Let's start with (c).

Def. Let $x \in \mathbb{R}$. Then:

- $\lfloor x \rfloor$ denotes the largest integer $\leq x$ (called the "floor of x", or "rounding down x");
- $\lceil x \rceil$ denotes the smallest integer $\geq x$ (called the "ceiling of x", or "rounding up x").

Ex. $\lfloor 3 \rfloor = 3$, $\lfloor \sqrt{2} \rfloor = 1$, $\lfloor \pi \rfloor = 3$, $\lfloor -\pi \rfloor = -4$,
$\lceil 3 \rceil = 3$, $\lceil \sqrt{2} \rceil = 2$, $\lceil \pi \rceil = 4$, $\lceil -\pi \rceil = -3$.

Prop. 1.34. Let $n \in \mathbb{N}$. The largest size of a lacunar subset of $[n]$ is $\lceil n/2 \rceil$.

Proof. The lacunar subset $\{1 < 3 < 5 < \cdots < (n \text{ or } n-1)\}$ has size $\lceil n/2 \rceil$. So we only need to check that no higher size is possible.

Assume the contrary. Thus, $\exists$ lacunar subset S of $[n]$

with size $> \lceil n/2 \rceil$. Hence, $|S| \geq \lceil n/2 \rceil + 1$

$\geq n/2 + 1 > (n+1)/2$. Thus, $2|S| > n+1$.

Now, let S^+ be the set $\{s+1 \mid s \in S\}$.

(If $S = \{2, 4, 7, 10\}$, then $S^+ = \{3, 5, 8, 11\}$.)

Then, S and S^+ are two subsets of $[n+1]$. Hence

$$|S \cup S^+| \leq |[n+1]| = n+1.$$

But S and S^+ are disjoint (since S is lacunar), so

$$|S \cup S^+| = |S| + \underbrace{|S^+|}_{=|S|} = |S| + |S| = 2|S| > n+1.$$

The latter 2 inequalities contradict each other. $\square$

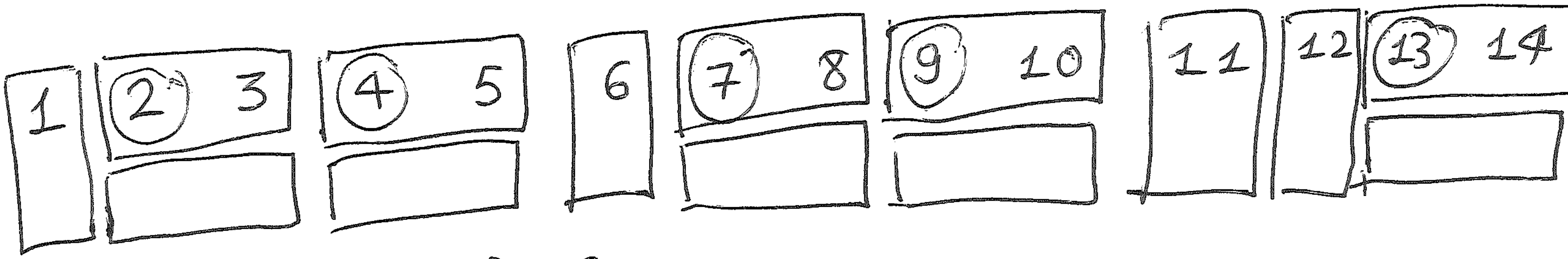
Prop. 1.35. Let $n \in \{-1, 0, 1, \ldots\}$. Then, the # of lacunar subsets of $[n]$ is f_{n+2}.

1st proof idea. Induction (or, rather, strong induction).

2nd proof idea.

lacunar subset $\{2<4<7<9<13\}$ of $[13]$

$\rightsquigarrow$



domino tiling of $R_{14,2}$.

This is a bijection

$\{\text{lacunar subsets of } [n]\} \to \{\text{domino tilings of } R_{n+1,2}\}$.

But Prop. 1.5 shows that # of domino tilings of $R_{n+1,2}$ is $d_{n+1,2} = f_{(n+1)+1} = f_{n+2}$. $\square$

<u>Prop. 1.36.</u> Let $n \in \mathbb{N}$ and $k \in \mathbb{N}$. Then, the # of k-elt. lacunar subsets of $[n]$ is $\dbinom{n+1-k}{k}$, if $k \leq n+1$.

<u>Proof.</u> Assume that $k \leq n+1$.

Let

~~A: {lacunar subset~~

$A : \{k\text{-elt. lacunar subsets of } [n]\} \to \{k\text{-elt. subsets of } [n+1-k]\}$

be the map that sends

$$\{s_1 < s_2 < \cdots < s_k\} \mapsto \{s_1 - 0 < s_2 - 1 < s_3 - 2 < \cdots < s_k - (k-1)\}.$$

This map A is a bijection: its inverse sends

$$\{t_1 < t_2 < \cdots < t_k\} \mapsto \{t_1 + 0 < t_2 + 1 < t_3 + 2 < \cdots < t_k + (k-1)\}.$$

Thus, $|\{k\text{-elt. lacunar subsets of } [n]\}|$

$$= |\{k\text{-elt. subsets of } [n+1-k]\}| \underset{\substack{\text{by Thm. 1.19,} \\ \text{since } n+1-k \geq 0}}{=} \binom{n+1-k}{k}. \qquad \square$$

Another proof uses induction on n.

Note that Proposition 1.34 can be reproven using Prop. 1.36.

Thm. 1.37. Let $m, n \in \mathbb{N}$. Then, $f_{m+n+1} = f_m f_n + f_{m+1} f_{n+1}$.

1st proof. Induction (exercise; see also last semester's 4707 hw/mt).

2nd proof. "Double counting": we count the same thing in 2 ways, and then conclude that the 2 results are equal.

Here: Count the lacunar subsets of $[m+n-1]$ (assuming WLOG $m>0$ and $n>0$).

1st method: Prop. 1.35 says that their # is f_{m+n+1}.

2nd method: Call a lacunar subset S of $[m+n-1]$

- red if $m \in S$;
- green if $m \notin S$.

A ~~black~~ green lacunar subset S of $[m+n-1]$ is just a union of a lacunar subset of $[m-1]$ with a lacunar subset of $\{m+1, m+2, \ldots, m+n-1\}$.

The converse also holds.

$$\Rightarrow \quad (\text{\# of green lacunar subsets of } [m+n-1])$$

$$= \underbrace{(\text{\# of lacunar subsets of } [m-1])}_{= f_{m+1} \text{ (by Prop. 1.35)}}$$

$$\cdot \underbrace{(\text{\# of lacunar subsets of } \{m+1, m+2, \ldots, m+n-1\})}_{\substack{= (\text{\# of lacunar subsets of } [n-1]) \\ = f_{n+1} \text{ (by Prop. 1.35)}}}$$

(3) $$= f_{m+1} f_{n+1}$$

A red lacunar subset S of $[m+n-1]$ is just a union of a lacunar subset of $[m-2]$ with a lacunar subset of $\{m+2, m+3, \ldots, m+n-1\}$ with $\{m\}$.

The converse also holds.

$$\Longrightarrow \quad (\text{\# of red lacunar subsets of } [m+n-1])$$

$$= \underbrace{(\text{\# of lacunar subsets of } [m-2])}_{= f_m \text{ (by Prop. 1.35)}}$$

$$\cdot \underbrace{(\text{\# of lacunar subsets of } \{m+2, m+3, \ldots, m+n-1\})}_{\substack{= (\text{\# of lacunar subsets of } [n-2]) \\ = f_n \text{ (by Prop. 1.35)}}}$$

$$= f_m f_n .$$

Adding this equality to (3), we get

(# of all lacunar subsets of $[m+n-1]$)

$= f_{m+1} f_{n+1} + f_m f_n$.

Comparing these 2 results, we get $f_{m+n+1} = f_{m+1} f_{n+1} + f_m f_n$. □

See [Benjamin & Quinn, "Proofs that really count"] for more examples of double counting.

<u>Proof of Prop. 1.25.</u> Let $n \in \mathbb{N}$.

Then, Prop. 1.35 says

f_{n+2}

$= (\#$ of lacunar subsets of $[n])$

$$= \sum_{k=0}^{n} \underbrace{(\# \text{ of } k\text{-elt. lacunar subsets of } [n])}_{= \binom{n+1-k}{k} \text{ (by Prop. 1.36)}}$$

$$= \sum_{k=0}^{n} \binom{n+1-k}{k}.$$

Substituting n for $n+1$, we get

$$f_{n+1} = \sum_{k=0}^{n-1} \binom{n-k}{k} = \sum_{k=0}^{n} \binom{n-k}{k} - \underbrace{\binom{n-n}{n}}_{= \binom{0}{n} = 0 \ \cancel{\text{if } n>0}}$$

$$= \sum_{k=0}^{n} \binom{n-k}{k}$$

whenever $n \geqslant 1$. The case $n=0$ is LTTR. $\square$

2. Binomial coefficients

In this chapter: a closer look at the binomial coefficients.

References:
- [GKP, Chapter 5] for the best source on bin. coeffs.;
- [detnotes, ch. 2] for a detailed treatment of the basic properties.

2.1. Basic identities

See §1.3:

- Definition: $\binom{n}{k} = \dfrac{n(n-1)\cdots(n-k+1)}{k!}$ for $k \in \mathbb{N}$;

  $\binom{n}{k} = 0$ for $k \notin \mathbb{N}$.

  (Here, n can be anything.)

- $\binom{n}{k} = \dfrac{n!}{k!\,(n-k)!}$ holds only for $n \in \mathbb{N}$, $k \in \mathbb{N}$ and $n \geq k$ (Thm. 1.28).

- Prop. 1.14, ..., Prop. 1.29.

Recall Thm. 1.20: Let $n \in \mathbb{N}$ and $k \in \mathbb{N}$. Then,

$$\binom{0}{k} + \binom{1}{k} + \dots + \binom{n}{k} = \binom{n+1}{k+1}.$$

Remark. The first k addends on the $\underbrace{\text{LHS}}_{=\text{left hand side}}$ are 0.

Thus, the eq. rewrites as $\binom{k}{k} + \binom{k+1}{k} + \dots + \binom{n}{k} = \binom{n+1}{k+1}$.

2nd proof of Thm. 1.20 (idea). Thm. 1.16 yields

$$\binom{n+1}{k+1} = \binom{n}{k} + \underbrace{\binom{n}{k+1}}_{=\binom{n-1}{k} + \binom{n-1}{k+1}}$$

$$= \binom{n}{k} + \binom{n-1}{k} + \underbrace{\binom{n-1}{k+1}}_{=\binom{n-2}{k} + \binom{n-2}{k+1}}$$

$$= \binom{n}{k} + \binom{n-1}{k} + \binom{n-2}{k} + \underbrace{\binom{n-2}{k+1}}_{= \cdots}$$

$$= \cdots$$

$$= \binom{n}{k} + \binom{n-1}{k} + \binom{n-2}{k} + \cdots + \binom{0}{k} + \underbrace{\binom{0}{k+1}}_{=0}$$

$$= \binom{n}{k} + \binom{n-1}{k} + \binom{n-2}{k} + \cdots + \binom{0}{k}$$

$$= \binom{0}{k} + \binom{1}{k} + \cdots + \binom{n}{k}.$$

$\square$

To make this rigorous, turn it into an induction.

<u>3rd proof of Thm. 1.20.</u> For each $i \in \mathbb{Z}$, we have

$$\binom{i}{k} = \binom{i+1}{k+1} - \binom{i}{k+1} \quad \left(\text{since } \binom{i+1}{k+1} = \binom{i}{k+1} + \binom{i}{k}\right).$$

Thus, $\binom{0}{k}+\binom{1}{k}+\dots+\binom{n}{k}$

$$= \left(\cancel{\binom{1}{k+1}} - \binom{0}{k+1}\right) + \left(\cancel{\binom{2}{k+1}} - \cancel{\binom{1}{k+1}}\right) + \left(\cancel{\binom{3}{k+1}} - \cancel{\binom{2}{k+1}}\right)$$

$$+ \cancel{\cdots} + \left(\cancel{\binom{n}{k+1}} - \cancel{\binom{n-1}{k+1}}\right)$$

$$+ \left(\binom{n+1}{k+1} - \cancel{\binom{n}{k+1}}\right)$$

$$= \binom{n+1}{k+1} - \underbrace{\binom{0}{k+1}}_{=0} = \binom{n+1}{k+1}. \qquad \square$$

The cancellations made in this proof are an instance of a general principle:

<u>Prop. 2.1. (telescoping sum principle).</u> Let u and v be integers with $u \leq v+1$. Let $a_u, a_{u+1}, \ldots, a_{v+1}$ be numbers. Then,

$$\sum_{j=u}^{v} (a_{j+1} - a_j) = a_{v+1} - a_u.$$

For general proof, see HW0 exercise 2 solution.

Prop. 2.1 is the discrete analogue of the fundamental theorem of calculus $\left(\int_u^v f'(t)\,dt = f(v) - f(u) \right)$.

<u>4-th proof of Thm. 1.20:</u> Thm. 1.19 yields

$$\binom{n+1}{k+1} = (\#\text{ of } (k+1)\text{-elt. subsets of } [n+1])$$

$$= \sum_{j=1}^{n+1} (\#\text{ of } (k+1)\text{-elt. subsets of } [n+1] \text{ whose largest element is } j).$$

(4)

Now, fix $j \in [n+1]$. How many $(k+1)$-elt. subsets of $[n+1]$ are there whose largest element is j?

The answer is $\binom{j-1}{k}$ (by Thm. 1.19), since the element j has already been chosen, and the remaining k elements must be chosen from $\{1, 2, \ldots, j-1\} = [j-1]$, which has $j-1$ elements.

[Rigorous version: There is a bijection

$$\{k\text{-elt. subsets of } [j-1]\} \to \{(k+1)\text{-elt. subsets of } [n+1] \text{ whose largest elt. is } j\},$$

$$S \mapsto S \cup \{j\}$$

(the inverse map sends $T \mapsto T \setminus \{j\}$). Thus,

$$|\{(k+1)\text{-elt. subsets of } [n+1] \text{ whose largest elt. is } j\}|$$
$$= |\{k\text{-elt. subsets of } [j-1]\}|$$
$$= (\# \text{ of } k\text{-elt. subsets of } [j-1]) = \binom{j-1}{k} \text{ (by Thm. 1.19)}.$$

This proves our answer.]

Thus, (4) becomes

$$\binom{n+1}{k+1} = \sum_{j=1}^{n+1} \binom{j-1}{k} = \binom{0}{k} + \binom{1}{k} + \cdots + \binom{n}{k}. \qquad \square$$

<u>Prop. 2.2. (Trinomial revision).</u> Let $n, a, b \in \mathbb{R}$. Then,

$$\binom{n}{a}\binom{a}{b} = \binom{n}{b}\binom{n-b}{a-b}.$$

<u>Proof.</u> Four cases:

Case 1: $b \notin \mathbb{N}$.

Case 2: $b \in \mathbb{N}$ and $a \notin \mathbb{N}$.

Case 3: $b \in \mathbb{N}$ and $a \in \mathbb{N}$ but $a < b$.

Case 4: $b \in \mathbb{N}$ and $a \in \mathbb{N}$ and $a \geq b$.

- In Case 1, we must prove $\binom{n}{a} 0 = 0 \binom{n-b}{a-b}$. True.
- In Case 2, we must prove $0 \binom{a}{b} = \binom{n}{b} 0$

(indeed, from $b \in \mathbb{N}$ and $a \notin \mathbb{N}$, we get $a-b \notin \mathbb{N}$, thus

$\binom{n-b}{a-b} = 0$). True.

- In Case 3, we must prove $\binom{n}{a} 0 = \binom{n}{b} 0$ (indeed, the conditions of Case 3 force $\binom{a}{b} = 0$ by Prop. 1.14, but also $a < b \Rightarrow a - b \notin \mathbb{N} \Rightarrow \binom{n-b}{a-b} = 0$).

True.

- Consider Case 4: Here, $a - b \in \mathbb{N}$. Now,

$$\binom{n}{b}\binom{n-b}{a-b} = \frac{n(n-1)\cdots(n-b+1)}{b!} \cdot \frac{(n-b)(n-b-1)\cdots((n-b)-(a-b)+1)}{(a-b)!}$$

$$= \frac{n(n-1)\cdots((n-\cancel{b})-(a-\cancel{b})+1)}{b!(a-b)!}$$

$$= \frac{n(n-1)\cdots(n-a+1)}{b!(a-b)!}.$$

Compared with

$$\binom{n}{a}\binom{a}{b} = \frac{n(n-1)\cdots(n-a+1)}{\cancel{a!}} \cdot \frac{\cancel{a!}}{b!\,(a-b)!}$$

(by Thm. 1.28, since $a \in \mathbb{N}$, $b \in \mathbb{N}$ and $a \geq b$)

$$= \frac{n(n-1)\cdots(n-a+1)}{b!\,(a-b)!},$$

we get $\binom{n}{a}\binom{a}{b} = \binom{n}{b}\binom{n-b}{a-b}$. $\square$