Fix a commutative ring K. (For example, $K = \mathbb{Z}$ or $\mathbb{Q}$ or $\mathbb{R}$ or $\mathbb{C}$.)

<u>Def.</u> A <u>formal power series (FPS)</u> (in 1 indeterminate x over K) is a sequence $(a_0, a_1, \ldots) \in K^\infty$ of elements of K.

This might answer "what is an FPS", but does not explain what we can do with FPSs. For example, why can we write $(a_0, a_1, a_2, \ldots)$ as $a_0 + a_1 x + a_2 x^2 + \ldots$? What is x?

<u>Def.</u> (a) The <u>sum</u> of two FPS $(a_0, a_1, \ldots)$ and $(b_0, b_1, \ldots)$ is

$$(a_0, a_1, \ldots) + (b_0, b_1, \ldots) = (a_0 + b_0, a_1 + b_1, \ldots).$$

(b) If $\lambda \in K$ and $(a_0, a_1, \ldots)$ is a FPS, then $\lambda (a_0, a_1, \ldots)$ is defined as $(\lambda a_0, \lambda a_1, \ldots)$.

(c) The <u>product</u> of two FPS $(a_0, a_1, \ldots)$ and $(b_0, b_1, \ldots)$ is the FPS $(c_0, c_1, \ldots)$, where

$$c_n = \sum_{i=0}^{n} a_i b_{n-i} = \sum_{\substack{i,j \in \mathbb{N};\\ i+j=n}} a_i b_j = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0 .$$

(d) For each $a \in K$, define the FPS $\underline{a} = (a, 0, 0, 0, \ldots)$.
We'll soon just call it a, but for now let's use $\underline{a}$.

(e) The set of all FPS is called $K[[x]]$. (Double square brackets)

<u>Thm. 1.</u> $K[[x]]$ is a comm. ring, with subring K (if we regard each $a \in K$ as the FPS $\underline{a}$). Specifically, this means:

(a) Addition in $K[[x]]$ is commutative & associative:
$\vec{a} + \vec{b} = \vec{b} + \vec{a}$, $\quad \vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$.

(b) $\underline{0} + \vec{a} = \vec{a} + \underline{0} = \vec{a}$.

(c) Multiplication in $K[[x]]$ is commutative & associative:
$\vec{a}\,\vec{b} = \vec{b}\,\vec{a}$, $\quad \vec{a}(\vec{b}\,\vec{c}) = (\vec{a}\,\vec{b})\vec{c}$.

(d) $\underline{1}\,\vec{a} = \vec{a}\,\underline{1} = \vec{a}$.

(e) $\underline{0}\,\vec{a} = \vec{a}\,\underline{0} = \underline{0}$.

(f) Distributivity holds: $\vec{a}(\vec{b} + \vec{c}) = \vec{a}\,\vec{b} + \vec{a}\,\vec{c}$, $\quad (\vec{a} + \vec{b}) \cdot \vec{c} = \vec{a}\,\vec{c} + \vec{b}\,\vec{c}$.

(g) $\forall\, \vec{a}, \vec{b} \in K[[x]]$ there is a unique $\vec{c}$ (called $\vec{a} - \vec{b}$) with $\vec{a} = \vec{b} + \vec{c}$. (This is quite easy: $(a_0, a_1, \ldots) - (b_0, b_1, \ldots) = (a_0 - b_0, a_1 - b_1, \ldots)$.)

(h) $\forall a, b \in K$ we have $\underline{a} + \underline{b} = \underline{a+b}$ and $\underline{a} \cdot \underline{b} = \underline{ab}$.

Furthermore, $K[[x]]$ is a K-module (same as K-vector space, except K doesn't have to be a field). That is,

(i) $\lambda(\vec{a} + \vec{b}) = \lambda\vec{a} + \lambda\vec{b}$,

(j) $(\lambda + \mu)\vec{a} = \lambda\vec{a} + \mu\vec{a}$,

(k) $(\lambda\mu)\vec{a} = \lambda(\mu\vec{a})$,

(l) $1\vec{a} = \vec{a}$.

Finally:

(m) $\lambda\vec{a} = \underline{\lambda} \cdot \vec{a} \quad \forall \lambda \in K$ and $\vec{a} \in K[[x]]$.

The purpose of Thm. 1 is to justify computing with FPS as with numbers, at least as far as $+$, $-$ and $\cdot$ are concerned.

Knowing that $K[[x]]$ is a comm. ring implies:

- subtraction works
- sums & products need no parentheses and don't care about the order: for example, $((\vec{a}\vec{b})\vec{c})\vec{d} = \vec{a}((\vec{b}\vec{c})\vec{d}) =$

$\vec{a}(\vec{b}(\vec{c}\,\vec{d}))$, so we can denote all of these by $\vec{a}\,\vec{b}\,\vec{c}\,\vec{d}$. Also, $\vec{a}\,\vec{b}\,\vec{c}\,\vec{d} = \vec{b}\,\vec{d}\,\vec{c}\,\vec{a} = \vec{c}\,\vec{a}\,\vec{b}\,\vec{d}$.

- Hence, finite sums & products work as usual: $\sum_{i=1}^{k} \vec{a_i}$, $\sum_{i\in I} \vec{a_i}$, $\prod_{i=1}^{k} \vec{a_i}$, $\prod_{i\in I} \vec{a_i}$. These can be computed in any order.
- In particular, powers exist: $\vec{a}^{\,n} = \underbrace{\vec{a}\,\vec{a}\cdots\vec{a}}_{n \text{ times}}$ $\forall n \in \mathbb{N}$.

  This includes $\vec{a}^{\,0} = \underline{1}$.
- Standard rules for exponents hold: $\vec{a}^{\,n+m} = \vec{a}^{\,n}\,\vec{a}^{\,m}$, $(\vec{a}\,\vec{b})^n = \vec{a}^{\,n}\,\vec{b}^{\,n}$, etc.
- The binomial formula holds: $(\vec{a}+\vec{b})^n = \sum_{k=0}^{n} \binom{n}{k} \vec{a}^{\,k}\,\vec{b}^{\,n-k}$.
- All other kinds of formulas hold, e.g. Vandermonde: $\binom{\vec{a}+\vec{b}}{n} = \sum_{k} \binom{\vec{a}}{k}\binom{\vec{b}}{n-k}$.

  (Recall $\binom{x}{n} = \frac{x(x-1)\cdots(x-n+1)}{n!}$). if $K = \mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$.

Def. $\forall n \in \mathbb{N}$ and $\vec{a} = (a_0, a_1, \dots) \in K[[x]]$, we set $[x^n]\vec{a} = a_n$.

This is called the coefficient of x^n in $\vec{a}$.

Thus, the definition of sum & product rewrite as follows:

(1) $[x^n](\vec{a}+\vec{b}) = [x^n]\vec{a} + [x^n]\vec{b}$,

(2) $[x^n](\vec{a}\vec{b}) = [x^0]\vec{a}\cdot[x^n]\vec{b} + [x^1]\vec{a}\cdot[x^{n-1}]\vec{b} + \dots\dots$

(3) $= \sum_{i=0}^{n} [x^i]\vec{a} \cdot [x^{n-i}]\vec{b}$

(4) $= \sum_{j=0}^{n} [x^{n-j}]\vec{a} \cdot [x^j]\vec{b}$.

Proof of Thm. 1. Most parts are strfwd.

(c) Associativity: $[x^n]((\vec{a}\vec{b})\vec{c}) \overset{(4)}{=} \sum_{j=0}^{n} \underbrace{[x^{n-j}](\vec{a}\vec{b})}_{\overset{(3)}{=} \sum_{i=0}^{n-j} [x^i]\vec{a}\cdot[x^{n-j-i}]\vec{b}} \cdot [x^j]\vec{c}$

$= \sum_{j=0}^{n} \sum_{i=0}^{n-j} [x^i]\vec{a} \cdot [x^{n-j-i}]\vec{b} \cdot [x^j]\vec{c}$,

whereas

$$[x^n]\left(\vec{a}\left(\vec{b}\,\vec{c}\right)\right) \overset{(3)}{=\!=} \sum_{i=0}^{n} [x^i]\vec{a} \cdot \underbrace{[x^{n-i}]\left(\vec{b}\,\vec{c}\right)}_{\overset{(4)}{=\!=} \sum_{j=0}^{n-i} [x^{n-i-j}]\vec{b} \cdot [x^j]\vec{c}}$$

$$= \sum_{i=0}^{n} \sum_{j=0}^{n-i} [x^i]\vec{a} \cdot [x^{n-i-j}]\vec{b} \cdot [x^j]\vec{c}.$$

The RHSes are equal, since

$$\sum_{j=0}^{n} \sum_{i=0}^{n-j} = \sum_{\substack{i,j \in \mathbb{N};\\ i+j \le n}} = \sum_{i=0}^{n} \sum_{j=0}^{n-i}$$

and $n-j-i = n-i-j$. Thus, $[x^n]\left(\left(\vec{a}\,\vec{b}\right)\vec{c}\right) = [x^n]\left(\vec{a}\left(\vec{b}\,\vec{c}\right)\right)$ $\forall n \in \mathbb{N}$. Hence, $\left(\vec{a}\,\vec{b}\right)\vec{c} = \vec{a}\left(\vec{b}\,\vec{c}\right)$, since a FPS is just the sequence of its coefficients. $\square$

Sometimes, ∞ sums also make sense in $K[[x]]$.

<u>Example:</u>

$$\begin{aligned} & (1,1,1,1,1,1,1,\ldots) \\ + & (0,1,1,1,1,1,1,\ldots) \\ + & (0,0,1,1,1,1,1,\ldots) \\ + & (0,0,0,1,1,1,1,\ldots) \\ + & \ldots \\ = & (1,2,3,4,5,6,7,8,\ldots) \end{aligned}$$

<u>Def.</u> A (possibly infinite) family $(\vec{a_i})_{i\in I}$ of FPSs is <u>summable</u> if

(5) $\forall n\in\mathbb{N}$, only finitely many $i\in I$ satisfy $[x^n]\vec{a_i}\neq 0$.

In this case, the sum $\sum_{i\in I}\vec{a_i}$ is defined as the FPS with

$$[x^n]\left(\sum_{i\in I}\vec{a_i}\right) = \underbrace{\sum_{i\in I}[x^n]\vec{a_i}}_{\substack{\text{a sum with only} \\ \text{finitely many} \\ \text{nonzero addends,} \\ \text{hence well-defined in } K}} \qquad \forall n\in\mathbb{N}.$$

Rmk.: (5) $\not\Leftrightarrow$ $\forall n \in \mathbb{N}$, infinitely many $i \in I$ satisfy $[x^n] a_i = 0$.

Prop. 2: Sums of summable families satisfy the usual rules for summation (as long as all families involved are summable), with the exception of "interchange of Σ signs":

EX. Here is this exception (in $\mathbb{Z}$):

Set $a_{i,j} = [i = j] - [i + 1 = j]$ for all $i, j \in \{1, 2, 3, \ldots\}$.

So

$$\begin{array}{c|cccccc} i \backslash j & 1 & 2 & 3 & 4 & 5 & \cdots \\ \hline 1 & 1 & -1 & & & & \\ 2 & & 1 & -1 & & & \\ 3 & & & 1 & -1 & & \\ 4 & & & & 1 & -1 & \\ \vdots & & & & & & \ddots \end{array}$$

Then $\sum\limits_{i=1}^{\infty} \underbrace{\sum\limits_{j=1}^{\infty} a_{i,j}}_{=0} = \sum\limits_{j} 0 = 0$ but $\sum\limits_{j=1}^{\infty} \underbrace{\sum\limits_{i=1}^{\infty} a_{i,j}}_{=[j=1]} = \sum\limits_{j} [j = 1] = 1$

But we can fix this if we require the whole family $(a_{i,j})_{(i,j)\in\mathbb{N}^2}$ to be summable.

"Discrete Fubini's theorem": If $(\overrightarrow{a_{i,j}})_{(i,j)\in I\times J}$ is a summable family of FPS, then

$$\sum_{i\in I}\sum_{j\in J}\overrightarrow{a_{i,j}} = \sum_{(i,j)\in I\times J}\overrightarrow{a_{i,j}} = \sum_{j\in J}\sum_{i\in I}\overrightarrow{a_{i,j}}.$$

Def. x denotes the FPS $(0,1,0,0,0,\ldots)$.

So this is our answer to "what is x?".

Prop. 3. $x^k = (\underbrace{0,0,\ldots,0}_{k \text{ zeroes}},1,0,0,0,\ldots)$ $\forall k\in\mathbb{N}$.

Proof. Induct over k, by observing that if $\vec{a}=(a_0,a_1,a_2,\ldots)$, then $x\vec{a}=(0,a_0,a_1,a_2,\ldots)$.

Cor. 4. Any FPS $(a_0,a_1,\ldots)\in K[[x]]$ satisfies

$$(a_0,a_1,\ldots) = a_0+a_1x+a_2x^2+\ldots = \sum_{n\in\mathbb{N}} a_n x^n.$$

In particular, the RHS is well-defined, i.e., the family $(a_n x^n)_{n\in\mathbb{N}}$ is summable.

Proof.

$$\begin{aligned} a_0 + a_1 x + a_2 x^2 + \cdots &= (a_0, 0, 0, 0, \ldots) \\ &+ (0, a_1, 0, 0, \ldots) \\ &+ (0, 0, a_2, 0, \ldots) \\ &+ (0, 0, 0, a_3, \ldots) \\ &= (a_0, a_1, a_2, a_3, \ldots). \end{aligned}$$

$\square$

No analysis was used to make these power series well-defined!

Now, we have learned

- what FPS are,
- how to do basic algebra with them $(+, -, \cdot)$,
- what x is,
- why we can compare coefficients (i.e., why $\sum_{n\in\mathbb{N}} a_n x^n = \sum_{n\in\mathbb{N}} b_n x^n$ implies $a_k = b_k \ \forall k$).

In particular, Example 3 is now justified.

We don't yet know:

- what we can substitute into a FPS;
- when & why can we do fancier algebra (like solving quadratic equations, or dividing).

So, the remaining examples are still hanging loose.

<u>Def.</u> Let $\vec{a} \in K[[x]]$. A <u>multiplicative inverse</u> of $\vec{a}$ means a FPS $\vec{b} \in K[[x]]$ such that $\vec{a}\vec{b} = \vec{b}\vec{a} = \underline{1}$.

<u>Thm. 5.</u> Let $\vec{a} \in K[[x]]$. Then, there is <u>at most 1</u> multiplicative inverse of $\vec{a}$.

<u>Proof.</u> Let $\vec{b}$ and $\vec{c}$ be two mult. inverses of $\vec{a}$. Then,

$$\vec{a}\vec{b} = \vec{b}\vec{a} = \underline{1} \quad \text{and} \quad \vec{a}\vec{c} = \vec{c}\vec{a} = \underline{1}.$$

Then,

$$\vec{b}(\vec{a}\vec{c}) = \vec{b}\,\underline{1} = \vec{b}$$

but

$$(\vec{b}\vec{a})\vec{c} = \underline{1}\,\vec{c} = \vec{c},$$

so by associativity the LHSs are equal. So the RHSs are equal,

so $\vec{b} = \vec{c}$. $\square$

<u>Def.</u> The mult. inverse of $\vec{a}$ (if it $\exists$) is called $\vec{a}^{-1}$ or $1/\vec{a}$.

Thm. 6. Let $a \in K[[x]]$. Then a has a multiplicative inverse $\iff$ $[x^0]a$ has a mult. inverse in K.

Rmk. What numbers have multiplicative inverses in K?

- If $K = \mathbb{Z}$, then $1, -1$ only.
- If $K = \mathbb{Q}$, then all nonzero numbers.
- If $K = \mathbb{R}$, then — // —
- If $K = \mathbb{C}$, then — // —

$\left(\frac{1}{a+bi} = \frac{a-bi}{a^2+b^2}. \right)$

$\Rightarrow \mathbb{Q}, \mathbb{R}, \mathbb{C}$ are fields

Proof of Thm. 6. $\Rightarrow$: easy:

Write a as $(a_0, a_1, a_2, \ldots)$.

We're assumed $\exists b = (b_0, b_1, b_2, \ldots)$ with $ab = 1$.

So $(1, 0, 0, \ldots) = 1 = ab = (a_0, a_1, \ldots)(b_0, b_1, \ldots)$

$= (a_0 b_0, \text{whatever})$.

So $1 = a_0 b_0 \Rightarrow \underset{[x^0]a}{\overset{\parallel}{a_0}}$ has a mult. inverse in K (namely, b_0). ✓

$\Leftarrow$: Write a as $a = (a_0, a_1, a_2, \ldots)$, and we're looking for a FPS $b = (b_0, b_1, b_2, \ldots)$ with $ab = 1$.

So we want

$$(1, 0, 0, 0, \ldots) = 1 = ab = (a_0, a_1, a_2, \ldots)(b_0, b_1, b_2, \ldots)$$
$$= (a_0 b_0,\ a_0 b_1 + a_1 b_0,\ a_0 b_2 + a_1 b_1 + a_2 b_0,\ \ldots).$$

So we want

$$\begin{cases} 1 = a_0 \underline{b_0}, \\ 0 = a_0 \underline{b_1} + a_1 b_0, \\ 0 = a_0 \underline{b_2} + a_1 b_1 + a_2 b_0, \\ 0 = a_0 \underline{b_3} + a_1 b_2 + a_2 b_1 + a_3 b_0, \\ \ldots \end{cases}$$

Solve this system by elimination: get b_0 from 1st eqn, then b_1 from the next, ...

This is possible, since $a_0 = [x^0]a$ has a mult. inverse, so it can be divided by. $\checkmark$

$\square$