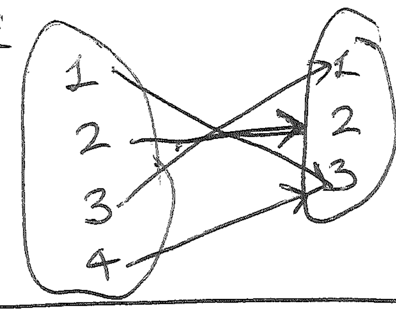
Let us look for a recursive formula for $\operatorname{sur}(m,n)$.

<u>1st approach:</u> Fix $m \in \mathbb{N}$ and $n > 0$.

Given a surjective map $f : [m] \to [n]$, we let $J_f$ be the set of all $i \in [m]$ such that $f(i) = n$.

Ex:




here, $J_f = \{1, 4\}$.

Clearly, $J_f \neq \varnothing$. Thus,

(# of all surjective maps $f : [m] \to [n]$)

$$= \sum_{\substack{J \subseteq [m]; \\ J \neq \varnothing}} \underbrace{(\text{\# of all surjective maps } f : [m] \to [n] \text{ with } J_f = J)}_{= (\text{\# of all surjective maps } [m] \setminus J \to [n-1])}$$

(because there is a bijection

$\{\text{surjective maps } f : [m] \to [n] \text{ with } J_f = J\}$

$\to \{\text{surjective maps } [m] \setminus J \to [n-1]\}$,

which sends each $f$ to $f|_{[m] \setminus J}$)

$$= \sum_{\substack{J \subseteq [m]; \\ J \neq \emptyset}} \underbrace{(\#\text{ of all surjective maps } [m]\setminus J \to [n-1])}_{\overset{\text{Prop. 2.9}}{=} \operatorname{sur}(|[m]\setminus J|,\ |[n-1]|) = \operatorname{sur}(|[m]\setminus J|,\ n-1)}$$

$$= \sum_{\substack{J \subseteq [m]; \\ J \neq \emptyset}} \operatorname{sur}(\underbrace{|[m]\setminus J|}_{= m - |J|},\ n-1)$$

$$= \sum_{\substack{J \subseteq [m]; \\ J \neq \emptyset}} \operatorname{sur}(m - |J|,\ n-1)$$

$$= \sum_{j=1}^{m} \sum_{\substack{J \subseteq [m]; \\ J \neq \emptyset \\ |J| = j}} \operatorname{sur}(m - \underbrace{|J|}_{=j},\ n-1)$$

$$= \sum_{j=1}^{m} \underbrace{\sum_{\substack{J \subseteq [m]; \\ J \neq \emptyset; \\ |J| = j}} \operatorname{sur}(m-j, n-1)}_{= \operatorname{sur}(m-j, n-1) \cdot \underbrace{(\text{\# of all } J \subseteq [m] \text{ such that } J \neq \emptyset \text{ and } |J| = j)}_{\overset{\text{since } j \geq 1}{=} (\text{\# of all } J \subseteq [m] \text{ such that } |J| = j) = \binom{m}{j} \text{ (by Thm. 1.19)}}}$$

$$= \sum_{j=1}^{m} \operatorname{sur}(m-j, n-1) \cdot \binom{m}{j} = \sum_{j=1}^{m} \binom{m}{j} \operatorname{sur}(m-j, n-1).$$

Since the LHS of this is $\operatorname{sur}(m,n)$, we have proven the following:

<u>Prop. 2.11.</u> Let $m \in \mathbb{N}$ and $n > 0$. Then,

$$\mathrm{sur}(m,n) = \sum_{j=1}^{m} \binom{m}{j} \mathrm{sur}(m-j, n-1)$$

$$= \sum_{j=0}^{m-1} \underbrace{\binom{m}{m-j}}_{=\binom{m}{j}} \mathrm{sur}(\underbrace{m-(m-j)}_{=j}, n-1)$$

(we substituted $m-j$ for $j$)

$$= \sum_{j=0}^{m-1} \binom{m}{j} \mathrm{sur}(j, n-1).$$

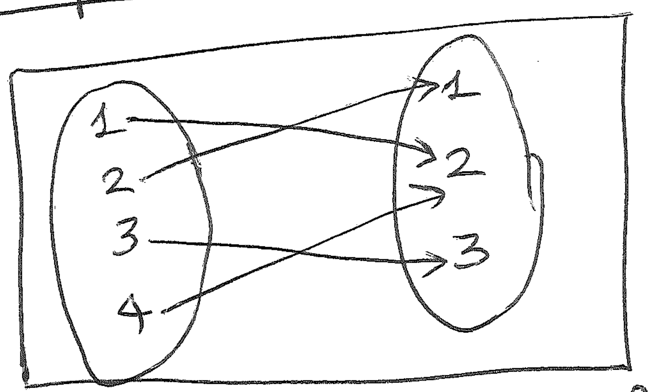
Using Prop. 2.11 and Prop. 2.10(a), we can compute $\mathrm{sur}(m,n)$ one by one, recursively.

<u>2nd approach.</u> Fix $m > 0$ and $n > 0$.
Classify the surjections according to the image of $m$.

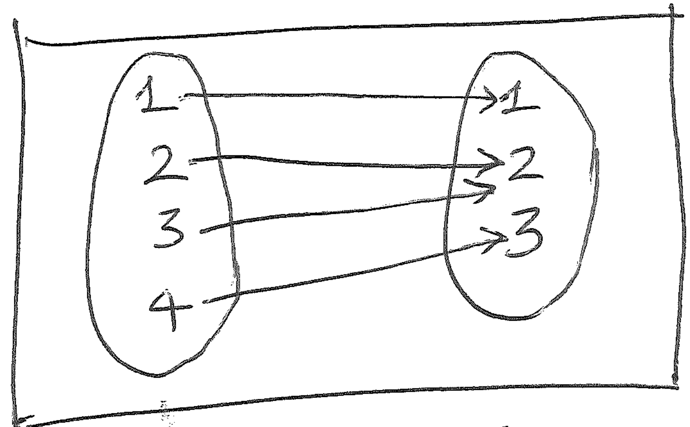
A surjection $f:[m]\to[n]$ is called

- red if $f(m)=f(i)$ for some $i<m$;
- green if it is not red (i.e., if $f(m)\neq f(i)$ for all $i<m$).

[Examples: If $m=4$ and $n=3$, then




is red (since $f(4)=f(1)$)




is green. ]

Thus, if a surjection $f:[m]\to[n]$ is red, then $f|_{[m-1]}$ is "still" a surjection $[m-1]\to[n]$.

But if a surjection $f:[m]\to[n]$ is green, then $f|_{[m-1]}$ has image $[n]\setminus\{f(m)\}$, so it can be viewed as a surjection $[m-1]\to[n]\setminus\{f(m)\}$.

Thus, we get the following algorithm for constructing a <u>red</u> surjection $f : [m] \to [n]$:

- first, we choose $f(m)$ (there are $n$ choices);
- then, we choose $f(1), f(2), \ldots, f(m-1)$; in other words, we choose $f|_{[m-1]}$ (there are $\operatorname{sur}(m-1, n)$ choices).

So there are $n \cdot \operatorname{sur}(m-1, n)$ <u>red</u> surjections $f : [m] \to [n]$.

We also get an algorithm for constructing a <u>green</u> surjection $f : [m] \to [n]$:

- first, we choose $f(m)$ (there are $n$ choices);
- then, we choose $f(1), f(2), \ldots, f(m-1)$; in other words, we choose $f|_{[m-1]}$ (there are $\operatorname{sur}(m-1, n-1)$ choices, since $f|_{[m-1]}$ needs to be a surjection $[m-1] \to [n] \setminus \{f(m)\}$),

So there are $n \cdot \operatorname{sur}(m-1, n-1)$ <u>green</u> surjections $f : [m] \to [n]$.

Hence,

$$\begin{aligned}
&\operatorname{sur}(m,n) \\
&= (\#\text{ of surjections } [m] \to [n]) \\
&= \underbrace{(\#\text{ of red surjections } [m]\to[n])}_{= n \cdot \operatorname{sur}(m-1,n)} + \underbrace{(\#\text{ of green surjections } [m]\to[n])}_{= n \cdot \operatorname{sur}(m-1,n-1)} \\
&= n \cdot (\operatorname{sur}(m-1,n) + \operatorname{sur}(m-1,n-1)).
\end{aligned}$$

So we obtain:

<u>Prop. 2.12.</u> Let $m$ and $n$ be positive integers. Then,

$$\operatorname{sur}(m,n) = n \cdot (\operatorname{sur}(m-1,n) + \operatorname{sur}(m-1,n-1)).$$

This also allows recursively computing $\operatorname{sur}(m,n)$ using Prop. 2.10 (a) & (d).

<u>Cor. 2.13.</u> (a) $\operatorname{sur}(n,n) = n!$ for all $n \in \mathbb{N}$.

(b) $\operatorname{sur}(m,n)$ is a multiple of $n!$ for all $n \in \mathbb{N}$ and $m \in \mathbb{N}$.

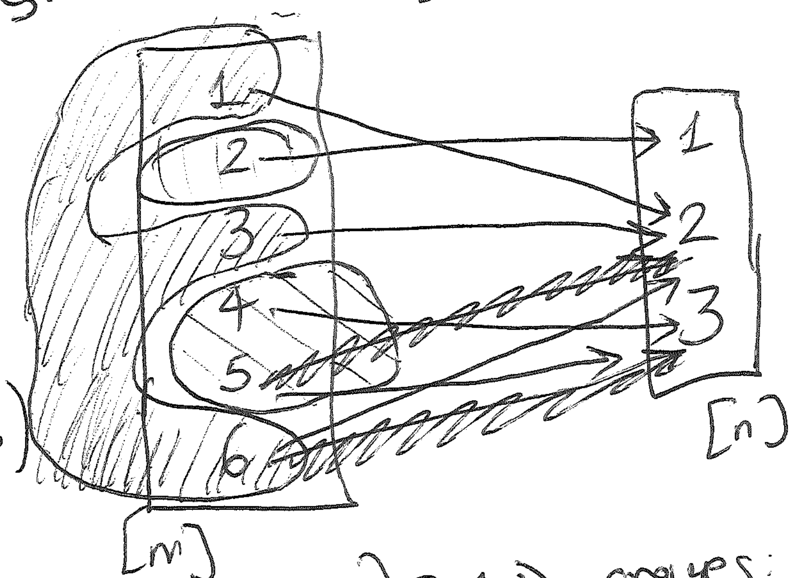
~~Proo~~ 1st proof: Both parts are easy by induction on $n$, using Prop. 2.12. $\square$

2nd proof. (a) The surjections $[n] \to [n]$ are ~~...~~ bijections (by the Pigeonhole Principle for surjections). Thus, they are precisely the permutations of $[n]$. Hence, their number is $n!$ (by Cor. 2.8).

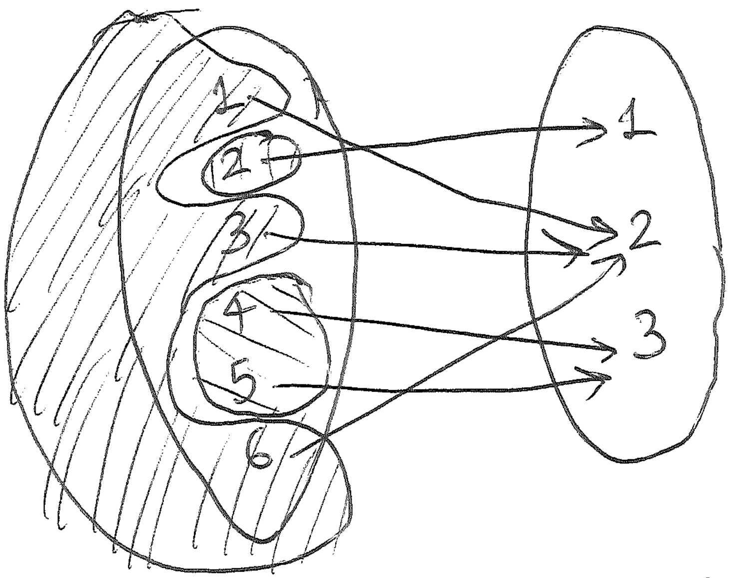
(b) What does $\operatorname{sur}(m,n)/n!$ count?

Rough idea: Each surjection $f : [m] \to [n]$ gives a way of "grouping" the elements of $[m]$ into $n$ nonempty (disjoint) groups.

(see next page for better drawing!)




$\rightsquigarrow$ groups: $\{2\}$, $\{1,3,6\}$, $\{4,5\}$.




If we disregard the order of the groups (i.e., if we forget about the values $f(i)$, but only keep track of ~~the #~~ which $i$ and $j$ have $f(i) = f(j)$), then the # of all possible groupings is $\operatorname{sur}(m,n)/n!$.

Hence, $\operatorname{sur}(m,n)/n! \in \mathbb{N}$.

(Details: see Spring 2018 Math 4707 HW 3, the section §0.3 on "set partitions".) $\square$

__Rmk:__ $\operatorname{sur}(m,n)/n!$ is denoted $\left\{ {m \atop n} \right\}$, and is called a __Stirling number of the 2nd kind__.

Here is the most explicit formula for $\operatorname{sur}(m,n)$ known:

<u>Thm. 2.14.</u> Let $m \in \mathbb{N}$ and $n \in \mathbb{N}$. Then,

$$\operatorname{sur}(m,n) = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} i^m.$$

<u>1st proof.</u> Use induction & Prop. 2.11.
(Details: Fall 2017 Math 4707 HW2 Exercise 4.)

<u>2nd proof.</u> Use the Principle of Inclusion & Exclusion (see later).
(Details: Spring 2018 Math 4707 HW 3 Exercise 2(b).)

Thus, for $n=3$, we get

$$\operatorname{sur}(m,3) = 3^m - 3 \cdot 2^m + 3 \cdot \cancel{1^m} - \underbrace{0^m}_{=[m=0]}.$$

<u>2.3. $1^m + 2^m + \cdots + n^m$</u>

Next goal: prove Thm. 1.13. First step:

**Thm. 2.15.** Let $k \in \mathbb{N}$ and $m \in \mathbb{N}$. Then,

$$k^m = \sum_{i=0}^{m} \operatorname{sur}(m,i) \binom{k}{i}.$$

**Proof.** Double counting. How many ways are there to choose a map $f : [m] \to [k]$?

1st answer: $k^m$.

2nd answer: We choose $f$ as follows:

- First, we choose $|f([m])|$ (this is the size of the image of $f$, i.e., the # of distinct values of $f$). This is an integer in $\{0, 1, \ldots, m\}$. Call this integer $i$.
- Then, choose $f([m])$ (this is the ~~size~~ set of all values of $f$). There are $\binom{k}{i}$ choices for this (since $f([m])$ must be an $i$-element subset of $[k]$).

- Finally, choose $f(1), f(2), \ldots, f(m)$. These $m$ numbers must be chosen from the already determined $i$-element set $f([m])$, and must cover this set. Thus, there are $\operatorname{sur}(m,i)$ choices here (since we are just choosing a surjection from $[m]$ to the $i$-element set $f([m])$).

So the total # of ways is $\sum_{i=0}^{m} \binom{k}{i} \operatorname{sur}(m,i)$.

Since the two answers answer the same question, we get

$$k^m = \sum_{i=0}^{m} \binom{k}{i} \operatorname{sur}(m,i).$$ $\square$

Thm. 2.16. Let $n \in \mathbb{N}$ and $m \in \mathbb{N}$. Then,

$$\sum_{k=0}^{n} k^m = \sum_{i=0}^{m} \operatorname{sur}(m,i) \binom{n+1}{i+1}.$$

Proof.

$$\sum_{k=0}^{n} \underbrace{k^m}_{=\sum_{i=0}^{m} \operatorname{sur}(m,i)\binom{k}{i}}$$

$$= \underbrace{\sum_{k=0}^{n} \sum_{i=0}^{m}}_{=\sum_{i=0}^{m}\sum_{k=0}^{n}} \operatorname{sur}(m,i)\binom{k}{i} = \sum_{i=0}^{m} \sum_{k=0}^{n} \operatorname{sur}(m,i)\binom{k}{i}$$

(by Thm. 2.17 below)

$$= \sum_{i=0}^{m} \operatorname{sur}(m,i) \underbrace{\sum_{k=0}^{n} \binom{k}{i}}_{\substack{=\binom{0}{i}+\binom{1}{i}+\cdots+\binom{n}{i} \\ =\binom{n+1}{i+1}}}$$

(by Thm. 1.20, applied to $k=i$)

$$= \sum_{i=0}^{m} \operatorname{sur}(m,i) \binom{n+1}{i+1}. \qquad \square$$

We have interchanged two $\Sigma$ signs in the above proof.
This relied on the following fact:

<u>Thm. 2.17 (interchanging $\Sigma$ signs / finite Fubini theorem).</u>
Let $X$ and $Y$ be finite sets. For any $x \in X$ and $y \in Y$, let $a_{x,y}$ be a number. Then,

$$\sum_{x \in X} \sum_{y \in Y} a_{x,y} = \sum_{(x,y) \in X \times Y} a_{x,y} = \sum_{y \in Y} \sum_{x \in X} a_{x,y}.$$

<u>Example:</u> If $X = [2]$ and $Y = [3]$, then this says

$$(a_{1,1} + a_{1,2} + a_{1,3}) + (a_{2,1} + a_{2,2} + a_{2,3})$$

$$= (\text{sum of all } a_{x,y})$$

$$= (a_{1,1} + a_{2,1}) + (a_{1,2} + a_{2,2}) + (a_{1,3} + a_{2,3}).$$

<u>Remark:</u> Thm. 2.17 is about <u>finite</u> sets $X$ and $Y$. If $X$ and $Y$ are infinite, then it may happen that

$$\sum_{x \in X} \sum_{y \in Y} a_{x,y} \neq \sum_{y \in Y} \sum_{x \in X} a_{x,y}$$

even if all of these sums are well-defined.

Example: "serial debtor": $X = Y = \mathbb{N}$

| $a_{x,y}$ | $y=0$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $x=0$ | 1 | $-1$ | | | | |
| 1 | | 1 | $-1$ | | | |
| 2 | | | 1 | $-1$ | | |
| 3 | | | | 1 | $-1$ | |
| 4 | | | | | 1 | $-1$ |

(empty entries are 0's)

(formula: $a_{x,y} = [x=y] - [x=y-1]$).

Then,

$$\sum_{x\in X} \underbrace{\sum_{y\in Y} a_{x,y}}_{=0} = \sum_{x\in X} 0 = 0$$

but

$$\sum_{y\in Y} \underbrace{\sum_{x\in X} a_{x,y}}_{=[y=0]} = \sum_{y\in Y} [y=0] = 1.$$

For Thm. 2.17 to hold without finiteness of $X$ and $Y$, we need to require that $a_{x,y}=0$ for all but finitely many pairs $(x,y)\in X\times Y$.

(There are also weaker conditions that make it hold — see, e.g., absolute convergence in analysis.)

Proof of Thm. 1.13. Rename $k$ as $m$. Thus, we must prove

$$1^m + 2^m + \cdots + n^m = \sum_{i=0}^{m} \operatorname{sur}(m,i)\cdot \binom{n+1}{i+1}.$$

But this is exactly what Thm. 2.16 claims, since

$$\sum_{k=0}^{n} k^m = \underbrace{0^m}_{\substack{=0 \\ (\text{since } m>0)}} + 1^m + 2^m + \cdots + n^m = 1^m + 2^m + \cdots + n^m. \qquad \square$$

## 2.4. Vandermonde convolution

**Thm. 2.18 ("Vandermonde convolution").** Let $n \in \mathbb{N}$, $x \in \mathbb{R}$ and $y \in \mathbb{R}$. Then,

$$\binom{x+y}{n} = \sum_{k=0}^{n} \binom{x}{k}\binom{y}{n-k} = \sum_{k} \binom{x}{k}\binom{y}{n-k}.$$

**Rmk.** The "$\sum_k$" in Thm. 2.18 means a sum over all $k \in \mathbb{Z}$.

Note that $\binom{x}{k}\binom{y}{n-k} = 0$ whenever $k \notin \{0, 1, \ldots, n\}$ (indeed, $\binom{y}{n-k} = 0$ if $k > n$, and $\binom{x}{k} = 0$ if $k < 0$).

Thus, the 2nd equality sign in Thm. 2.18 is clear.

We'll first prove Thm. 2.18 under extra assumptions:

1st proof of Thm. 2.18 for the case when $x \in \mathbb{N}$:

For any $u \in \mathbb{R}$ and $n \in \mathbb{N}$, we have

$$\binom{u}{n} = \underbrace{\binom{u-1}{n-1}}_{=\binom{u-2}{n-2}+\binom{u-2}{n-1}} + \underbrace{\binom{u-1}{n}}_{=\binom{u-2}{n-1}+\binom{u-2}{n}}$$

$$= \underbrace{\binom{u-2}{n-2}}_{=\binom{u-3}{n-3}+\binom{u-3}{n-2}} + 2\underbrace{\binom{u-2}{n-1}}_{=\binom{u-3}{n-2}+\binom{u-3}{n-1}} + \underbrace{\binom{u-2}{n}}_{=\binom{u-3}{n-1}+\binom{u-3}{n}}$$

$$= \binom{u-3}{n-3} + 3\binom{u-3}{n-2} + 3\binom{u-3}{n-1} + \binom{u-3}{n}$$

$$= \binom{u-4}{n-4} + 4\binom{u-4}{n-3} + 6\binom{u-4}{n-2} + 4\binom{u-4}{n-1} + \binom{u-4}{n}$$

$$= \dots$$

after $v$ steps

$$\overset{}{=} \binom{u-v}{n-v} + \dots + \binom{v}{k}\binom{u-v}{n-k} + \dots + \binom{u-v}{n}$$

(convince yourself that the coefficients follow the same recursion as Pascal's triangle, and thus are $\binom{v}{k}$)

$$= \sum_{k=0}^{v} \binom{v}{k}\binom{u-v}{n-k}.$$

Applying this to $u = x+y$ and $v = x$, we get

$$\binom{x+y}{n} = \sum_{k=0}^{x} \binom{x}{k}\binom{(x+y)-x}{n-k} = \sum_{k=0}^{x} \binom{x}{k}\binom{y}{n-k}$$

$$\overset{\circ}{=} \sum_{k\in\mathbb{Z}} \binom{x}{k}\binom{y}{n-k} \overset{\circ}{=} \sum_{k=0}^{n} \binom{x}{k}\binom{y}{n-k}.$$