3.5. THE VANDERMONDE IDENTITY

Thm. 12. Let $n \in \mathbb{N}$, $x \in \mathbb{N}$ and $y \in \mathbb{N}$. Then,

$$\binom{x+y}{n} = \sum_{k=0}^{n} \binom{x}{k}\binom{y}{n-k} = \sum_{k} \binom{x}{k}\binom{y}{n-k}.$$

This is the "Vandermonde convolution identity",

1st proof.

$$\binom{u}{n} = \binom{u-1}{n-1} + \binom{u-1}{n}$$

$$= \binom{u-2}{n-2} + \underbrace{\binom{u-2}{n-1} + \binom{u-2}{n-1}} + \binom{u-2}{n}$$

$$= \binom{u-2}{n-2} + 2\binom{u-2}{n-1} + \binom{u-2}{n}$$

$$= \binom{u-3}{n-3} + 3\binom{u-3}{n-2} + 3\binom{u-3}{n-1} + \binom{u-3}{n}$$

$$= \cdots$$

$$= \binom{u-v}{n-v} + \underset{\substack{\uparrow \\ \text{general term}}}{\cdots \binom{v}{k}\binom{u-v}{n-k} \cdots} + \binom{u-v}{n}$$

$$= \sum_{k=0}^{v} \binom{v}{k}\binom{u-v}{n-k} \qquad \forall u \in \mathbb{Q},\ v \in \mathbb{N}$$

Now, apply this to $v = x$ and $u = x+y$. $\square$

<u>2nd proof.</u> How many ways are there to choose an n-elt. subset of $\{1, 2, \ldots, x\} \cup \{-1, -2, \ldots, -y\}$?

— This is $\binom{x+y}{n}$,

— ~~Choose~~ Decide first how many positive integers will be in the subset; call this number k.

Now, choose the k pos. integers ($\binom{x}{k}$ choices) and choose the $n-k$ neg. integers ($\binom{y}{n-k}$ choices).

Total # of possibilities is $\sum_{k=0}^{n} \binom{x}{k}\binom{y}{n-k}$.

$\square$

<u>3rd proof.</u> Induction over n. See [detnotes], Thm. 2.25. $\square$

<u>Rmk.</u> We can replace $\sum_{k=0}^{n}$ by $\sum_{k=0}^{x}$.

<u>Cor. 14.</u> $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$.

<u>Proof.</u> $\sum_{k=0}^{n} \binom{n}{k}^2 = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{k} \overset{\text{symm}}{=} \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k} = \binom{n+n}{n}$,

~~since~~ by Thm. 12 (applied to $x=n$ & $y=n$). $\square$

3.7. Inclusion / Exclusion

- If A is a finite set, then $|A| = |A|$.
- If A and B are finite sets, then $|A \cup B| = |A| + |B| - |A \cap B|$.
- If A, B and C are finite sets, then

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

How do we generalize this?

Theorem 35. (Principle of Inclusion / Exclusion) or Sylvester sieve. Let $A_1, A_2, \ldots, A_n$ be finite sets.

(a) We have

$$\begin{aligned} |A_1 \cup A_2 \cup \cdots \cup A_n| &= |A_1| + |A_2| + \cdots + |A_n| \\ &\quad - |A_1 \cap A_2| - |A_1 \cap A_3| - \cdots - |A_{n-1} \cap A_n| \\ &\quad + |A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| \\ &\quad + \cdots\cdots \\ &\quad - |A_1 \cap A_2 \cap A_3 \cap A_4| - \cdots \\ &\quad \cdots\cdots \\ &\quad + (-1)^{n-1} |A_1 \cap A_2 \cap \cdots \cap A_n| \end{aligned}$$

Rigorously:

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{\substack{I \subseteq [n];\\ I \neq \emptyset}} (-1)^{|I|-1} \left|\bigcap_{i \in I} A_i\right|$$

(For example, $\bigcap_{i \in \{2,3,6\}} A_i = A_2 \cap A_3 \cap A_6$.)

(b) Let U be a finite set containing all A_i's as subsets. Then,

$$\left|U \setminus \bigcup_{i=1}^{n} A_i\right| = \sum_{I \subseteq [n]} (-1)^{|I|} \left|\bigcap_{i \in I} A_i\right|,$$

where $\bigcap_{i \in \emptyset} A_i$ means U.

<u>Proof.</u> First, we notice that (a) $\Leftrightarrow$ (b). In fact,

LHS of (b) $= |U| -$ LHS of (a), but also

RHS of (b) $= \underbrace{(-1)^{|\emptyset|}}_{=1} \underbrace{\left|\bigcap_{i \in \emptyset} A_i\right|}_{=U} + \sum_{\substack{I \subseteq [n];\\ I \neq \emptyset}} (-1)^{|I|} \left|\bigcap_{i \in I} A_i\right|$

$$= |U| + \sum_{\substack{I \subseteq [n]; \\ I \neq \emptyset}} \underbrace{(-1)^{|I|}}_{= -(-1)^{|I|-1}} \left| \bigcap_{i \in I} A_i \right|$$

$$= |U| - \underbrace{\sum_{\substack{I \subseteq [n]; \\ I \neq \emptyset}} (-1)^{|I|-1} \left| \bigcap_{i \in I} A_i \right|}_{= \text{RHS of (a)}}$$

$$= |U| - \text{RHS of (a)}.$$

So it remains to prove one of (a) and (b).

Let's prove (b),

For any $x \in U$, ~~and~~ we have

$$[x \in \underbrace{U \setminus \bigcup_{i=1}^{n} A_i}_{= \bigcap_{i=1}^{n} (U \setminus A_i)}] = \left[x \in \bigcap_{i=1}^{n} (U \setminus A_i)\right]$$

$$= [x \in U \setminus A_i \ \forall i \in [n]]$$

$$= [x \in U \setminus A_1 \wedge x \in U \setminus A_2 \wedge \ldots \wedge x \in U \setminus A_n]$$

$$= \prod_{i=1}^{n} [x \in U \setminus A_i]$$

(since $[S_1 \wedge S_2 \wedge \dots \wedge S_n] = \prod_{i=1}^{n} [S_i]$)

$$= \prod_{i=1}^{n} (1 - [x \in A_i])$$

$$= \sum_{I \subseteq [n]} (-1)^{|I|} \underbrace{\prod_{i \in I} [x \in A_i]}_{}$$

$$= \Big[\bigwedge_{i \in I} (x \in A_i) \Big]$$

$$= [x \in A_i \ \forall i \in I]$$

$$= \Big[x \in \bigcap_{i \in I} A_i \Big]$$

(1) $$= \sum_{I \subseteq [n]} (-1)^{|I|} \Big[x \in \bigcap_{i \in I} A_i \Big].$$

If P is any subset of U, then

(2) $$|P| = \sum_{x \in U} [x \in P].$$

Apply this to $P = U \setminus \bigcup_{i=1}^{n} A_i$, and obtain

$$\left| U \setminus \bigcup_{i=1}^{n} A_i \right| = \sum_{x \in U} \left[x \in U \setminus \bigcup_{i=1}^{n} A_i \right]$$

$$\overset{(1)}{=} \sum_{x \in U} \sum_{I \subseteq [n]} (-1)^{|I|} \left[x \in \bigcap_{i \in I} A_i \right]$$

$$\overset{\text{switch sums}}{=} \sum_{I \subseteq [n]} (-1)^{|I|} \underbrace{\sum_{x \in U} \left[x \in \bigcap_{i \in I} A_i \right]}_{\overset{(2)}{=} \left| \bigcap_{i \in I} A_i \right|}$$

$$= \sum_{I \subseteq [n]} (-1)^{|I|} \left| \bigcap_{i \in I} A_i \right| .$$

$\square$

This proves (b). Hence, (a) follows.
(See also Galvin §16 for this proof.)

Example 1: Derangements.

A *derangement* of a set X means a permutation of X that has no fixed points.

Let D_n be the # of derangements of $[n]$.

What is D_n?

Let $U = \{\text{all permutations of } [n]\}$.

For each $i \in [n]$, let $A_i = \{\pi \in U \mid \pi(i) = i\}$.

Then, $D_n = \left| U \setminus \bigcup_{i=1}^{n} A_i \right|$

$$\overset{\text{Thm. 35(b)}}{=} \sum_{I \subseteq [n]} (-1)^{|I|} \left| \bigcap_{i \in I} A_i \right|.$$

But for any $I \subseteq [n]$, we have

$$\left| \bigcap_{i \in I} A_i \right| = \left| \{\pi \in U \mid \pi(i) = i \;\; \forall i \in I\} \right|$$
$$= (n - |I|)(n - |I| - 1) \cdots 1$$
$$= (n - |I|)!,$$

so this becomes

$$D_n = \sum_{I \subseteq [n]} (-1)^{|I|} (n - |I|)! = \sum_{k=0}^{n} (-1)^k (n-k)! \binom{n}{k}.$$

So we have

$$D_n = \sum_{k=0}^{n} (-1)^k \underbrace{\binom{n}{k}(n-k)!}_{=\frac{n!}{k!}} = \sum_{k=0}^{n} (-1)^k \frac{n!}{k!}$$

$$= n! \sum_{k=0}^{n} \frac{(-1)^k}{k!},$$

thus

$$\frac{D_n}{n!} = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \simeq \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} = e^{-1}$$

Actually, $D_n = \mathrm{round}(n!/e)$ for $n \geq 1$.

Example 2: Surjections, again.

Fix $n \in \mathbb{N}$, $k \in \mathbb{N}$. Compute $\operatorname{sur}(n,k) = \#(\text{surj from } [n] \text{ to } [k])$.

Set $U = \{\text{maps } [n] \to [k]\}$.

For each $i \in [k]$, set $A_i = \{\text{maps } [n] \to [k] \text{ which miss } i\}$

(we say that a map f misses i if i is not in its image).

Then, $U \setminus \bigcup_{i=1}^{k} A_i = \{\text{surjs from } [n] \to [k]\}$.

But Thm. 35 (b) yields

$$\left| U \setminus \bigcup_{i=1}^{k} A_i \right| = \sum_{I \subseteq [k]} (-1)^{|I|} \Big| \underbrace{\bigcap_{i \in I} A_i}_{\substack{= \{\text{maps missing all } i \in I\} \\ \cong \{\text{maps } [n] \to [k] \setminus I\}}} \Big|$$

$$= \sum_{I \subseteq [k]} (-1)^{|I|} \underbrace{\left| \{\text{maps } [n] \to [k] \setminus I\} \right|}_{= (k - |I|)^n}$$

$$= \sum_{I \subseteq [k]} (-1)^{|I|} (k-|I|)^n$$

$$= \sum_{i=0}^{k} (-1)^i (k-i)^n \binom{k}{i}$$

$$= \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n .$$

So $\mathrm{sur}(n,k) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^n .$

This was a HW2 problem.

Example 3:
Euler's ϕ-function. called Euler's totient function (also written φ)

Define a function $\phi : \{1,2,3,\ldots\} \to \mathbb{N}$ by

$\phi(n) = \#$ of all $m \in [n]$ coprime to n.

Examples: $\phi(2) = |\{1, \cancel{2}\}| = 1.$

$\phi(4) = |\{1, \cancel{2}, 3, \cancel{4}\}| = 2.$

$\phi(6) = |\{1, \cancel{2}, \cancel{3}, \cancel{4}, 5, \cancel{6}\}| = 2.$ (2)

$\phi(12) = |\{1, \cancel{2}, \cancel{3}, \cancel{4}, 5, \cancel{6}, 7, \cancel{8}, \cancel{9}, \cancel{10}, 11, \cancel{12}\}| = 4.$

What about $\phi(n)$ in general?

Let $p_1, p_2, \dots, p_k$ be the distinct prime factors of n. Then, let $U = [n]$, and for each $i \in [k]$, let $A_i = \{x \in [n] \mid \underbrace{x \in p_i \mathbb{Z}}_{\text{this says } p_i \mid x}\}$.

Then $\{m \in [n] \text{ coprime to } n\} = U \setminus \bigcup_{i=1}^{k} A_i$.

Hence, $\phi(n) = \left| U \setminus \bigcup_{i=1}^{k} A_i \right| \overset{\text{Thm. 35(b)}}{=} \sum_{I \subseteq [k]} (-1)^{|I|} \underbrace{\left| \bigcap_{i \in I} A_i \right|}$

$= \#$ of all $m \in [n]$ that are multiples of all p_i (for $i \in I$)

$= \#$ of all $m \in [n]$ that are multiples of $\prod_{i \in I} p_i$

$= \dfrac{n}{\prod_{i \in I} p_i}$

$$= \sum_{I \subseteq [k]} (-1)^{|I|} \frac{n}{\prod_{i \in I} p_i}$$

$$= n \underbrace{\sum_{I \subseteq [k]} (-1)^{I} \prod_{i \in I} \frac{1}{p_i}}_{= \prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right)} = n \prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right).$$

3.8. More about Derangements

We have seen:

Cor. 36. $D_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} \quad \forall n \in \mathbb{N}$.

We'll say more.

Prop. 41. $D_n = (n-1)(D_{n-1} + D_{n-2}) \quad \forall n \geq 2$.