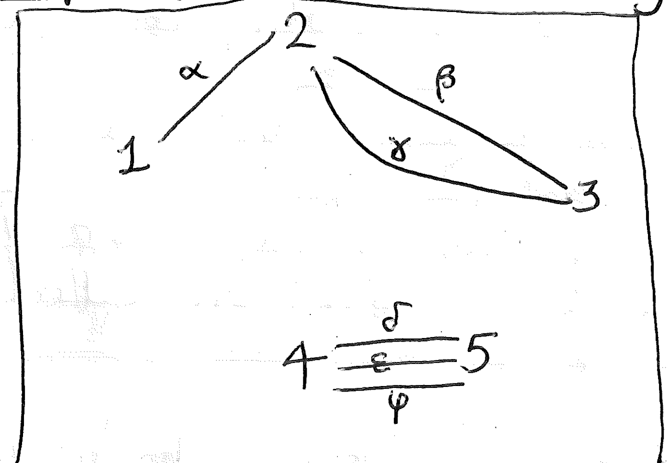
5707 Spring 2017 Lecture 6 A

Multigraphs

We have so far been working with simple graphs. But there are also other kinds of graphs, one of which we shall now introduce. (See also the solutions to hw2 for these definitions.)

Def. A *multigraph* is a triple (V, E, ϕ), where V and E are two finite sets, and $\phi : E \to \mathcal{P}_2(V)$ is a map.

Examples: Here is a multigraph



Formally speaking, it is the multigraph (V, E, ϕ), where $V = \{1, 2, 3, 4, 5\}$,

$E = \{\alpha, \beta, \gamma, \delta, \varepsilon, \varphi\}$ and $\phi : E \to \mathcal{P}_2(V)$ is the map sending $\alpha, \beta, \gamma, \delta, \varepsilon, \varphi$ to $\{1,2\}, \{2,3\}, \{2,3\}, \{4,5\}, \{4,5\}, \{4,5\}$, respectively.

This suggests further notations:

Def. Let $G = (V, E, \phi)$ be a multigraph.

(a) The elements of V are called the vertices of G.

The set V is called the vertex set of G. It is denoted $V(G)$.

(b) The elements of E are called the edges of G. It is denoted $E(G)$.

The set E is called the edge set of G.

(c) If e is an edge of G, then the two elements of $\phi(e)$ are called the endpoints of G.

(d) The degree ~~of~~ $\deg v$ (also written $\deg_G v$) of a vertex v of G is defined to be the number of edges of G having v as an endpoint.

In other words,

$$\deg v = \deg_G v = \left| \{ e \in E \mid v \in \phi(e) \} \right| .$$

Note that (unlike for simple graphs) $\deg v$ is not always the number of neighbors of v, since two edges can lead to the same neighbor!

(e) We say that an edge e <u>contains</u> a vertex v if v is an endpoint of e (that is, $v \in \phi(e)$).

(f) Two edges e and f of G are said to be <u>parallel</u> if $\phi(e) = \phi(f)$.

(In the above example, δ is parallel to each of $\delta, \varepsilon, \rho$, for instance.)

(g) We say that G has <u>no parallel edges</u> if $\forall$ edges e and f of G:

If e and f are parallel, then $e = f$.

(h) A <u>walk</u> in G means a list of the form $(v_0, e_1, v_1, e_2, v_2, \ldots, e_k, v_k)$ where $v_0, v_1, \ldots, v_k$ are vertices of G,

and where e_i (for each $i \in \{1, 2, \ldots, k\}$) is an edge of G having endpoints e_i and

e_{i+1}.

Note that we have to record both the vertices and the edges, since we want a walk to "know" which edges it traverses.
(E.g., in our above example, we count the walks $(1, \alpha, 2, \beta, 3)$ and $(1, \alpha, 2, \gamma, 3)$ as different.)

The vertices of such a walk are

$v_0, v_1, \ldots, v_k$.

(i) The notions of "connected" and $\simeq_G$ and "connected component" are defined as for simple graphs.

(j) A path is a walk whose vertices are distinct.

(k) A circuit is a walk $(v_0, e_1, v_1, e_2, v_2, \ldots, e_k, v_k)$ with $v_0 = v_k$.

(l) A cycle means a ~~walk~~ circuit $(v_0, e_1, v_1, e_2, v_2, \ldots, e_k, v_k)$ such that the vertices $v_0, v_1, \ldots, v_{k-1}$ are distinct, and the edges $e_1, e_2, \ldots, e_k$ are distinct, and $k \geq 2$.

(Note that we are not requiring $k \geq 3$ unlike for simple graphs.
Thus, in our above example, $(2, \beta, 3, \gamma, 2)$ counts as a cycle, But $(2, \beta, 3, \beta, 2)$ does not, since the edges have to be distinct.)

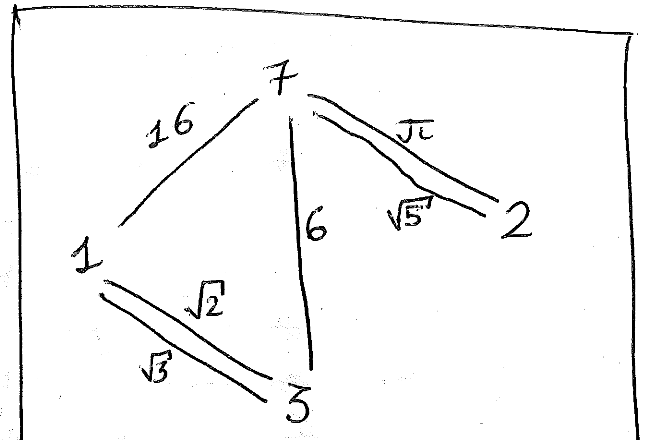
~~(m) Notions such as neighbors~~

(m) Hamiltonian paths and cycles are defined as for simple graphs.

(n) The <u>underlying simple graph</u> G^{simp} of G means the simple graph $(V, \phi(E))$.
In other words, it is the simple graph with the same vertex set as G, in which two vertices u and v are adjacent if and only if there is at least one edge of G with endpoints u and v.

(For example, the underlying simple graph of the multigraph

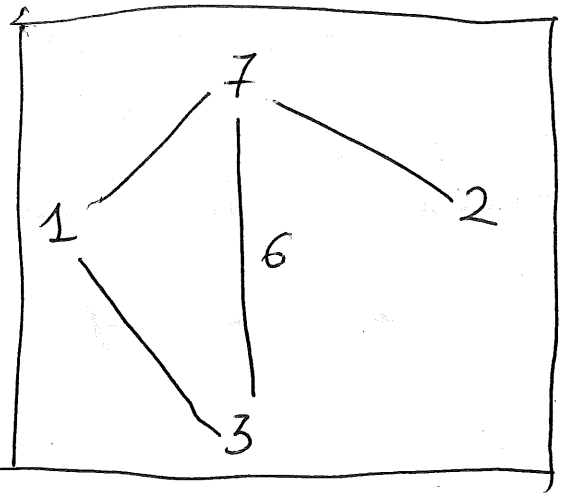
$G =$



(see the next page for how to interpret such a drawing)

is

$G^{simp} =$



.

Any parallel edges in G get "collapsed" to one edge in G^{simp}, since simple graphs have "no room" for parallel edges.)

I like to say that the difference between simple graphs and multigraphs is the following: In a simple graph, an edge e __is__ a set of two vertices,

while in a multigraph, an edge e __has__ a set of two vertices (assigned to it by the map ϕ), but otherwise is free to have "its own identity". This not only allows ~~so, in a mult~~ parallel edges, but also allows edges to carry extra information.

We draw a multigraph (V, E, ϕ) by drawing the vertices as points, and then, for each edge $e \in E$, drawing a curve connecting the two endpoints of e; then, labelling this curve by e.
The labels are often omitted when we are not interested in them.

Some things about multigraphs are different than the corresponding things about simple graphs; some are analogous; some are ~~actually~~ exactly the same. Let me show some examples:

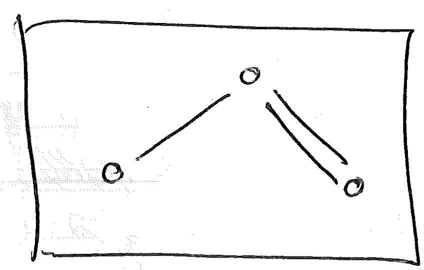
- Proposition 2.5.8 in the notes (the "handshaking lemma") says that if $G = (V, E)$ is a simple graph, then

nogra.pdf

the number of $v \in V$ such that $\deg v$ is odd is even.
Exactly the same theorem holds for multigraphs (V, E, ϕ), and essentially the same proof applies (with just a trivial modification: $v \in e$ becomes $v \in \phi(e)$).
Same holds for Proposition 2.5.6 (which says that

$$\sum_{v \in V} \deg v = 2|E|).$$

- Proposition 2.5.9 in nogra.pdf says that if $G = (V, E)$ is a simple graph with $\geqslant 2$ vertices, then G has two distinct vertices with the same degree. This does NOT hold for multigraphs. For instance,



has a vertex of degree 1, a vertex of degree 2, and a vertex of degree 3.

- A multigraph G has a hamp (= Hamiltonian path) if and only if the simple graph G^{simp} has a hamp. But the hamps of G are not in a 1-to-1 correspondence with the hamps of G^{simp}; ~~for example,~~ the "collapsing" of parallel edges can render two distinct paths identical.

- If you define adjacency and dominating sets for multigraphs in the obvious way, then the dominating sets of G are just the dominating sets of G^{simp} (where G is any multigraph).
~~Thus, for the purpose of~~
Thus, as long as we are studying dominating sets, we don't lose any generality by keeping to simple graphs.

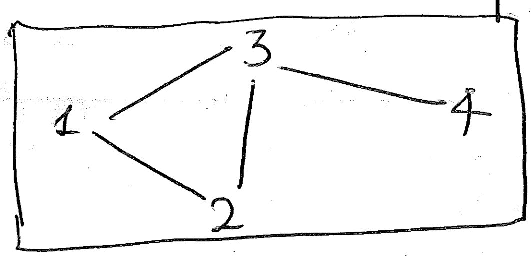
- We defined graph isomorphisms between simple graphs as certain ~~b~~ bijections between vertex sets. Such a definition would be inappropriate for multigraphs, since since it would ignore parallel edges (and thus, isomorphic multigraphs would no longer necessarily have the same number of edges). Instead, we use the following definition:

<u>Def.</u> Let G and H be two multigraphs. An <u>isomorphism</u> (or, more precisely, a <u>multigraph isomorphism</u>) from G to H

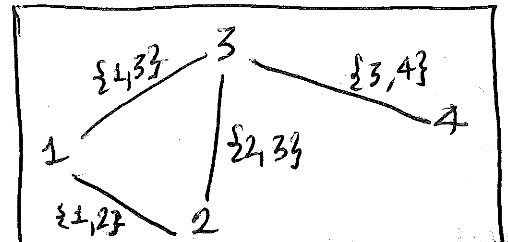
is a pair of bijections $\alpha : V(G) \to V(H)$ and $\beta : E(G) \to E(H)$ such that for each edge $e \in E(G)$, the following holds:
The endpoints of e are sent by α to the endpoints of $\beta(e)$.

Using this definition, the concept of multigraph isomorphisms has the same properties as the previously defined concept of graph isomorphisms.

<u>Def.</u> If $G = (V, E)$ is a simple graph, then we can define a multigraph G^{mult} by $G^{mult} = (V, E, \iota)$, where $\iota : E \to \mathcal{P}_2(V)$ sends each edge $e \in E$ to e itself. For example, if

$G =$ 

, then

$G^{mult} =$ 

.

Prop. (a) If G is a simple graph, then

$$(G^{mult})^{simp} = G.$$

(b) If G is a multigraph that has no parallel edges, then

$$(G^{simp})^{mult} \cong G.$$

(Only an isomorphism, not an equality – because G^{simp} has "forgotten" the identities of the edges of G, and nothing can recover them).

(c) If G is a multigraph that has parallel edges, then

$$(G^{simp})^{mult} \not\cong G$$

(and in fact, $(G^{simp})^{mult}$ has fewer edges than G).

Eulerian circuits and walks

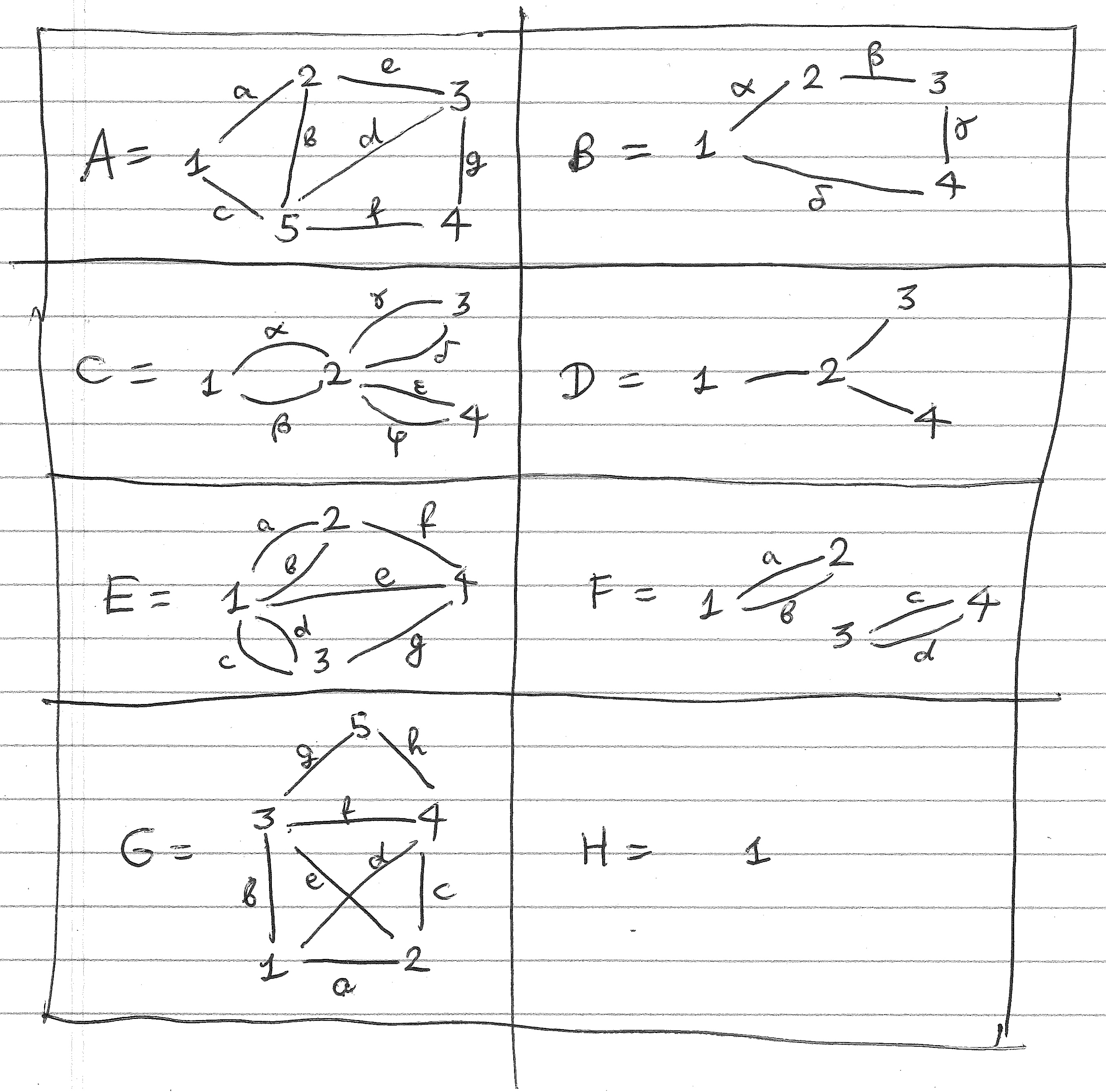
A multigraph G has a Hamiltonian ~~cycl~~ cycle if & only if the simple graph G^{simp} does. Same holds for Hamiltonian paths. Thus, there is no need to study Hamiltonian ~~circuits~~ cycles & paths again for multigraphs. But if we replace "containing each vertex" by "containing each edge", then the difference between multigraphs and simple graphs becomes significant. This leads to ~~two~~ a notion ~~s~~ better defined for multigraphs:

Def. Let G be a multigraph. A walk in G is said to be **Eulerian** if each edge of G appears uniquely on this walk. (I.e., a walk $(v_0, e_1, v_1, e_2, v_2, \ldots, e_k, v_k)$ in G is said to be **Eulerian** if ~~there~~ for each edge e of ~~G~~ G, there exists exactly one $i \in \{1, 2, \ldots, k\}$ such that $e = e_i$.)

We don't need to separately define Eulerian circuits; they are just circuits that are Eulerian (regarded as walks).

Eulerian walks rarely tend to be cycles or paths. (Nevertheless, some authors use the notations "Eulerian paths" and "Eulerian cycles" for "Eulerian walks" and "Eulerian circuits", respectively. This is quite confusing.)

<u>Example:</u> Consider the following multigraphs:



The multigraph A has an Eulerian walk: for example, (2, e, 3, g, 4, f, 5, c, 1, a, 2, b, 5, d, 3). It is not the only one.
But A has no Eulerian circuit. Do you see why, without trying out many cases?

The multigraph B has an Eulerian circuit $(1, \alpha, 2, \beta, 3, \gamma, 4, \delta, 1)$. This, of course, is an Eulerian walk. It is actually one of the few cases of an ~~graph~~ Eulerian circuit that is a cycle.

The multigraph C has an Eulerian circuit $(1, \alpha, 2, \gamma, 3, \delta, 2, \varepsilon, 4, \varphi, 2, \beta, 1)$.

The multigraph D has neither an Eulerian circuit nor an Eulerian walk.

The multigraph E has ~~an~~ no Eulerian walk. It actually is a famous multigraph: see the Wikipedia page on "Seven Bridges of Königsberg". In a nutshell: In 1736, Euler wondered whether it is possible to make a tour of Königsberg, walking through each of the 7 bridges that Königsberg had at his time exactly once. The multigraph E is a model for the town of Königsberg:

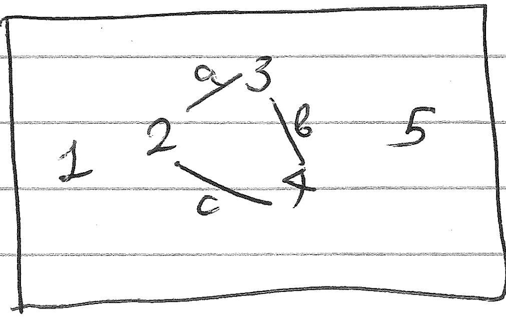
The vertices of E stand for the islands and river banks; the edges stand for the bridges. So Euler was looking for an Eulerian circuit or an Eulerian walk (depending on whether the tour should end where it starts or not) in E.

The multigraph F has no Eulerian walk, since it has two connected components each containing at least one edge. (An Eulerian walk would have to contain both edges b and c, but there is no way to walk between them.)

The multigraph G has an Eulerian walk $(1, b, 3, g, 5, h, 4, d, 1, a, 2, e, 3, f, 4, c, 2)$, but no Eulerian circuit.

The multigraph H has an Eulerian circuit (1).

Rmk. Note that a multigraph that is not connected can still have an Eulerian walk if all but one of its connected components are isolated vertices.
For example, the multigraph



has an Eulerian circuit $(2, a, 3, b, 4, c, 2)$.

How hard is it to find an Eulerian walk in a multigraph, or to check if there is any? Surprisingly, it is a lot easier than the same questions for Hamiltonian paths.
The second question is fully answered (for connected multigraphs) by the Euler - Hierholzer theorem:

~~Def~~ Thm. (Euler, Hierholzer): Let G be a connected multigraph.

(a) The multigraph G has an Eulerian circuit if & only if each vertex of G has even degree.

(b) The multigraph G has an Eulerian walk if & only if all but at most two vertices of G have even degree.

There are various proofs of this Thm.:

- a proof in [Guicha16, Thm. 5.22 and Thm. 5.2.3] (beware that Guichard has weird notations);

~~- a proof in [LeLeMe16, Problem 10.[illegible]]~~

- a proof (sketch) in [LeLeMe16, Problem 12.35];

- yet another proof can be obtained with some work using the BEST theorem (to be considered later).

I don't think Euler had a proof; so the first proof might be by Hierholtzer in 1873.

Digraphs

On to two other kinds of graphs.

<u>Def.</u> A <u>digraph</u> is a pair (V, A) with V being a finite set and A being a subset of $V \times V$.
The elements of V are called the <u>vertices</u> of (V, A). The set V is called the <u>vertex set</u> of (V, A), and is denoted by ~~$V((V,A))$~~, $V(D)$, where $D = (V, A)$.
The elements of A are called the <u>arcs</u> of (V, A). The set A is called the <u>arc set</u> of (V, A), and is denoted by $A(D)$, where $D = (V, A)$.
Some authors say "edges" instead of "arcs", but I will stick to "arcs" to better separate the setting of a digraph from that of a graph.

If (u, v) is an arc of D (or, more generally, a pair in $V \times V$), then u is called the <u>source</u> of this arc, and v is called the ~~sink of this arc.~~ ~~The~~ <u>target</u> of this arc.

Def. A <u>multidigraph</u> is a triple (V, A, ϕ), where V and A are finite sets, and $\phi : A \to V \times V$ is a map.
The elements of V are called the <u>vertices</u> of ~~V(D), where D=~~ (V, A).
The set V is called the <u>vertex set</u> of (V, A), and is denoted by $V(D)$, where $D = (V, A)$.
The elements of A are called the <u>arcs</u> of (V, A). The set A is called the <u>arc set</u> of (V, A), and is denoted by $A(D)$, where $D = (V, A)$.
Again, some authors speak of "edges" instead of "arcs".

~~If (u, v) is an~~
If a is an arc of D, and if $\phi(a) = (u, v)$, then the vertex u is called the <u>source</u> of this arc a, and the vertex v is called the <u>target</u> of this arc a.

So digraphs and *multidigraphs are analogues of simple graphs and multigraphs (respectively), in which the edges have been replaced by arcs ("endowed with a direction").
*Digraphs are often called <u>directed graphs</u> for this reason.