5707 Spring 2017 Lecture 8

One more fact about tournaments:

<u>Prop. 1.</u> Let $D = (V, A)$ be a tournament.

Let (u, v, w) be a 3-cycle of D.

(A "3-cycle" of D means a triple (u, v, w) of vertices of D such that $uv, vw, wu \in A$.)

Let D' be the tournament obtained from D by reorienting the arcs uv, vw, wu (this means replacing them by vu, wv, uw).

Then, # of 3-cycles of D'
= # of 3-cycles of D.

Let's give two proofs:

<u>1st proof.</u> HW2 ~~pr~~ exercise 5(b)

shows that the # of 3-cycles of D depends only on $|V|$ and the in-~~out~~degrees $\deg^- v$ of the vertices $v \in V$. But these do not change when we reorient our arcs uv, vw, wu (since each of u, v, w

loses 1 outgoing arc and gains another). Hence, # of 3-cycles also doesn't change. $\square$

<u>2nd proof.</u> It is easy to prove the claim in the case $|V| \leq 4$ (just check all cases). Hence, $\forall x \in V$, we have:

(1) # of ~~vertic~~ 3-cycles of D' whose vertices belong to $\{u, v, w, x\}$
= # of 3-cycles of D whose vertices belong to $\{u, v, w, x\}$

(because the induced subdigraph on the subset $\{u, v, w, x\}$ of a tournament is again a tournament).

Now, ~~split~~ the 3-cycles of D can be of the following 3 types:

TYPE 1: 3-cycles that ~~to~~ contain <u>at most 1</u> of the vertices u, v, w.

TYPE 2: 3-cycles that contain ~~at most 2~~ <u>precisely 2</u> of the vertices u, v, w.

TYPE 3: 3-cycles that contain <u>all</u> of the vertices u, v, w.

The 3-cycles of Type 2 can be ~~all~~ classified further: Each of them has

TYPE 2_x: 3-cycles that contain precisely 2 of the vertices u, v, w, and also the vertex x

for a <u>unique</u> $x \in V \setminus \{u, v, w\}$.

Now,

# of 3-cycles of D' of Type 1
= # of 3-cycles of D of Type 1

(since 3-cycles of Type 1 are preserved when we reorient arcs uv, vw, wu);

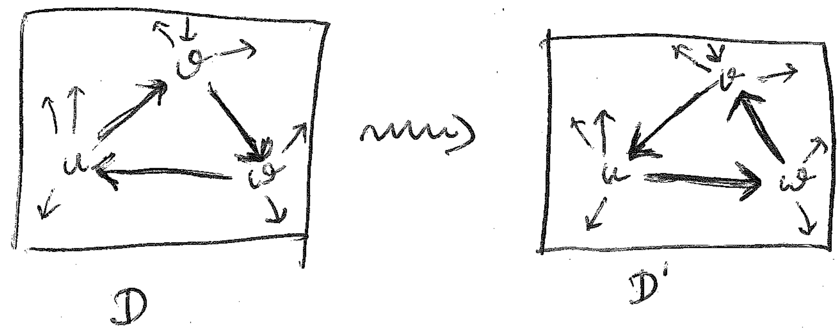
# of 3-cycles of D' of Type 2_x
= # of 3-cycles of D of Type 2_x
$\forall x \in V \setminus \{u, v, w\}$

(by (1));

$\#$ of 3-cycles of D' of Type 3
$= \#$ of 3-cycles of D of Type 3
(since both numbers are 3).

Adding these equalities together, we get

$\#$ of 3-cycles of D'
$= \#$ of 3-cycles of D. $\square$



Reorienting the arcs uv, vw, wu

Now, a few reminders about permutations.

Def. A permutation of a set X is a bijection $X \to X$.

Def. For each $n \in \mathbb{N}$, we let S_n be the set of all permutations of $\{1, 2, \ldots, n\}$. Note that $|S_n| = n!$.

There are several ways to write a permutation $\sigma \in S_n$:

- as the n-tuple $[\sigma(1), \sigma(2), \ldots, \sigma(n)]$ ("one-line notation").
- as the $2 \times n$-table $\begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{pmatrix}$ ("two-line notation").
- as a digraph $(\{1, 2, \ldots, n\}, \{(i, \sigma(i)) \mid i \in \{1, 2, \ldots, n\}\})$.

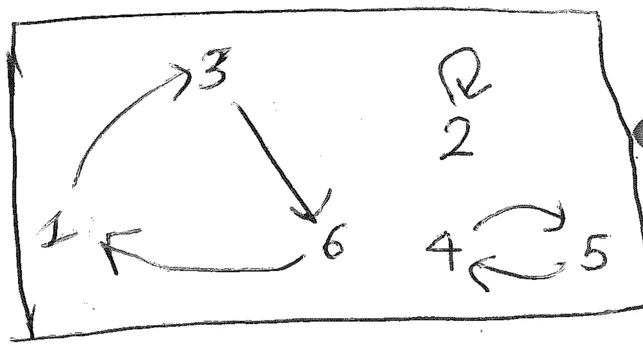
Examples: Let σ be the permutation of $\{1, 2, 3, 4, 5, 6\}$ ~~sending~~ sending $1 \mapsto 3$, $2 \mapsto 2$, $3 \mapsto 6$, $4 \mapsto 5$, $5 \mapsto 4$, $6 \mapsto 1$.

Then,

- the one-line notation for σ is
  $[3,2,6,5,4,1]$.

- the two-line notation for σ is
  $$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 6 & 5 & 4 & 1 \end{pmatrix}.$$

- the digraph for σ is



<u>Rmk:</u> The digraph for $\sigma \in S_n$ has the property that $\forall$ vertex v we have

$$\deg^- v = 1 \quad \text{and} \quad \deg^+ v = 1.$$

This allows one to prove that this digraph is a disjoint union of cycles (incl. 1-vertex cycles).

This is quite useful (although not

for us right now).

<u>Def.</u> Let $n \in \mathbb{N}$ and $\sigma \in S_n$.

~~The sign of the permutation~~

An <u>inversion</u> of σ means a pair (i, j) of integers i, j such that

$$1 \leq i < j \leq n \text{ and } \sigma(i) > \sigma(j).$$

The <u>length</u> of σ is the number of inversions of σ. It is written $\ell(\sigma)$.

The <u>sign</u> of σ is $(-1)^{\ell(\sigma)}$. It is denoted by $(-1)^{\sigma}$ or $\operatorname{sign} \sigma$ or $\operatorname{sgn} \sigma$ or $\varepsilon(\sigma)$.

Properties of the sign:

- $\operatorname{sign}(\sigma) \in \{1, -1\}$.
- $\operatorname{sign}(\mathrm{id}) = 1$
- $\operatorname{sign}(\text{a transposition}) = -1$.

- $\operatorname{sign}(\sigma \circ \tau) = \operatorname{sign}\sigma \cdot \operatorname{sign}\tau$ $\forall \sigma, \tau \in S_n$.
- $\operatorname{sign}(\sigma^{-1}) = \operatorname{sign}\sigma$.
- If the digraph for σ has r cycles, then $\operatorname{sign}\sigma = (-1)^{n-r}$.
- $\operatorname{sign}\sigma = \prod_{1 \le i < j \le n} \frac{\sigma(i) - \sigma(j)}{i - j}$.
- If you write down the one-line notation for σ, and sort it into increasing order by repeatedly swapping adjacent entries ("bubblesort", or rather a more general version thereof), then $\ell(\sigma)$ is the smallest # of swaps you need. (Actually, it is the exact # of swaps you need if you don't waste time by swapping pairs that already are increasing.)

For proofs, see references cited in the Introduction of negra.pdf, especially [Day 16, Chapter 6, B], [Grinbe16, §4.1–§4.3], [Conrad].

Having all this out of our way, we can define the determinant.

<u>Def.</u> Let A be an $n \times n$-matrix (say, with real entries — not that it matters).

For all i, j, let $a_{i,j}$ be the (i,j)-th entry of A (i.e., the entry in row i & column j).

The <u>determinant</u> $\det A$ of A is defined by

$$\det A = \sum_{\sigma \in S_n} \operatorname{sign} \sigma \cdot \prod_{i=1}^{n} a_{i, \sigma(i)} .$$

This is called the Leibniz formula. Among many definitions of the determinant, it is the most explicit one.

<u>Thm. 2 (Vandermonde).</u> Let $n \in \mathbb{N}$.

Let $x_1, x_2, \ldots, x_n$ be n numbers.

Let V be the $n\times n$-matrix whose (i,j)-th entry is x_j^{i-1}

(thus,
$$V = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{pmatrix}$$
).

Then,
$$\det V = \prod_{1\leq i<j\leq n} (x_j - x_i).$$

("Vandermonde determinant", in one of its many forms.)

We ~~...~~ shall prove this using tournaments.

Proof of Thm. 2 (Ira Gessel, [Gessel79])

Let $\mathcal{T}$ be the set of tournaments with vertex set $\{1,2,\ldots,n\}$. (Thus, $|\mathcal{T}| = 2^{n(n-1)/2}$.)

For each $D \in \mathcal{T}$, define the following:

- For each arc $a = ij$ of D, define the weight $w(a)$ of a to be
$$(-1)^{[i>j]} x_j$$
(where we use Iverson bracket notation).

- Define the weight $w(D)$ of D to be $\prod_{a \text{ is an arc of } D} w(a) = \prod_{ij \text{ is an arc of } D} \left((-1)^{[i>j]} x_j\right)$.

Then,

$$\prod_{1 \leq i < j \leq n} (x_j - x_i) = \sum_{D \in \mathcal{T}} w(D),$$

because expanding the left hand side results in a sum of lots and lots of products, each of which corresponds to choosing either the x_j or the $-x_i$ addend from each factor $x_j - x_i$, which we can encode by a tournament $D \in \mathcal{T}$. (Namely: If we choose the x_j addend, then we let the tournament have an arc ij; otherwise, let it have an arc ji.)

Hence, it suffices to show that

$$\det V = \sum_{D \in \mathcal{T}} \cancel{\#} w(D).$$

To do so, we study the # of 3-cycles in a tournament.

~~Let~~ For each $k \in \mathbb{N}$, let

$$w_k = \sum_{\substack{D \in \mathcal{T}; \\ D \text{ has exactly} \\ k \text{ 3-cycles}}} w(D).$$

Then $\sum_{D \in \mathcal{T}} w(D) = w_0 + w_1 + w_2 + \dots$

(this infinite sum is well-defined, since

$w_k = 0$ for any large enough k). Hence, it remains to prove

$$(4) \qquad \det V = w_0 + w_1 + w_2 + \dots,$$

(Note that ~~the~~ the way we defined 3-cycles, the # of 3-cycles in a tournament D is always a multiple of 3, since each 3-cycle (u, v, w) yields two others (v, w, u) and (w, u, v), and we don't equate them. But that's not a problem.)

Let us first study the tournaments without 3-cycles. I claim that they correspond to permutations:

<u>Def.</u> Let $\sigma \in S_n$. Then, define a tournament $T_\sigma \in \mathcal{T}$ as follows: Its arcs should be $(\sigma(i), \sigma(j))$ for $1 \leq i < j \leq n$.

<u>Lem. 3.</u> (a) The tournaments $D \in \mathcal{T}$ having 0 3-cycles are precisely those of the form T_σ for $\sigma \in S_n$.

(b) Any $\sigma \in S_n$ can be reconstructed uniquely from T_σ.

(c) Any $\sigma \in S_n$ satisfies

$$w(T_\sigma) = \operatorname{sign} \sigma \cdot \prod_{i=1}^{n} x_{\sigma(i)}^{i-1}.$$

<u>Proof of Lem. 3.</u> (a) Clearly, T_σ ~~never~~ has no 3-cycles

(because if we had a 3-cycle, then we could write it as $(\sigma(u), \sigma(v), \sigma(w))$, and thus we would have $u < v$, $v < w$ and $w < u$ $\Rightarrow$ $\lightning$).

Conversely: Let $D \in \mathcal{T}$ be a tournament with no 3-cycles. We must find a $\sigma \in S_n$ such that $D = T_\sigma$.

We know from Lecture 7 that every tournament has a Hamiltonian path. Thus, D has one. Let it be $(\sigma(1), \sigma(2), \ldots, \sigma(n))$. Thus, $\sigma \in S_n$. We claim that $D = T_\sigma$. Why?

We know that

$$(\sigma(i), \sigma(i+1)) \text{ is an arc of } D \quad \forall i$$

(since $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ is a Hamiltonian path). Thus,

$$(\sigma(i), \sigma(i+2)) \text{ is an arc of } D \ \forall i$$

(because otherwise, $(\sigma(i+2), \sigma(i))$ would be an arc of D instead, but then, $(\sigma(i), \sigma(i+1), \sigma(i+2))$ would be a 3-cycle, which we know D has not). Thus,

$$(\sigma(i), \sigma(i+3)) \text{ is an arc of } D \ \forall i$$

(because otherwise, $(\sigma(i+3), \sigma(i))$ would be an arc of D instead, but then, $(\sigma(i), \sigma(i+2), \sigma(i+3))$ would be a 3-cycle, which we know D has not).

Continuing the same logic, we find that

$$(\sigma(i), \sigma(i+k)) \text{ is an arc of } D \ \forall i \ \forall k>0.$$

In other words, $(\sigma(i), \sigma(j))$ is an arc of D $\forall i<j$.
In other words, D has all the arcs of T_σ. And no further arcs, since D is a tournament and cannot have more than 1 arc between two given vertices.

So $D = T_\sigma$. This completes the ~~proof~~ proof of (a).

(b) I claim that $(\sigma(1), \sigma(2), \dots, \sigma(n))$ is the <u>only</u> Hamiltonian path of T_σ.

Once this is proven, reconstruction of σ from T_σ will be trivial.

~~This is~~ Let $(\tau(1), \tau(2), \dots, \tau(n))$ be any Hamiltonian path of T_σ. We must prove $\tau = \sigma$.

Clearly, $\tau \in S_n$. If $\tau \neq \sigma$, then

$\sigma^{-1} \circ \tau \neq \mathrm{id}$, thus $\exists\, k \in \{1, 2, \dots, n-1\}$ for which $(\sigma^{-1} \circ \tau)(k) > (\sigma^{-1} \circ \tau)(k+1)$.

Consider this k. Then,

$$(\tau(k), \tau(k+1)) \text{ is an arc of } T_\sigma$$

~~(by the definition of T_σ, since $\sigma^{-1}(\tau(k)) >$~~

(since $(\tau(1), \tau(2), \dots, \tau(n))$ is a Hamiltonian path), but also

$$(\tau(k+1), \tau(k)) \text{ is an arc of } T_\sigma$$

(since $\sigma^{-1}(\tau(k+1)) = (\sigma^{-1} \circ \tau)(k+1)$

$$< (\sigma^{-1} \circ \tau)(k) = \sigma^{-1}(\tau(k)),$$

and by the construction of T_σ),
This is a contradiction, since T_σ is a tournament and cannot have 2 arcs between any given 2 vertices.

So $\tau \neq \sigma$ cannot happen. Hence, $\tau = \sigma$.

(Alternatively, look at T_σ:

$$\sigma(1) \to \sigma(2) \to \sigma(3) \to \cdots \to \sigma(n)$$

This is how T_σ looks like.

All arcs go from left to right, so a Hamiltonian path must traverse $\sigma(1), \sigma(2), \ldots, \sigma(n)$ in this order.

(c) $$w(T_\sigma) = \prod_{\substack{a \text{ is an} \\ \text{arc of } T_\sigma}} w(a)$$

$$= \prod_{1 \le i < j \le n} w(\sigma(i), \sigma(j))$$

(by the definition of T_σ)

$$= \prod_{1 \le i < j \le n} \left((-1)^{[\sigma(i) > \sigma(j)]} x_{\sigma(j)} \right)$$

(by the definition of weights)

$$= \underbrace{\left(\prod_{1 \le i < j \le n} (-1)^{[\sigma(i) > \sigma(j)]} \right)}_{\substack{= (-1)^{\sum_{1 \le i < j \le n} [\sigma(i) > \sigma(j)]} \\ = (-1)^{\#\text{ of inversions of } \sigma} \\ = (-1)^{\ell(\sigma)} = \operatorname{sign} \sigma}} \underbrace{\prod_{1 \le i < j \le n} x_{\sigma(j)}}_{\substack{= \prod_{j=1}^{n} \prod_{i=1}^{j-1} x_{\sigma(j)} \\ = \prod_{j=1}^{n} x_{\sigma(j)}^{j-1} \\ = \prod_{i=1}^{n} x_{\sigma(i)}^{i-1}}}$$

$$\cancel{= \operatorname{sign} \sigma \prod^{n}}$$

$$= \operatorname{sign} \sigma \cdot \prod_{i=1}^{n} x_{\sigma(i)}^{i-1} .$$

Thus, Lemma 3 (c) is proven. $\square$

Cor. 4, $\quad \det V = w_0$.

Proof of Cor. 4. The definition of a determinant yields

$$\det V = \sum_{\sigma \in S_n} \underbrace{\operatorname{sign} \sigma \cdot \prod_{i=1}^{n} x_{\sigma(i)}^{i-1}}_{\substack{= w(T_\sigma) \\ \text{(by Lemma 3(c))}}}$$

$$= \sum_{\sigma \in S_n} w(T_\sigma) = \sum_{\substack{D \in \mathcal{T}; \\ D \text{ has exactly} \\ 0 \text{ 3-cycles}}} w(D)$$

(by Lemma 3 (a) & (b))

$$= w_0. \qquad \square$$

Recall that our goal is to prove (4), In light of Cor. 4, it suffices to show that $w_k = 0 \quad \forall k > 0$.

Thus, let us fix $k>0$. We want to prove $\omega_k = 0$.

We shall prove something slightly stronger:

Lem. 5. Let $(d_1, d_2, \ldots, d_n) \in \mathbb{N}^n$.

~~Then~~ Then, $$\sum_{\substack{D \in T; \\ D \text{ has exactly} \\ k \text{ 3-cycles;} \\ \deg_D^-(i) = d_i \ \forall i}} \operatorname{sign} D = 0,$$

where $\operatorname{sign} D = \prod_{\substack{(i,j) \text{ is an} \\ \text{arc of } D}} (-1)^{[i>j]} \in \{1, -1\}$.

Proof of Lem. 5. A ~~neat pa~~ *flippy pair* will mean a pair (D, α) where D ~~is a~~ $\in T$ is a tournament with exactly k 3-cycles and having $\deg_D^-(i) = d_i \ \forall i$, and where α is a 3-cycle of D.

If (D, α) is a flippy pair, then $\operatorname{flip}(D, \alpha)$ shall mean the pair

(D', α') defined as follows:

- Let (u, v, w) be the 3-cycle α.
- Let D' be the tournament obtained from D by reorienting the arcs uv, vw, wu.
- Let α' be the 3-cycle (u, w, v) of D'.

To see that this (D', α') is indeed a flippy pair, we must observe that D' has exactly k 3-cycles (by Prop. 1) and satisfies $\deg^-_{D'}(i) = d_i$ $\forall i$ (since the indegrees do not change from D to D').

Moreover, $\operatorname{flip}(D', \alpha') = (D, \alpha)$

and $\operatorname{sign}(D') = \operatorname{sign} D$ (since in the definition of $\operatorname{sign} D$, precisely 3 factors in the product change their sign when we pass from D to D').

Thus, we have two mutually inverse bijections

$$\{\text{flippy pairs } (D,\alpha) \text{ with sign } D = 1\}$$
$$\to \{\text{flippy pairs } (D,\alpha) \text{ with sign } D = -1\},$$
$$(D,\alpha) \mapsto \text{flip}(D,\alpha)$$

and

$$\{\text{flippy pairs } (D,\alpha) \text{ with sign } D = -1\}$$
$$\to \{\text{flippy pairs } (D,\alpha) \text{ with sign } D = 1\},$$
$$(D,\alpha) \mapsto \text{flip}(D,\alpha).$$

Hence, the sets

$$\{\text{flippy pairs } (D,\alpha) \text{ with sign } D = -1\}$$

and

$$\{\text{flippy pairs } (D,\alpha) \text{ with sign } D = 1\}$$

have the same size. Thus,

$$\sum_{\substack{(D,\alpha) \text{ is a} \\ \text{flippy pair}}} \text{sign } D = \left(\begin{array}{l}\text{a sum of several} \\ 1\text{s and equally} \\ \text{many } -1\text{s}\end{array}\right)$$
$$= 0.$$

But the left hand side of this equality is

$$k \cdot \sum_{\substack{D \in T; \\ D \text{ has exactly} \\ k \text{ 3-cycles;} \\ \deg_D^-(i) = d_i \ \forall i}} \operatorname{sign} D$$

(because each $D \in T$ having exactly k 3-cycles and satisfying $\deg_D^-(i) = d_i \ \forall i$ is part of precisely k flippy pairs — namely, 1 for each of its k 3-cycles).

Hence, dividing by k, we ~~find~~ obtain the claim of Lem. 5.

Now,

$$w_k = \sum_{\substack{D \in T; \\ D \text{ has exactly} \\ k \text{ 3-cycles}}} w(D)$$

$$= \sum_{(d_1, d_2, \ldots, d_n) \in \mathbb{N}^n} \ \sum_{\substack{D \in T; \\ D \text{ has exactly} \\ k \text{ 3-cycles;} \\ \deg_D^-(i) = d_i \ \forall i}} w(D)$$

$$= \sum_{(d_1, d_2, \ldots, d_n) \in \mathbb{N}^n} \ \sum_{\substack{D \in T; \\ D \text{ has exactly} \\ k \text{ 3-cycles;} \\ \deg_D^-(i) = d_i \ \forall i}} \overbrace{(\operatorname{sign} D)}^{\substack{\text{defined as in} \\ \text{Lem. 5}}} \cdot \prod_{j=1}^{n} x_j^{d_j}$$

(because for each $D \in T$ with $\deg_D^-(i) = d_i \ \forall i$, we have

$$w(D) = \prod_{\substack{ij \text{ is an} \\ \text{arc of } D}} \left((-1)^{[i>j]} x_j \right)$$

$$= \underbrace{\left(\prod_{\substack{ij \text{ is an} \\ \text{arc of } D}} (-1)^{[i>j]} \right)}_{= \operatorname{sign} D} \underbrace{\left(\prod_{\substack{ij \text{ is an} \\ \text{arc of } D}} x_j \right)}_{= \prod_{j=1}^{n} \prod_{\substack{1 \leq i \leq n; \\ ij \text{ is an} \\ \text{arc of } D}} x_j}$$

$$= (\operatorname{sign} D) \cdot \prod_{j=1}^{n} \underbrace{\prod_{\substack{1 \leq i \leq n; \\ ij \text{ is an} \\ \text{arc of } D}} x_j}_{= x_j^{d_j}}$$

$$= (\operatorname{sign} D) \cdot \prod_{j=1}^{n} x_j^{d_j} \qquad)$$

$$= \sum_{(d_1, d_2, \ldots, d_n) \in \mathbb{N}^n} \underbrace{\left(\sum_{\substack{D \in \mathcal{T}; \\ D \text{ has exactly} \\ k \text{ 3-cycles}; \\ \deg^-_D(i) = d_i \ \forall i}} \operatorname{sign} D \right)}_{\substack{=0 \\ \text{(by Lem. 5)}}} \prod_{j=1}^{n} x_j^{d_j}$$

$= 0$, and so we're done. $\square$