Proof of Prop. 2.31:

$$\binom{n}{\equiv 0 \bmod 2} - \binom{n}{\equiv 1 \bmod 2} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} = [n=0];$$

$$\binom{n}{\equiv 0 \bmod 2} + \binom{n}{\equiv 1 \bmod 2} = \sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

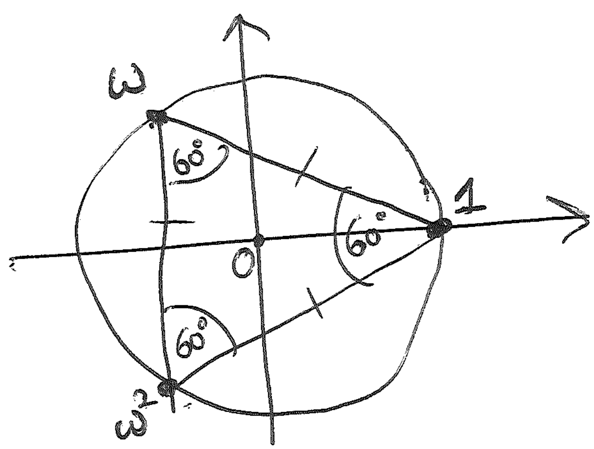
Add/subtract these.

Rmk: We've used the binomial formula for $(1-1)^n$ and for $(1+1)^n$.

Proof of Prop. 2.32 (outline).

Rename i as j. Set $i = \sqrt{-1} \in \mathbb{C}$.

Let $\omega = e^{2\pi i/3}$. Then, the 3 roots of the polynomial $X^3 - 1$ are $1, \omega, \omega^2$. Note also $\omega^2 + \omega + 1 = 0$.



(1st proof: $\Delta 1 \omega \omega^2$ is equilateral

$\Rightarrow$ 0 is its centroid

$\Rightarrow \frac{\omega^2 + \omega + 1}{3} = 0 \Rightarrow \omega^2 + \omega + 1 = 0.$

2nd proof: $\omega^3 = 1$, but $\omega \neq 1$.

Now $\omega^2 + \omega + 1 = \frac{\omega^3 - 1}{\omega - 1} = \frac{1-1}{\text{nonzero}} = 0$))

The binomial formula yields

$$(1+\omega)^n = \sum_k \binom{n}{k} \underbrace{\omega^k}_{=\begin{cases}1 \text{ if } k\equiv 0 \bmod 3\\ \omega \text{ if } k\equiv 1 \bmod 3\\ \omega^2 \text{ if } k\equiv 2 \bmod 3\end{cases}} = \sum_{k\equiv 0 \bmod 3} \binom{n}{k} 1 + \sum_{k\equiv 1 \bmod 3} \binom{n}{k} \omega + \sum_{k\equiv 2 \bmod 3} \binom{n}{k} \omega^2$$

$$= 1 \binom{n}{\equiv 0 \bmod 3} + \omega \binom{n}{\equiv 1 \bmod 3} + \omega^2 \binom{n}{\equiv 2 \bmod 3}.$$

Similarly,

$$(1+\omega^2)^n = 1 \binom{n}{\equiv 0 \bmod 3} + \omega^2 \binom{n}{\equiv 1 \bmod 3} + \omega \binom{n}{\equiv 2 \bmod 3}$$

and

$$(1+1)^n = 1 \binom{n}{\equiv 0 \bmod 3} + 1 \binom{n}{\equiv 1 \bmod 3} + 1 \binom{n}{\equiv 2 \bmod 3}.$$

Thus,

$$\begin{pmatrix} (1+1)^n \\ (1+\omega)^n \\ (1+\omega^2)^n \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}}_{=:\, T} \begin{pmatrix} \binom{n}{\equiv 0 \bmod 3} \\ \binom{n}{\equiv 1 \bmod 3} \\ \binom{n}{\equiv 2 \bmod 3} \end{pmatrix}$$

(the discrete Fourier transform)

Known: $T^{-1} = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix}$.

Thus,

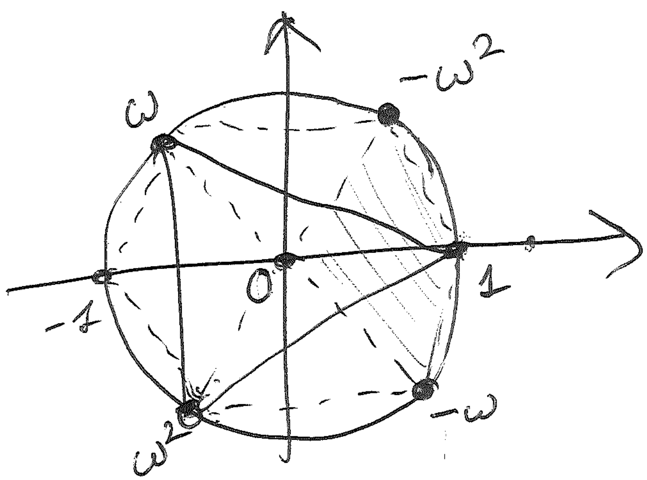
$$\begin{pmatrix} \binom{n}{\equiv 0 \bmod 3} \\ \binom{n}{\equiv 1 \bmod 3} \\ \binom{n}{\equiv 2 \bmod 3} \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix} \begin{pmatrix} (1+1)^n \\ (1+\omega)^n \\ (1+\omega^2)^n \end{pmatrix}$$

Hence,

$$\binom{n}{\equiv 0 \bmod 3} = \frac{1}{3}\Big(1\,\underbrace{(1+1)^n}_{=2^n} + 1\,(\underbrace{1+\omega}_{\substack{=-\omega^2 \\ (\text{since } \omega^2+\omega+1=0)}})^n + (\underbrace{1+\omega^2}_{\substack{=-\omega \\ (\text{since } \omega^2+\omega+1=0)}})^n\Big)$$

$$= \frac{1}{3}\Big(2^n + \underbrace{(-\omega^2)^n + (-\omega)^n}_{= \begin{cases} 2 & \text{if } n \equiv 0 \bmod 6 \\ 1 & \text{if } n \equiv 1 \bmod 6 \\ -1 & \text{if } n \equiv 2 \bmod 6 \\ -2 & 3 \\ -1 & 4 \\ 1 & 5 \end{cases}}\Big) \qquad \Big(\approx \frac{1}{3} 2^n \text{ by the way}\Big)$$

(here, we use $\omega^3 = 1$ and $(-\omega)^6 = 1$ and $\omega^2+\omega+1=0$)



$$= \frac{1}{3}\Big(2^n + \begin{cases} 2 & \text{if } n \equiv 0 \bmod 6 \\ 1 & 1 \\ -1 & 2 \\ -2 & 3 \\ -1 & 4 \\ 1 & 5 \end{cases}\Big)$$

by checking all cases

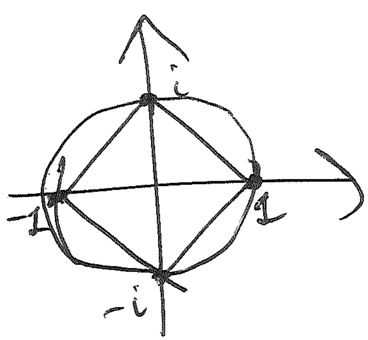
$$= \frac{2^n - (-1)^n}{3} + (-1)^n [n \equiv 0 \bmod 3].$$

Similarly for $\binom{n}{\equiv 1 \bmod 3}$ & $\binom{n}{\equiv 2 \bmod 3}$. $\square$

<u>Prop. 2.33.</u> For any $n>0$, we have

$$\binom{n}{\equiv 0 \bmod 4} = \begin{cases} 2^{n-2} & \text{if } n \equiv 2 \bmod 4 \quad (\leftarrow \text{neat exercise}) \\ 2^{n-2} + (-4)^{n/4}/2 & \text{if } n \equiv 0 \bmod 4 \\ 2^{n-2} + 2^{(n-3)/2} & \text{if } n \equiv 1 \bmod 8 \text{ or } n \equiv 7 \bmod 8 \\ 2^{n-2} - 2^{(n-3)/2} & \text{if } n \equiv 3 \bmod 8 \text{ or } n \equiv 5 \bmod 8 \end{cases}$$

(Proof somewhat similar using $e^{2\pi i/4} = i$.)



2.9. Compositions, weak compositions & multisets

How many ways are there to write 5 as a sum of 3 positive integers, if the order matters? 6 ways, namely:

$$5 = 1+1+3 = 1+3+1 = 3+1+1$$
$$= 2+2+1 = 2+1+2 = 1+2+2.$$

Theorem 2.34. Let $\mathbb{P} = \{1,2,3,\ldots\}$. Let $n, k \in \mathbb{N}$. Then,

$$(\# \text{ of } (x_1,\ldots,x_k) \in \mathbb{P}^k \text{ satisfying } x_1+\cdots+x_k = n)$$
$$= \begin{cases} \binom{n-1}{k-1} & \text{if } n \geq 1; \\ [k=0] & \text{if } n = 0 \end{cases}$$

Rmk. A composition is a tuple of positive integers. A composition into k parts is a k-tuple of positive integers. A composition of n is a tuple of positive integers whose sum is n. Thus, Thm. 2.34 says:

(# of compositions of n into k parts)

$$= \begin{cases} \binom{n-1}{k-1} & \text{if } n \geq 1; \\ [k=0] & \text{if } n=0. \end{cases}$$

<u>Proof of Thm. 2.34.</u> WLOG assume $n \geq 1$ (else, trivial).

For each ~~comp~~ k-tuple $(x_1, x_2, \ldots, x_k) \in P^k$ satisfying $x_1 + \ldots + x_k = n$, set

$$D(x_1, \ldots, x_k) := \{x_1 + \cdots + x_j \mid j \in [k-1]\}$$
$$= \{x_1,$$
$$x_1 + x_2,$$
$$\vdots$$
$$x_1 + x_2 + \ldots + x_{k-1}\};$$

this is a subset of $[n-1]$, and has $k-1$ elements.

The map

~~$\{$~~ $\{k$-tuples $(x_1, \ldots, x_k) \in P^k$ satisfying $x_1 + \cdots + x_k = n\}$

$\to \{(k-1)$-elt. subsets of $[n-1]\}$,

$(x_1, \ldots, x_k) \mapsto D(x_1, \ldots, x_k)$

is a bijection (the inverse map sends any $(k-1)$-elt. subset $\{s_1 < \cdots < s_{k-1}\}$ of $[n-1]$ to $(s_1 - s_0, s_2 - s_1, s_3 - s_2, \ldots, s_k - s_{k-1})$, where $s_0 = 0$ and $s_k = n$). (See HW0 for details.)

Thus,

(# of k-tuples $(x_1, \ldots, x_k) \in P^k$ satisfying $x_1 + \cdots + x_k = n$)

$=$ (# of $(k-1)$-elt. subsets of $[n-1]$)

$= \binom{n-1}{k-1}$ (by comb. interpr. of binom. coeffs., since $n-1 \in \mathbb{N}$). $\square$

<u>Theorem 2.35.</u> Let $n, k \in \mathbb{N}$. Then,

(# of $(x_1, x_2, \ldots, x_k) \in \{0,1\}^k$ satisfying $x_1 + \cdots + x_k = n$)

$= \binom{k}{n}$.

Proof. To construct such a k-tuple, we just need to choose which n indices $i \in [k]$ will have $x_i = 1$.
There are $\binom{k}{n}$ many options. $\square$

Theorem 2.36. Let $n, k \in \mathbb{N}$. Then,

$$(\#\text{ of } (x_1, x_2, \ldots, x_k) \in \mathbb{N}^k \text{ satisfying } x_1 + \cdots + x_k = n)$$

$$= \binom{n+k-1}{n} = \begin{cases} \binom{n+k-1}{k-1} & \text{if } k > 0; \\ [n=0] & \text{if } k = 0 \end{cases}.$$

Rmk. Tuples of nonnegative integers are called weak compositions.

Ex: For $n = 2$ and $k = 3$, the # of weak compositions of 2 into 3 parts is $\binom{2+3-1}{2} = \binom{4}{2} = 6$, and they are

$$2 = 0 + 0 + 2 = 0 + 2 + 0 = 2 + 0 + 0$$
$$= 1 + 1 + 0 = 1 + 0 + 1 = 0 + 1 + 1$$

Proof of Thm. 2.36. WLOG assume that $k > 0$ and $n > 0$ (the other cases are easy).

The map

$$\{(x_1, \ldots, x_k) \in \mathbb{N}^k \mid x_1 + \cdots + x_k = n\}$$
$$\to \{(x_1, \ldots, x_k) \in \mathbb{P}^k \mid x_1 + \cdots + x_k = n + k\},$$
$$(x_1, \ldots, x_k) \mapsto (x_1 + 1, \ldots, x_k + 1)$$

is a bijection. Thus,

$$(\# \text{ of } (x_1, \ldots, x_k) \in \mathbb{N}^k \text{ satisfying } x_1 + \cdots + x_k = n)$$
$$= (\# \text{ of } (x_1, \ldots, x_k) \in \mathbb{P}^k \text{ satisfying } x_1 + \cdots + x_k = n + k)$$
$$\underset{\text{instead of } n)}{\overset{\text{Thm. 2.34 (for } n+k}{=}} \begin{cases} \dbinom{n+k-1}{k-1} & \text{if } n+k \geq 1; \\ [k=0] & \text{if } n+k=0 \end{cases}$$
$$= \binom{n+k-1}{k-1} \overset{\text{symmetry}}{=} \binom{n+k-1}{n}. \qquad \square$$

<u>Def.</u> Let S be a set.

A <u>finite multisubset of S</u> is a map $S \to \mathbb{N}$.

We regard such a map $f : S \to \mathbb{N}$ as a "set with multiplicities", in which each $s \in S$ appears $f(s)$ many times.

For example, "the multisubset $\{1,4,4,5,7,7,7\}$ of $[8]$" is encoded as the map $[8] \to \mathbb{N}$ with the following values.

i	1	2	3	4	5	6	7	8
$f(i)$	1	0	0	2	1	0	3	0

.

<u>Corollary 2.37.</u> Let $n, k \in \mathbb{N}$. Let S be a k-elt. set.

Then, the # of multisubsets of S having size n is $\binom{n+k-1}{n}$.

(Here, the <u>size</u> of a multisubset f is $\sum_{s \in S} f(s)$.)

<u>Proof.</u> Let $s_1, s_2, \ldots, s_k$ be the k elements of S.

Then, the map

$$\{\text{multisubsets of } S \text{ having size } n\} \to \{(x_1, \ldots, x_k) \in \mathbb{N}^k \mid x_1 + \cdots + x_k = n\},$$

$$f \mapsto (f(s_1), \ldots, f(s_k))$$ is a bijection.

Now, use Theorem 2.36. $\square$

<u>Exercise.</u> Let $m \in \mathbb{N}$. Let $a, b \in \{0, 1, \ldots, m\}$.

Prove that the # of lacunar subsets of $[2m]$ with exactly a even & b odd elements is

$$\binom{m-a}{b} \cdot \binom{m-b}{a}.$$

Solution sketch using multisets:

Let S be a lacunar subset of $[2m]$ with exactly a even and b odd elts. Write S as $S = \{s_1 < s_2 < \cdots < s_{a+b}\}$.

Then,
$$s_1 - 0 \leq s_2 - 2 \leq s_3 - 4 \leq \cdots \leq s_{a+b} - 2(a+b-1).$$

Consider the multi~~set~~ subset

$$M_S := \{s_1 - 0, s_2 - 2, s_3 - 4, \ldots, s_{a+b} - 2(a+b-1)\}_{\text{multiset}}$$

of $[2m - 2(a+b-1)]$. It has exactly ~~a~~ a even & b odd elements. So we can split M_S into a "multiset union"

$M_{S,\text{even}} \mathbin{\dot\cup} M_{S,\text{odd}}$ (like a union of sets, but multiplicities get added), where

$M_{S,\text{even}}$ is a size-a multisubset of $\{2,4,6,\ldots,2m-\cancel{\ }2(a+b-1)\}$,

and where
$M_{S,\text{odd}}$ is a size-b multisubset of $\{1,3,5,\ldots,2m-2(a+b-1)-1\}$.

Moreover, this encoding

$\{\text{lacunar subsets of } [2m] \text{ with exactly } a \text{ even } \& \ b \text{ odd elts}\}$
$\to \{\text{size-}a \text{ multisubsets of } \{2,4,6,\ldots,2m-2(a+b-1)\}\}$
$\times \{\text{size-}b \text{ multisubsets of } \{1,3,5,\ldots,2m-2(a+b-1)-1\}\}$,

$S \mapsto (M_{S,\text{even}}, M_{S,\text{odd}})$

is a bijection. Thus,
$|\{\text{lacunar subsets of } [2m] \text{ with exactly } a \text{ even } \& \ b \text{ odd elts}\}|$

$$= \underbrace{\left|\left\{\text{size-}a \text{ multisubsets of } \{2,4,6,\ldots,2m-2(a+b-1)\}\right\}\right|}_{\overset{\text{Cor. 2.37}}{=} \binom{a+(m-a-b+1)-1}{a} = \binom{m-b}{a}}$$

$$\cdot \underbrace{\left|\left\{\text{size-}b \text{ multisubsets of } \{1,3,5,\ldots,2m-2(a+b-1)-1\}\right\}\right|}_{\overset{\text{Cor. 2.37}}{=} \binom{b+(m-a-b+1)-1}{b} = \binom{m-a}{b}}$$

$$= \binom{m-b}{a}\binom{m-a}{b}.$$

For other proofs, see Math 4707 Spring 2018 HW2 exercise 3(2).

2.9. Multinomial coefficients.

Def. Let $n, n_1, \dots, n_k \in \mathbb{N}$ be such that $n_1 + n_2 + \dots + n_k = n$.

Then,
$$\binom{n}{n_1, \dots, n_k} := \frac{n!}{n_1! \dots n_k!}.$$

Rmk:
- We are not defining this for negative or non-integer n.
- If $m \in \{0, 1, \dots, n\}$, then $\binom{n}{m} = \binom{n}{m, n-m}$.

Prop. 2.38. Let $n, n_1, \dots, n_k \in \mathbb{N}$ be such that $n_1 + \dots + n_k = n$.

(2)
$$\binom{n}{n_1, \dots, n_k} = \prod_{i=1}^{k} \binom{n - n_1 - n_2 - \dots - n_{i-1}}{n_i}$$
$$= \binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3} \cdot \dots \cdot \underbrace{\binom{n-n_1-\dots-n_{k-1}}{n_k}}_{=1}$$
$$= \prod_{i=1}^{k-1} \binom{n - n_1 - n_2 - \dots - n_{i-1}}{n_i}.$$

(b) $\binom{n}{n_1, \ldots, n_k} \in \mathbb{N}$.

(c) The # of maps $f : [n] \to [k]$ such that
$|f^{-1}(i)| = n_i$ for each $i \in [k]$

is $\binom{n}{n_1, \ldots, n_k}$.

(d) Let α be ~~an n-tuple~~ the n-tuple
$$(\underbrace{1, 1, \ldots, 1}_{n_1 \text{ times}}, \underbrace{2, 2, \ldots, 2}_{n_2 \text{ times}}, \ldots, \underbrace{k, k, \ldots, k}_{n_k \text{ times}}).$$

Then, the # of distinct permutations (= anagrams) of α

is $\binom{n}{n_1, \ldots, n_k}$.

<u>Proof.</u> (a) easy.

(b) follows from (a).

(c) We choose such a map as follows:

- Choose $f^{-1}(1)$. (Here we have $\binom{n}{n_1}$ choices.)
- Choose $f^{-1}(2)$. (Here we have $\binom{n-n_1}{n_2}$ choices.)
- Choose $f^{-1}(3)$. (— // — $\binom{n-n_1-n_2}{n_3}$ choices.)
- etc.

$\Rightarrow$ The total # of f's is $\displaystyle\prod_{i=1}^{k} \binom{n-n_1-n_2-\cdots-n_{i-1}}{n_i}$

$$= \binom{n}{n_1,\ldots,n_k} \quad \text{(by part (a))}.$$

(d) Bijection

$\{\text{anagrams of } \alpha\} \longrightarrow \{\text{maps } f:[n]\to[k] \text{ as in part (b)}\}$,

$(\beta_1,\ldots,\beta_n) \longmapsto ([n]\to[k],\ j\mapsto\beta_j)$.

Then, use (c). $\square$

Thm. 2,39. (recurrence relation of multinomial coeffs).

Let $n, n_1, \ldots, n_k \in \mathbb{N}$ be such that $n_1 + \cdots + n_k = n > 0$. Then,

$$\binom{n}{n_1, \ldots, n_k} = \sum_{i=1}^{k} \underbrace{\binom{n}{n_1, \ldots, n_{i-1}, n_i - 1, n_{i+1}, \ldots, n_k}}_{\text{This should be interpreted as } 0 \text{ if } n_i = 0.}$$

Proof. LTTR. $\square$

(= Pascal's pyramid.)

Thm. 2,40 (multinomial formula). Let $x_1, \ldots, x_k$ be k numbers. Let $n \in \mathbb{N}$. Then,

$$(x_1 + \cdots + x_k)^n = \sum_{\substack{(n_1, n_2, \ldots, n_k) \in \mathbb{N}^k; \\ n_1 + \cdots + n_k = n}} \binom{n}{n_1, \ldots, n_k} x_1^{n_1} \cdots x_k^{n_k}.$$

Proof. Use ~~the~~ Prop. 2.38 (c) (see [Galvin, Thm. 11.7]). Or induction on n. Or induction on k. LTTR. $\square$