Math 5705: Enumerative combinatorics

Three main threads:

- counting
- proving polynomial identities
- finding bijections/maps between finite structures.

1. Introduction

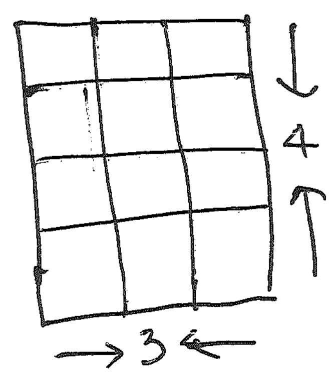
Let us give some concrete examples of enumerative combinatorics.

1.1. Domino tilings

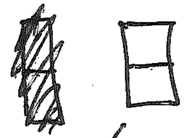
Let $n, m \in \mathbb{N} = \{0, 1, 2, \ldots\}$.

Let $R_{n,m}$ be a rectangle with width n & height m.

Ex:

$R_{3,4} =$ 

A <u>domino</u> is a 1×2-rectangle or a 2×1-rectangle.

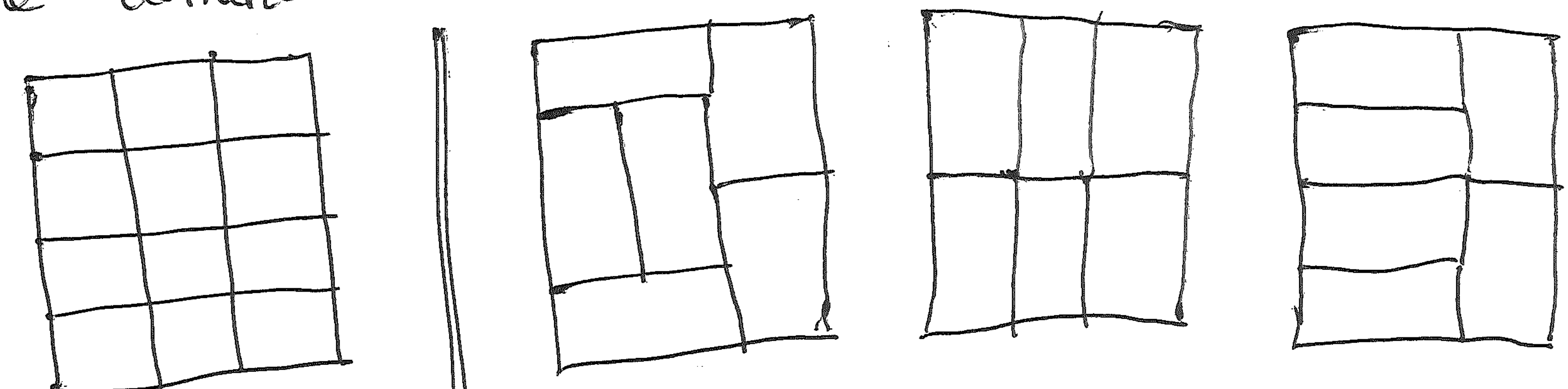


(vertical)

(horizontal)

A <u>domino tiling</u> of $R_{n,m}$ is a way to subdivide $R_{n,m}$ into nonintersecting dominos.

E.g., some domino tilings of $R_{3,4}$

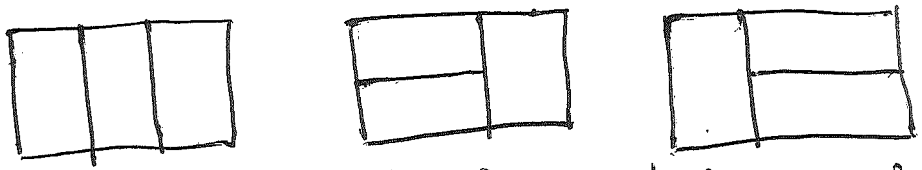


$R_{3,4}$

tilings

How many domino tilings does $R_{n,m}$ have?

E.g. domino tilings of $R_{3,2}$:



How to define domino tilings rigorously?

Geometrically: we'd have to make sense of "subdivide". This is hard.

Combinatorially: Redefine $R_{n,m}$ as the set $[n]\times[m]$,
where $[k] = \{1,2,\dots,k\}$ for each $k\in\mathbb{N}$.
Its elements are the pairs (i,j) with $i\in[n]$ and $j\in[m]$;
we call them "squares". So $|R_{n,m}| = nm$.

A vertical domino is a set of the form $\{(i,j),\ (i,j+1)\}$ for some $i,j\in\mathbb{Z}$.

A horizontal domino is a set of the form $\{(i,j),\ (i+1,j)\}$ for some $i,j\in\mathbb{Z}$.

A domino tiling of a set S is a set of dominos (i.e., vertical or horizontal dominos) whose union is S

and which are disjoint.

This is the model we will use.

Let $d_{n,m}$ = # of all domino tilings of $R_{n,m}$.

<u>Problem:</u> What is $d_{n,m}$?

<u>Prop. 1.1.</u> If n and m are odd, then $d_{n,m} = 0$.

<u>Proof.</u> Assume that n and m are odd. Thus $|R_{n,m}| = nm$ is odd. But each domino has even size. If $R_{n,m}$ had a domino tiling, then the sum of the sizes of all the dominos would be even, but it would also be $|R_{n,m}|$, which is odd. ↯

← this means "contradiction"

$\rightarrow\leftarrow$

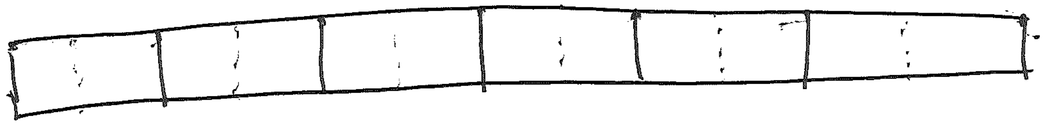
So $\nexists$ a domino tiling. $\square$

We have used the following basic fact:

<u>Thm. 1.2.</u> If a finite set S is the union of k disjoint sets $S_1, S_2, \dots, S_k$, then $|S| = |S_1| + |S_2| + \dots + |S_k|$.

Prop. 1.3. If $m=1$ and n is even, then $d_{n,m}=1$.

Proof.

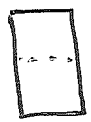


$\square$

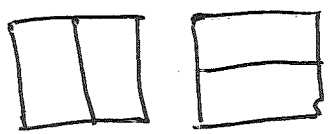
Next case: $m=2$

$d_{0,2}=1$

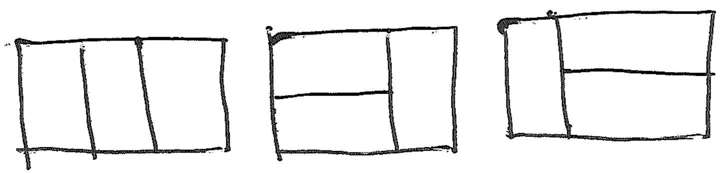
$d_{1,2}=1$



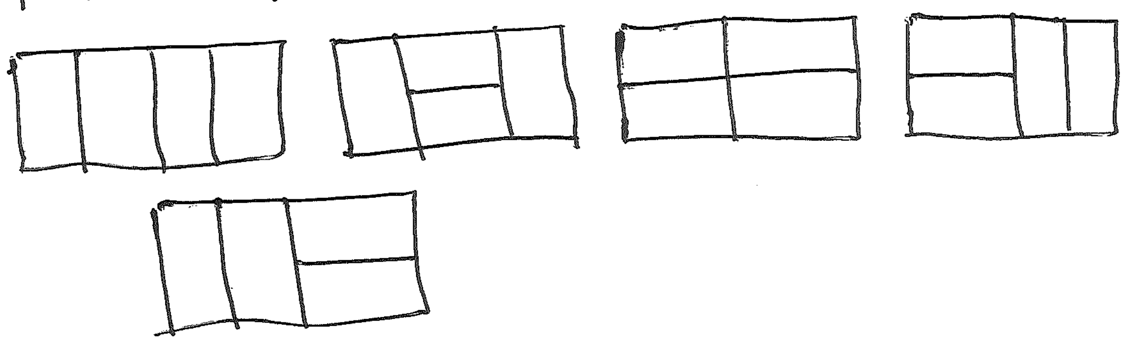
$d_{2,2}=2$



$d_{3,2}=3$



$d_{4,2}=5$



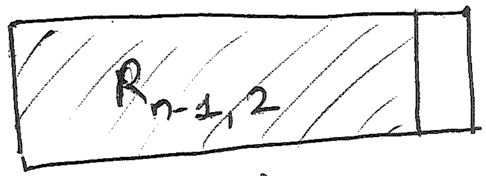
$d_{5,2}=\ ?$

<u>Prop. 1.4.</u> For each $n \geq 2$, we have $d_{n,2} = d_{n-1,2} + d_{n-2,2}$.

<u>Proof.</u> Given a domino tiling T of $R_{n,2}$, consider its last column (i.e., $\{(n,1), (n,2)\}$).

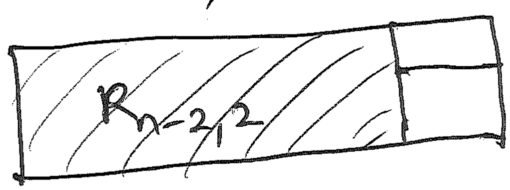
This column is <u>either</u> covered by 1 vertical domino,
<u>or</u> covered by 2 horizontal dominos.

In the first case, T consists of the vertical domino covering the last column, and a domino tiling of $R_{n-1,2}$:



← There are $d_{n-1,2}$ such tilings.

In the second case, T consists of the horizontal dominos covering the last column, and a domino tiling of $R_{n-2,2}$:



← There are $d_{n-2,2}$ such tilings.

So the total # of T's is $d_{n-1,2} + d_{n-2,2}$.
But this # is $d_{n,2}$. Thus, $d_{n,2} = d_{n-1,2} + d_{n-2,2}$. $\square$

Ex: $d_{5,2} = d_{4,2} + d_{3,2} = 5+3 = 8$,

$d_{6,2} = d_{5,2} + d_{4,2} = 8+5 = 13$.

Def: The Fibonacci sequence is a sequence $(f_0, f_1, f_2, \dots)$ of nonnegative integers defined recursively by

$$f_0 = 0, \quad f_1 = 1, \quad f_n = f_{n-1} + f_{n-2} \quad \forall n \geq 2.$$

Ex:

n	0	1	2	3	4	5	6	7	8	9	....
f_n	0	1	1	2	3	5	8	13	21	34	....

Prop. 1.5. $d_{n,2} = f_{n+1} \quad \forall n \in \mathbb{N}$.

Proof. Strong induction on n:

Base: True for $n=0$ and for $n=1$. (Check it.)

Step: Fix $N \in \mathbb{N}$. Assume Prop. 1.5 is true for all $n < N$.
This is the induction hypothesis (short "IH").
We must show that Prop. 1.5 also is true for $n = N$.

Assume $N \geqslant 2$ (else, this follows from Base).

Prop. 1.4 thus yields $d_{N,2} = \underbrace{d_{N-1,2}}_{\substack{= f_N \\ \text{(by IH)}}} + \underbrace{d_{N-2,2}}_{\substack{= f_{N-1} \\ \text{(by IH)}}}$

$$= f_N + f_{N-1} = f_{N+1}. \quad \square$$

How do we compute $d_{n,2}$ or f_n faster than by the above recursion?

<u>Thm. 1.6. (Binet's formula).</u> For each $n \in \mathbb{N}$, we have

$$f_n = \frac{1}{\sqrt{5}}(\varphi^n - \psi^n), \text{ where } \varphi = \frac{1+\sqrt{5}}{2} \approx 1{,}618\ldots$$
$$\text{and } \psi = \frac{1-\sqrt{5}}{2} \approx -0.618\ldots. \quad \square$$

<u>Proof.</u> Induction, as above.

What ~~...~~ about $d_{n,m}$ for $m > 2$?

Thm. 1.7. (Kasteleyn). Let m be even and $n \geq 1$. Then

$$d_{n,m} = 2^{mn/2} \prod_{j=1}^{m/2} \prod_{k=1}^{n} \sqrt{\left(\cos \frac{j\pi}{m+1}\right)^2 + \left(\cos \frac{k\pi}{n+1}\right)^2}.$$

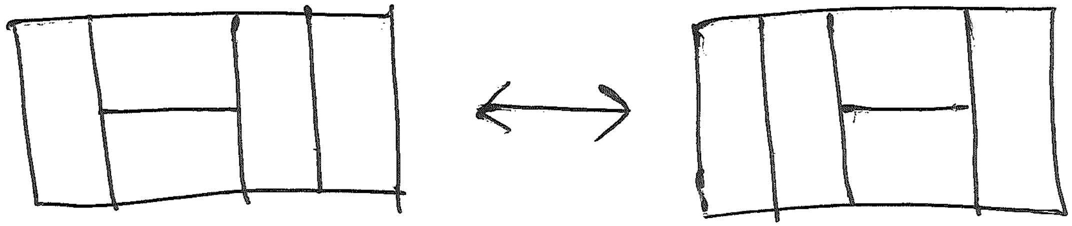
[Rmk: If $a_1, a_2, \ldots, a_p$ are numbers, then $\prod_{i=1}^{p} a_i$ means $a_1 \cdot a_2 \cdot \cdots \cdot a_p$.]

Proof. see [Loehr, Thm. 12.85].

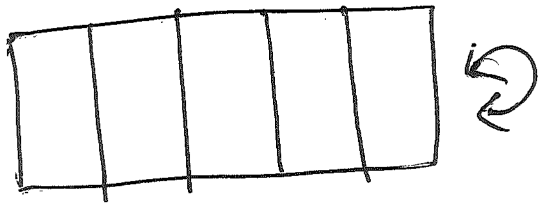
references like this are explained in the syllabus

Ex. $d_{8,8} = 12\ 988\ 816$.

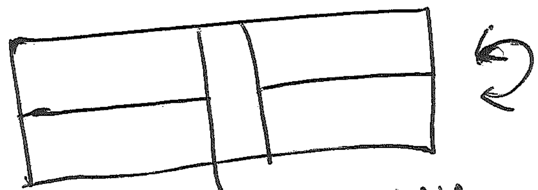
Exercise 1.8. Say that a domino tiling T of $R_{n,2}$ is axisymmetric if reflecting it across the vertical axis of $R_{n,2}$ leaves it unchanged (i.e., for every domino $\{(i,j), (i',j')\} \in T$, the "mirror domino" $\{(n+1-i, j), (n+1-i', j')\}$ is also in T).



not 2xisymmetric

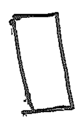


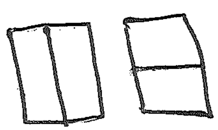
2xisymmetric

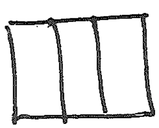


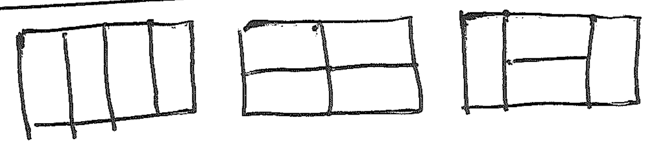
2xisymmetric

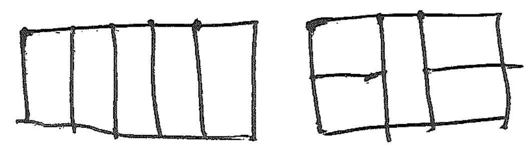
How many 2xisymmetric domino tilings does $R_{n,2}$ have?

$n=1$:  1

$n=2$:  2

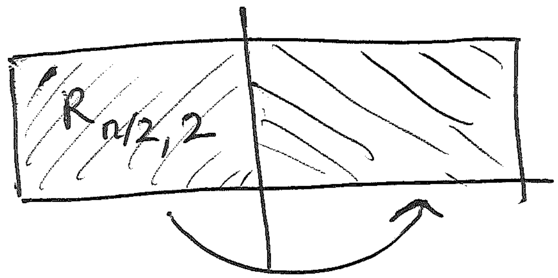
$n=3$:  1

$n=4$:  3

$n=5$:  2

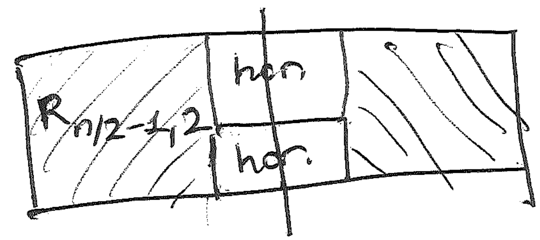
Ideas: Cases "n even" & "n odd" behave differently.

If n is even: – either



$\#\text{options} = d_{n/2,2} = f_{n/2+1}$
(by Prop. 1.5)

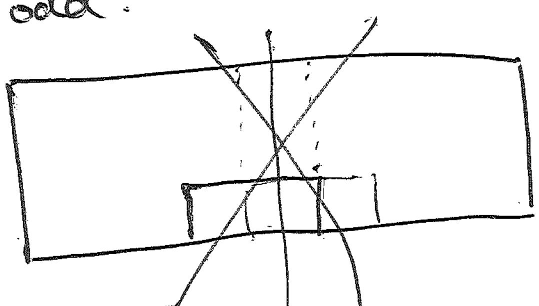
or

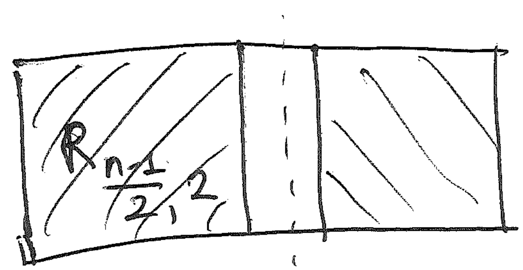


$\#\text{options} = d_{n/2-1,2}$
$= f_{n/2}$
by Prop 1.5

$\Rightarrow$ total # is $f_{n/2+1} + f_{n/2} = f_{n/2+2}$

If n is odd:

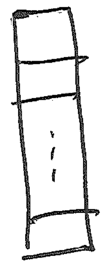


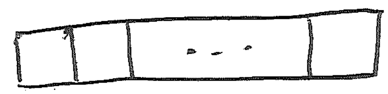


$\Rightarrow$ total # is $d_{\frac{n-1}{2},2} = f_{\frac{n+1}{2},2}$

See Math 4707 Spr '18 HW 1 sol to Exe 5 for details. □

~~A k~~ Fix $k > 0$. A <u>k-polymino</u> is a rectangle of shape $1 \times k$ or $k \times 1$

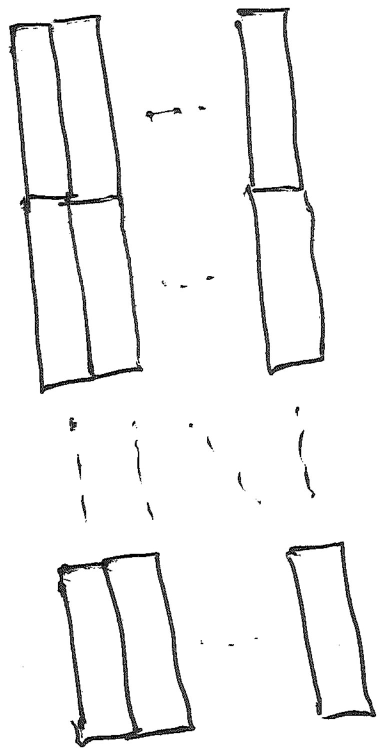




<u>Prop. 1.9.</u> Let ~~n~~ n, m, k be positive integers. Then, $R_{n,m}$ can be tiled with k-polyminos if & only if $k \mid m$ or $k \mid n$.

<u>Proof:</u> $\Longleftarrow$: Assume $k \mid m$ or $k \mid n$.

If $k \mid m$, then $R_{n,m}$ can be tiled with vertical k-polyminos:



If $k | n$, then by horizontal k-polyminos.

$\Rightarrow$: Assume ~~k~~ $R_{n,m}$ can be tiled with k-polyminos.

Assume (for contradiction) that $k \nmid m$ and $k \nmid n$.

Use notation: $a // b$ = quotient of a divided by b;

$a \% b$ = remainder —— // ——

(where a, b are integers, and $b > 0$).

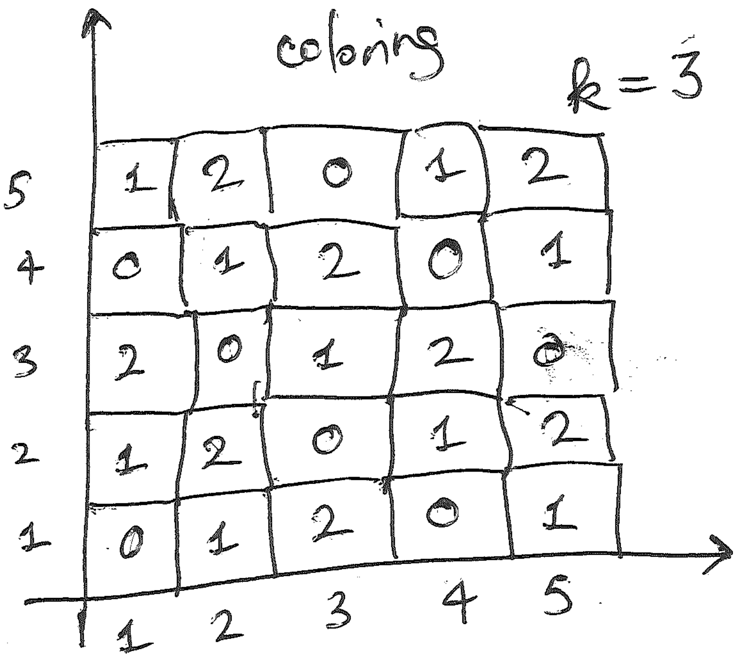
Consider k colors $0, 1, \ldots, k-1$.

Color each square $(i,j) \in \mathbb{Z}^2$ with the color

$$(i+j-2) \,\%\, k,$$

Each k-polymino has exactly 1 square of each color,

$\boxed{u \quad u+1 \quad \cdots \quad k-1 \quad 0 \quad \cdots \quad u-1}$

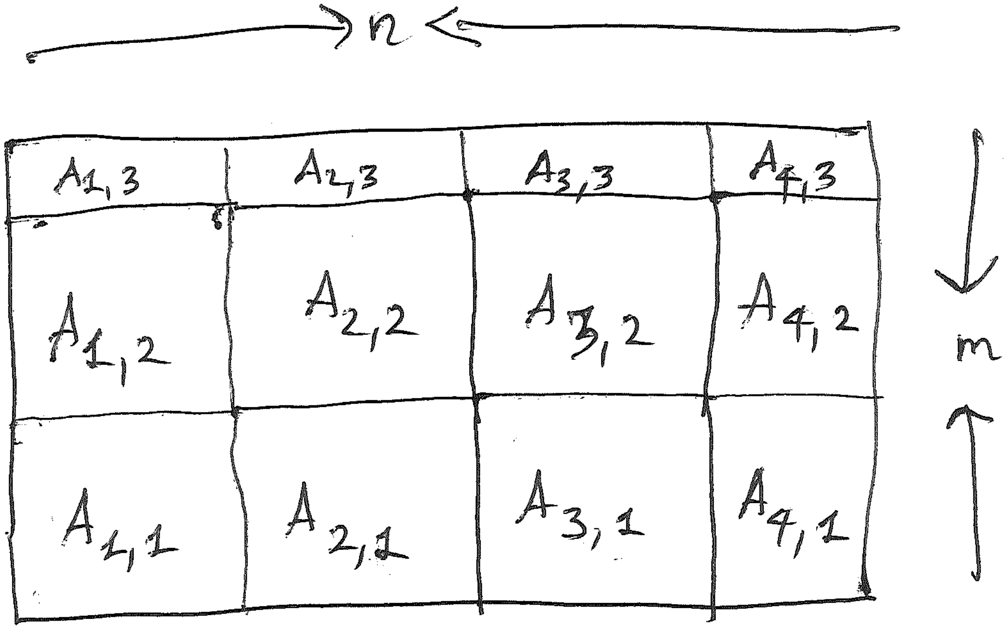


We call a set S of squares <u>balanced</u> if it has equally many squares of each color.

Thus, any k-polymino is balanced.

~~Note that~~ Hence, $R_{n,m}$ is balanced (since it is tiled by k-polyminos).

On the other hand, we split $R_{n,m}$ into the following sub-rectangles:



$$A_{u,v} = \{ (i,j) \in R_{n,m} \mid (i-1)//k = u,\ (j-1)//k = v \}.$$

Note that each $A_{u,v}$ has at least one side divisible by k, unless it is ~~the~~ $A_{(n-1)//k,\ (m-1)//k}$.

~~Meanwhile, $A_{(n-1)//k,\ (m}$~~ So all of these $A_{u,v}$'s are balanced. Thus, $A_{(n-1)//k,\ (m-1)//k}$ must also be balanced (since each color ~~#~~ h satisfies

$(\#\text{ of squares with color } h \text{ in } A_{(n-1)//k,\,(m-1)//k})$

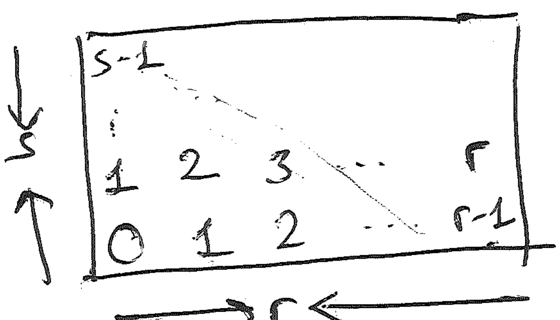
$$= \underbrace{(\#\text{ of squares with color } h \text{ in } R_{n,m})}_{\text{independent on } h} - \sum_{(u,v) \neq ((n-1)//k,\,(m-1)//k)} \underbrace{(\#\text{ of squares with color } h \text{ in } A_{u,v})}_{\text{independent on } h}$$

which is independent on h).

Let $r = n \% k$ and $s = m \% k$.

Then $0 < r < k$ and $0 < s < k$.

But $A_{(n-1)//k,\,(m-1)//k}$ is a rectangle of width r & height s.



WLOG $s \leq r$.

The # of squares with color $s-1$ in $A_{(n-1)//k,\,(m-1)//k}$ is s.

(There is exactly one such square in each row.)

The # of squares with color 0 in $A_{(n-1)//k,\ (m-1)//k}$ is $< s$. (There is at most one such square in each row, but the 2nd row does not contain any.)

So $A_{(n-1)//k,\ (m-1)//k}$ is NOT balanced. $\lightning$ $\square$