 2)$, $(n-3, 3)$, ..., $(\underbrace{\lceil n/2 \rceil}_{\text{the ceiling function}}, \lfloor n/2 \rfloor)$.

□

Prop. 3.14. (# of surjective $U \to U$ placements $A \to X$) $= p_{|X|}(|A|)$.

Proof idea. Encode a surjective $U \to U$ placement as a partition ~~of~~ of $|A|$ into $|X|$ parts: namely, the partition

(# of balls in the box with the largest # of balls,
—— // —— second-largest —//—,
—— // —— third-largest —//—,
⋮
).

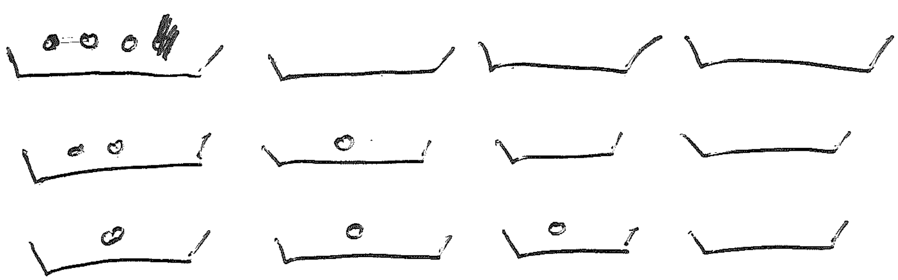
This is a bijection. □

Prop. 3.15. (# of $U \to U$ placements) $= p_0(|A|) + p_1(|A|) + \dots + p_{|X|}(|A|)$

$$= p_{|X|}(|A| + |X|).$$

thanks Kastubh

Ex: $|A| = 3$, $|X| = 4$:



$$p_4(3+4) = \cancel{p_4(3)} + p_3(6)$$

$$= p_3(6) = \underbrace{p_3(3)}_{=1} + p_2(5)$$

$$= 1 + p_2(5) = 1 + \lfloor 5/2 \rfloor = 3$$

Proof. Exercise. $\square$

Prop. 3.16. (# of injective $U \to U$ placements) $= [\,|A| \leqslant |X|\,]$.

Proof. Exercise. $\square$

3.7. Odds & ends

Exercise: Given n persons $(n>0)$ and k tasks $(k>0)$.

(a) What is the # of ways to assign a task to each person such that each task has at least 1 person working on it?

(b) What if we additionally want to choose a leader for each task (among the people assigned to it)?

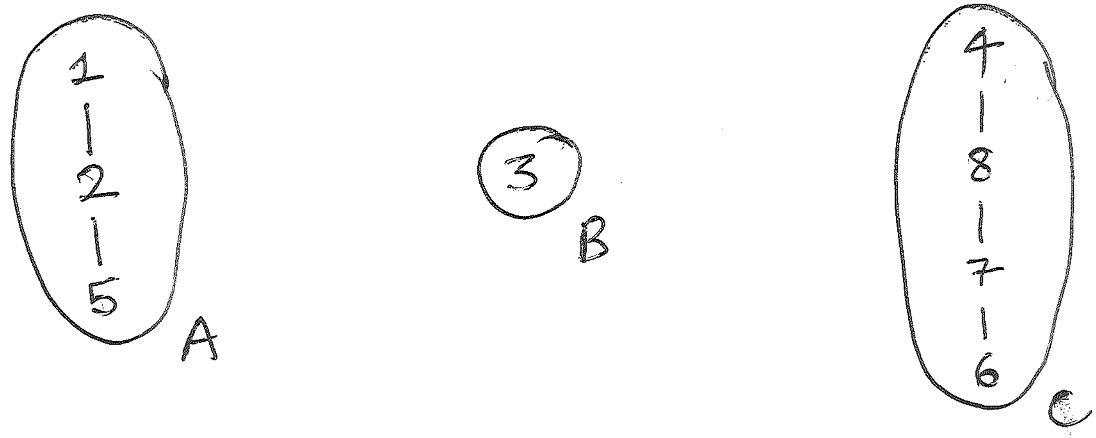
(c) What if, instead, we want to choose a vertical hierarchy (between all people working on the task) for each task?

Example: 8 people $(1, 2, 3, \ldots, 8)$ and 3 tasks (A, B, C).

(a): $\{1\ 2\ 5\}_A$ $\quad \{3\}_B$ $\quad \{4\ 6\ 7\ 8\}_C$

(b): $\{1\ 2\ 5\}_A$ $\quad \{3\}_B$ $\quad \{4\ 6\ 7\ 8\}_C$

(c):



Answers:

(a) $\mathrm{sur}(n, k)$.

(b) $n^{\underline{k}} \cdot k^{n-k}$.

(Pick the $n^{\underline{k}}$ leaders first, then assign the rest of the people arbitrarily.)

(c) $n! \cdot \binom{n-1}{k-1}$.

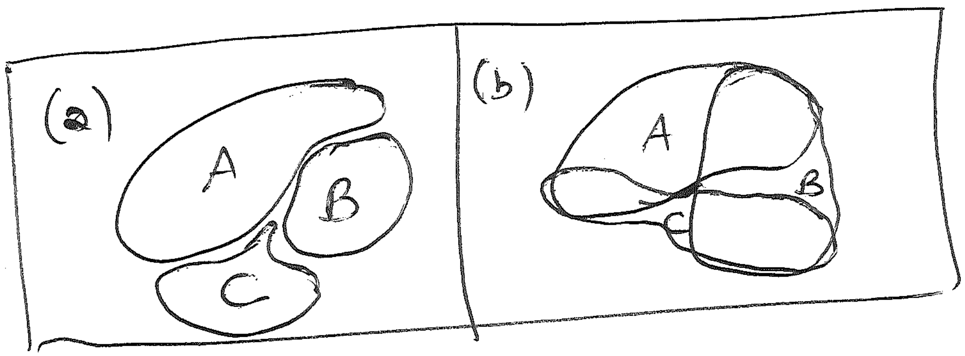
(First, ~~sort~~ order the entire n people, then split the ordering into k nonempty chunks.)

(See ~~also~~ Fall 2017 Math 4990 for details.)

Exercise: Let X be an n-element set.

(a) What is the # of triples (A, B, C) of subsets of X such that $B \cap C = \emptyset$ and $C \cap A = \emptyset$ and $A \cap B = \emptyset$?

(b) What is the # of triples (A, B, C) of subsets of X such that $A \cap B \cap C = \emptyset$?



(See also fall 2017 Math 4550 hw #3 Exercise 1.)

Answers: (a) 4^n.

(For each element $x \in X$, we choose to put x in A or in B or in C or in none of A, B, C; these are 4 options.)

(b) 7^n.

(Now there are 7 options for each $x \in X$.)

Exercise: Let $n \in \mathbb{N}$.

A subset S of $[2n]$ is called shadowed if $\forall$ ~~all~~ $i \in S$ we have $i+1 \in S$.

How many shadowed subsets does $[2n]$ have?

Answer: 3^n.

(For each $k \in [n]$, we choose between having

- $2k-1 \notin S$, $2k \notin S$;
- $2k-1 \notin S$, $2k \in S$;
- $2k-1 \in S$, $2k \in S$.)

Exercise: Let $n \in \mathbb{N}$. How many compositions of n have the property that all entries of the composition are in $\{1,2\}$?

($n = 5 \Rightarrow$ $(1,1,1,1,1)$, $(1,1,1,2)$, $(1,1,2,1)$, $(1,2,1,1)$, $(2,1,1,1)$, $(2,2,1)$, $(2,1,2)$, $(1,2,2)$.)

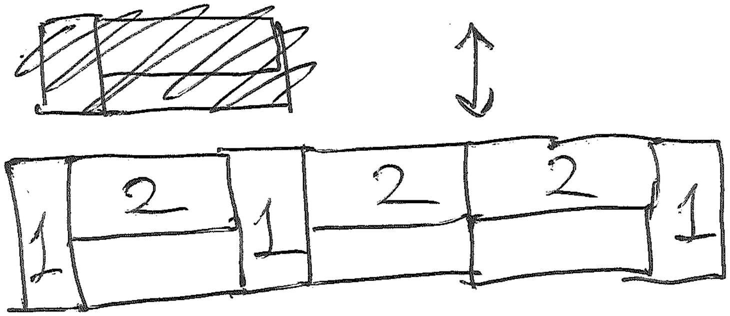
Answer: the Fibonacci number f_{n+1}.

(Proof 1: induction on n.

Proof 2: bijection to lacunar subsets.

Proof 3:

$$8 = 1 + 2 + 1 + 2 + 2 + 1$$



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<u>Exercise:</u> Let $n \in \mathbb{N}$ and $d > 0$.
An n-tuple $(x_1, x_2, \ldots, x_n) \in [d]^n$ is called <u>1-even</u> if the # of i's satisfying $x_i = 1$ is even.
(e.g. $(5, 4, 1, 2, 1)$ and $(5, 4, 2)$ are 1-even, but $(1, 2, 4, 2)$ is not.)

What is the # of 1-even n-tuples?

(This is Spring 2018 Math 4707 homework set #3 exercise 5.)

Answer: $\displaystyle\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} (d-1)^{n-2k}$

(because first choose how many 1's the n-tuple will have;

then choose where to put them;
then choose each of the remaining n-2k entries from d-1 choices)

$$= \sum_{k \text{ even}} \binom{n}{k} (d-1)^{n-k} = \frac{1}{2}\left(d^n + (d-2)^n\right).$$

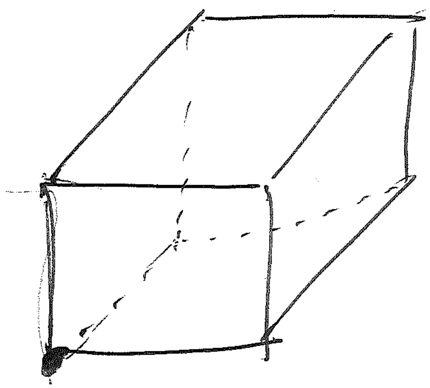
More generally:

$$\sum_{k \text{ even}} \binom{n}{k} x^k y^{n-k} = \frac{1}{2}\left((x+y)^n + (-x+y)^n\right)$$

(by adding $\displaystyle\sum_{k\in\mathbb{Z}} \binom{n}{k} x^k y^{n-k} = (x+y)^n$

with $\displaystyle\sum_{k\in\mathbb{Z}} \binom{n}{k} (-x)^k y^{n-k} = (-x+y)^n$)

Next hw will have an exercise on counting
<u>all-even</u> n-tuples: those in which each number
occurs an even # of times.



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(see also Stanley's "Algebraic Combinatorics".)