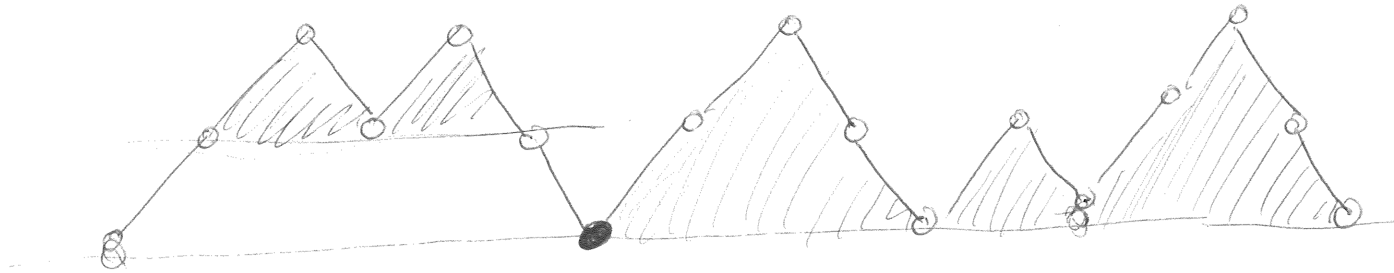
What is a recursion for c_n?



Given $n>0$, we want to see how many Dyck paths $(0,0) \to (2n,0)$ first return to the x-axis at $(2k,0)$ for a given $k \in [n]$.
(Note: The first return <u>must</u> be of the form $(2k,0)$ for some $k \in [n]$.)

To construct such a path, we

- first choose its first $2k$ steps:
  they have to form a Dyck path that never returns to x-axis until $(2k,0]$;
  in other words, they begin with a $\nearrow$, end with a $\searrow$, and inbetween proceed as a Dyck path $(0,0) \to (2(k-1),0)$.
  $\Rightarrow$ c_{k-1} choices

- then choose its last $2(n-k)$ steps;

  they have to proceed as a Dyck path

  $(0,0) \to (2(n-k), 0)$

  $\Rightarrow c_{n-k}$ choices.

$$\Rightarrow \quad c_n = \sum_{k=\cancel{0}\,1}^{n} c_{k-1} c_{n-k}, \qquad (1)$$

Let $C(x) = \sum_{n \geq 0} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$.

Thus,
$$\begin{aligned} C(x) &= 1 + (c_0 c_0) x \\ &\quad + (c_0 c_1 + c_1 c_0) x^2 \\ &\quad + (c_0 c_2 + c_1 c_1 + c_2 c_0) x^3 \\ &\quad + \cdots \\ &= 1 + x \underbrace{(c_0 + c_1 x + c_2 x^2 + \cdots)}_{=C(x)}^2, \end{aligned}$$

so $C(x) = 1 + x\,(C(x))^2$.

This is a quadratic in $C(x)$. Solve it (whyever it's allowed). Get

$$C(x) = \frac{1 \pm \sqrt{1-4x}}{2x}.$$

The $\pm$ cannot be a $+$, since with a $+$, the power series on top of the fraction would not be divisible by $2x$. So

$$C(x) = \frac{1-\sqrt{1-4x}}{2x} = \frac{1}{2x}\left(1-(1-4x)^{1/2}\right).$$

But recall $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k = \sum_k \binom{n}{k} x^k$.

Let's pretend this works for $n = 1/2$, too.

Thus, $(1+x)^{1/2} = \sum_k \binom{1/2}{k} x^k$.

Now, substitute $-4x$ for x here, to get

$$(1-4x)^{1/2} = \sum_k \binom{1/2}{k} (-4x)^k$$

$$= \sum_k \binom{1/2}{k} (-4)^k x^k.$$

So our formula for $C(x)$ becomes

$$C(x) = \frac{1}{2x}\underbrace{\left(1 - \sum_k \binom{1/2}{k}(-4)^k x^k\right)}_{= -\sum_{k\geq 1}\binom{1/2}{k}(-4)^k x^k}$$

$$= -\sum_{k\geq 1}\binom{1/2}{k}\frac{(-4)^k}{2}x^{k-1}$$

$$= -\sum_{k\geq 0}\binom{1/2}{k+1}\frac{(-4)^{k+1}}{2}x^k .$$

Now, comparing coeffs before x^n gives

$$c_n = -\underbrace{\binom{1/2}{n+1}}_{=\frac{1/2}{n+1}\underbrace{\binom{-1/2}{n}}_{=\binom{2n}{n}\left(-\frac{1}{4}\right)^n \text{ (by HW2 exe 1)}}}\frac{(-4)^{n+1}}{2}$$

$$= \cancel{-}\,\frac{\cancel{1/2}}{n+1}\binom{2n}{n}\cancel{\left(-\frac{1}{4}\right)^n}\,\frac{\cancel{(-4)^{n+1}}}{\cancel{2}}$$

$$= \frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n-1}.$$

Ex. 3. Thm. 3. 12 (= Thm 12 in Ch. 3) says that $\binom{a+b}{n} = \sum_{k=0}^{n} \binom{a}{k}\binom{b}{n-k}$ $\forall a, b \in \mathbb{N}$, $n \in \mathbb{N}$.

We can reprove it using FPS.

Namely, $\underbrace{(1+x)^a}_{= \sum_k \binom{a}{k} x^k} \underbrace{(1+x)^b}_{= \sum_k \binom{b}{k} x^k} = \underbrace{(1+x)^{a+b}}_{= \sum_k \binom{a+b}{k} x^k}$

$$= \left(\sum_k \binom{a}{k} x^k\right)\left(\sum_k \binom{b}{k} x^k\right)$$

$$= \sum_{k,l} \binom{a}{k} x^k \binom{b}{l} x^l$$

$$= \sum_{k,l} \binom{a}{k}\binom{b}{l} x^{k+l}$$

$$= \sum_n \left(\sum_{k=0}^{n} \binom{a}{k}\binom{b}{n-k}\right) x^n$$

So $\sum_n \left(\sum_{k=0}^{n} \binom{a}{k}\binom{b}{n-k}\right) x^n = \sum_n \binom{a+b}{n} x^n$.

Comparing coeffs yields $\sum_{k=0}^{n} \binom{a}{k}\binom{b}{n-k} = \binom{a+b}{n}$.

Ex. 4. (from Wilf's "generatingfunctionology")

Define a sequence $(a_0, a_1, a_2, \dots)$ recursively by

$$a_0 = 1, \quad a_{n+1} = 2a_n + n \quad \forall n \geq 0.$$

(So it starts out $1, 2, 5, 12, 27, 58, 121, \dots$)

(OEIS: A000325, shifted.)

Set $A(x) = a_0 + a_1 x + a_2 x^2 + \cdots$.

Now, $A(x) = 1 + (2a_0 + 0)x + (2a_1 + 1)x^2 + (2a_2 + 2)x^3 + \dots$

$$= 1 + 2(a_0 x + a_1 x^2 + a_2 x^3 + \dots)$$

$$+ (0x + 1x^2 + 2x^3 + \dots)$$

$$= 1 + 2xA(x) + x\underbrace{(0 + 1x + 2x^2 + 3x^3 + \dots)}_{=?}.$$

Now,

$$0 + 1x + 2x^2 + 3x^3 + \dots = ?$$

Two ways to compute it:

1. $$0 + 1x + 2x^2 + 3x^3 + \dots = x\underbrace{(1 + 2x + 3x^2 + \dots)}$$

$$= (\cancel{x}\, 1 + x + x^2 + \dots)'$$

$$= \left(\frac{1}{1-x}\right)' = \frac{1}{(1-x)^2}$$

$$= x \cdot \frac{1}{(1-x)^2}.$$

2. $$\begin{aligned} 0 + 1x + 2x^2 + 3x^3 + \dots = {} & x + x^2 + x^3 + x^4 + \dots \\ & + x^2 + x^3 + x^4 + \dots \\ & + x^3 + x^4 + \dots \\ & + x^4 + \dots \\ & + \dots \end{aligned}$$

$$= x \cdot \frac{1}{1-x} + x^2 \cdot \frac{1}{1-x} + x^3 \cdot \frac{1}{1-x} + \dots$$

$$= \underbrace{\left(x + x^2 + x^3 + \dots\right)}_{= x \cdot \frac{1}{1-x}} \cdot \frac{1}{1-x}$$

$$= x \cdot \left(\frac{1}{1-x}\right)^2 .$$

[But be careful with ∞ sums.]

So our formula for $A(x)$ becomes

$$A(x) = 1 + 2x\,A(x) + x \cdot x \cdot \frac{1}{(1-x)^2} .$$

This is a linear eqn. in $A(x)$. Solving it yields

$$A(x) = \frac{1 - 2x + 2x^2}{(1-x)^2(1-2x)} \overset{\text{PFD}}{=} \underbrace{\frac{-1}{(1-x)^2}}_{= -(1+2x+3x^2+\dots) \text{ (by above)}} + \underbrace{\frac{2}{1-2x}}_{= 2\sum_k (2x)^k = \sum_k 2^{k+1} x^k}$$

$$= -\left(1 + 2x + 3x^2 + \dots\right) + \sum_k 2^{k+1} x^k = \sum_k \left(2^{k+1} - (k+1)\right) x^k .$$

Comparing coefficients yields $a_n = 2^{n+1} - (n+1)$.

4.2. THE CONCEPT OF A FPS (FORMAL POWER SERIES)

We want to give a definition of a FPS and also justifications for the things we did to FPS (dividing, quadratic eqns, ∞ sums, ...).

First things first: FPS are not functions. You cannot substitute $x=2$ into $\frac{1}{1-x} = 1+x+x^2+\dots$ and "obtain" $\frac{1}{-1} = 1+2+4+8+16+\dots$.

Let us go back to abstract algebra.

<u>Def.</u> A <u>commutative ring</u> K, informally, a set K equipped with binary operations $\oplus$, $\ominus$, and $\odot$ and elements 0 and 1 that "behave" like ~~th~~ addition, subtraction, multiplication of numbers, and the numbers 0 and 1, respectively. For example, they have to satisfy rules like ~~(a⊕b)~~

$$(a \oplus b) \odot c = a \odot c \oplus \cancel{}\, b \odot c.$$

Formally, a comm. ring is a set K equipped with maps $\oplus : K \times K \to K$, $\ominus : K \times K \to K$, $\odot : K \times K \to K$ and elts $0 \in K$ and $1 \in K$ satisfying the following axioms.

(a) Commutativity of $\oplus$: $a \oplus b = b \oplus a$.

(b) Associativity of $\oplus$: $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.

(c) Neutrality of 0: $a \oplus 0 = 0 \oplus a = a$.

Most authors do this part differently, but the end-result is the same.

(d) $\ominus$ undoes $\oplus$: $a \oplus b = c \iff a = c \ominus b$.

(e) Commutativity of $\odot$: $a \odot b = b \odot a$.

(f) Associativity of $\odot$: $a \odot (b \odot c) = (a \odot b) \odot c$.

(g) Distributivity: $a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c)$;
$(a \oplus b) \odot c = (a \odot b) \oplus (b \odot c)$

(h) Neutrality of 1: $a \odot 1 = 1 \odot a = a$.

(i) Annihilation: $a \odot 0 = 0 \odot a = 0$.

Examples: $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$.

Non-examples: $\mathbb{N}$ (semiring); $\mathbb{Q}^{\cancel{2 \times 2}\, n \times n}$ (noncomm. ring), for $n > 1$.

Usually, drop the circles, so write $\oplus, \ominus, \odot$ as $+, -, \cdot$.

See [detnotes], start of Ch. 5.

Good news: In an arbitrary commutative ring, the standard rules of computation apply:

- You can compute finite sums without caring about the order of summation or parenthesization:

$$((a+(b+c))+d)+e = (a+b)+(c+(d+e))$$

("general associativity"),

so you $\cancel{}$ can just write $a+b+c+d+e$.

Also, $a+b+c+d+e = d+b+a+e+c$

("general commutativity").

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Ex 0. Counting compositions using FPS.

Let ~~k_n~~ $k(n)$ be the # of compositions of n.

Want $k(n)$.

Set $K(x) = k(0) + k(1)x + k(2)x^2 + \cdots$,

Thus, $K(x) = \sum_{n \geq 0} k(n) x^n = \sum_{n \geq 0} \sum_{\substack{(a_1, \dots, a_k) \\ \text{comp of } n}} x^n$,

$$= \sum_{\substack{(a_1, a_2, \dots, a_k) \\ \text{composition} \\ \text{(i.e. tuples of} \\ \text{pos ints)}}} x^{a_1 + a_2 + \dots + a_k}$$

$$= \sum_{k \geq 0} \underbrace{\sum_{(a_1, \dots, a_k) \in \{1,2,3,\dots\}^k} x^{a_1 + a_2 + \dots + a_k}}_{\underset{\text{product rule}}{=} \left(\sum_{a=1}^{\infty} x^a \right)^k}$$

$$= \sum_{k \geq 0} \left(\sum_{a=1}^{\infty} x^a \right)^k = \frac{1}{1 - \sum_{a=1}^{\infty} x^a}$$

$$\leqq \qquad \left(\text{since } \sum_{k\geq 0} y^k = \frac{1}{1-y}\right)$$

$$= \frac{1}{1-\left(\frac{1}{1-x}-1\right)} \qquad \left(\text{since } \sum_{a=1}^{\infty} x^a = \sum_{a\geq 0} x^a - 1 = \frac{1}{1-x}-1\right)$$

$$= \frac{1}{2-\frac{1}{1-x}} = \frac{1-x}{2(1-x)-1} = \frac{1-x}{1-2x}$$

$$= \cancel{\frac{1}{1-2x} - x\cdot\frac{1}{1-2x}} \quad 1 + \frac{x}{1-2x}$$

$$= 1 + x\sum_{k\geq 0}(2x)^k \qquad \left(\text{since } \frac{1}{1-y} = \sum_{k\geq 0} y^k\right)$$

$$= 1 + \sum_k 2^k x^{k+1} = \sum_k \begin{cases} 2^{k-1} & \text{if } k>0 \\ 1 & \text{if } k=0 \end{cases} \cdot x^k$$

Compare coeffs

$$\Rightarrow \quad k(n) = \begin{cases} 2^{n-1} & \text{if } n>0 \\ 1 & \text{if } n=0 \end{cases}.$$