3. The twelvefold way

3.1. What is it?

The twelvefold way is a table of $4 \cdot 3 = 12$ standard counting problems that tend to appear often.

Informal description: Given a set A of balls, and a set X of boxes.

A placement is a way to distribute the balls into the boxes.

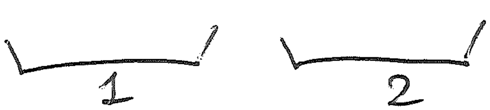
Rigorously: a placement is a map $A \to X$.

At least, this is what we will call an "$L \to L$ placement".

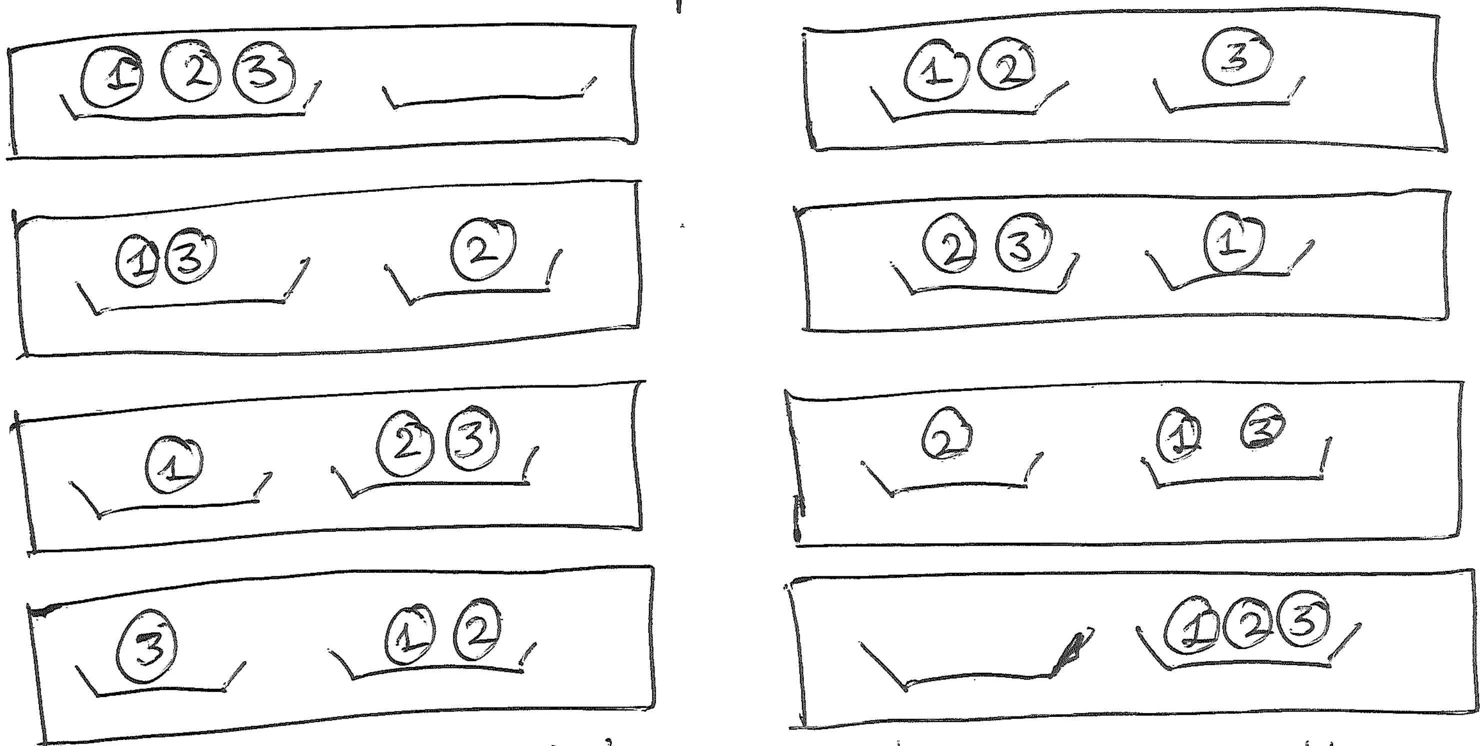
How many such placements are there? $|X|^{|A|}$.

Example: $|X| = 2$, $|A| = 3$. For example, $X = [2]$ and $Y = [3]$.

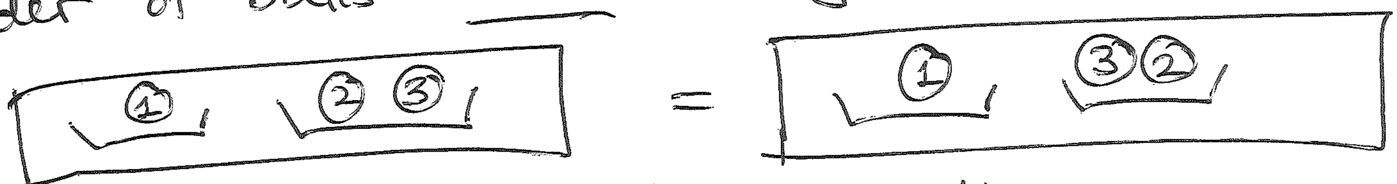
Always draw boxes in increasing order:



Here are the 8 L→L placements:



The order of balls __within__ a single box doesn't matter:

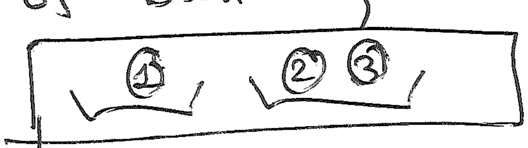
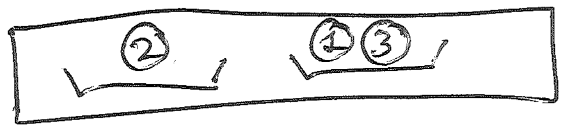


This suggests the following variations:

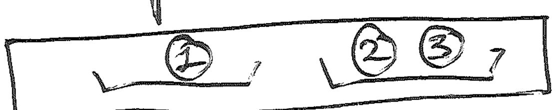
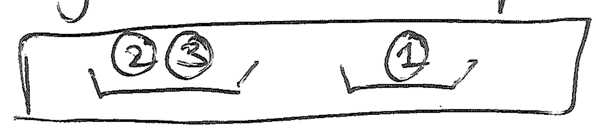
– What if we require $f: A \to X$ to be injective (i.e., each box contains ≤ 1 ball) or surjective (i.e, each box contains ≥ 1 ball)?

- What if the balls are unlabelled (i.e., indistinguishable)?

To make this rigorous, we will use equivalence classes: We'll say that two maps $f : A \to X$ are "ball-equivalent" if one is sent to the other by a permutation of the balls. Then, "$U \to L$ placements" (= placements of unlabelled balls in labelled boxes) will be equivalence classes of ball-equivalence.

E.g.:  and  are ball-equivalent, thus contributing to only 1 equivalence classes.

- What if the boxes are unlabelled? Similar answer, but now we permute boxes. This gives "$L \to U$ placements".

E.g.:  and  are box-equivalent.

- What if both boxes & balls are indistinguishable?

So we get $3 \cdot 4 = 12$ different counting problems.

We list them in a table:

$A \to X$ \ f is	arbitrary	injective	surjective
$L \to L$	$\lvert X \rvert^{\lvert A \rvert}$		
$U \to L$			
$L \to U$			
$U \to U$			

Here:

- $L \to L$ means "balls are labelled, boxes are labelled", so we are just counting maps $f: A \to X$
- $U \to L$ means "balls are unlabelled, boxes are labelled".
- $L \to U$ —//— labelled —//— unlabelled
- $U \to U$ —//— unlabelled —//— unlabelled

Plan: fill in the remaining 11 entries of the table.

Ex: $|X| = 2$, $|A| = 3$.

$A \to X$ \ f is	arbitrary	injective	surjective
$L \to L$	8	0	6
$U \to L$	4	0	2
$L \to U$	4	0	3
$U \to U$	2	0	1

In general: Not each of the 12 question has a closed-form solution. But there are good recurrences at least.

3.2. $L \to L$

$L \to L$ placements are just maps $A \to X$.

Prop. 3.1. (# of $L \to L$ placements $A \to X$) $= |X|^{|A|}$.

Proof. This is Thm. 2.4. $\square$

Prop. 3.2. (# of injective $L \to L$ placements $A \to X$)
$=$ (# of injective maps $A \to X$) $= |X|^{\underline{|A|}}$.

<u>Proof.</u> This is Thm. 2.5. $\square$

<u>Prop. 3.3.</u> (# of surjective L→L placements $A \to X$)
$= $ (# of surjective maps $A \to X$) $= \operatorname{sur}(|A|, |X|)$.

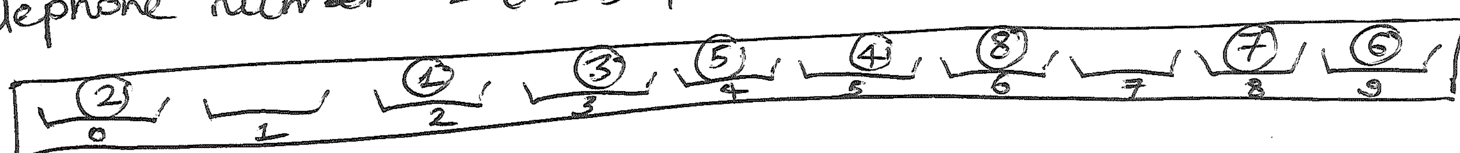
<u>Proof.</u> This is Prop. 2.9. $\square$

Typical applications of L→L placements:

- assigning grades (from a finite set X) to students (from a finite set A) : L→L placements (arbitrary).
- assigning IP addresses to computers: injective L→L placements.
- How many 8-digit telephone numbers are there with no 2 equal digits?

$\hat{=}$ injective L→L placements with $A = [8]$ and $X = \{0, 1, \ldots, 9\}$

(telephone number 20354986

$\hat{=}$ )

$\Rightarrow$ the # of such numbers is $10^{\underline{8}} = 10 \cdot 9 \cdot \ldots \cdot 3$
$= 10!/2!$.

<u>Remark:</u> Here is a quick problem NOT from the above table:
How many 8-digit telephone numbers have no two adjacent equal digits?
(e.g., 31315315 is okay, but 12<u>33</u>4567 is not.)

Answer:

$$\underbrace{10}_{\substack{\text{options}\\\text{for 1st}\\\text{digit}}} \cdot \underbrace{9}_{\substack{\text{options}\\\text{for 2nd}\\\text{digit}}} \cdot \underbrace{9}_{\substack{\text{options}\\\text{for 3rd}\\\text{digit}}} \cdot 9 \cdot \ldots \cdot 9$$

$= 10 \cdot 9^7$.

(These are the "smords" (Smirnov words, or Carlitz words) from MT1 Exercise 6.)

3.3. Unlabelled objects

What does it mean for balls, or boxes, to be unlabelled?

Rigorously, it means that we are counting

NOT the maps $f : A \to X$,

BUT their equivalence classes wrt. some relation.
with respect to

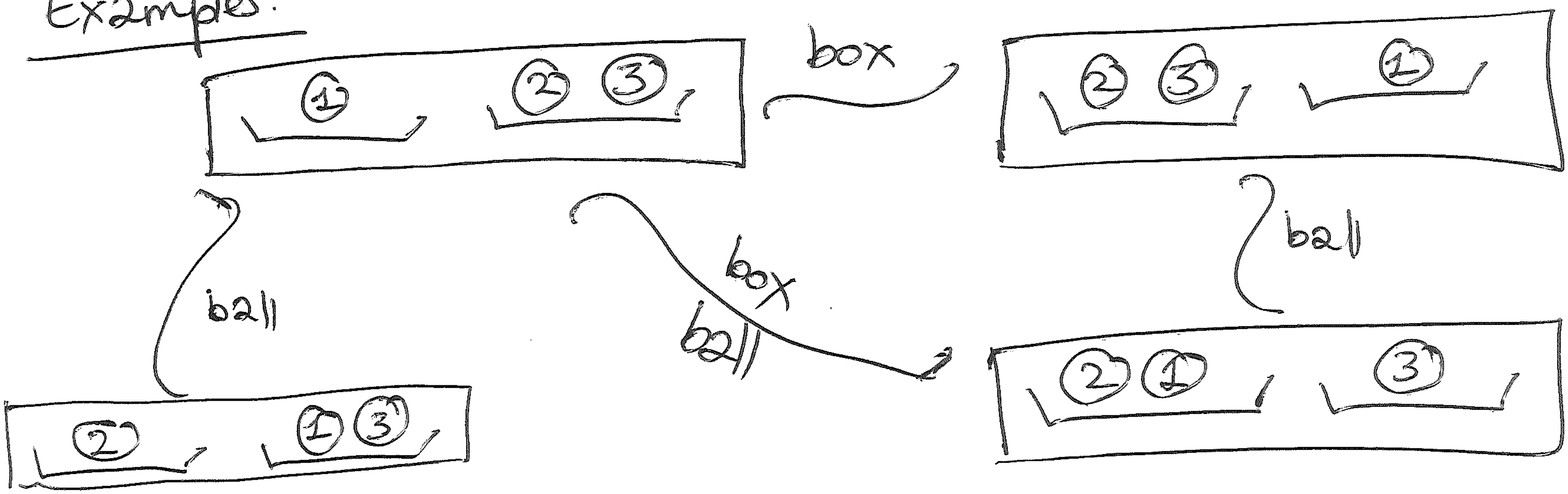
What relation?

Def. Let $f, g : A \to X$. Then we say that

- f is box-equivalent to g (written $f \overset{\text{box}}{\sim} g$) if & only if $\exists$ permutation σ of X such that $f = \sigma \circ g$ (in other words, f can be obtained from g by permuting boxes).
- f is ball-equivalent to g (written $f \overset{\text{ball}}{\sim} g$) if & only if $\exists$ permutation τ of A such that $f = g \circ \tau$ (in other words, f can be obtained from g by permuting balls).

- f is <u>box-ball-equivalent to</u> g (written $f \underset{\text{ball}}{\overset{\text{box}}{\sim}} g$) if & only if $\exists$ a permutation σ of X & a permutation τ of A such that $f = \sigma \circ g \circ \tau$.

<u>Examples:</u>



All of $\overset{\text{box}}{\sim}$, $\overset{\text{ball}}{\sim}$ and $\underset{\text{ball}}{\overset{\text{box}}{\sim}}$ are equivalence relations.

Now, we define:

- <u>U→L placements</u> as $\overset{\text{ball}}{\sim}$-equivalence classes;
- <u>L→U placements</u> as $\overset{\text{box}}{\sim}$-equivalence classes;
- <u>U→U placements</u> as $\underset{\text{ball}}{\overset{\text{box}}{\sim}}$-equivalence classes.

Ex: In our running example with $X=[2]$ and $A=[3]$, the $\overset{\text{box}}{\sim}$-equivalence class of $\boxed{\underline{①}, \ \underline{②③}}$ is

$$\left\{ \boxed{\underline{①}, \ \underline{②③}} , \ \boxed{\underline{②③}, \ \underline{①}} \right\},$$

whereas the $\overset{\text{ball}}{\sim}$-equivalence class of $\boxed{\underline{①}, \ \underline{②③}}$ is

$$\left\{ \boxed{\underline{①}, \ \underline{②③}} , \ \boxed{\underline{②}, \ \underline{①③}} , \ \boxed{\underline{③}, \ \underline{①②}} \right\}.$$

Recall the following crucial fact about equivalence classes:

Prop. 3.4. Let $\sim$ be an equivalence relation ~~[illegible]~~ on a set S. Let $x \in S$ and $y \in S$. Then, $x \sim y$ if & only if the $\sim$-equivalence classes of x and y are identical.

Thus, "counting elements of S up to equivalence" means counting

$\sim$-equivalence classes.

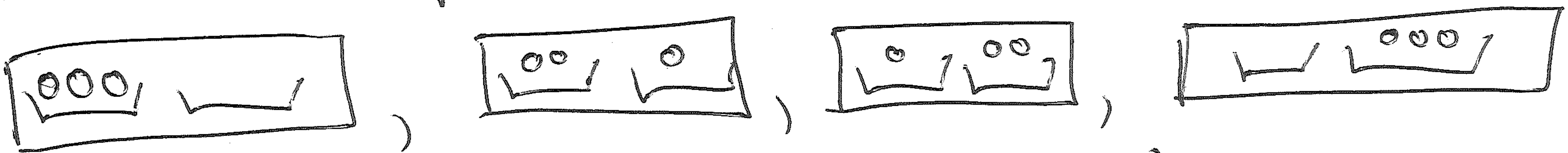
3.4. $\mathcal{U} \to \mathcal{L}$

Recall: $\mathcal{U} \to \mathcal{L}$ placements are $\overset{\text{ball}}{\sim}$-equivalence classes.

Ex: For $X = [2]$ and $A = [3]$, here are the $\mathcal{U} \to \mathcal{L}$ placements (drawn as blobs):

Notation: When visualizing a U→L placement, we just draw the balls as circles, without putting any numbers in them.

So the 4 U→L placements for $X=[2]$ and $A=[3]$ are



Prop. 3.5. (# of U→L placements $A \to X$)

$= (\#\text{ of } (x_1, x_2, \ldots, x_{|X|}) \in \mathbb{N}^{|X|}$ satisfying $x_1 + x_2 + \cdots + x_{|X|} = |A|)$

$$= \binom{|A|+|X|-1}{|A|}.$$

Proof. 1st equality: First, we WLOG assume that $X = [|X|]$.

Now, consider the bijection

$\{U \to L \text{ placement}\} \to \{\text{weak composition of } |A| \text{ into } |X| \text{ parts}\}$,

$$\boxed{\underbrace{\quad}_{a_1 \text{ balls}}, \underbrace{\quad}_{a_2 \text{ balls}}, \ldots, \underbrace{\quad}_{a_{|X|} \text{ balls}}} \longmapsto (a_1, a_2, \ldots, a_{|X|}).$$

(More rigorously: The $\overset{\text{ball}}{\sim}$-equivalence of $f : A \to X$ is sent to the weak composition $(|f^{-1}(1)|, |f^{-1}(2)|, \ldots, |f^{-1}(|X|)|)$.

To see that this is a bijection, one needs to show that if $f, g : A \to X$ are two maps such that

$$|f^{-1}(x)| = |g^{-1}(x)| \quad \text{for all } x \in X,$$

then $f \overset{\text{ball}}{\sim} g$.)

Thus, (# of $U \to L$ placements $A \to X$)

$=$ (# of weak compositions of $|A|$ into $|X|$ parts)

$=$ (# of $(x_1, x_2, \ldots, x_{|X|}) \in \mathbb{N}^{|X|}$ satisfying $x_1 + x_2 + \cdots + x_{|X|} = |A|$).

<u>2nd equality:</u> Thm. 2.36. $\square$

Prop. 3.6. (# of surjective $U \to L$ placements)

= (# of $(x_1, x_2, \ldots, x_{|X|}) \in \mathbb{P}^{|X|}$ satisfying $x_1 + x_2 + \ldots + x_{|X|} = |A|$)

$$= \begin{cases} \dbinom{|A|-1}{|X|-1} & \text{if } |A| \geq 1; \\ [|X| = 0] & \text{if } |A| = 0 \end{cases} = \dbinom{|A|-1}{|A|-|X|}.$$

(Here, $\mathbb{P} = \{1, 2, 3, \ldots\}$.)

Proof. 1st equality: same argument as in Prop. 3.5.

2nd equality: Thm. 2.34.

3rd equality: If $|A| = 0$, this is easy to verify. If not, use the symmetry of Pascal's triangle. $\square$

Prop. 3.7. (# of injective $\mathcal{U}\to\mathcal{L}$ placements)

$= \left(\#\text{ of } (x_1, x_2, \dots, x_{|X|}) \in \{0,1\}^{|X|} \text{ satisfying } x_1+x_2+\dots+x_{|X|} = |A|\right)$

$= \binom{|X|}{|A|}.$

Proof. 1st equality: same argument as in Prop. 3.5.

2nd equality: Thm. 2.35. $\square$

3.5. $\mathcal{L}\to\mathcal{U}$

Recall: an $\mathcal{L}\to\mathcal{U}$ placement is a $\overset{\text{box}}{\sim}$-equivalence class.

EX: With $X=[2]$ and $A=[3]$, the $\mathcal{L}\to\mathcal{U}$ placements are

Prop. 3.8. (# of injective $L \to U$ placements) $= [|A| \leq |X|]$.

Proof. If $|A| \leq |X|$, then such placements exist, and are identical; e.g.

(since they each consist of 1 box with "ball 1", 1 box with "ball 2", ..., 1 box with "ball $|A|$", and $|X| - |A|$ empty boxes); thus, the # is 1.

If $|A| > |X|$, then no such placements exist (by Pigeonhole). $\square$

Recall: If $n \in \mathbb{N}$ and $k \in \mathbb{N}$, then $\left\{ {n \atop k} \right\} := \dfrac{\operatorname{sur}(n,k)}{k!}$ is called a <u>Stirling number of the 2nd kind</u>.

<u>Prop. 3.9.</u> (# of surjective $L \to U$ placements)

$$= \left\{ {|A| \atop |X|} \right\} = \frac{\operatorname{sur}(|A|, |X|)}{|X|!} .$$

<u>Proof.</u> <u>Claim 1:</u> Each ~~box~~ $\overset{\text{box}}{\sim}$-equivalence class of surjective maps contains exactly $|X|!$ many maps $A \to X$.

[<u>Proof:</u> Consider the box-equivalence class of some surjective map $g : A \to X$. Then, the elements of this class are all maps of the form $\theta \circ g$ with θ a permutation of X. There are $|X|!$ many permutations of X, and they lead to <u>distinct</u> maps $\theta \circ g$

$$\left(\underset{g}{\boxed{\underline{14},\ \underline{2},\ \underline{356}}} \longmapsto \underset{\theta \circ g}{\boxed{\underline{2},\ \underline{356},\ \underline{14}}} \right.$$

can only happen if $\theta = (3, 1, 2)$ in one-line notation $\Big)$.

Thus, there are exactly $|X|!$ many distinct maps $\sigma \circ g$ in the class of g. This proves Claim 1.]

Now,

$$\operatorname{sur}(|A|, |X|)$$

$$= (\# \text{ of surjections } A \to X)$$

$$= \sum_{\substack{\overset{\text{box}}{\sim}\text{-equivalence} \\ \text{classes } C}} \underbrace{|C|}_{\substack{= |X|! \\ (\text{by Claim 1})}}$$

(since each surjection $A \to X$ lies in exactly one $\overset{\text{box}}{\sim}$-equivalence class)

$$= \sum_{\substack{\overset{\text{box}}{\sim}\text{-equivalence} \\ \text{classes } C}} |X|! = |X|! \cdot (\# \text{ of } \overset{\text{box}}{\sim}\text{-equivalence classes}).$$

Hence,

$$(\#\text{ of } \overset{\text{box}}{\sim}\text{-equivalence classes}) = \frac{\operatorname{sur}(|A|, |X|)}{|X|!} = \left\{ \begin{matrix} |A| \\ |X| \end{matrix} \right\}.$$ $\square$

<u>Prop. 3.10.</u> $(\#\text{ of all } L \to U \text{ placements})$

$$= \left\{ \begin{matrix} |A| \\ 0 \end{matrix} \right\} + \left\{ \begin{matrix} |A| \\ 1 \end{matrix} \right\} + \cdots + \left\{ \begin{matrix} |A| \\ |X| \end{matrix} \right\}.$$

<u>Proof.</u> ~~More~~ Better: $\forall k \in \mathbb{N}$, the # of all $L \to U$ placements that ~~leave exa~~ use exactly k boxes is $\left\{ \begin{matrix} |A| \\ k \end{matrix} \right\}$.

(Details LTTR.) $\square$

A bit more on Stirling numbers of the 2nd kind:

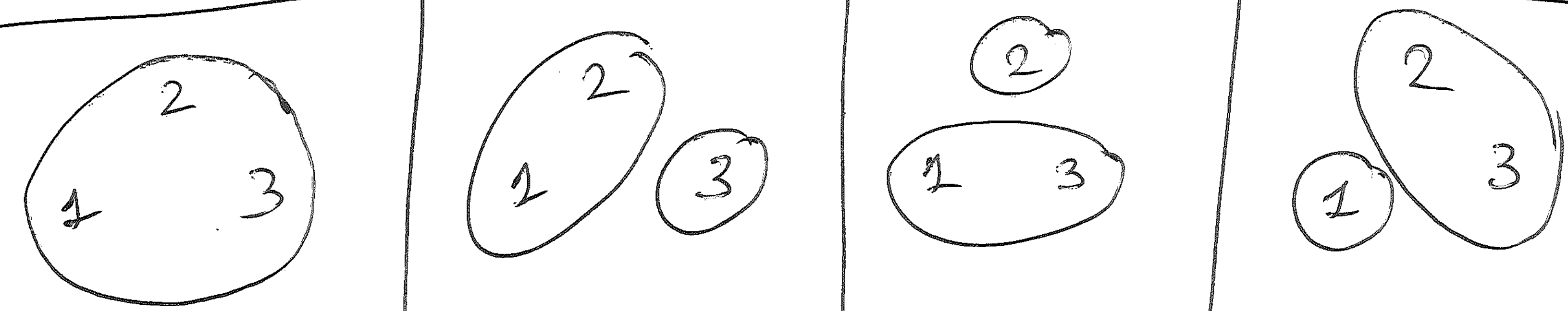
<u>Def.</u> Let S be a set.

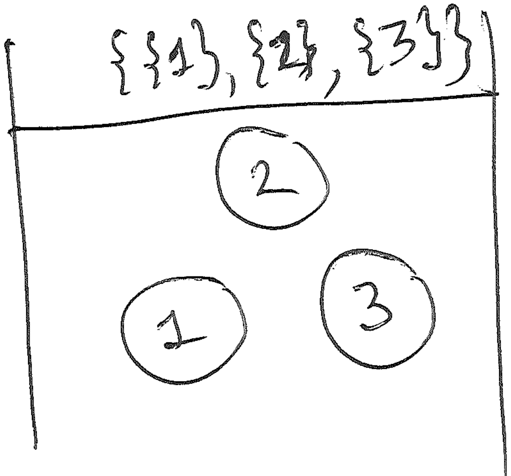
A <u>set partition</u> of S is a set ~~A~~ $\mathcal{F}$ of nonempty disjoint subsets of S such that the union of these subsets in S.

In other words, a set partition of S is a set $\{S_1, S_2, S_3, \ldots\}$ of nonempty subsets of S such that each element of S lies in exactly one S_i.

Ex: The set partitions of $[3]$ are

$\{\{1,2,3\}\}$	$\{\{1,2\},\{3\}\}$	$\{\{1,3\},\{2\}\}$	$\{\{2,3\},\{1\}\}$





Def: If ~~S~~ $\mathcal{F}$ is a set partition of S, then the elements of $\mathcal{F}$ are called the *blocks* (or *parts*) of $\mathcal{F}$.
They are subsets of S.

Prop. 3.11. Let X be an n-element set. Let $k \in \mathbb{N}$.
Then, the # of set partitions of X into k parts is $\left\{ {n \atop k} \right\}$.

Proof. These set partitions "are" (= are in bijection with) surjective L→U placements $X \to [k]$.
(The blocks of any set partition are the sets of balls in the boxes.) $\square$