5707 Spring 2017 Lecture 10

The centers of a graph / of a tree

<u>Def.</u> Let v be a vertex of a multigraph $G = (V, E, \phi)$. Let the <u>eccentricity</u> of v (with respect to G) be defined as the number

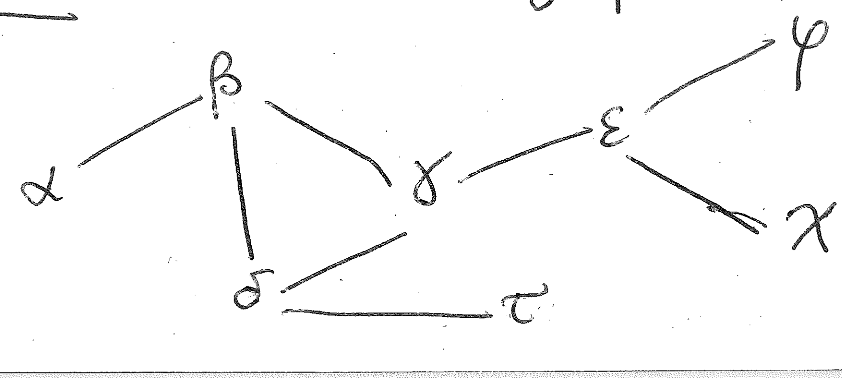
$$\max\{d(v,u) \mid u \in V\} \in \mathbb{N} \cup \{\infty\}$$

(where $d(v,u)$ = (distance from v to u) = (length of the shortest path $v \to u$), which is ∞ if no paths $v \to u$ exist).

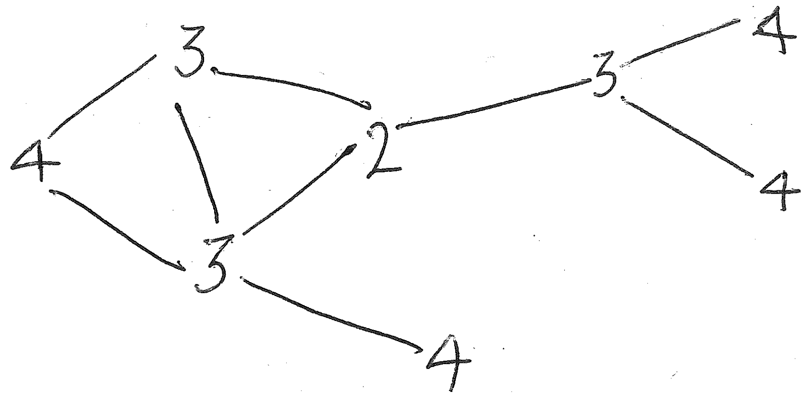
This eccentricity is denoted by $\operatorname{ecc}(v)$ or by $\operatorname{ecc}_G(v)$.

A <u>center</u> of G means a vertex of G whose eccentricity is minimum (among all vertices).

<u>Examples:</u> (a) Here is a graph:

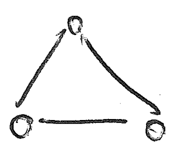


and here are the eccentricities of its vertices (each drawn atop of the corresponding vertex, instead of the name of the vertex):



So this graph has a unique center, namely γ (with eccentricity 2).

(b) Here is a graph with 3 centers:

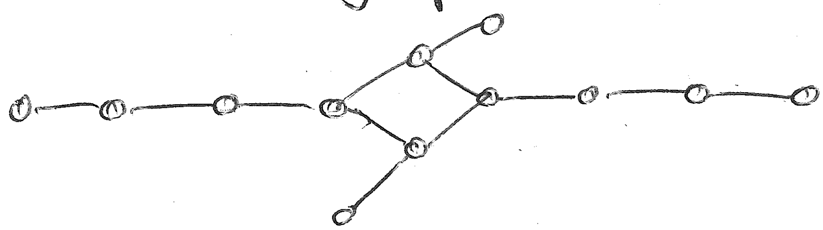


$(= K_3)$.

In fact, each of its vertices is a center, having eccentricity 1.

Similarly, K_n has n centers, each with eccentricity 1.

(c) Another graph with 2 centers:



Find them!

Rmk: Some people use the word "center" not for what we call a "center", but for what we would call the set of the centers.
So their "center" is a subset of V.

Thm. 1. Let T be a tree. Then, T has either exactly 1 center, or exactly 2 centers and these 2 centers are adjacent.

To prove this, we recall that a leaf of a graph means a vertex having degree 1. Now:

Lem. 2. Let T be a tree with ~~...~~ ≥ 3 vertices. Let L be the set of all leaves of T. Let $T \setminus L$ be the ~~...~~ multigraph obtained from T by removing the vertices in L and all edges adjacent to them.

(a) The graph $T \setminus L$ is a tree.

(b) Each vertex v of $T \setminus L$ satisfies
$$\operatorname{ecc}_T v = \operatorname{ecc}_{T\setminus L} v + 1.$$

(c) Each $v \in L$ satisfies
$$\operatorname{ecc}_T v = \operatorname{ecc}_T w + 1,$$
where w is the unique neighbor of v in T.

(d) The centers of T are precisely the centers of $T \setminus L$.

Proof. (a) As we know, removing a leaf from a tree always yields a tree ~~or a (unless the tree~~ or a graph with 0 vertices (since a leaf cannot lie on any path, unless it is the starting or ending point of that path).

Hence, $T \setminus L$ (being the result of consecutively removing many leaves from T) must be a tree or a graph with 0 vertices. Thus, we only need to rule out the "graph with 0 vertices" case.

So assume (for contradiction) that $T \setminus L$ is a graph with 0 vertices. Thus,

$L = V(T)$. In other words, each vertex of T is a leaf. Now, pick any three distinct vertices u, v and w of T. (This can be done, since T has ≥ 3 vertices.)
~~Thus, u is a leaf.~~

There is a unique path $u \to v$ in T (since T is a tree). If this path had length ≥ 2, then one of the ~~points betw~~ vertices between u and v on this path would not be a leaf (since it has ~~both an~~ at least two edges containing it: the one along which the path enters it, & the one along which the path exits it). So this path has length < 2. Thus, it has length 1 (since $u \neq v$). Hence, $uv \in E(T)$. Similarly, $vw \in E(T)$.
Now, the vertex v cannot be a leaf (since uv and vw are two edges containing this vertex).
Contradiction! This finishes (a).

(b) Let v be a vertex of $T \setminus L$. Thus, $v \notin L$, so that v is not a leaf of T.

Let w be a vertex of $T \setminus L$ maximizing

$d(v,w)$. (We write d for both distances in T and distances in $T \setminus L$, because for vertices in $T \setminus L$ they are equal.) Then, $d(v,w) = \operatorname{ecc}_{T \setminus L} v$.

The vertex w of T is not a leaf of T (since it is a vertex of $T \setminus L$, thus not in L). Hence, it has a neighbor apart from the one lying on the path $v \to w$. Let u be such a neighbor. Then, $d(v,u) = d(v,w) + 1$ (since concatenating the path $v \to w$ with the edge $w \to u$ yields a backtrack-free walk, thus a path (since T is a tree), and this must be the unique path $v \to u$, so its length is $d(v, \cancel{w} u)$).

But, of course,

$$\operatorname{ecc}_T v \geq d(v,u) = \underbrace{d(v,w)}_{=\operatorname{ecc}_{T \setminus L} v} + 1 = \operatorname{ecc}_{T \setminus L} v + 1.$$

It remains to show that

$$\operatorname{ecc}_T v \leq \operatorname{ecc}_{T \setminus L} v + 1.$$

This I leave to the reader (hint: a path from v to a vertex of T can ~~only~~ be longer than $\operatorname{ecc}_{T \setminus L} v$ only if said vertex is a leaf of T,

but in this case the penultimate vertex on this path is not a leaf and thus no farther than $\operatorname{ecc}_{T \setminus L} v$ away from v).
Thus, (b) is proven.

(c) Let $v \in L$.
~~Let p be~~ If w was a leaf of T, then each of the two vertices v and w would be the other's only neighbor, which would contradict the fact that T is connected (since $|T| \geq 3$, so T has at least one vertex besides v and w, but there is no way to reach such a vertex from v if v and w are each other's only neighbor).
Thus, w is not a leaf of T. ~~Hence, w is a vertex of $T \setminus L$. So part (b) (applied to w instead of v) yields~~

$$\cancel{\operatorname{ecc}_T w \leq \operatorname{ecc}_{T \setminus L} w + 1.}$$

~~Let p be a vertex of T maximizing $d(w,p)$.~~

But each vertex p of T distinct from v satisfies $d(v,p) = d(w,p) + 1$ (since the path $v \to p$ must begin by going from v to its only neighbor w).
Hence, $\operatorname{ecc}_T v = \operatorname{ecc}_T w + 1$, unless the only vertex q of T maximizing $d(w,q)$

is v. But ~~the on~~ it is impossible that the only vertex q of T maximizing $d(w,q)$ is v (because in this case, ~~w would be a leaf of T,~~ v would be the only neighbor of w, so that w would be a leaf of T, but we know that w is not a leaf of T). Hence,

$$\mathrm{ecc}_T\, v = \mathrm{ecc}_T\, w + 1.$$

(d) The centers of T are the vertices v of T minimizing $\mathrm{ecc}_T\, v$. By part (c), these vertices cannot be in L (since vertices $v \in L$ have larger eccentricity than their neighbors). So they are the vertices v of $T \setminus L$ minimizing $\mathrm{ecc}_T\, v$. According to part (b), a vertex v of $T \setminus L$ minimizes $\mathrm{ecc}_T\, v$ if and only if it minimizes $\mathrm{ecc}_{T \setminus L}\, v$. Hence, the centers of T are the vertices v of $T \setminus L$ minimizing $\mathrm{ecc}_{T \setminus L}\, v$. But these are clearly the centers of $T \setminus L$. $\square$

Proof of Thm. 1. Strong induction over $|V(T)|$.

If $|V(T)| \leq 2$, then it's obvious.
Hence, WLOG assume $|V(T)| > 2$. Hence, T has ≥ 3 vertices. Let L be the set of all leaves of T. Define $T \setminus L$ as in Lem. 2.

Lem. 2 (a) yields that $T \setminus L$ is a tree. This tree has fewer vertices than T (since $|L| > 0$ (since T has at least 1 leaf)). Hence, by the induction assumption, $T \setminus L$ has either exactly 1 center, or exactly 2 centers and these 2 centers are adjacent. But Lem. 2 (d) shows that the centers of T are the centers of $T \setminus L$; hence, we conclude that exactly the same holds for T. Thm. 1 is proven. $\square$

Rmk. Our proof of Thm. 1 yields the following algorithm for finding the centers of a tree:

Remove all leaves. If the resulting tree has ≥ 3 vertices, remove all leaves again. And so on, until at most 2 vertices remain. Those are the centers.

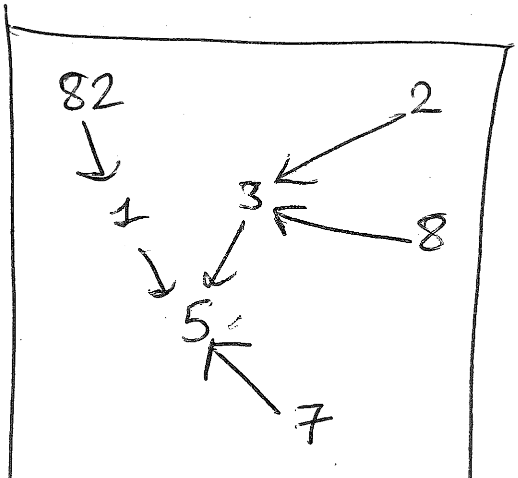
Oriented spanning trees

Def. Let $D = (V, A, \phi)$ be a multidigraph. Let v be a vertex of D.

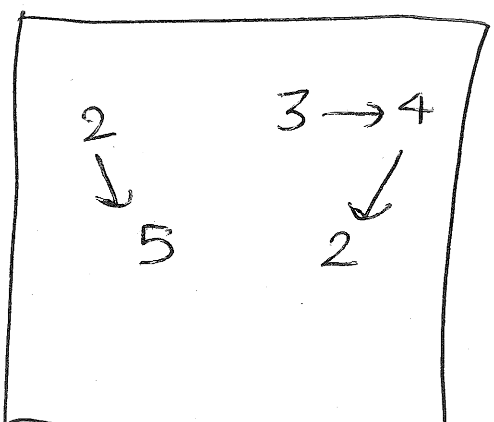
We say that D is an <u>oriented tree rooted at v</u> (short: <u>$OT \to v$</u>) if it has the following property:

For every $u \in V$, there is a unique walk $u \to v$ in D.

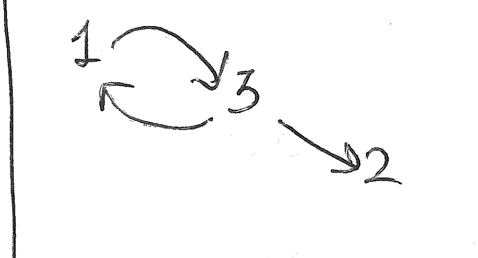
Examples:



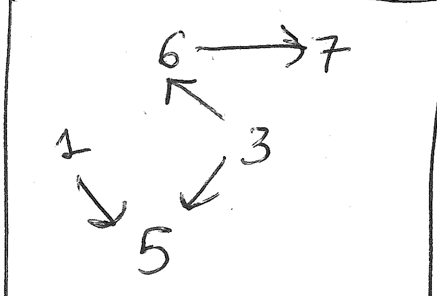
an $OT \to 5$



not an $OT \to 5$ (as there is no walk $2 \to 5$)



not an OT$\to 2$
(since $\exists$ two walks $1 \to 2$: for example, $(1,3,2)$ and $(1,3,1,3,2)$)



not an OT$\to 5$
(since $\nexists$ walk $6 \to 5$)

Again, we have several equivalent criteria:

<u>Thm. 6.</u> Let $D = (V, A, \phi)$ be a multidigraph. Let v be a vertex of D.

Then, TFAE (= <u>t</u>he <u>f</u>ollowing <u>a</u>re <u>e</u>quivalent):

<u>Statement $\mathcal{O}_1$:</u> D is an OT$\to v$.

<u>Statement $\mathcal{O}_2$:</u> (a) We have $\deg^+ u = 1$ $\forall u \in V$ with $u \neq v$,

AND (b) We have $\deg^+ v = 0$,
AND (c) $\forall u \in V$ $\exists$ walk $u \to v$ in D.

<u>Statement $\mathcal{O}_3$:</u> (a) We have $\deg^+ u = 1$ $\forall u \in V$ with $u \neq v$,

AND ~~(b)~~ (d) D has no cycles.

<u>Statement $\mathcal{O}_4$</u>: (e) The ~~digr~~ multidigraph D has no loops. ~~and if~~

AND (f) If we forget the directions of its arcs (i.e., replace each arc ~~arc~~ $p \to q$ by an edge connecting p with q, thus obtaining a multigraph), then the resulting multigraph is a tree.

AND (g) Each arc of D is directed "towards v" (i.e., its target is closer to v than its source in the ~~unde~~ tree obtained by forgetting the directions of the arcs).

<u>Rmk.</u> Statement $\mathcal{O}_4$ is clumsy, but its purpose is simple: It shows that an $OT \to v$ is simply a tree (in the usual sense) whose edges are directed towards v. So an $OT \to v$ carries the same information as a tree (in the usual sense) plus the choice of the vertex v. The directions of the arcs ~~are~~ are uniquely determined

by this data.

Proof of ~~Pr~~ Thm. 6.

Proof of $\mathcal{O}_1 \Rightarrow \mathcal{O}_3$: Assume $\mathcal{O}_1$.

Now, we must prove $\mathcal{O}_3$.
In other words, we must prove (a) and (d).

To verify (d), assume the contrary.
Thus, $\exists$ cycle. Let u be any vertex on this cycle.
Since $\mathcal{O}_1$ holds, there is a unique walk $u \to v$. ~~But~~ If we concatenate ~~this~~ our cycle with this walk,

we find a new ~~longer~~ (longer) walk $u \to v$. But this contradicts the uniqueness of the walk $u \to v$.
So (d) holds.

To prove (a), pick $u \in V$ with $u \neq v$.
If $\deg^+ u = 0$, then $\nexists$ walk $u \to v$, contradicting $\mathcal{O}_1$. Hence, $\deg^+ u \neq 0$.
So $\deg^+ u \geq 1$.

If $\deg^+ u \geq 2$, then we can find two different walks $u \to v$ (in fact, we have two arcs with source u, and from the target of each of these two arcs we can keep walking to v according to $\mathcal{O}_1$, thus obtaining two different walks $u \to v$).

So $\deg^+ u < 2$. ~~deg~~
Hence, $\deg^+ u = 1$ (since $\deg^+ u \geq 1$).
This proves (a).

Hence, $\mathcal{O}_1 \Rightarrow \mathcal{O}_3$ is proven.

<u>Proof of $\mathcal{O}_3 \Rightarrow \mathcal{O}_2$:</u> Assume $\mathcal{O}_3$.

We need to prove $\mathcal{O}_2$. In other words, we need to prove (a), (b) and (c). We already know (a), so only (b) and (c) remain.

To prove (c), start at u and keep walking along arcs. You will eventually get stuck, since (d) says that $\nexists$ cycles. But (a) shows that you cannot get stuck at any vertex other than v.
So you will have to get stuck at v. Thus, you found a walk $u \to v$. This proves (c).

Finally, we need to prove (b). Assume the contrary. Thus, $\exists$ arc with source v. Let u be its ~~target~~.

From (c), we know that $\exists$ walk $u \to v$. Hence, $\exists$ path $u \to v$. Combine it with the arc from v to u to obtain a cycle. But this contradicts (d).
So (b) is proven. Hence, $\mathcal{O}_3 \Rightarrow \mathcal{O}_2$ is proven.

<u>Proof of $\mathcal{O}_2 \Rightarrow \mathcal{O}_1$:</u> Assume $\mathcal{O}_2$. We need to prove $\mathcal{O}_1$. So let us fix $u \in V$ and $v \in V$. We must then show that there is a unique walk $u \to v$. The existence of this walk follows from (c). So why is it unique?

Well, if we had two such walks, then they would have to diverge at some vertex. This vertex would then have outdegree ≥ 2. But each vertex has outdegree 1 or 0 (by (a) & (b)). Contradiction. Thus, $\mathcal{O}_2 \Rightarrow \mathcal{O}_1$ is proven.

<u>Proof of $\mathcal{O}_2 \Rightarrow \mathcal{O}_4$:</u> Let $\underline{D}$ be the multigraph obtained by forgetting the directions of the arcs of D. Then,

$$|E(\underline{D})| = |A(D)| = \sum_{u \in V} \deg^+ u$$

(since each arc of D has a

unique source, and thus counts into $\deg^+ u$ for a unique $u \in V$)

$$= \underbrace{\deg^+ v}_{\substack{=0 \\ \text{(by (b))}}} + \sum_{u \in V \setminus \{v\}} \underbrace{\deg^+ u}_{\substack{=1 \\ \text{(by (a))}}}$$

$$= \sum_{u \in V \setminus \{v\}} 1 = |V \setminus \{v\}| = |V| - 1.$$

Also, $\underline{D}$ is connected (by (c)).

Hence, $\underline{D}$ satisfies Statement T_4 of the tree equivalence theorem. Thus, $\underline{D}$ satisfies Statement T_1 as well, i.e., is a tree. So (f) is clear.

(e) follows from (d) (which, as we have seen in our proof of $O_2 \Rightarrow O_3$, holds).

(f) follows from (c), since the walk $u \to v$ is the path $u \to v$ in $\underline{D}$.

Thus, $O_2 \Rightarrow O_4$ is proven.

Proof of $\mathcal{O}_4 \Rightarrow \mathcal{O}_2$: ~~For each $u \in V$~~

Again, let $\underline{D}$ be as in the proof of $\mathcal{O}_2 \Rightarrow \mathcal{O}_4$. For each $u \in V$, the unique path $u \to v$ in $\underline{D}$ is actually a walk in D (by (g)). Hence, (c) holds.

(b) ~~Follows~~ follows from (g) as well (since no vertex can be closer to v than v itself).

(a) follows from (g), too (since $\underline{D}$ is a tree, so there is only one vertex closer to v than u).

So $\mathcal{O}_4 \Rightarrow \mathcal{O}_2$ is ~~pr~~ proven.

~~Alto~~ Altogether, we have now proven the following implications:

$$\begin{tikzcd}
 & \mathcal{O}_3 \arrow[dd, Rightarrow] & \\
\mathcal{O}_1 \arrow[ur, Rightarrow] & & \mathcal{O}_4 \arrow[dl, Rightarrow, bend left] \\
 & \mathcal{O}_2 \arrow[ul, Rightarrow] \arrow[ur, Rightarrow, bend left] &
\end{tikzcd}$$

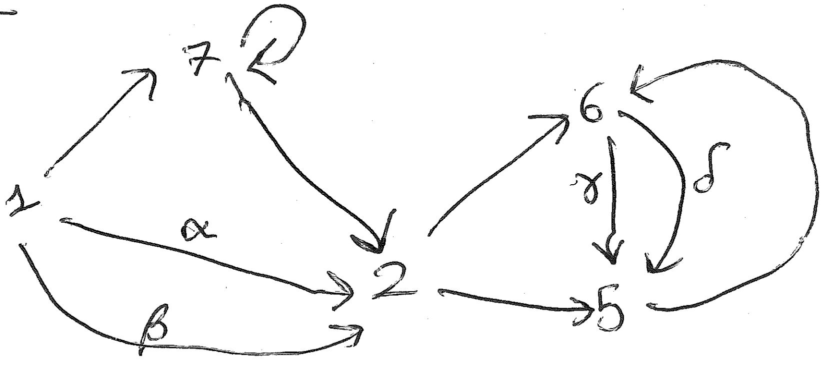
Hence, $\mathcal{O}_1 \Longleftrightarrow \mathcal{O}_2 \Longleftrightarrow \mathcal{O}_3 \Longleftrightarrow \mathcal{O}_4$. $\square$

Def. Let D be a multidigraph.

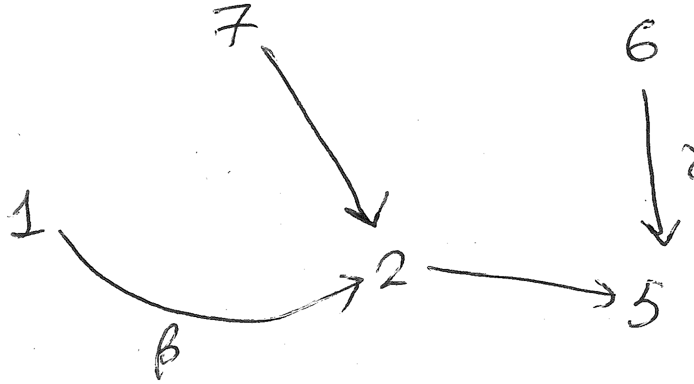
(a) A *spanning subdigraph* of D means a sub-multidigraph of D whose vertex set is the vertex set of D.
Thus, it is a ~~digraph of the~~ multidigraph of the form
$$(V, B, \phi\mid_B), \text{ where } D = (V, A, \phi) \text{ and } B \subseteq A.$$

(b) Let $v \in V$. An *oriented spanning tree (of D) rooted at v* (short: *$OST \to v$ (of D)*) means an $OT \to v$ which is a spanning subdigraph of D.

Example: Here is a multidigraph D:

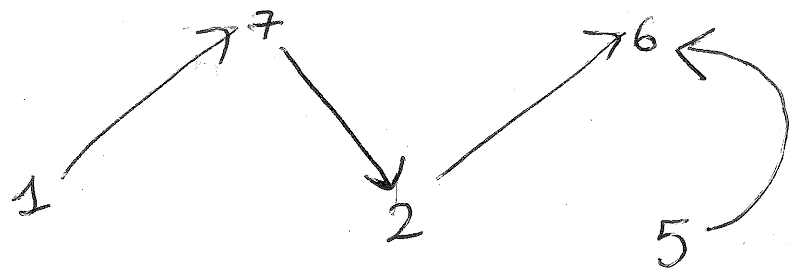


And here is an OST$\to 5$ of D:



There are several others, too.

Here is an OST $\to 6$ of D:



There is no OST$\to 2$ of D, since $\nexists$ walk $5 \to 2$.

Now, let me state something barely believable: the "BEST theorem" (named so after de <u>B</u>ruijn, van Aardenne-<u>E</u>hrenfest, <u>S</u>mith & <u>T</u>utte, of which arguably the last two have little to do with the theorem):

Thm. 7 (BEST theorem). Let

$D = (V, A, \phi)$ be a ~~strongly connected~~ multidigraph such that each $v \in V$ satisfies $\deg^- v = \deg^+ v$.

Fix an arc e of D, and let v be its source.

Let $\tau(D, v)$ be the # of OST $\to v$ of D.

Let $\varepsilon(D, e)$ be the # of Eulerian circuits of D whose first arc is e.

Then:

$$\varepsilon(D, e) = \tau(D, v) \cdot \prod_{u \in V} (\deg^+ u - 1)!$$

For the proof and (nontrivial) applications to counting Eulerian circuits in multidigraphs (sadly, not in multigraphs), see [Stanley 13, Chapter 10]. (Details on the course website.)