5707 Spring 2017 Lecture 11

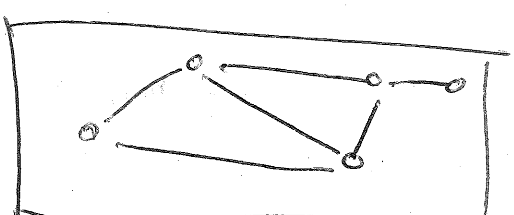
Independent sets & probabilistic proofs.

Let us now change the topic.

Recall that an <u>independent ~~set~~ set</u> of a graph G ~~G~~ is a subset S of V(G) such that no two elements of S are adjacent.

Finding a maximal independent set is easy
Finding a maximum independent set is hard.
But can we find, at least, a reasonably large one?

<u>Thm. 1.</u> Let $G = (V, E)$ be a simple graph.
Then, $\exists$ an independent set of G having size $\geq \sum_{v \in V} \frac{1}{1 + \deg v}$.

<u>Example:</u> G = 

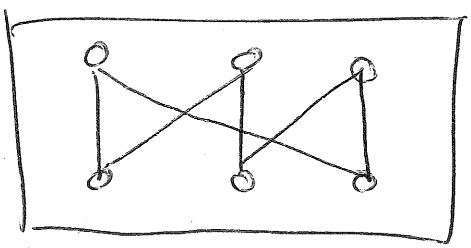
Thm. 1 predicts an independent set of size

$$\geq \frac{1}{1+2} + \frac{1}{1+3} + \frac{1}{1+3} + \frac{1}{1+3} + \frac{1}{1+1} = \frac{19}{12},$$

hence ≥ 2.

And that's ~~[illegible]~~ actually a maximum independent set.

But here:

$G =$ 

Thm. 1 predicts an indep. set of size $\geq 6 \cdot \frac{1}{1+2} = 2$, which exists but is not maximum.

So Thm. 1 does not always yield a maximum indep. set.

We shall give two proofs of Thm. 1. The first will illustrate the ~~[illegible]~~ method of probabilistic proof.

First proof of Thm. 1.

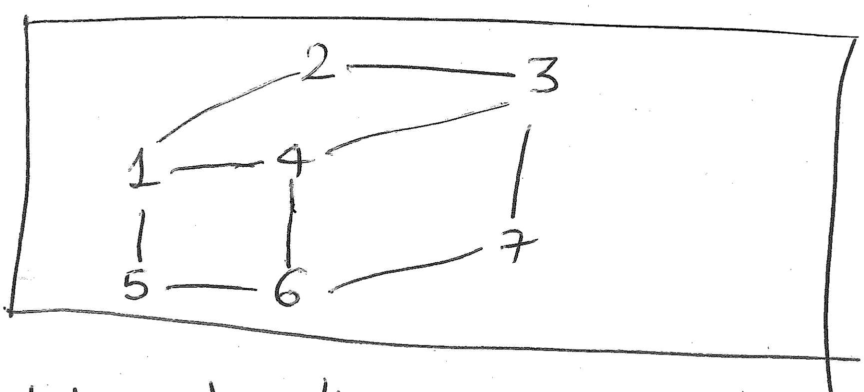
Assume the contrary. Thus, each independent set S of G has size

$$(1) \qquad |S| < \sum_{v \in V} \frac{1}{1 + \deg v}.$$

Let a V-listing mean a list of all vertices in V, each occurring exactly once. If σ is a V-listing, then we define a subset J_σ of V as follows:

$$J_\sigma = \{ v \in V \mid v \text{ occurs before all neighbors of } v \text{ in the V-listing } \sigma \}.$$

[Example: Let G be the graph



and let σ be the V-listing $(1, 2, 7, 5, 3, 4, 6)$. Then, 1 occurs before all its neighbors (2, 4 and 5) in σ, and 7 also occurs before all its neighbors (3 and 6) in σ. But no other vertex does (e.g., 4 does not, since it occurs after 1). So $J_\sigma = \{1, 7\}$.]

The set J_σ is an independent set of G (since any two adjacent elements of J_σ would have to ~~occur~~ occur before each other in σ, which is absurd). $\Rightarrow$ (1) yields

$$(2) \qquad |J_\sigma| < \sum_{v \in V} \frac{1}{1+\deg v}.$$

This holds for each σ. Thus, summing up, we see that

$$\sum_{\substack{\sigma \text{ is a} \\ V\text{-listing}}} |J_\sigma| < \sum_{\substack{\sigma \text{ is a} \\ V\text{-listing}}} \sum_{v \in V} \frac{1}{1+\deg v}$$

$$(3) \qquad = (\# \text{ of all } V\text{-listings } \sigma) \cdot \sum_{v \in V} \frac{1}{1+\deg v}.$$

But let me prove something different: I claim that for each $v \in V$, we have

(# of all V-listings σ such that v occurs before all neighbors of v in σ)

(4) $$= \frac{1}{1+\deg v} \cdot (\text{# of all } V\text{-listings } \sigma).$$

[<u>Proof of (4):</u> Fix $v \in V$. Whenever τ is a V-listing, we can create a new V-listing $\hat{\tau}$ in which ~~occurs~~ v occurs before all neighbors of v, simply by switching v with the first neighbor of v occurring in τ (or doing nothing if v already occurs before all its neighbors in τ).

~~Thus,~~ Each V-listing σ in which v occurs before all neighbors of v is obtained as $\hat{\tau}$ for exactly $1+\deg v$ different V-listings τ (in fact, ~~these τ are obt~~ one of these τ is σ itself, whereas the remaining $\deg v$ of these τ are obtained from σ by switching v with each neighbor of v). Hence, we have a many-~~to~~-one (actually, a $(1+\deg v)$-to-one) correspondence between the V-listings in which v occurs before all neighbors of v and all V-listings. Therefore (4).]

$\rightarrow\!\!\!\!\rightarrow$

~~Hence,~~ Now, (3) yields

$$(\# \text{ of all } V\text{-listings } \sigma) \cdot \sum_{v \in V} \frac{1}{1+\deg v}$$

$$\geq \sum_{\sigma \text{ is a } V\text{-listing}} \underbrace{|J_\sigma|}_{\substack{= \sum_{v \in V} [v \in J_\sigma] \\ \text{(using Iverson bracket notation)}}}$$

$$= \sum_{\sigma \text{ is a } V\text{-listing}} \sum_{v \in V} [v \in J_\sigma]$$

$$= \sum_{v \in V} \underbrace{\sum_{\sigma \text{ is a } V\text{-listing}} [v \in J_\sigma]}_{\substack{= (\# \text{ of all } V\text{-listings } \sigma \text{ such that } v \in J_\sigma) \\ = (\# \text{ of all } V\text{-listings } \sigma \text{ such that } v \text{ occurs before all neighbors of } v \text{ in } \sigma) \\ = \frac{1}{1+\deg v} \cdot (\# \text{ of all } V\text{-listings } \sigma)}}$$

$$= \sum_{v \in V} \frac{1}{1+\deg v} \cdot (\# \text{ of all } V\text{-listings } \sigma)$$

$= (\#\text{ of all } V\text{-listings } \sigma) \cdot \sum_{v \in V} \frac{1}{1+\deg v}.$

This is absurd. Thus, Thm. 1 is proven. $\square$

Rmk. Why is this called a probabilistic proof? Because what we really showed is that the average ~~a~~ V-listing σ has $|J_\sigma| \geq \sum_{v \in V} \frac{1}{1+\deg v}$.

~~This~~ (We worked with sums instead of averages, but those are just a factor away.)

Our auxiliary equality (4) could be restated as follows:

"The probability for ~~a~~ v to occur before all neighbors of v in a randomly (more precisely: uniformly) chosen V-listing is $\frac{1}{1+\deg v}$", we could have rewritten the whole argument in terms of averages, probabilities and expected values.

Note that the proof is very useless as it comes to actually finding an independent set of size $\geq \sum_{v \in V} \frac{1}{1+\deg v}$.

It doesn't give any better algorithm than ~~try all possible subsets, one at~~ "try the ~~lists~~ subsets J_σ for all possible V-listings σ", which is even more work than trying all subsets of V!

Note also that it doesn't show that more than half of the V-listings σ will satisfy $|J_\sigma| \geq \sum_{v \in V} \frac{1}{1+\deg v}$. The mean is not the median!

Let's give another proof of Thm. 1 now, which does give a good algorithm:

Second proof of Thm. 1. We proceed by strong induction over $|V|$. Thus, assume Thm. 1 is proven for all graphs G with fewer vertices.

If $|V|=0$, then $\emptyset$ works. Thus, WLOG $|V| \neq 0$. Hence, $\exists u \in V$ with ~~deg u minimum.~~ $\deg u$ minimum. Pick such a u.

Let ~~X~~ $U = \{u\} \cup \{\text{all neighbors of } u\}$.
Thus, $U \subseteq V$ and $|U| = \cancel{\deg}\ 1 + \deg u$.

~~Consider~~ Let G' be the induced subgraph of G on the set $V \setminus U$. This is the graph obtained from G by removing all vertices belonging to U (that is, u and its neighbors) and all edges that require these vertices.

Then, G' has fewer vertices than G. Hence, by the induction assumption, Thm. 1 holds for G'.

$\Rightarrow \exists$ an independent set T of G' with size $|T| \geq \sum_{v \in V \setminus U} \frac{1}{1 + \deg_{G'} v}$.

Consider this T. Set $S = \{u\} \cup T$.

Then, S is an independent set of G (since $T \subseteq V \setminus U$, so T contains no neighbors of u). Moreover,

$$\sum_{v \in V} \frac{1}{1 + \deg_G v}$$

$$= \sum_{v \in U} \underbrace{\frac{1}{1+\deg_G v}}_{\substack{\leq \frac{1}{1+\deg_G u} \\ \text{(since } \deg_G v \geq \deg_G u \\ \text{(since we picked} \\ u \text{ to have minimum} \\ \text{degree))}}} + \sum_{v \in V \setminus U} \underbrace{\frac{1}{1+\deg_G v}}_{\substack{\leq \frac{1}{1+\deg_{G'} v} \\ \text{(since } \deg_G v \\ \geq \deg_{G'} v \\ \text{(since } G' \text{ is a} \\ \text{subgraph of } G))}}$$

$$\leq \sum_{v \in U} \underbrace{\frac{1}{1+\deg_G u}}_{\substack{= |U| \cdot \frac{1}{1+\deg_G u} \\ = 1 \\ \text{(because} \\ |U| = 1+\deg_G u)}} + \underbrace{\sum_{v \in V \setminus U} \frac{1}{1+\deg_{G'} v}}_{\substack{\leq |T| \\ \text{(by definition of } T)}}$$

$\leq 1 + |T| = |S|$ (since $S = \{u\} \cup T$).

In other words, $|S| \geq \sum_{v \in V} \frac{1}{1+\deg_G v}$. So Thm. 1 holds for our G. $\Rightarrow$ Inductive proof complete.

$\square$

Cor. 2. Let $G=(V,E)$ be a simple graph. Let $n=|V|$ and $d=\frac{1}{n}\sum_{v\in V}\deg v$. Then, G has an independent set of size $\geq \frac{n}{1+d}$.

To derive this from Thm. 1, we need the following inequality:

Lem. 3. Let $a_1, a_2, \ldots, a_n$ be n positive reals. Then,

$$(a_1+a_2+\cdots+a_n)\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}\right)\geq n^2.$$

Proof of Lem. 3. Lem. 3 can be shown in many ways:

- Apply Jensen's inequality to the convex function $\mathbb{R}^+\to\mathbb{R}^+$, $x\mapsto\frac{1}{x}$.
- Apply the Cauchy–Schwarz inequality.
- Apply the AM–HM inequality.
- Apply the AM–GM inequality twice:

$$a_1 + a_2 + \dots + a_n \geq n \sqrt[n]{a_1 a_2 \cdots a_n},$$

$$\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} \geq n \sqrt[n]{\frac{1}{a_1} \cdot \frac{1}{a_2} \cdots \frac{1}{a_n}},$$

then multiply.

- Notice first that

(7) $$\frac{u}{v} + \frac{v}{u} \geq 2 \quad \forall\, u, v > 0$$

$$\left(\text{since } \frac{u}{v} + \frac{v}{u} - 2 = \frac{(u-v)^2}{uv} \geq 0\right).$$

Now,

$$(a_1 + a_2 + \dots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n}\right)$$

$$= \left(\sum_{i=1}^{n} a_i\right)\left(\sum_{j=1}^{n} \frac{1}{a_j}\right)$$

$$= \sum_{i,j} a_i \cdot \frac{1}{a_j} = \sum_{i,j} \frac{a_i}{a_j}$$

$$= \frac{1}{2}\left(\sum_{i,j} \frac{a_i}{a_j} + \sum_{i,j} \frac{a_j}{a_i}\right)$$

$$\left(\text{since } \sum_{i,j} \frac{a_i}{a_j} = \sum_{i,j} \frac{a_j}{a_i}\right)$$

$$= \frac{1}{2} \sum_{i,j} \underbrace{\left(\frac{a_i}{a_j} + \frac{a_j}{a_i}\right)}_{\geq 2 \text{ (by (7))}}$$

$$\geq \cancel{4}\,\frac{1}{2} \underbrace{\sum_{i,j} 2}_{=2n^2} = \frac{1}{2} \cdot 2n^2 = n^2. \qquad \square$$

<u>Proof of Cor. 2.</u> This will follow straight from Thm. 1 once we show that

$$(8) \qquad \sum_{v \in V} \frac{1}{1+\deg v} \geq \frac{n}{1+d}.$$

So let's prove (8).

Let $v_1, v_2, \ldots, v_n$ be the vertices of G.

Then, Lem. $\cancel{4}$ 3 (applied to $a_i = 1 + \deg(v_i)$) yields

$$\left((1+\deg(v_1)) + (1+\deg(v_2)) + \cdots + (1+\deg(v_n))\right) \cdot \left(\frac{1}{1+\deg(v_1)} + \frac{1}{1+\deg(v_2)} + \cdots + \frac{1}{1+\deg(v_n)}\right)$$

$$\geq n^2.$$

In other words,

$$\left(\sum_{v \in V} (1+\deg v)\right)\left(\sum_{v \in V} \frac{1}{1+\deg v}\right) \geq n^2,$$

Since

$$\sum_{v \in V} (1+\deg v) = n + \underbrace{\sum_{v \in V} \deg v}_{= nd}$$

$$= n + nd = n(1+d),$$

this rewrites as

$$n(1+d) \cdot \sum_{v \in V} \frac{1}{1+\deg v} \geq n^2,$$

Hence, (8). This proves Cor. 2. $\square$

Cor. 2 can be used to prove Turán's theorem (see HW3, Exercise 7).

See the Wikipedia page on the "Method of conditional probabilities" for more.

Introduction to bipartite graphs

Forests ~~Trees~~ are multigraphs with no cycles. What are the multigraphs with no odd-length cycles?

<u>Thm. 4.</u> Let $G = (V, E, \phi)$ be a multigraph.

TFAE (the following are equivalent):

<u>Statement $\mathcal{B}_1$:</u> G has no cycles of odd length.

<u>Statement $\mathcal{B}_2$:</u> G has no circuits of odd length.

<u>Statement $\mathcal{B}_3$:</u> G has a proper 2-coloring.

Here, we are using the following definition:

<u>Def.</u> Let $G = (V, E, \phi)$ be a multigraph.

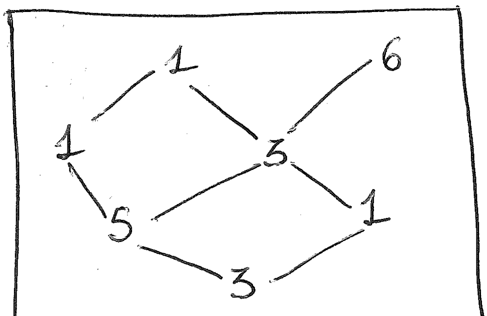
Let $k \in \mathbb{N}$. A <u>k-coloring</u> of G just means a map from V to $\{1, 2, \ldots, k\}$.

If f is a k-coloring of G, then the image of a vertex $v \in V$ under f is called the <u>color</u> of v (with respect to f).

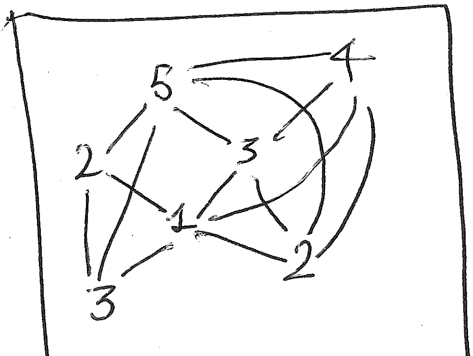
A k-coloring f is said to be **proper** if no two adjacent vertices have the same color (i.e., no two adjacent vertices u and v satisfy $f(u) = f(v)$).

Rmk. Visualize a k-coloring of G by pretending that $1, 2, \ldots, k$ are actual colors. ~~and so~~

Examples (here, the colors are ~~drawn~~ written on the vertices):



a 7-coloring that is not proper (since two vertices with color 1 are adjacent)



a proper 5-coloring

Note that a k-coloring needs not be a surjective map (i.e., some colors may remain unused).

Proof of Thm. 4.

Proof of $\mathcal{B}_1 \Rightarrow \mathcal{B}_2$:

Assume the contrary. Thus, $\mathcal{B}_1$ but not $\mathcal{B}_2$. So $\exists$ a circuit of odd length.

Let $(v_0, e_1, v_1, e_2, v_2, \ldots, e_k, v_k)$ be such a circuit of minimum length. Thus, k is odd, but any shorter circuit has even length.

Clearly, this circuit is not a cycle (since that would contradict $\mathcal{B}_1$).

$\Rightarrow$ ~~$\exists$ positive~~ $\exists\ 0 \leq i < j < k$ such that $v_i = v_j$.

Consider such i and j.

The two circuits

$(v_0, e_1, v_1, e_2, v_2, \ldots, e_i, v_i = v_j, e_{j+1}, v_{j+1}, \ldots, e_k, v_k)$ and $(v_i, e_{i+1}, v_{i+1}, e_{i+2}, v_{i+2}, \ldots, e_j, v_j)$ both are shorter than our original circuit (in fact, they have lengths $k-(j-i)$ and $j-i$, which are both $< k$), and thus have even lengths (since any

circuit shorter than the original one has even length). That is,
$k-(j-i)$ and $j-i$ is even.

$$\Rightarrow k = \underbrace{(k-(j-i))}_{\text{even}} + \underbrace{(j-i)}_{\text{even}} \text{ is even, too.}$$

Contradiction (since k is odd)!
So $\mathcal{B}_1 \to \mathcal{B}_2$ is proven.

<u>Proof of $\mathcal{B}_2 \Rightarrow \mathcal{B}_3$:</u> Assume $\mathcal{B}_2$.

We must prove $\mathcal{B}_3$.

WLOG assume that G is connected (otherwise, treat each connected component of G separately, and then combine the 2-colorings).

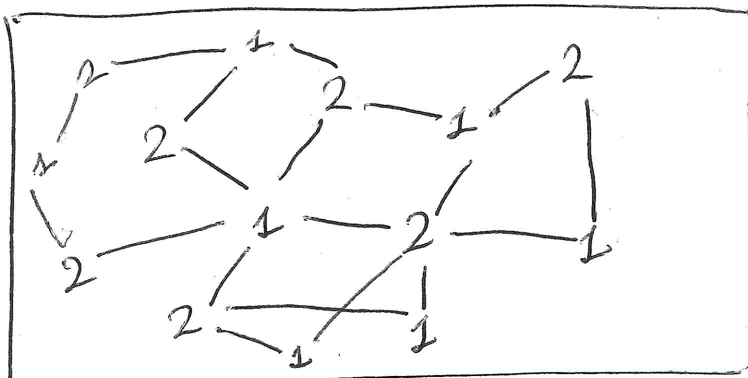
Pick any vertex u. Define a 2-coloring $f: V \to \{1,2\}$ of G by

$$f(v) = \begin{cases} 1, & \text{if } d(u,v) \text{ is even;} \\ 2, & \text{if } d(u,v) \text{ is odd.} \end{cases}$$

Here, $d(u,v)$ = distance between u and v.

(This is well-defined, since $d(u,v) \neq \infty$, because G is connected.)

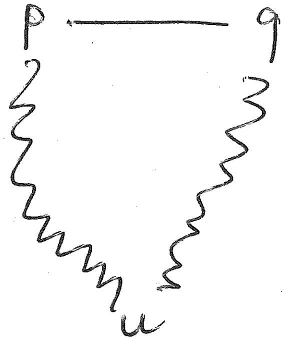
I claim that f is a proper 2-coloring.



example of an f as above

Indeed, assume the contrary. Thus, $\exists$ two adjacent vertices p and q of the same color. Hence, $d(u,p)$ and $d(u,q)$ are either both even or both odd. So ~~...~~ there are paths $u \to p$ and $u \to q$ that either both have even length or both have odd length.

Combine these two paths, with an edge that connects p with q. The result is a circuit



that has odd length, contradicting $\mathcal{B}_2$.

Hence, our assumption was false.
So $\mathcal{B}_2 \Rightarrow \mathcal{B}_3$ is proven.

Proof of $\mathcal{B}_3 \Rightarrow \mathcal{B}_1$: Let $\mathcal{B}_3$ hold.

Thus $\exists$ proper 2-coloring f of G.

Consider a cycle
$(v_0, e_1, v_1, e_2, v_2, \ldots, e_k, v_k)$ of G.
~~We want to show that this cycle has even length~~ having odd length. We want to obtain a contradiction.

So k is odd.
WLOG $f(v_0) = 1$ (otherwise, $f(v_0) = 2$, and the proof works analogously).

Then, $f(v_1) \neq f(v_0)$ (since v_0 and v_1 are adjacent and f is a proper coloring)
$\Rightarrow f(v_1) \neq 1 \Rightarrow f(v_1) = 2$.

Similarly, $f(v_2) \neq f(v_1) \Rightarrow f(v_2) \neq 2 \Rightarrow f(v_2) = 1$.

And so on. Thus, we get

(9) $$f(v_i) = \begin{cases} 1, & \text{if } i \text{ is even;} \\ 2, & \text{if } i \text{ is odd.} \end{cases}$$

(Rigorously, you prove (9) by

induction over i, using the adjacency of v_{i-1} and v_i in the induction step to find $f(v_i) \neq f(v_{i-1})$.)

Applying (9) to $i=k$, we find $f(v_k)=2$ (since k is odd). But this contradicts $f(\underbrace{v_k}_{=v_0}) = f(v_0) = 1$.

Thus, our assumption was false. So $\nexists$ cycle of odd length. $\Longrightarrow$ $\mathcal{B}_1$ holds.

This proves $\mathcal{B}_3 \Longrightarrow \mathcal{B}_1$.

$$\begin{tikzcd} & \mathcal{B}_1 \arrow[dr, Rightarrow] & \\ \mathcal{B}_3 \arrow[ur, Rightarrow] & & \mathcal{B}_2 \arrow[ll, Rightarrow] \end{tikzcd}$$

$\square$

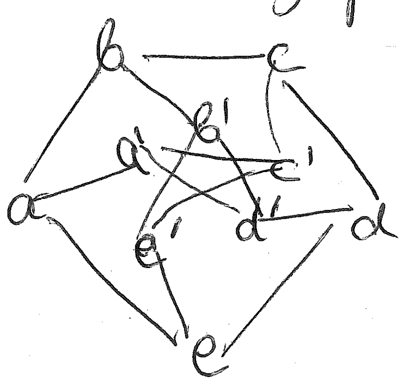
<u>Rmk.</u> Usually, a graph G satisfying the three conditions $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$ of Thm. 4 is called "bipartite".
We shall be a little bit more precise and reserve the name "bipartite graph" for

a graph G equipped with a proper 2-coloring, not just a graph which is known to have one. This is a little bit different, since a graph can have several 2-colorings.

Rmk. Our proof of Thm. 4 shows that $\exists$ a fast (polynomial-time) algorithm to check if a graph has a proper 2-coloring and (if it has one) to construct one.

Nothing comparable exists for k-colorings when $k \geq 3$.

Exa: Does this graph



(the Petersen graph)

have a proper 3-coloring?

Can you find it?