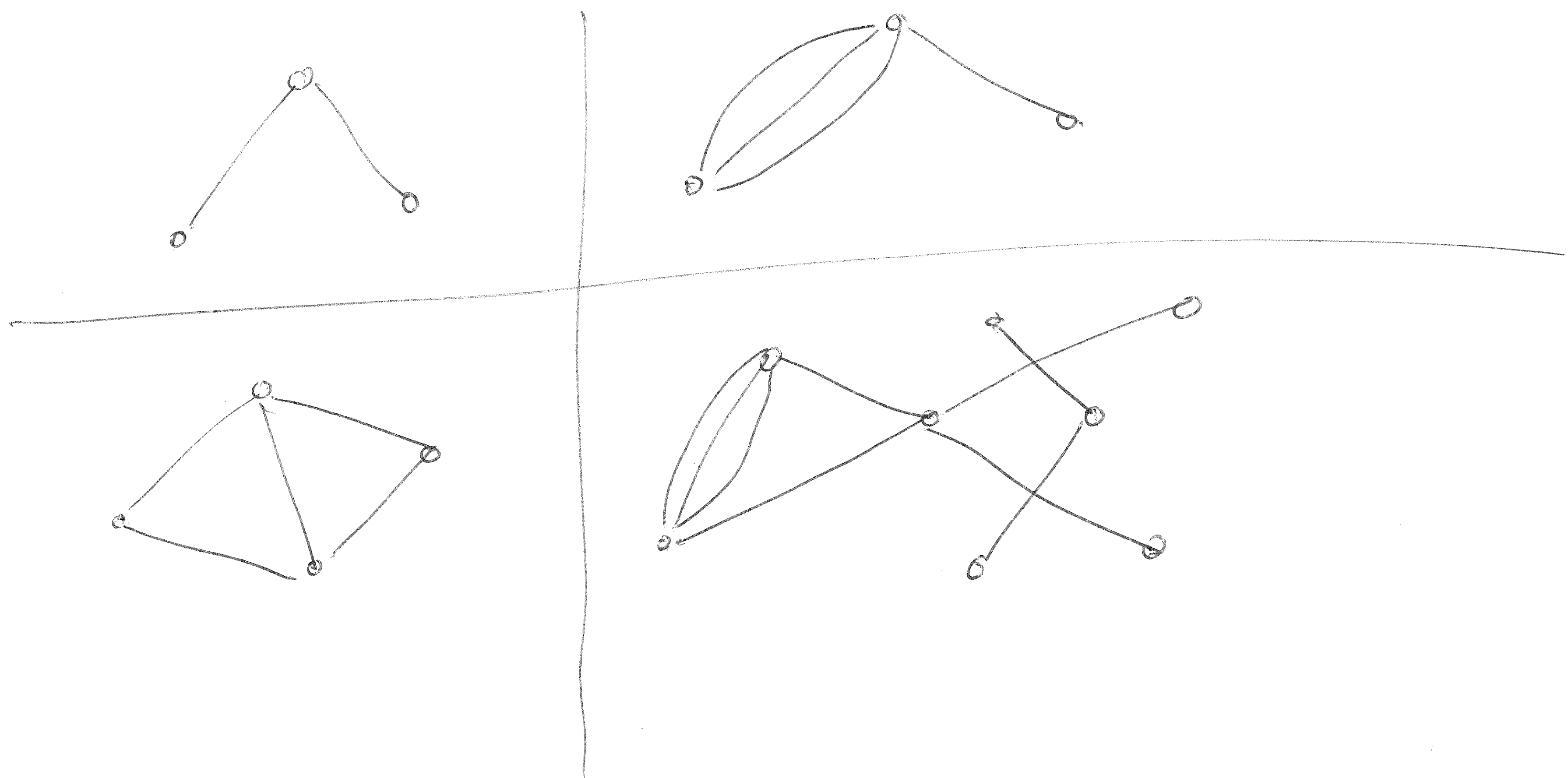
7. Graphs

7.1. Basics

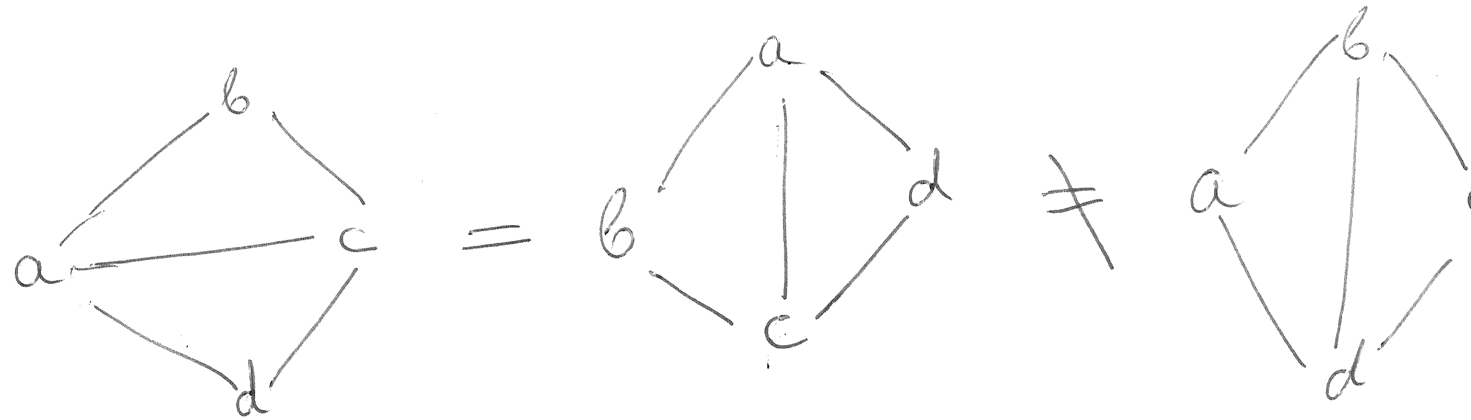
[LeLeMe, Ch. 12], [Galvin, §4-§8], [Guichard], nogra (my 5707 Spring 2017).

Idea: a graph is a collection of "vertices" and "edges", where ~~ex~~ each edge joins 2 vertices.



But the vertices are abstract objects, not points in plane; the pictures just visually represent the graphs.

~~Edg~~



Rigorous def:

Def. Let $\mathcal{P}_2(S)$ ~~be~~ denote ~~set~~ the set of all 2-element subsets of a given set S.

Def. A *graph* (or, better, *multigraph*) is a triple (V, E, φ), where V and E are finite sets, and $\varphi : E \to \mathcal{P}_2(V)$ is a map.

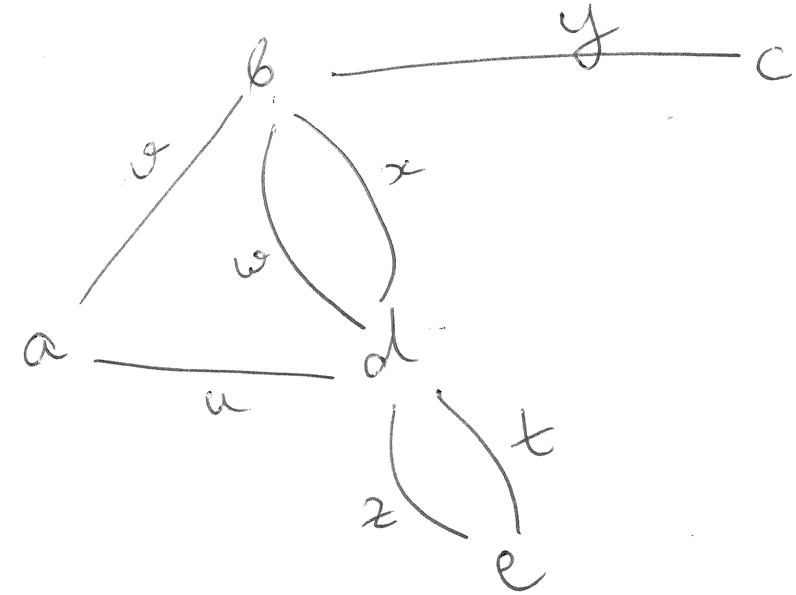
The *vertices* of (V, E, φ) are the elts of V.

The *edges* of (V, E, φ) are the elts of E.

For each edge e, the two elements of $\varphi(e)$ are called the *endpoints* of e, and we say that e *joins* these two elements.

An edge $e \in E$ contains / passes through a vertex v if $v \in \varphi(e)$.

Ex.



Here $V = \{a, b, c, d, e\}$,

$E = \{u, v, w, x, y, z, t\}$,

$\varphi(y) = \{b, c\}$,

$\varphi(z) = \{d, e\}$,

$\varphi(t) = \{d, e\}$,

$\varphi(v) = \{a, b\}$, ...

To be fully precise, these graphs are called multigraphs.
There is a notion of simple graphs, too, which are pairs (V, E) with $E \subseteq \mathcal{P}_2(V)$.

Note: Multigraphs ~~are able to~~ support "parallel edges" (= multiple edges joining the same 2 vertices), whereas simple graphs don't.

<u>Def.</u> Two vertices u and v of a graph are <u>adjacent</u> if there's an edge with endpoints u and v.

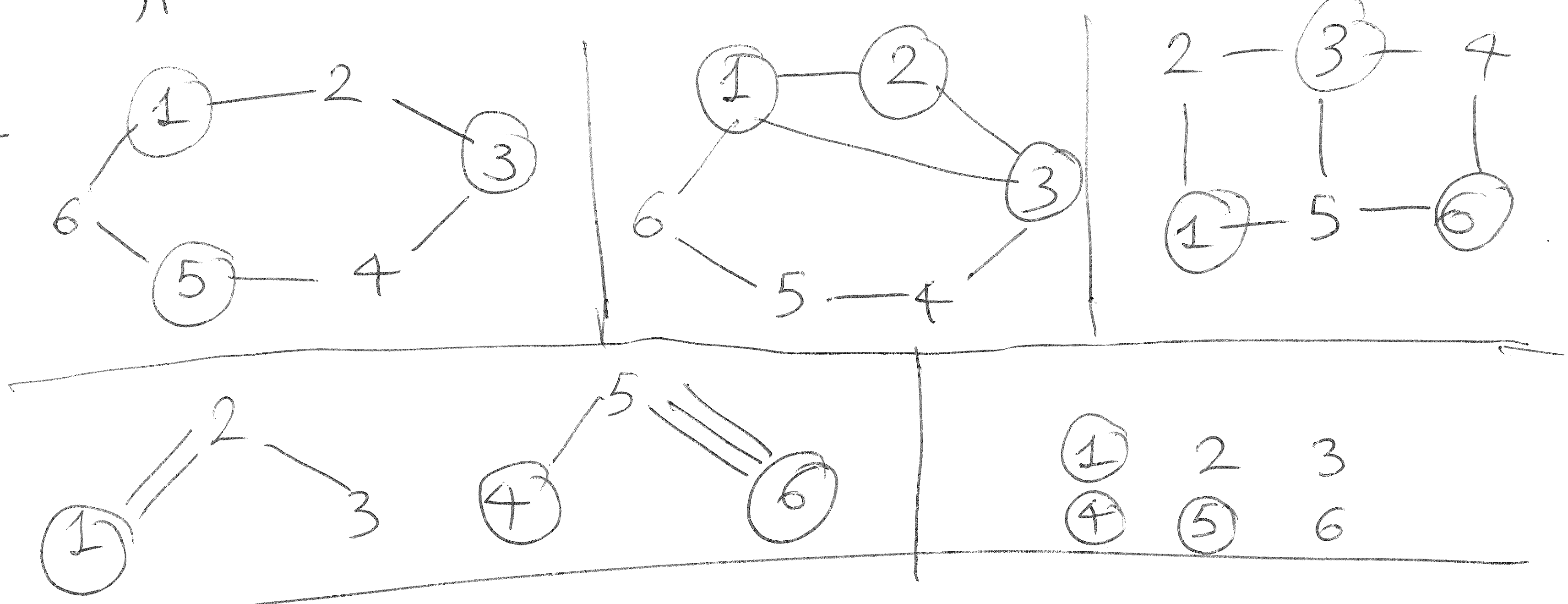
<u>Prop. 1. ("$R(3,3) \leq 6$").</u> Let $G = (V, E, \varphi)$ be a graph with ≥ 6 vertices. Then,

EITHER $\exists$ 3 distinct mutually adjacent vertices in G

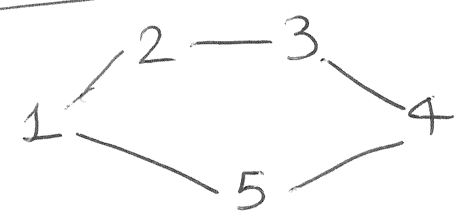
OR $\exists$ 3 distinct mutually non-adjacent vertices in G

(or both).

<u>Examples:</u>



<u>Non-example:</u>



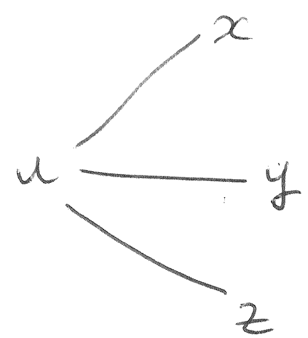
Def. A <u>neighbor</u> of a vertex u of a graph G is any vertex adjacent to u.

<u>Proof of Prop. 1.</u> Fix any vertex u of G.

Then, u has either ≥ 3 neighbors or ≥ 3 non-neighbors (not counting u), since there are ≥ 5 vertices $\neq u$.

CASE 1: u has ≥ 3 neighbors.

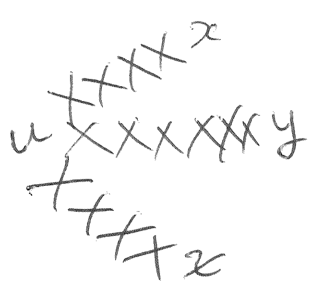
Choose 3 distinct neighbors x, y, z of u.



If two of x, y, z are adjacent to each other then we get 3 adjacent vertices.

If not, then we get 3 non-adjacent vertices.

CASE 2: u has ≥ 3 non-neighbors.



Same as in Case 1, except with the roles of "adjacent" and "non-adjacent" interchanged.

$\square$

More generally:

Prop. 2 (Ramsey's theorem for 2 colors). Let r, s be pos integers. Let G be a graph with $\geq \binom{r+s-2}{r-1}$ vertices.

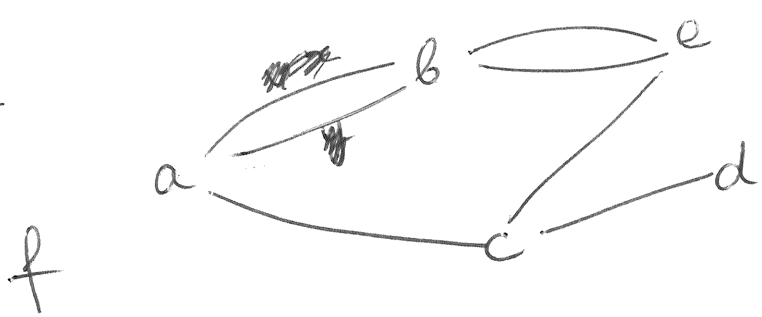
Then, EITHER $\exists$ r distinct mutually adjacent vertices
OR $\exists$ s distinct mutually non-adjacent vertices.

See texts on Ramsey theory.
$\binom{r+s-2}{r-1}$ is usually not optimal bound.

7.2. DEGREES

Def. The degree ~~of a vert~~ $\deg v$ of a vertex v of a graph G is the # of edges containing v.

Example:



$\deg a = 3$, $\deg b = 4$,
$\deg c = 3$, $\deg d = 1$,
$\deg e = 3$, $\deg f = 0$.

<u>Prop. 3 ("handshaking lemma").</u> Let $G = (V, E, \varphi)$ be a graph.

Then, $\sum_{v \in V} \deg v = 2|E|$.

<u>Proof.</u>

$$\sum_{v \in V} \underbrace{\deg v}_{= \#(\text{edges } e \text{ such that } v \in \varphi(e))}$$

$$= \sum_{v \in V} \#(\text{edges } e \text{ such that } v \in \varphi(e))$$

$$= \#(\text{pairs } (v, e) \text{ with } v \in \varphi(e))$$

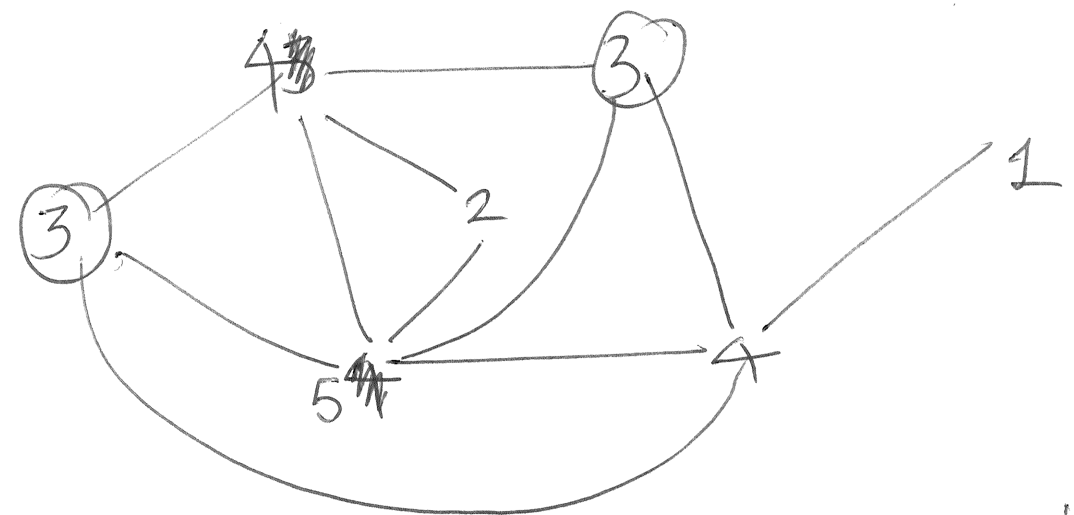
$$= \sum_{e \in E} \underbrace{\#(\text{vertices } v \text{ with } v \in \varphi(e))}_{=2} = \sum_{e \in E} 2 = 2|E|. \qquad \square$$

<u>Prop. 4.</u> Let $G = (V, E)$ be a <u>simple</u> graph (i.e., no parallel edges) with ≥ 2 vertices.

Then, G has 2 vertices of equal degree.

(I regard simple graphs (V, E) as multigraphs $(V, E, \text{"id"})$.)

Ex:



Proof of Prop. 4. Assume the contrary. Let $n = |V|$.

Thus, the map $\deg : V \to \{0, 1, \ldots, n-1\}$ is injective.

Hence, this map is bijective (by pigeonhole, since V and $\{0, 1, \ldots, n-1\}$ both have size n).

Thus, $\exists$ vertex p with $\deg p = 0$,

and $\exists$ vertex q with $\deg q = n-1$.

Note $p \neq q$, since $n = |V| \geq 2$.

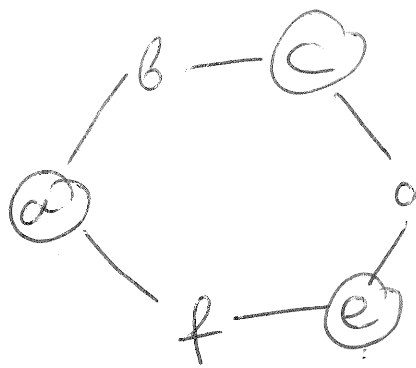
Thus, q is adjacent to each vertex, in particular to p.

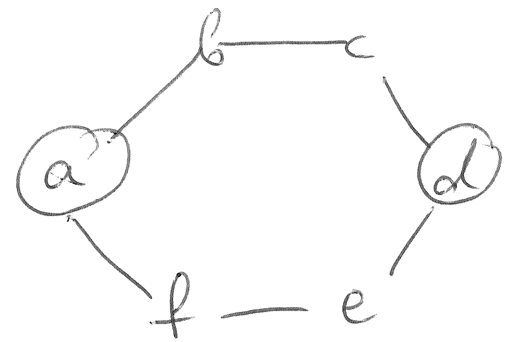
$\Rightarrow \deg p \geq 1$. $\lightning$ $\deg p = 0$. $\square$

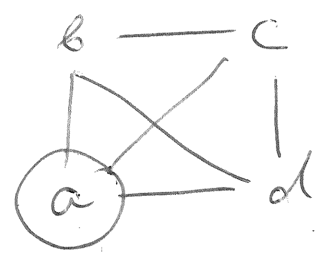
7.3. DOMINATING SETS

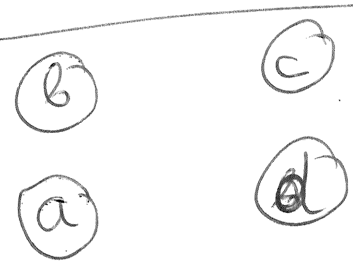
Def. A dominating set of a graph $G=(V,E,\varphi)$ is a subset S of V such that each vertex not in S has a neighbor in S.

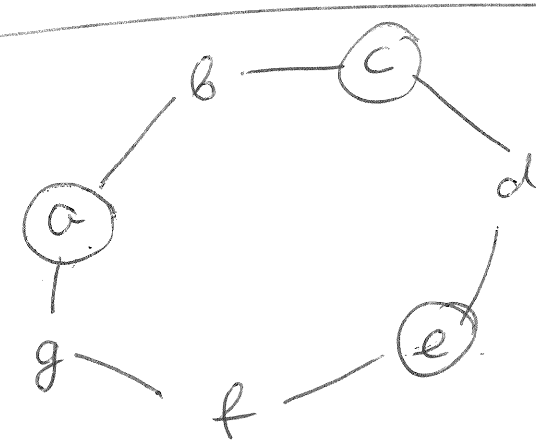
Exa:











Prop. 5. Let $G = (V, E, \varphi)$ be a graph such that each vertex v has $\deg v \geq 1$.

(a) There are two disjoint dominating sets of G with union V.

(b) $\exists$ dominating set of size $\leq |V|/2$.

Proof. (a) Define an *independent set* of G to be a set of mutually non-adjacent vertices.

Pick a maximum-size independent set A of G.

Let $B = V \setminus A$.

Claim that A and B are disjoint and dominating.

Of course $A \cup B = V$.

(b) follows from (a). $\square$

Thm. 6. (Brouwer 2009). Let G be a graph. Then, # of dominating sets of G is odd.

See [Enogra (5707 notes), Ch. 3].

7.4. PATHS, WALKS, CONNECTIVITY

Def. Let $G = (V, E, \varphi)$ be a graph. Let $p, q \in V$.

(a) A walk from p to q means a list $(v_0, e_1, v_1, e_2, v_2, \ldots, v_{k-1}, e_k, v_k)$, where $v_i \in V$, $e_i \in E$, $\varphi(e_i) = \{v_{i-1}, v_i\}$, $v_0 = p$, $v_k = q$.

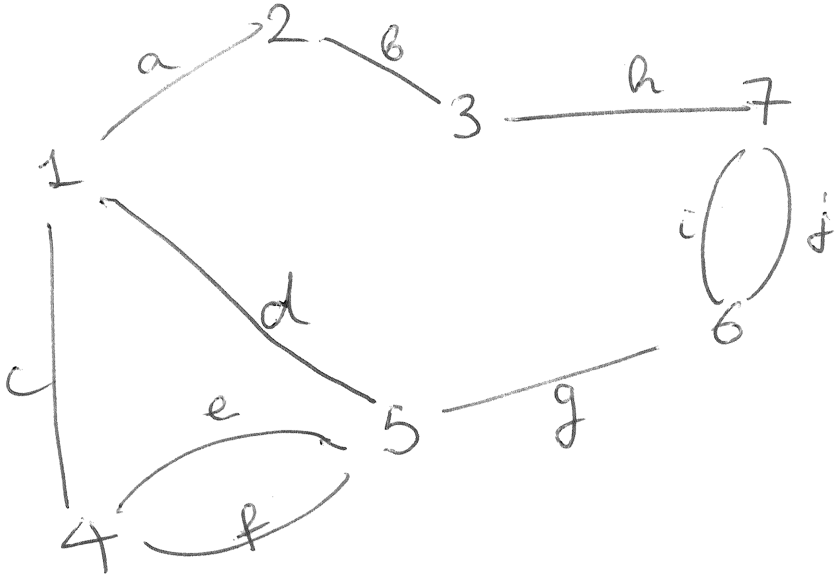
~~(b) A walk~~ (We abbreviate "walk from p to q" as "walk $p \to q$".)

(b) A walk is called a path if all v_i are distinct.

(c) A walk $p \to q$ is called a circuit (or closed walk) if $p = q$.

(d) A walk $p \to q$ is called a cycle if $p = q$, $e_1, e_2, \ldots, e_k$ are distinct, but $v_0, v_1, \ldots, v_{k-1}$ are distinct, and $k \geq 1$.

Ex2:



$(1, a, 2, b, 3, h, 7, i, 6, j, 7)$ is a walk $1 \to 7$, but not a path.

$(1, c, 4, f, 5)$ is a path $1 \to 5$.

$(4, e, 5, f, 4)$ is a circuit & a cycle.

$(4, e, 5, e, 4)$ is a circuit but not a cycle.

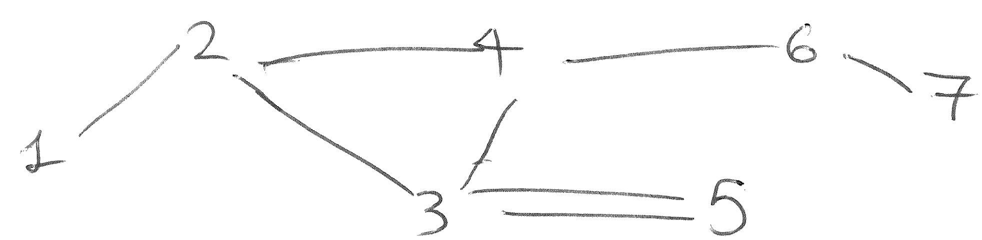
(3) is a circuit, ~~but~~ and a path $3 \to 3$, but not a cycle.

Prop. 7. Let u, v be two vertices of a graph G.
If $\exists$ a walk $u \to v$, then $\exists$ path $u \to v$.

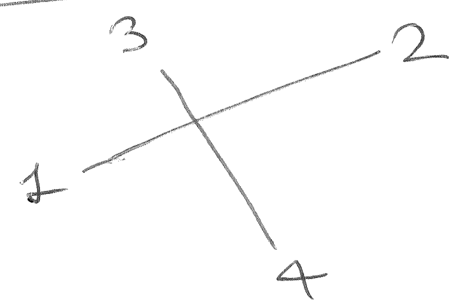
Proof. A walk that is not a path can be shortened by removing a circuit.

<u>Def.</u> A graph G is <u>connected</u> if $\forall$ vertices u, v $\exists$ walk $u \to v$, and $\exists$ at least 1 vertex.

<u>Ex:</u>



is connected

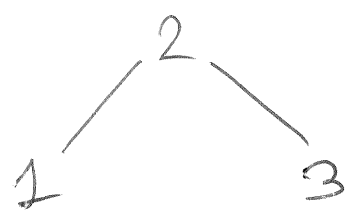


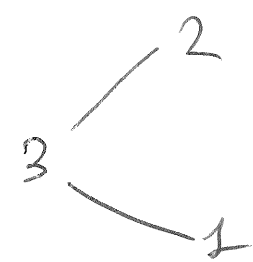
is not connected

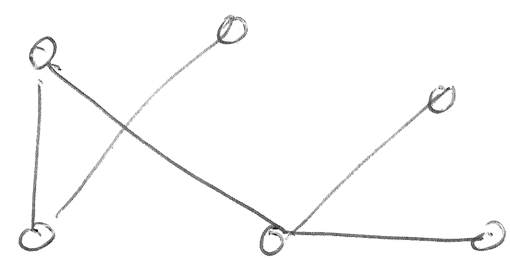
7.5. TREES

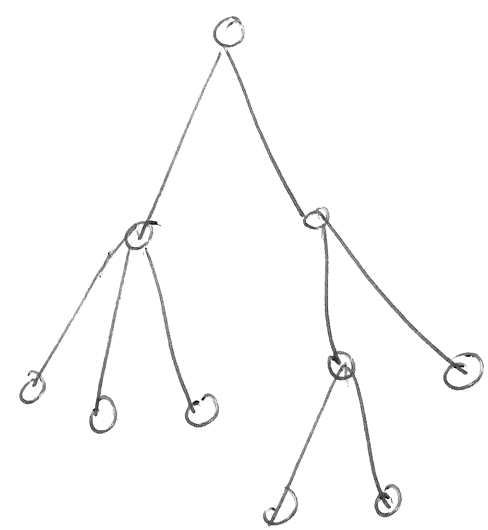
<u>Def.</u> A <u>tree</u> is a connected graph that has no cycles.

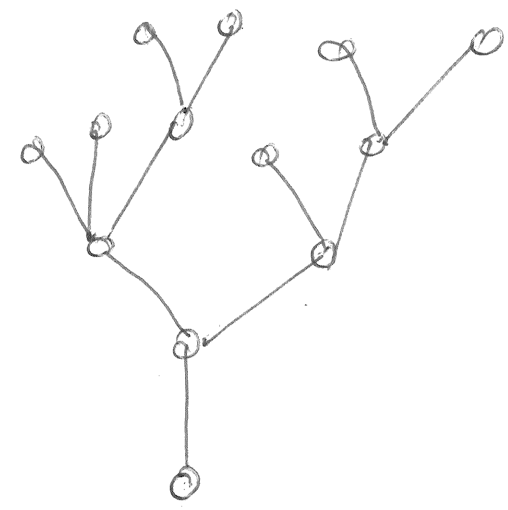
<u>Ex.</u>











Thm. 8. (tree equivalence thm.) Let $G=(V,E,\varphi)$ be a graph.

TFAE (= the following are equivalent):

T_1: G is a tree.

T_2: $V\neq\emptyset$, and $\forall\ u\in V$ and $v\in V$ $\exists$ unique path $u\to v$,

T_3: $V\neq\emptyset$, and $\forall\ u\in V$ and $v\in V$
$\exists$ unique backtrack-free walk $u\to v$
(= walk $u\to v$ with no ~~two~~ two consecutive edges identical).

T_4: G is connected, and $|E|=|V|-1$.

$\underline{T_5}$: G is connected, and $|E| < |V|$.

$\underline{T_6}$: G has no cycles, but adding any new edge produces a cycle.

$\underline{T_7}$: G is connected, but removing any edge disconnects it.

$\underline{T_8}$: EITHER $|V| = 1$
OR $\exists\, v \in V$ with $\deg v = 1$ and such that removing v from G (along with edge containing it) produces a tree.

$\underline{T_9}$: G has no cycles, and $|E| \geq |V| - 1$ and $V = \emptyset$,

__Proof.__ See Thm. 13 in 5707 Spr '17.

__Thm. 9. (Cayley).__ Let n be a positive integer. Then, # of trees ~~on vert~~ with vertex set $[n]$ (as simple graphs) is n^{n-2}.

[__Proofs:__ [Galvin, §4-§8]]