4990 10 Oct 2017

2.10. Odds & Ends

Exercise. Given $n>0$ and $k>0$.

We have n ~~the~~ people & k tasks (all labeled).

(a) What is the # of ways to assign to each person a task such that ~~no task~~ each task gets someone working on it?

(b) What if each task should also have a "leader"?

(c) What if each task should also have the people working on it organized in a vertical ~~hea~~ hierarchy?

Example: 1 2 3 4 5 6 7 8

(a) 1 2 5   3   4 6 7 8 $\mathrm{sur}(n,k)$

τ_1 τ_2 τ_3

(b) 1 2 5   3   4 6 7 8 $n(n-1)\cdots(n-k+1)\cdot k^{n-k}$

1   2   3

(c) 1 – 5 – 2   3   4 – 8 – 7 – 6 $n!\binom{n-1}{k-1}$

Given $a_1, \dots, a_k \in \mathbb{N}$,

the # of ways of putting n people into k groups (labelled) of sizes $a_1, \dots, a_k$ is (assuming $a_1 + \dots + a_k = n$)

$$\binom{n}{a_1}\binom{n-a_1}{a_2}\binom{n-a_1-a_2}{a_3} \cdots \binom{n-a_1-a_2-\dots-a_{k-1}}{a_k}$$

$$= \frac{n!}{a_1!\cancel{(n-a_1)!}} \cdot \frac{\cancel{(n-a_1)!}}{a_2!\cancel{(n-a_1-a_2)!}} \cdot \frac{\cancel{(n-a_1-a_2)!}}{a_3!\cancel{(n-a_1-a_2-a_3)!}} \cdots \frac{\cancel{(n-a_1-\dots-a_{k-1})!}}{a_k!\underbrace{(n-a_1-\dots-a_k)!}_{=0!=1}}$$

$$= \frac{n!}{a_1! \cdots a_k!}$$

("multinomial coefficient", since it is the coeff before $x_1^{a_1} x_2^{a_2} \cdots x_k^{a_k}$ in $(x_1 + \dots + x_k)^n$)

Shortcut for (c):

$n!$ ways to order the n people:

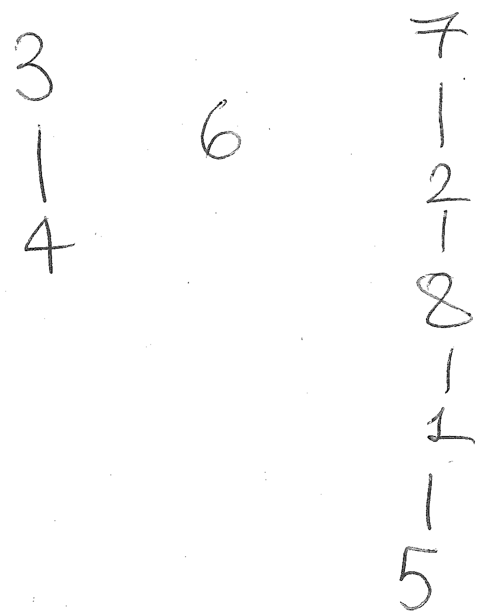
3 4 6 7 2 8 1 5

$\binom{n-1}{k-1}$ ways to put $k-1$ dividers

1 into this arrangement:

34 | 6 | 72815

$\Rightarrow$



3. BINOMIAL COEFFICIENTS

3.1. EXTENDING THE DEFINITION

Definition. Set $\binom{n}{k} = 0$ whenever $k \notin \mathbb{N}$.

(Slightly ~~controversiaa~~ controversial.)

Reminders:

- $\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}$ $\forall n \in \mathbb{Q}, k \in \mathbb{Z}$,

- $\binom{n}{k}$ = # of k-elt subsets of a given n-elt. set $\forall n \geq 0$.

- $\binom{n}{k} = 0$ if $n < k$ and $n \in \mathbb{N}$.

- $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ if $n \geq k \geq 0$, $n, k \in \mathbb{N}$

- $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ $\forall k > 0$

Prop. 1. $\binom{n}{k} \in \mathbb{Z}$ for all $n \in \mathbb{Z}$ and $k \in \mathbb{Z}$.

Proof. Case 1: $k \notin \mathbb{N}$.

Clear, since $\binom{n}{k} = 0$.

Case 2: $k \in \mathbb{N}$, $n \geq 0$.

Follows from $\binom{n}{k} = \#(k\text{-elt. subsets of } [n])$.

Case 3: $k \in \mathbb{N}$, $n < 0$.

HW1 exe 2(a) says $\binom{n}{k} = (-1)^k \binom{k-n-1}{k}$ ("upper negation"). But $\binom{k-n-1}{k} \in \mathbb{Z}$ by Case 2 (since

$$k - \underbrace{n}_{<0} - 1 > k-1 \geq -1 \Rightarrow k-n-1 \in \mathbb{N}). \quad \square$$

Prop. 2. ("Trinomial revision") $\binom{n}{a}\binom{a}{b} = \binom{n}{b}\binom{n-b}{a-b}$

$\forall n, a, b \in \mathbb{Q}$.

Proof. If $b \notin \mathbb{N}$, then both sides are 0.

Hence WLOG assume that $b \in \mathbb{N}$.

Similarly, WLOG assume that $a \in \mathbb{N}$.

If $a<b$, then both sides are 0 (since $\binom{a}{b}=0$). Hence WLOG assume $a \geq b$. Now, the claim follows from HW 1 exe 2 (c). $\square$

<u>Prop. 3. ("Symmetry").</u> If $n \in \mathbb{N}$ and $k \in \mathbb{Z}$, then $\binom{n}{k}=\binom{n}{n-k}$.

(But not for $n \notin \mathbb{N}$.)

<u>Proof.</u>
- If $k<0$, then both sides are 0.
- If $k>n$, then both sides are 0.
- If $0 \leq k \leq n$, then

$$\binom{n}{k}=\frac{n!}{k!\,(n-k)!}=\frac{n!}{(n-k)!\,k!}=\binom{n}{n-k} \quad \square$$

Now to more interesting identities.

Follow Graham/Knuth/Patashnik Ch. 5:

<u>3.2. HOCKEY-STICK IDENTITY</u>

Prop. 4. (recursion).

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

$$\forall\, n \in \mathbb{Q},\ k \in \mathbb{Z}.$$

Proof.
- Known if $k>0$.
- Easy if $k=0$.
- Clear if $k<0$. $\square$

1

1

$$
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
\boxed{1} \quad \boxed{4} \quad 6 \quad 4 \quad 1 \\
1 \quad \boxed{5} \quad \boxed{10} \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad \boxed{15} \quad \boxed{20} \quad 15 \quad 6 \quad 1 \\
7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1
\end{array}
$$

$$
\begin{aligned}
\binom{7}{3} &= \binom{6}{3} + \binom{6}{2} \\
&= \binom{6}{3} + \binom{5}{2} + \binom{5}{1} \\
&= \binom{6}{3} + \binom{5}{2} + \binom{4}{1} + \underbrace{\binom{4}{0}} \\
&= \binom{6}{3} + \binom{5}{2} + \binom{4}{1} + \binom{3}{0} + \cancel{\binom{3}{-1}}
\end{aligned}
$$

Thm. 5. Let $n \in \mathbb{Z}$ and $r \in \mathbb{Q}$. Then,

$$\binom{r+n+1}{n} = \sum_{k=0}^{n} \binom{r+k}{k} = \sum_{k \le n} \binom{r+k}{k}.$$

(An infinite sum of 0's is still a 0. Hence, $\sum_{k \le n} \binom{r+k}{k}$ is well-defined.)

Proofs of Thm 5.

WLOG $n \in \mathbb{N}$, else $0=0$.

1st proof. Induction over n, formalizing the above process.

2nd proof. We have

$$\binom{r+k}{k} = \binom{r+k+1}{k} - \binom{r+k}{k-1} \quad \forall k$$

(by recursion).

Hence,

$$\sum_{k=0}^{n} \binom{r+k}{k} = \sum_{k=0}^{n} \left(\binom{r+k+1}{k} - \binom{r+k}{k-1} \right)$$

$$= \left(\binom{r+n+1}{n} - \cancel{\binom{r+n+1}{n-1}} \right)$$

$$+ \left(\cancel{\binom{r+n}{n-1}} - \cancel{\binom{r+n-1}{n-2}} \right)$$

$$+ \left(\cancel{\binom{r+n-1}{n-2}} - \cancel{\binom{r+n-2}{n-3}} \right)$$

$$+ \ldots + \left(\cancel{\binom{r+1}{0}} - \binom{r}{-1} \right)$$

$$= \binom{r+n+1}{n} - \underbrace{\binom{r}{-1}}_{=0} = \binom{r+n+1}{n}.$$

Generally: $\sum_{k=0}^{n}(a_{k+1}-a_k) = a_{n+1}-a_0$

("telescoping sum").

<u>3rd proof.</u> Bijective proof; for $r \in \mathbb{N}$.

$\binom{r+n+1}{n}$ is the # of ways to choose an n-element subset of $[r+n+1]$.

How to count this differently?

First, decide on the <u>largest</u> $k \geq 0$ such that $r+k+1 \notin$ subset.

Now, $r+k+1 \notin$ subset but $r+k+2, r+k+3, \ldots, r+n+1 \in$ subset.

So what remains to be done is to choose the remaining k elts out of $[r+k]$, for which there are $\binom{r+k}{k}$ options.

So $\binom{r+n+1}{n} = \sum_{k=0}^{n}\binom{r+k}{k}$.

we'll be able to "fix" the $r \in \mathbb{N}$ assumption: the "polynomial identity trick" shows that if ~~$\mathbb{N}$~~ Thm. 5 holds for $r \in \mathbb{N}$, then it holds for $r \in \mathbb{Q}$. $\square$

<u>Cor. 6.</u> Let $n \in \mathbb{N}$ and $m \in \mathbb{N}$. Then,

$$\binom{n+1}{m+1} = \sum_{k=0}^{n} \binom{k}{m}.$$

<u>Proof.</u> Apply Thm. 5 to m and $n-m$ instead of n and r.

Get $$\binom{n+1}{n-m} = \sum_{k=0}^{n-m} \binom{m+k}{k} = \sum_{k=m}^{n} \underbrace{\binom{k}{k-m}}_{= \binom{k}{m} \text{ by symmetry}}$$

(we sub'd $k-m$ for k)

$$= \sum_{k=m}^{n} \binom{k}{m} \overset{0}{=} \sum_{k=0}^{n} \binom{k}{m}.$$

(My notation $\overset{0}{=}$ stands for "equals, because the only different addends are all 0.")

So $\sum_{m=0}^{n} \binom{k}{m} = \binom{n+1}{n-m} \overline{\overline{\text{Symmetry}}} \binom{n+1}{n+1-(n-m)}$

$$= \binom{n+1}{m+1}. \qquad \square$$

Remark: Apply Cor. 6 to $m=1$. Get

$$\frac{n(n+1)}{2} = \binom{n+1}{2} = \sum_{k=0}^{n} \binom{k}{1} = \sum_{k=0}^{n} k$$

$$= 0+1+\cdots+n.$$

Apply Cor. 6 to $m=2$. Get

$$\binom{n+1}{3} = \sum_{k=0}^{n} \binom{k}{2} = \sum_{k=0}^{n} \frac{k(k-1)}{2},$$

so $\sum_{k=0}^{n} k(k-1) = 2\binom{n+1}{3}$.

Thus, $2\binom{n+1}{3} = \sum k(k-1) = \sum k^2 - \underbrace{\sum k}_{=\binom{n+1}{2}}$,

so $\sum k^2 = 2\binom{n+1}{3} + \binom{n+1}{2} = \frac{n(n+1)(2n+1)}{6}$.

This way, we can get a formula for $\sum_{k=0}^{n} k^m$ for each $m \geq 1$. But we can do better.

3.3. $1^m + 2^m + \dots + n^m$

<u>Thm. 7.</u> $$k^m = \sum_{i=0}^{m} \underbrace{\left\{ {m \atop i} \right\} i!}_{= sur(m,i)} \binom{k}{i}$$

$\forall k \in \mathbb{N}$ and $m \in \mathbb{N}$.

<u>Proof.</u> How many ways are there to choose a map $[m] \to [k]$?

(a) k^m.

(b) First, choose the image of the map: a subset of $[k]$.
Let this be an i-element subset; it can then be chosen in $\binom{k}{i}$ ways.

Then, choose ~~a~~ our map from $[m]$ to this image in $sur(m,i)$ ways.
$\Rightarrow$ The number is $\sum_{i=0}^{m} sur(m,i) \binom{k}{i}$.

$\square$

Thm. 8. $\displaystyle\sum_{k=0}^{n} k^m = \sum_{i=0}^{m} \left\{ {m \atop i} \right\} i! \binom{n+1}{i+1}$

$\forall n \in \mathbb{N},\ m \in \mathbb{N}.$

Proof. $\displaystyle\sum_{k=0}^{n} k^m \overset{\text{Thm. 7}}{=\!=} \sum_{k=0}^{n}\sum_{i=0}^{m} \left\{ {m \atop i} \right\} i! \binom{k}{i}$

$$= \sum_{i=0}^{m}\sum_{k=0}^{n} \left\{ {m \atop i} \right\} i! \binom{k}{i}$$

$$= \sum_{i=0}^{m} \left\{ {m \atop i} \right\} i! \underbrace{\sum_{k=0}^{n} \binom{k}{i}}_{\overset{\text{Cor. 6}}{=\!=} \binom{n+1}{i+1}}$$

$$= \sum_{i=0}^{m} \left\{ {m \atop i} \right\} i! \binom{n+1}{i+1}. \qquad \square$$

3.4. THE BINOMIAL FORMULA

Thm. 9. Let $n \in \mathbb{N}$ and $x, y \in \mathbb{Q}$. Then,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Proof.

$$(x+y)^n = (x+y)(x+y)\cdots(x+y)$$

$$= xx\cdots x + xx\cdots y$$

$$+ xy\cdots x + \cdots\cdots$$

(2^n terms).

$$= \sum_{k=0}^{n} a_k x^k y^{n-k}$$

where a_k = # of terms that simplify to $x^k y^{n-k}$

= # of ways to ~~choose x~~ ~~from~~ pick x out of k of the n parentheses and y out of the remaining $n-k$ parentheses

= # of ways to chose k numbers out of $[n]$

$= \binom{n}{k}$. $\square$

Cor. 10. $\sum_{k=0}^{n} \binom{n}{k} = 2^n \quad \forall n \in \mathbb{N}.$

Proof. Set $x = 1$, $y = 1$ in Thm. 9. $\square$

Cor. 11. $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = [n=0].$

Proof. Set $x = -1$, $y = 1$ in Thm. 9, get

$$0^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \cancel{1^{n-k}}$$

$$\| $$

$$\begin{cases} 0 & \text{if } n>0, \\ 1 & \text{if } n=0 \end{cases} = [n=0]. \quad \square$$

Rmk. Cor. 11 says that if $n>0$, then

# of even-size subsets of $[n]$
= # of odd-size subsets of $[n]$.

We can prove it by constructing the bijection

$$\begin{Bmatrix} \text{even-size subsets} \\ \text{of } [n] \end{Bmatrix} \longrightarrow \begin{Bmatrix} \text{odd-size subsets} \\ \text{of } [n] \end{Bmatrix},$$

$$S \longmapsto \begin{cases} S \cup \{n\}, & \text{if } n \notin S; \\ S \setminus \{n\}, & \text{if } n \in S \end{cases}$$

Rmk. Thm. 9 ("the binomial formula") holds also when x, y are complex numbers, or polynomials, or commuting matrices.

Exercise. Let $n = 4k+2$ for $k \in \mathbb{N}$. Prove that exactly 1 in 4 subsets of $[n]$ have size divisible by 4.

Proof. Let $i = \sqrt{-1}$. Then,

$$\cancel{(1+i)^n = \sum_{k=0}^{n} \binom{n}{k} 1^k i^{n-k}}$$

$$(i+1)^n = \sum_{k=0}^{n} \binom{n}{k} \underbrace{i^k}_{= \begin{cases} 1 \text{ if } k \equiv 0 \bmod 4 \\ i \text{ if } k \equiv 1 \bmod 4 \\ -1 \text{ if } k \equiv 2 \bmod 4 \\ -i \text{ if } k \equiv 3 \bmod 4 \end{cases}} \cancel{1^{n-k}}$$

$$= \sum_{k \equiv 0 \bmod 4} \binom{n}{k} + i \sum_{k \equiv 1 \bmod 4} \binom{n}{k} - \sum_{k \equiv 2 \bmod 4} \binom{n}{k}$$

$$- i \sum_{k \equiv 3 \bmod 4} \binom{n}{k}.$$

$\xRightarrow{\text{real part}}$ $\underbrace{\mathrm{Re}}_{\substack{=\text{real part}\\ =x\text{-coordinate}}}\left((i+1)^n\right) = \sum_{k\equiv 0 \bmod 4} \binom{n}{k} - \sum_{k\equiv 2 \bmod 4} \binom{n}{k}.$

We want to show that RHS $\equiv 0$.

We want to prove that $\arg((i+1)^n) = \frac{\pi}{2}$ or $-\frac{\pi}{2}$.

But $\arg(i+1) = 45^\circ = \frac{\pi}{4}$, so

$$\arg((i+1)^n) = n\,\underbrace{\arg((i+1))}_{=\frac{\pi}{4}}$$

$$= n\cdot\frac{\pi}{4} = (4k+2)\cdot\frac{\pi}{4}$$

$$= k\pi + \frac{\pi}{2} \equiv \frac{\pi}{2} \text{ or } -\frac{\pi}{2} \mod 2\pi, \quad \square$$