<u>1st proof.</u> Just compute the D_i using Cor. 36.

$$\text{LHS} = D_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} = n!\left(\sum_{k=0}^{n-1} \frac{(-1)^k}{k!} + \frac{(-1)^n}{n!}\right);$$

$$\text{RHS} = (n-1)(D_{n-1} + D_{n-2})$$

$$= (n-1)\left((n-1)! \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} + (n-2)! \underbrace{\sum_{k=0}^{n-2} \frac{(-1)^k}{k!}}_{= \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} - \frac{(-1)^{n-1}}{(n-1)!}}\right)$$

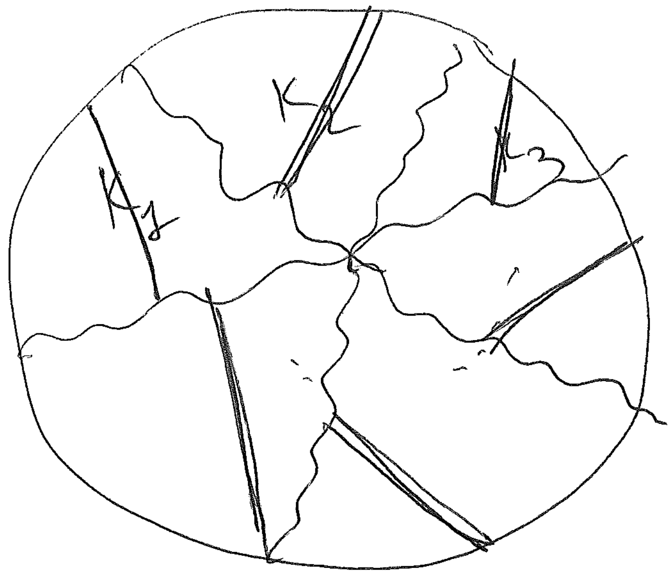
$$= (n-1)(n-1)! \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} + \underbrace{(n-1)(n-2)!}_{=(n-1)!} \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} - \underbrace{(n-1)(n-2)! \frac{(-1)^{n-1}}{(n-1)!}}_{=(-1)^{n-1}}$$

$$= \underbrace{(n-1)(n-1)! \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} + (n-1)! \sum_{k=0}^{n-1} \frac{(-1)^k}{k!}}_{= n(n-1)! \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} = n! \sum_{k=0}^{n-1} \frac{(-1)^k}{k!}} - (-1)^{n-1}$$

$$= n! \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} - \underbrace{(-1)^{n-1}}_{= -n! \frac{(-1)^n}{n!}} = n!\left(\sum_{k=0}^{n-1} \frac{(-1)^k}{k!} + \frac{(-1)^n}{n!}\right),$$

which is the same. $\square$

2nd proof. Classify the D_n derangements of $[n]$ into $n-1$ "kingdoms", each of which will contain 2 "genera".



For each $i \in [n-1]$, the i-th kingdom K_i consists of the derangements π of $[n]$ with $\pi(n) = i$. Then each derangement lies in exactly 1 kingdom (since $\pi(n) \neq n$).

The 1st genus $G_{i,1}$ of the i-th kingdom consists of those $\pi \in K_i$ with $\pi(i) = n$.

The 2nd genus $G_{i,2}$ of the i-th kingdom consists of those $\pi \in K_i$ with $\pi(i) \neq n$.

Claim 1: Each 1st genus $G_{i,1}$ contains D_{n-2} derangements.

Claim 2: Each 2nd genus $G_{i,2}$ contains D_{n-1} derangements.

Proof of Claim 1: How can we construct a derangement in $G_{i,1}$?

We take a derangement of the $(n-2)$-element set $[n] \setminus \{n, i\}$, then extend it to the set $[n]$ by sending $n \mapsto i$ and $i \mapsto n$.

Thus, D_{n-2} options (since $(n-2)$-elt. set has D_{n-2} derangements).

Proof of Claim 2: How can we construct a derangement in $G_{i,2}$?

We ~~start~~ choose a derangement γ of $[n-1]$, and then we "stick in" n "between" $\gamma^{-1}(i)$ and i.

Formally: We start with a derangement γ of $[n-1]$, and then define $\pi : [n] \to [n]$ by

$$\pi(k) = \begin{cases} \gamma(k) & \text{if } k \neq n, \ \& \ k \neq \gamma^{-1}(i) \\ n & \text{if } k = \gamma^{-1}(i) \\ i & \text{if } k = n \end{cases}$$

Can check: π is a derangement in $G_{i,2}$ (since $\gamma^{-1}(i) \neq i$).

Conversely, If π is a derangement in $G_{i,2}$, then the map $\gamma : [n-1] \to [n-1]$ defined by

$$\gamma(k) = \begin{cases} \pi(k) & \text{if } k \neq \pi^{-1}(n) \\ i & \text{if } k = \pi^{-1}(n) \end{cases}$$

is a derangement of $[n-1]$.

The maps $\pi \mapsto \gamma$ and $\gamma \mapsto \pi$ are mutually inverse, so bijections.

Now, $D_n = \sum_i \underbrace{|K_i|}_{=|G_{i,1}|+|G_{i,2}|} = \sum_i (|G_{i,1}| + |G_{i,2}|)$

$$\underset{\text{Claim 2}}{\overset{\text{Claim 1,}}{=}} \sum_i (D_{n-2} + D_{n-1}) = (n-1)(D_{n-2} + D_{n-1}). \quad \square$$

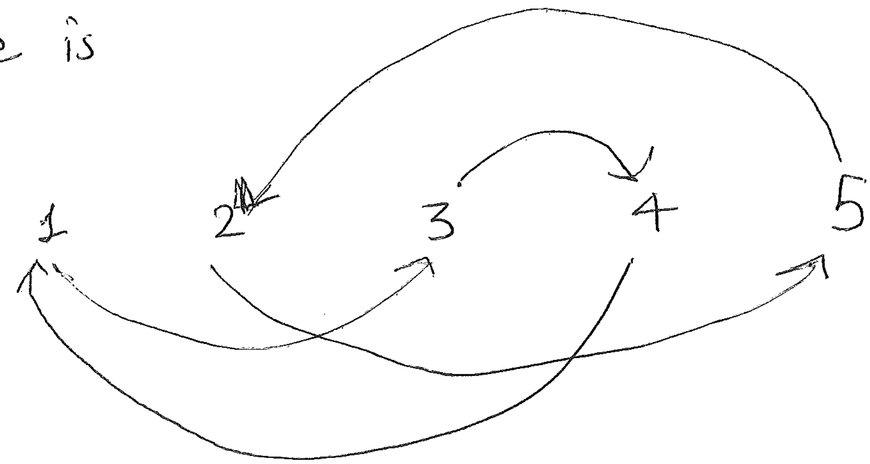
What did actually happen in the proof of Claim 2?

Idea: <u>cycle decomposition</u> of a permutation.

Short introduction:

Given a perm. ξ of a set X, we represent it by a "picture": Draw a point ("node") for each $x \in X$, and draw an arrow ("arc") from each node $x \in X$ to the node $\xi(x)$.

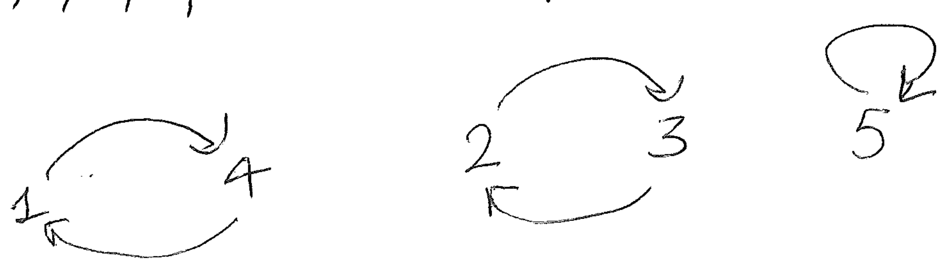
Examples: (a) If ρ is the perm. of $[5]$ sending $1, 2, 3, 4, 5$ to $3, 5, 4, 1, 2$, then the picture is



~~represents~~ or

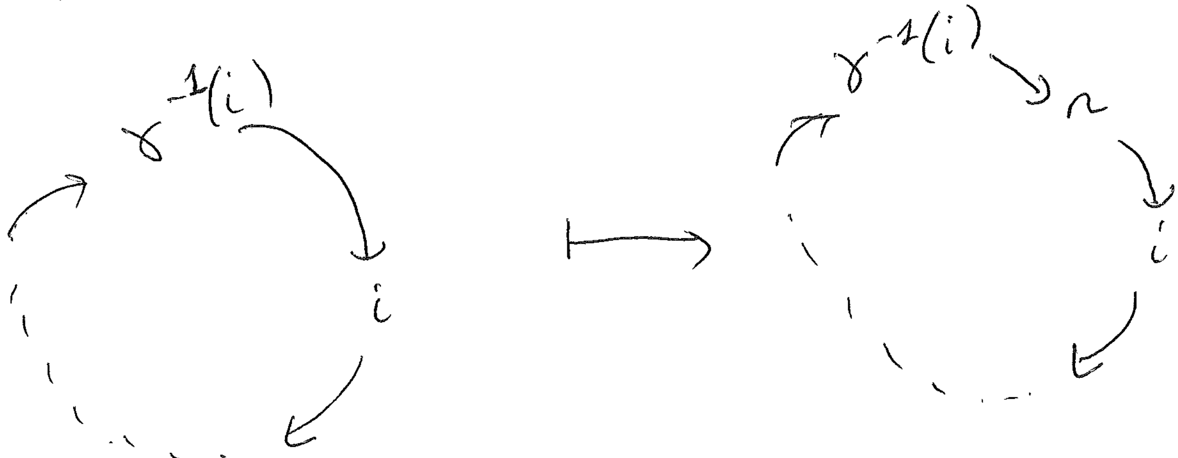
(placement~~s~~ of nodes doesn't matter).

(b) If ρ is the perm. of $[5]$ sending $1, 2, 3, 4, 5$ to $4, 3, 2, 1, 5$, then the picture is

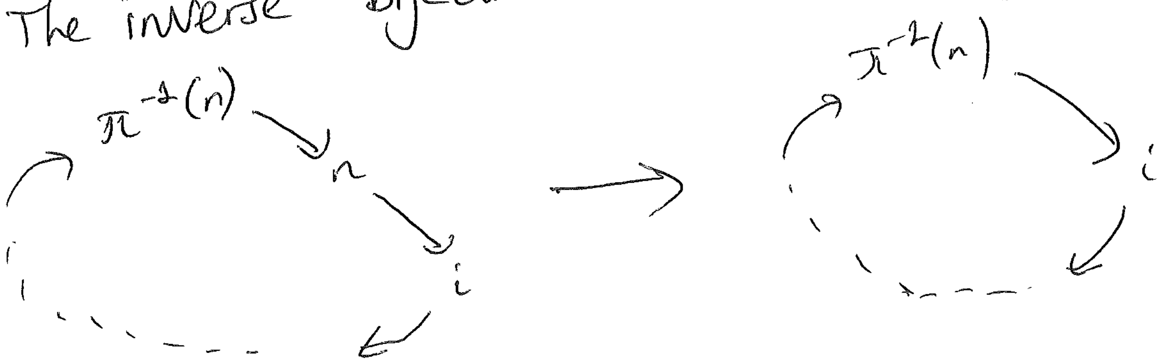


This picture is called the <u>cycle digraph</u> of ρ.
It is / represents a digraph (= directed graph), which we'll study later.

The perm. ξ can be uniquely reconstructed from this picture.
Each node has exactly 1 outgoing arc and exactly 1 incoming arc. If X is finite, then the picture consists of a bunch of disjoint cycles ("disjoint" = having no nodes in common).
A permutation ξ is a derangement iff it has no 1-node cycles.
In terms of cycle digraphs, the bijection $\gamma \mapsto \pi$ in the proof of Claim 2 inserts n into a cycle just before i:



The inverse bijection removes n from its cycle:



~~Rmk: Cor. 36 & Prop. 4~~

Prop. 40. $n! = \sum_{k=0}^{n} \binom{n}{k} D_k, \quad \forall n \in \mathbb{N}.$

Proof. Construct a perm. π of $[n]$ in the following way:

- Choose the # of non-fixed points of π. Call it k.
- Choose the k non-fixed points of π;
  There are $\binom{n}{k}$ ways to do this.
- Choose what π does to these ~~remaining~~ k points.
  There are D_k ways to do this.

$\Longrightarrow n! = \sum_{k=0}^{n} \binom{n}{k} D_k.$ $\square$

Rmk. Cor. 36 $\Longleftrightarrow$ Prop. 40.

Indeed,

Cor. 36 $\Longleftrightarrow D_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)!$

$\overset{\text{subs.}}{\underset{i \text{ for } n-k}{\Longleftrightarrow}} D_n = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} i!$

$\Longleftrightarrow (-1)^n D_n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} i!$

$\overset{\text{HW4}}{\underset{\text{exe. 5}}{\Longleftrightarrow}} n! = \sum_{i=0}^{n} \cancel{(-1)^i} \binom{n}{i} \cancel{(-1)^i} D_i$

$\Longleftrightarrow n! = \sum_{i=0}^{n} \binom{n}{i} D_i \Longleftrightarrow$ Prop. 40.

Prop. 42. $D_n = n D_{n-1} + (-1)^n \quad \forall n \geq 1.$

Proof. Easy using Cor. 36 (exercise).

Bijective proof: much trickier.

(Benjamin & Ornstein:
A Bijective Proof of a Derangement Recurrence.)

3.9. PS on Inclusion & Exclusion

Prop. 43. For any $n \in \mathbb{N}$ and $m \in \mathbb{N}$ and $x \in \mathbb{Q}$, set
$$z_{m,n}(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (x^{n-k} - 1)^m.$$

Then, $z_{m,n}(x) = z_{n,m}(x)$.

This is Add. exe 5 in [detnotes].

But it has a combinatorial proof for $x \in \mathbb{N}$:

Namely, $z_{m,n}(x) = \#$ of all $m \times n$-matrices with entries from $\{0, 1, \ldots, x-1\}$ with no zero rows & no zero columns.

(This is because we can set $U = \{m \times n\text{-matrices with entries from } \{0, 1, \ldots, x-1\} \text{ with no zero rows}\}$,

$A_i = \{\text{matrices in } U \text{ whose } i\text{-th column is } 0\}$.

Then, #(matrices with no zero columns & no zero rows)

$$= \left|U \setminus (A_1 \cup A_2 \cup \dots \cup A_n)\right| = \left|U \setminus \bigcup_{i=1}^{n} A_i\right|$$

$$\underset{\text{incl/excl}}{=} \sum_{I \subseteq [n]} (-1)^{|I|} \underbrace{\left(\#\text{ matrices in } U \text{ having } i\text{-th column} = 0 \ \forall i \in I\right)}_{= (x^{n-|I|} - 1)^m}$$

(choose each row separately)

$$= \sum_{I \subseteq [n]} (-1)^{|I|} (x^{n-|I|} - 1)^m = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (x^{n-k} - 1)^m$$

$$= z_{m,n}(x).$$

Get a bijection by transposing the matrices. $\square$

4. GENERATING FUNCTIONS

4.1. EXAMPLES

Let me show what *generating functions* ("gfs") are good for. Then, in §4.2, I'll explain what they are.
Suspend your disbelief for §4.1, please.

Basic idea: Any sequence $(a_0, a_1, a_2, \dots)$ of numbers gives rise to a "power series" $a_0 + a_1 x + a_2 x^2 + \dots$, called its generating function. What does this mean? See later. (See also [Loehr].)

Before we make this rigorous, some examples:

Ex. 1. Recall the Fibonacci sequence $(f_0, f_1, f_2, \dots)$ with $f_0 = 0$ & $f_1 = 1$ & $f_n = f_{n-1} + f_{n-2}$.

Consider its gf
$$F(x) = f_0 + f_1 x + f_2 x^2 + \dots$$
$$= 0 + 1x + 1x^2 + 2x^3 + 3x^4 + \dots$$

Then,
$$F(x) = 0 + 1x + (f_0 + f_1)x^2 + (f_1 + f_2)x^3 + (f_2 + f_3)x^4 + \dots$$
$$= x + (\cancel{f_0 x^2} + f_1 x^3 + f_2 x^4 + \dots)$$
$$+ (f_1 x^2 + f_2 x^3 + f_3 x^4 + \dots)$$
$$= x + x^2 (f_0 + f_1 x + f_2 x^2 + \dots)$$
$$+ x (f_0 + f_1 x + f_2 x^2 + \dots) \quad (\text{since } f_0 = 0)$$

$$= x + x^2F(x) + xF(x).$$

Solving this for $F(x)$ yields

$$F(x) = \frac{x}{1-x-x^2}$$

(1) by partial fraction decomp

$$= \frac{1}{\sqrt{5}} \cdot \frac{1}{1-\phi_1 x} - \frac{1}{\sqrt{5}} \cdot \frac{1}{1-\phi_2 x},$$

where $\phi_1 = (1+\sqrt{5})/2$ and $\phi_2 = (1-\sqrt{5})/2$ are the "golden ratios",

What is $\frac{1}{1-\alpha x}$?

Well: $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$

(since $(1-x)(1+x+x^2+x^3+\cdots)$

$$= (1 + \cancel{x} + \cancel{x^2} + \cancel{x^3} + \cdots) - (\cancel{x} + \cancel{x^2} + \cancel{x^3} + \cancel{x^4} + \cdots)$$

$= 1$),

so if we substitute αx for x where $\alpha \in \mathbb{C}$, we get

$$\frac{1}{1-\alpha x} = 1 + \alpha x + (\alpha x)^2 + (\alpha x)^3 + \cdots$$

$$= 1 + \alpha x + \alpha^2 x^2 + \alpha^3 x^3 + \cdots$$

Thus, (1) becomes

$$F(x) = \frac{1}{\sqrt{5}} (1 + \phi_1 x + \phi_1^2 x^2 + \cdots) - \frac{1}{\sqrt{5}} (1 + \phi_2 x + \phi_2^2 x^2 + \cdots)$$

$$= \left(\frac{1}{\sqrt{5}} 1 - \frac{1}{\sqrt{5}} 1\right)$$
$$+ \left(\frac{1}{\sqrt{5}} \phi_1 - \frac{1}{\sqrt{5}} \phi_2\right) x$$
$$+ \left(\frac{1}{\sqrt{5}} \phi_1^2 - \frac{1}{\sqrt{5}} \phi_2^2\right) x^2$$
$$+ \cdots,$$

Now, compare coefficients before x^n to get

$$\boxed{f_n = \frac{1}{\sqrt{5}} \phi_1^n - \frac{1}{\sqrt{5}} \phi_2^n}.$$

(Binet's formula)

<u>EX. 2.</u> A <u>Dyck word</u> of length $2n$ is a $2n$-tuple that contains n 0's and n 1's and has the property that for each k, the # of 0's among its first k entries is $\leq$ the # of 1's among its first k entries.

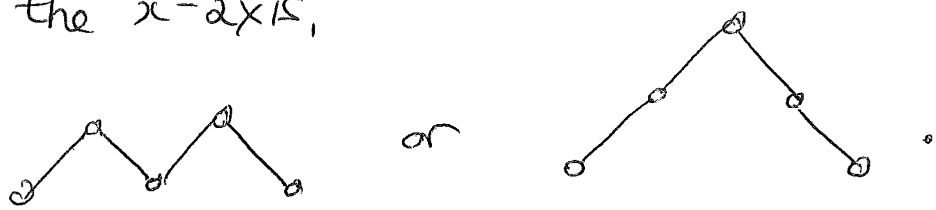
<u>Examples:</u> ~~(0,1,0,1)~~ $(1,0,1,0)$, $(1,1,0,0)$, $(1,1,0,1,0,0)$, $()$, $(1,0)$.

We often write D for 0 and U for 1.

Non-Dyck words: $(1,0,0,1)$, $(0,1)$, $(1,1,0)$, (1).

A <u>Dyck path</u> is a path $(0,0) \to (2n,0)$ that moves by "NE steps" (= steps $(1,1)$) and "SE steps" (steps $(1,-1)$) and never falls below the x-axis.

<u>Examples:</u>



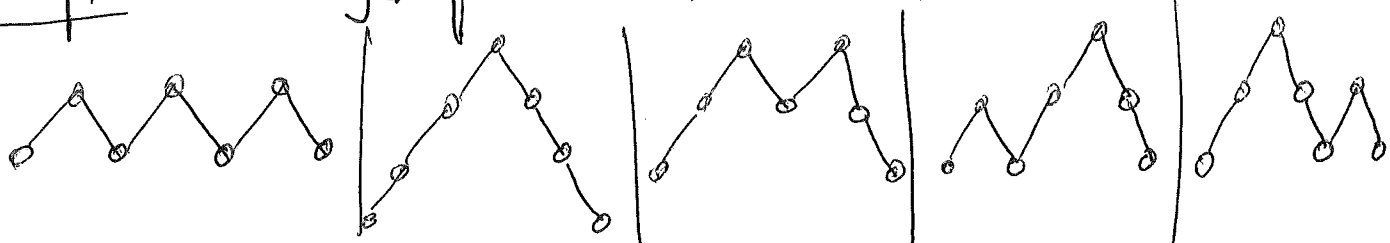
The Dyck paths $(0,0) \to (2n,0)$ can be viewed as "mountain ranges" made of n $/$-steps and n $\backslash$-steps.

There is a simple bijection between Dyck words of length $2n$ and Dyck paths $(0,0) \to (2n,0)$:

send each 1 to a $/$-step, 0 to a $\backslash$-step.

But how many are there?

<u>Example:</u> The Dyck paths $(0,0) \to (6,0)$:



So there are 5 of them.

Let $c_n = \#(\text{Dyck paths } (0,0) \to (2n,0))$

$= \#(\text{Dyck words of length } 2n)$.

Then, $c_1 = 1$,

$c_2 = 2$,

$c_3 = 5$,

$c_4 = 14$,

$c_n = ?$

These numbers c_n are called <u>Catalan numbers</u>,

[Stanley, "Catalan numbers".]