1.2. Sums of powers

Thm. 1.10 ("Little Gauss"). Let $n \in \mathbb{N}$. (keep in mind: $0 \in \mathbb{N}$.)

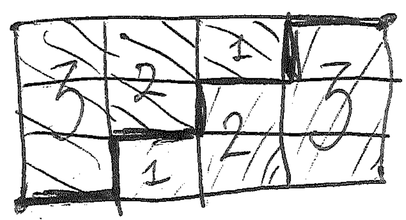
Then,
$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

Rmk: If $n = 0$, then $1 + 2 + \cdots + n = 1 + 2 + \cdots + 0 = (\text{empty sum}) = 0$.

1st proof. Induction on n. LTTR (= left to the reader). $\square$

2nd proof.
$$\begin{aligned}
&2(1 + 2 + \cdots + n) \\
&= (1 + 2 + \cdots + n) \\
&\quad + (1 + 2 + \cdots + n) \\
&= (1 + 2 + \cdots + n) \\
&\quad + (n + (n-1) + \cdots + 1) \\
&= \underbrace{(n+1) + (n+1) + \cdots + (n+1)}_{n \text{ times}} = n(n+1).
\end{aligned}$$
$\square$

3rd proof (outline): For $n = 3$,



$\Rightarrow 3 \cdot 4$
$= (1+2+3)$
$+ (1+2+3)$. $\square$

Rmk: Why is $1+2+\cdots+n$ well-defined?

E.g., why is $(\cdots((1+2)+3)+\cdots)+n$

$= 1+(\cdots+((n-2)+((n-1)+n))\cdots)$?

This is nontrivial ("general commutativity"); for a proof, see Math 4707 Spring 2018 February 7th.

Prop. 1.11. Let $n\in\mathbb{N}$. Then, $1^2+2^2+\cdots+n^2 = \dfrac{n(n+1)(2n+1)}{6}$.

Proof. Induction, LTTR. $\square$

Prop. 1.12. Let $n\in\mathbb{N}$. Then, $1^3+2^3+\cdots+n^3 = \dfrac{n^2(n+1)^2}{4}$.

Proof. Induction, LTTR. $\square$

Question: Given $k\in\mathbb{N}$, what is $1^k+2^k+\cdots+n^k$? polynomial in n?

A $(k+1)$-th degree polynomial in n?

Def. If $a_p, a_{p+1}, \ldots, a_q$ (for some integers p,q) are numbers, then $\sum_{i=p}^{q} a_i$ means $a_p + a_{p+1} + \cdots + a_q$.

So $1^k + 2^k + \cdots + n^k = \sum_{i=1}^{n} i^k$.
(not the i from complex numbers)

Thm. 1.13. (18th century?). Let $n \in \mathbb{N}$, $k \in \mathbb{N}$. Assume $k > 0$.

Then,
$$1^k + 2^k + \cdots + n^k = \sum_{i=0}^{k} \operatorname{sur}(k,i) \cdot \binom{n+1}{i+1},$$

where:

- $\operatorname{sur}(k,i) := (\#$ of $\underbrace{\text{surjective}}_{= \text{onto}}$ maps $\{1,2,\ldots,k\} \to \underbrace{\{1,2,\ldots,i\}}_{\text{This is } \varnothing \text{ if } i=0})$;

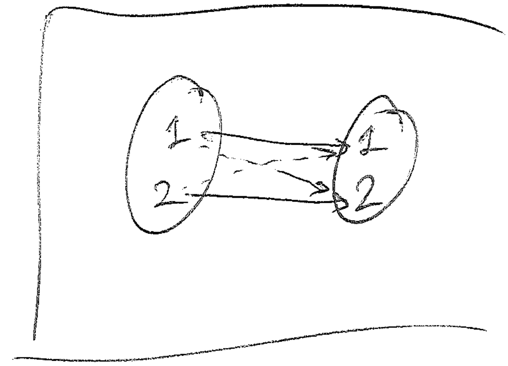
- $$\binom{x}{j} = \frac{x(x-1)\cdots(x-j+1)}{j(j-1)\cdots 1} \quad \forall x \in \mathbb{R} \text{ and } j \in \mathbb{N}.$$

(We won't prove it right now; see [Galvin, Prop. 23.2].)

Example: ~~Be~~ Let $k=2$. Then, Thm. 1.13 yields
$$1^2 + 2^2 + \cdots + n^2 = \sum_{i=0}^{2} \operatorname{sur}(2,i) \cdot \binom{n+1}{i+1}$$

$$= \underbrace{\mathrm{sur}(2,0)}_{=0} \cdot \binom{n+1}{0+1} + \underbrace{\mathrm{sur}(2,1)}_{=1} \cdot \binom{n+1}{1+1}$$

$$+ \underbrace{\mathrm{sur}(2,2)}_{=2} \cdot \binom{n+1}{2+1}$$



$$= 0 \cdot \binom{n+1}{1} + 1 \cdot \underbrace{\binom{n+1}{2}}_{=\frac{(n+1)n}{2\cdot 1}} + 2 \cdot \underbrace{\binom{n+1}{3}}_{=\frac{(n+1)n(n-1)}{3\cdot 2\cdot 1}}$$

$$= 1 \cdot \frac{(n+1)n}{2\cdot 1} + 2 \cdot \frac{(n+1)n(n-1)}{3\cdot 2\cdot 1} = \frac{n(n+1)(2n+1)}{6},$$

so Prop. 1.11 follows. Similarly, get Thm. 1.10 & Prop. 1.12.

1.3 Factorials & binomial coefficients

<u>Def.</u> For any $n \in \mathbb{N}$, set $n! = 1 \cdot 2 \cdot \ldots \cdot n$.

This is understood to be 1 if $n=0$, since generally, empty products are understood to be 1. (Comp: $0^0 = 1$.)

We refer to $n!$ as "n factorial".

<u>Rmk:</u> If n is a positive int., then

$$n! = 1 \cdot 2 \cdot \ldots \cdot n = \underbrace{\big(1 \cdot 2 \cdot \ldots \cdot (n-1)\big)}_{=(n-1)!} \cdot n = (n-1)! \cdot n,$$

<u>Ex:</u> $0! = 1$, $1! = 1$, $2! = 1 \cdot 2 = 2$, $3! = 1 \cdot 2 \cdot 3 = 6$,
$4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$, $5! = 120$, $6! = 720$, $7! = 5040$,
$8! = 40\,320$, ...

Def. Let n be any number ($n \in \mathbb{N}$ or $n \in \mathbb{Z}$ or $n \in \mathbb{R}$ or $n \in \mathbb{C}$).

(a) For any $k \in \mathbb{N}$, set

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} = \frac{n^{\underline{k}}}{k!}.$$

see HW0 Exercise 2

(b) If $k \notin \mathbb{N}$, set $\binom{n}{k} = 0$. (This is the convention of [GrKnPa].)

We call $\binom{n}{k}$ a "<u>binomial coefficient</u>", and refer to it as "n choose k".

<u>Examples:</u> $\binom{n}{0} = \frac{(\text{empty product})}{0!} = \frac{1}{1} = 1,$

$$\binom{n}{1} = \frac{n^{\underline{1}}}{1!} = \frac{n}{1} = n.$$

$$\binom{n}{2} = \frac{n(n-1)}{2!} = \frac{n(n-1)}{2}.$$

$$\binom{n}{3} = \frac{n(n-1)(n-2)}{3!} = \frac{n(n-1)(n-2)}{6}.$$

$$\binom{-1}{5} = \frac{(-1)(-2)(-3)(-4)(-5)}{5!} = -1.$$

$$\binom{-1}{k} = \frac{(-1)(-2)\cdots(-k)}{k!} = (-1)^k = \begin{cases} -1 & \text{if } k \text{ is odd} \\ 1 & \text{if } k \text{ is even} \end{cases} \quad \forall k \in \mathbb{N}.$$

$$\binom{\sqrt{2}}{2} = \frac{\sqrt{2}(\sqrt{2}-1)}{2} = \frac{2-\sqrt{2}}{2}.$$

$$\binom{2}{\sqrt{2}} = 0 \quad \text{since } \sqrt{2} \notin \mathbb{N}.$$

Prop. 1.14. If $n \in \mathbb{N}$ and $k > n$, then $\binom{n}{k} = 0$.

Proof. $\binom{n}{k} = \dfrac{n(n-1)\cdots(n-k+1)}{k!} = \dfrac{0}{k!}$, since one of the factors $n, n-1, \dots, n-k+1$ is $n-n=0$. $\square$

Prop. 1.15. ("upper negation"). Let $n \in \mathbb{R}$ and $k \in \mathbb{Z}$.

Then, $\binom{-n}{k} = (-1)^k \binom{n+k-1}{k}$.

Proof. Case 1: $k \notin \mathbb{N}$. Then, both sides are 0, qed.

Case 2: $k \in \mathbb{N}$. Then,

$$\binom{-n}{k} = \frac{(-n)(-n-1)\cdots(-n-k+1)}{k!} = \frac{(-1)^k n(n+1)\cdots(n+k-1)}{k!}$$

$$= \frac{(-1)^k (n+k-1)(n+k-2)\cdots n}{k!} = (-1)^k \underbrace{\frac{(n+k-1)(n+k-2)\cdots n}{k!}}_{= \binom{n+k-1}{k}}$$

$$= (-1)^k \binom{n+k-1}{k}.$$

$\square$

Here are some more results, to be proven later.

Thm. 1.16 (recurrence of the BC). Let $n \in \mathbb{R}$ and $k \in \mathbb{Z}$. Then,

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Thm. 1.17. (symmetry of the BC). Let $n \in \mathbb{N}$ and $k \in \mathbb{Z}$. Then,

$$\binom{n}{k} = \binom{n}{n-k}.$$

<u>Thm. 1.18. (integrality of the BC)</u> Let $n \in \mathbb{Z}$ and $k \in \mathbb{Z}$. Then, $\binom{n}{k} \in \mathbb{Z}$.

<u>Thm. 1.19. (combinatorial interpretation of the BC)</u>, Let $n \in \mathbb{N}$ and $k \in \mathbb{Z}$. Let S be an n-element set. Then,

$$\binom{n}{k} = (\#\text{ of } k\text{-element subsets of } S).$$

<u>Ex:</u> Let $n = 4$ and $k = 2$ and $S = \{1, 2, 3, 4\}$. Then, the 2-$\underbrace{\text{elt.}}_{=\text{element}}$ subsets of S are

$$\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}.$$

Their number is $6 = \binom{4}{2}$, as Thm. 1.19 predicts.

We'll prove these later (see also [detnotes, Ch. 2] or [GrKnPa]).

<u>Warning:</u> Thm. 1.17 says nothing about $\binom{n}{k}$ when $n \notin \mathbb{N}$. ~~Thm 1.18~~ Neither does Thm. 1.19.

Thm. 1.20 ("hockey-stick identity"). Let $n \in \mathbb{N}$ and $k \in \mathbb{N}$. Then,

$$\binom{0}{k} + \binom{1}{k} + \binom{2}{k} + \dots + \binom{n}{k} = \binom{n+1}{k+1}.$$

$$\left[\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}\right]$$

Proof. HW0 Exercise 2 showed that

$$0^{\underline{k}} + 1^{\underline{k}} + \dots + n^{\underline{k}} = \frac{1}{k+1}(n+1)^{\underline{k+1}}.$$

Dividing it by $k!$ gives

$$\binom{0}{k} + \binom{1}{k} + \dots + \binom{n}{k} = \frac{1}{(k+1) \cdot k!}(n+1)^{\underline{k+1}}$$

$$= \frac{1}{(k+1)!}(n+1)^{\underline{k+1}} = \binom{n+1}{k+1}. \qquad \square$$

Thm. 1.21. (binomial formula). Let $x, y \in \mathbb{R}$. Let $n \in \mathbb{N}$.

Then,
$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Cor. 1.22. Let $n \in \mathbb{N}$. Then, $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

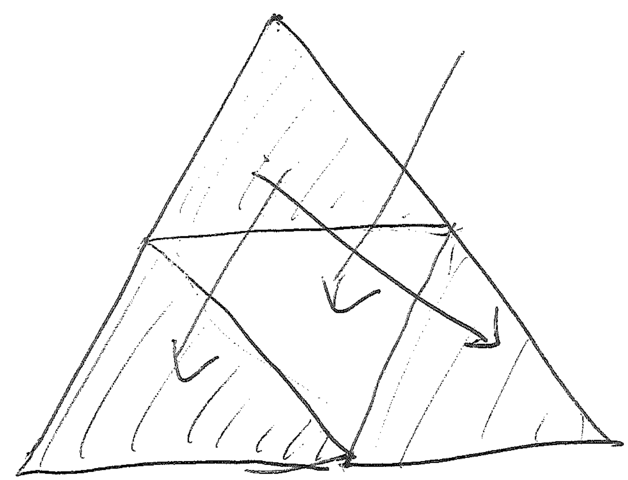
Proof. Apply Thm. 1.21 to $x=1$ & $y=1$. $\square$

Prop. 1.23. Let $n \in \mathbb{N}$. Let $a, b \in \{0, 1, \ldots, 2^n - 1\}$. Then:

(a) $\binom{2^n+a}{b} \equiv \binom{a}{b} \mod 2$.

(b) $\binom{2^n+a}{2^n+b} \equiv \binom{a}{b} \mod 2$.

(For a proof, see Exercise 4 of Spring '18 Math 4707 HW 1.)



Thm. 1.24. Let p be a prime. Let $k \in \{1, 2, \ldots, p-1\}$. Then, $p \mid \binom{p}{k}$.

Proof. We have $\binom{p}{k} = \dfrac{p(p-1)\cdots(p-k+1)}{k!}$, thus $k!\binom{p}{k} = p(p-1)\cdots(p-k+1)$. Hence $p \mid k!\binom{p}{k}$. But p is coprime to $k!$, Hence, $p \mid \binom{p}{k}$. (We tacitly relied on Thm. 1.18.). $\square$

Prop. 1.25. Let $n \in \mathbb{N}$. Then, the Fibonacci number f_{n+1} is

$$f_{n+1} = \sum_{k=0}^{n} \binom{n-k}{k} = \binom{n-0}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots + \binom{n-n}{n}.$$

Thm. 1.26 (Cauchy's binomial formula). Let $x, y, z \in \mathbb{R}$. Then,

$$\sum_{k=0}^{n} \binom{n}{k} (x+kz)^k (y-kz)^{n-k} = \sum_{k=0}^{n} \frac{n!}{k!} (x+y)^k z^{n-k}.$$

Thm. 1.27 (Abel's binomial formula). Let $x, y, z \in \mathbb{R}$. Then

$$\sum_{k=0}^{n} \binom{n}{k} \underbrace{x(x+kz)^{k-1}}_{\substack{\text{This should be} \\ \text{read as 0 if } k=0.}} (y-kz)^{n-k} = (x+y)^n .$$

Proof of Thm. 1.16. We are in one of the following 3 cases:

CASE 1: $k \in \{1, 2, 3, \ldots\}$.

CASE 2: $k = 0$.

CASE 3: $k \notin \mathbb{N}$.

Case 3 is easy: If $k \notin \mathbb{N}$, then $k-1 \notin \mathbb{N}$, so $\binom{n-1}{k-1} = 0$, but also $\binom{n}{k} = 0$ & $\binom{n-1}{k} = 0$, so we have to prove that $0 = 0 + 0$.

Case 2: ~~then~~ Assume $k=0$. Then $\binom{n}{k} = \binom{n}{0} = 1$, and similarly $\binom{n-1}{k} = 1$. Also, $\binom{n-1}{k-1} = \binom{n-1}{-1} = 0$. So we have to prove $1 = 0 + 1$.

Case 1: Let $k \in \{1,2,3,\dots\}$. Then,

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \frac{(n-1)(n-2)\cdots(n-k+1)}{(k-1)!} + \frac{(n-1)(n-2)\cdots(n-k)}{k!}$$

$$= \frac{(n-1)(n-2)\cdots(n-k+1)}{(k-1)!} \underbrace{\left(1 + \frac{n-k}{k}\right)}_{=\frac{n}{k}} \quad (\text{since } k! = k\cdot(k-1)!)$$

$$= \frac{n(n-1)(n-2)\cdots(n-k+1)}{k(k-1)!} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}$$

$$= \binom{n}{k}.$$

$\square$

Thm. 1.28: Let $n \in \mathbb{N}_0$, $k \in \mathbb{N}$ be such that $n \geq k$. Then,

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Proof. $(n-k)!\binom{n}{k} = (n-k)! \cdot \frac{n(n-1)\cdots(n-k+1)}{k!} = \frac{n(n-1)\cdots 1}{k!} = \frac{n!}{k!}.$

$\square$