6*. PARTITIONS

6.1. BASICS

<u>Recall</u>: ~~A~~ A <u>partition</u> of $n \in \mathbb{Z}$ is a weakly decreasing tuple of pos. ints. with sum n.

The entries of a partition are called its <u>parts</u>.

$p(n) := \#$ partitions of n.

$p_k(n) := \#$ partitions of n into k parts.

We have proven: $p_k(n) = p_k(n-k) + p_{k-1}(n-1)$.

$\forall k \geq 1 \ \forall n \in \mathbb{N}$.

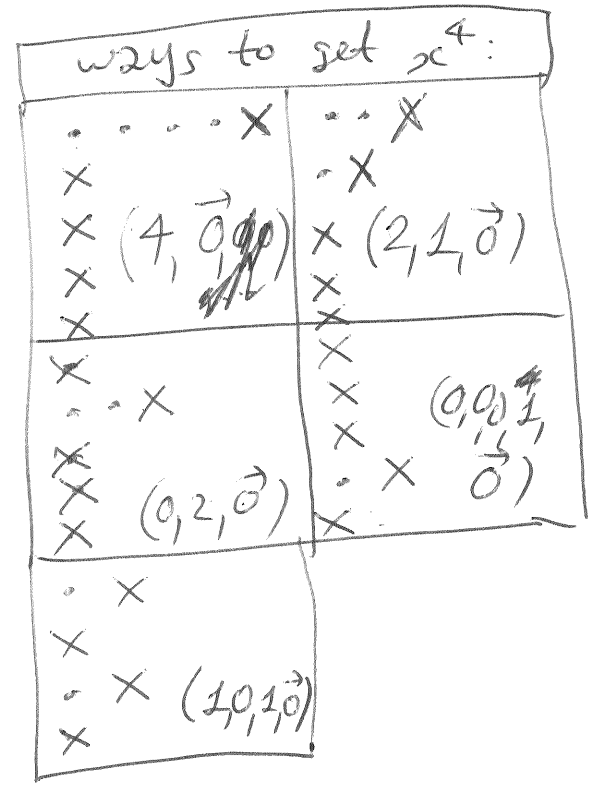
Also, $p(n) = p_0(n) + p_1(n) + \cdots + p_n(n)$,

<u>Thm. 1.</u> $\sum_{n \geq 0} p(n) x^n = \prod_{k=1}^{\infty} \frac{1}{1-x^k}$.

<u>Example. / proof.</u> $\prod_{k=1}^{\infty} \frac{1}{1-x^k} = \prod_{k=1}^{\infty} \left(1 + x^k + (x^k)^2 + (x^k)^3 + \cdots\right)$

$$= \prod_{k=1}^{\infty} (1 + x^k + x^{2k} + x^{3k} + \dots)$$

$$= (1 + x + x^2 + x^3 + x^4 + \dots)$$
$$(1 + x^2 + x^4 + x^6 + \dots)$$
$$(1 + x^3 + x^6 + x^9 + \dots)$$
$$(1 + x^4 + x^8 + x^{12} + \dots)$$
$$(1 + x^5 + x^{10} + x^{15} + \dots)$$
$$\dots$$



$$= 1 + x + 2x^2 + 3x^3 + 5x^4 + \dots$$

What is the ~~x^n term~~ coeff. of x^n?
It is the # of ways to assemble x^n from an addend picked from each factor.
IOW: it is the # of all $(m_1, m_2, m_3, \dots) \in \mathbb{N}^\infty$ such that $1m_1 + 2m_2 + 3m_3 + \dots = n$.

Now,

$$\{\text{partitions of } n\} \to \{(m_1, m_2, m_3, \dots) \in \mathbb{N}^\infty \mid 1m_1 + 2m_2 + 3m_3 + \dots = n\},$$

$$\lambda \longmapsto (\#\text{of parts } 1 \text{ in } \lambda,\ \#\text{of parts } 2 \text{ in } \lambda,\ \#\text{of parts } 3 \text{ in } \lambda,\ \dots).$$

is a bijection.

E.g. for $n=4$:

(4), $(3,1)$, $(2,2)$, $(2,1,1)$, $(1,1,1,1)$

$\longmapsto (0,0,0,1,\vec{0})$, $(1,0,1,\vec{0})$, $(0,2,\vec{0})$, $(2,1,\vec{0})$, $(4,\vec{0})$.

$\square$

Next, a result of Euler.

<u>Def.</u> $p_{odd}(n) :=$ # of partitions of n into odd parts.

$p_{dist}(n) :=$ # of partitions of n into distinct parts.

<u>Ex.</u> $p_{odd}(7) = |\{(7), (3,3,1), (3,1,1,1,1), (5,1,1), (1,1,1,1,1,1,1)\}| = 5$,

$p_{dist}(7) = |\{(7), (6,1), (5,2), (4,3), (4,2,1)\}| = 5$.

Thm. 2 (Euler). $p_{odd}(n) = p_{dist}(n) \quad \forall n \in \mathbb{N}$.

1st proof.
$$\sum_{n \geq 0} p_{odd}(n) x^n = \prod_{\substack{k \geq 1 \\ odd}} \frac{1}{1-x^k}.$$

$$\sum_{n \geq 0} p_{dist}(n) x^n = \prod_{k \geq 1} (1+x^k).$$

Goal:
$$\prod_{\substack{k \geq 1 \\ odd}} \frac{1}{1-x^k} = \prod_{k \geq 1} (1+x^k). \qquad (1)$$

Warmup:
$$\frac{1}{1-x} = (1+x)(1+x^2)(1+x^4)(1+x^8)\cdots \qquad (2)$$

[1st proof of (2):
$$RHS = \sum_k x^k \cdot \underbrace{(\#\text{ of ways write } k \text{ as a sum of distinct powers of } 2)}_{=1}$$
$$= \sum_k x^k = \frac{1}{1-x}.$$

2nd proof of (2):
$$\underbrace{(1-x) \cdot (1+x)}(1+x^2)(1+x^4)(1+x^8)\cdots$$
$$= \underbrace{(1-x^2) \cdot (1+x^2)}(1+x^4)(1+x^8)\cdots$$

$$= \frac{\underbrace{(1-x^4)\cdot(1+x^4)}(1+x^8)\cdots}{(1-x^8)(1+x^8)\cdots}$$

$$=$$

$$= \cdots = 1.\ ]$$

In other words, $\displaystyle \frac{1}{1-x} = \prod_{i\geq 0} \left(1+x^{2^i}\right)$.

Thus, $$\prod_{\substack{k\geq 1\\ \text{odd}}} \frac{1}{1-x^k} = \prod_{\substack{k\geq 1\\ \text{odd}}} \prod_{i\geq 0} \left(1+(x^k)^{2^i}\right) = \prod_{\substack{k\geq 1\\ \text{odd}}} \prod_{i\geq 0} \left(1+x^{2^i k}\right)$$

$$= \prod_{m\geq 1} (1+x^m)$$ (since each positive integer m can be uniquely written as $2^i k$ for $k \geq 1$ odd and $i \geq 0$)

$$= \prod_{k\geq 1} (1+x^k).$$

$\square$

2nd proof. Construct a bijection

A: {partitions of n into odd parts} $\to$ {partitions of n into distinct parts}, which transforms a partition by repeatedly merging 2 equal parts until no more equal parts can be found.

Ex: $(5,5,3,1,1,1) \rightsquigarrow (10,3,1,1,1) \rightsquigarrow (10,3,2,1)$

Ex: $(5,3,1,1,1,1) \rightsquigarrow (5,3,2,1,1) \rightsquigarrow (5,3,2,2) \rightsquigarrow (5,4,3)$

Conversely, A^{-1} transforms a ~~bijection~~ partition by repeatedly splitting even parts into 2 equal pieces. $\square$

6.2. EULER'S ⬠ # THM. [Ueckerdt, §4.2]

Def. $\forall k \in \mathbb{Z}$ define $\omega_k \in \mathbb{N}$ by $\omega_k = \frac{(3k-1)k}{2}$.

This is called a pentagonal number.

Thm. 4. (Euler's pentagonal number thm.)

$$\prod_{k=1}^{\infty} (1-x^k) = \sum_{k \in \mathbb{Z}} (-1)^k x^{\omega_k}.$$

Thus,

$$\prod_{k=1}^{\infty}(1-x^k) = \cdots + x^{26} - x^{15} + x^7 - x^2 + 1 - x + x^5 - x^{12} + x^{22} - x^{35} \pm \cdots$$

$$= 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - x^{35} \pm \cdots$$

Cor. 5. $p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + p(n-12) + p(n-15) \pm \cdots$

$$= \sum_{k \neq 0} (-1)^{k-1} p(n - \omega_k) \qquad \forall n > 0.$$

Proof of Cor. 5. using Thm. 4.

$$\underbrace{\left(\sum_{n\geq 0} p(n) x^n\right)}_{= \prod_{k\geq 1} \frac{1}{1-x^k} \text{ by Thm. 1}} \underbrace{\left(\sum_{k\in \mathbb{Z}} (-1)^k x^{\omega_k}\right)}_{= \prod_{k\geq 1}(1-x^k) \text{ by Thm. 4}} = 1.$$

Now, the x^n - coefficient on the LHS is

$$\sum_{\substack{m \geq 0, \\ k \in \mathbb{Z}; \\ m + \omega_k = n}} p(m)\,(-1)^k = \sum_{k \in \mathbb{Z}} p(n-\omega_k)(-1)^k$$

$$= \sum_{k \in \mathbb{Z}} (-1)^k\, p(n-\omega_k)$$

$$= \cancel{(-1)^0}\, p(n-\cancel{\omega_0}) + \sum_{k \neq 0} (-1)^k\, p(n-\omega_k).$$

~~Solve for p(n),~~

But (for $n > 0$) the x^n - coeff. on the RHS is 0.

Comparing yields

$$p(n) + \sum_{k \neq 0} (-1)^k\, p(n-\omega_k) = 0.$$

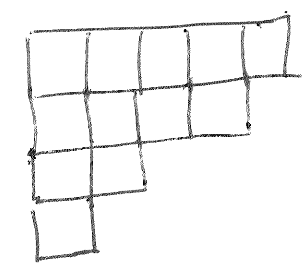
Solve for $p(n)$. $\square$

Proof of Thm. 4. (following Ueckerdt)

Let $SP_n = \{\text{partitions of } n \text{ into distinct parts}\}$.

Every $\lambda = (\lambda_1, \dots, \lambda_k) \in SP_n$ can be represented as a Young diagram (a table with k rows, left-aligned, with the i-th row having λ_i boxes).

For example: $(5, 4, 2, 1)$ is represented by



For $\lambda \in SP_n$, each row is shorter than the row above it.

Define $\ell(\lambda)$ to be the number of parts of λ.

1st step. $[x^n](\mathrm{LHS}) = \sum_{\lambda \in SP_n} (-1)^{\ell(\lambda)}$.

(Remember $[x^n]\left(\prod_{k=1}^{\infty}(1+x^k)\right) = |SP_n|$.

This is similar.)

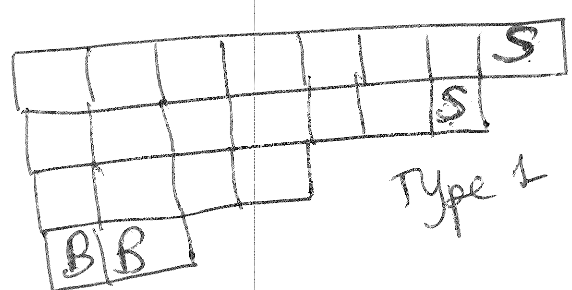
$$\boxed{\begin{aligned} \mathrm{LHS} &= \prod_{k=1}^{\infty} (1-x^k) \\ \mathrm{RHS} &= \sum_{k \in \mathbb{Z}} (-1)^k x^{\omega_k} \end{aligned}}$$

<u>2nd step:</u> ω_k are distinct,

$(\omega_k = \omega_\ell \Rightarrow k = \ell$ for $k, \ell \in \mathbb{Z}.)$

<u>3rd step:</u> $[x^n](\mathrm{RHS}) = \begin{cases} (-1)^k, & \text{if } n = \omega_k \text{ for some } k \in \mathbb{Z}; \\ 0, & \text{otherwise} \end{cases}$.

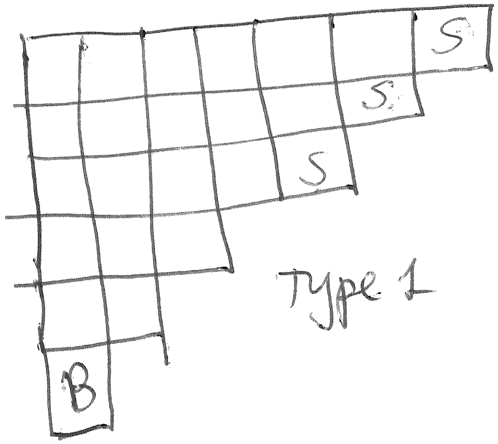
<u>4th step:</u> For any $\lambda \in SP_n$, draw the Young diagram of λ:



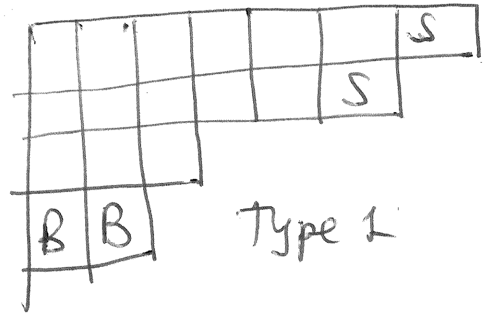
Type 1

Define the <u>base</u> ("B") of λ as the bottommost row of this diagram.

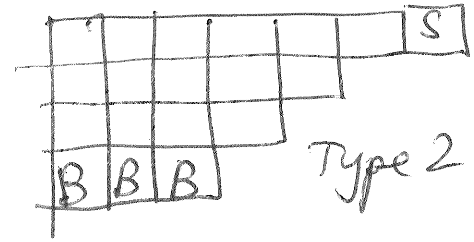
Define the <u>slope</u> ("S") of λ as the boxes starting with the rightmost box and the boxes due SW of it.



Type 1



Type 1



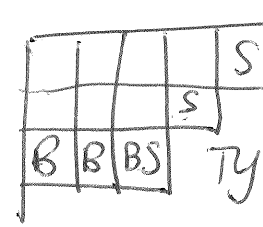
Type 2

Now, classify partitions $\lambda \in SP_n$ into 3 types.

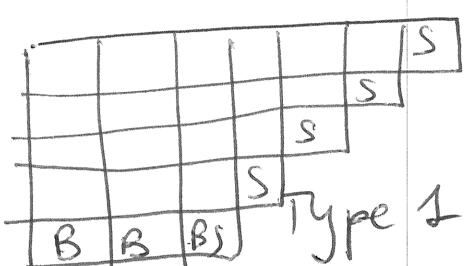
<u>Type 1:</u> $|S| \geq |B| + |S \cap B|$

<u>Type 2:</u> $|S| < |B| - |S \cap B|$

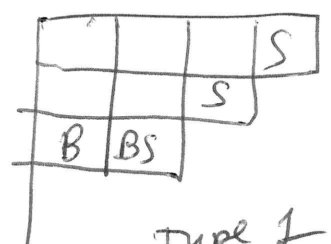
<u>Type 3:</u> $|B| - |S \cap B| \leq |S| < |B| + |S \cap B|$



Type 3



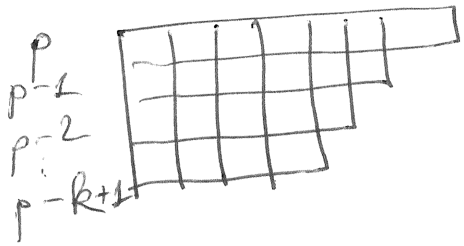
Type 1



Type 1

<u>5th step</u> <u>Claim:</u> (1) Type 3 partitions exist only if $n = \omega_k$ for some $k \in \mathbb{Z}$.

(2) If $n = \omega_k$, then there is exactly one Type 3 partition.



$$n = p + (p-1) + \dots + (p-k+1) = pk - \binom{k}{2}$$

$$|S| = k, \quad |B| = p-k+1, \quad |S \cap B| = 1$$

$$p-k+\cancel{1}+1 \leq k < p-k+1+1$$

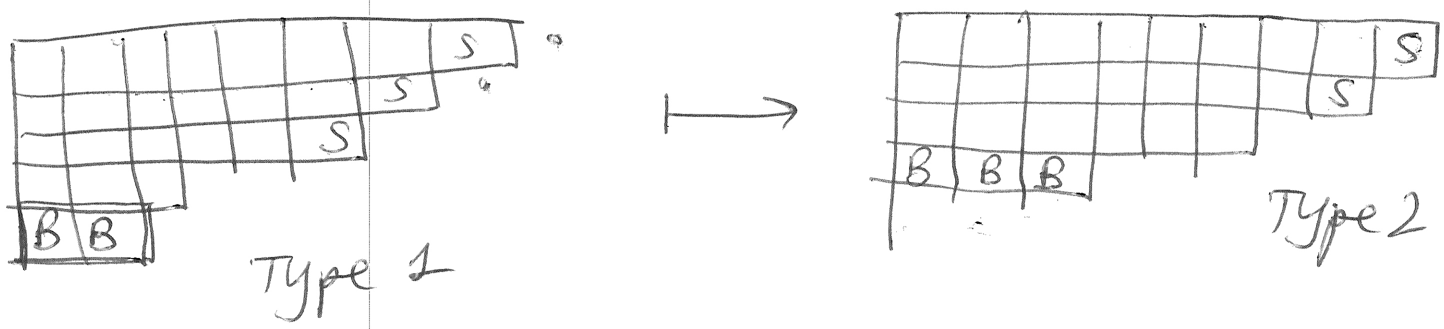
$$p \leq 2k < p+2$$

$$p \in \{2k, 2k-1\}$$

6th step: There is a bijection

$\{\text{Type 1 partitions}\} \longrightarrow \{\text{Type 2 partitions}\}$,

turn the base into a new slope



(inverse map turns the slope into a new base).

This bijection removes 1 part from the partition.

Hence, $$\sum_{\substack{\lambda \in SP_n \\ \text{of Type 1}}} (-1)^{\ell(\lambda)} = \sum_{\substack{\lambda \in SP_n \\ \text{of Type 2}}} (-1)^{\ell(\lambda)+1} = -\sum_{\substack{\lambda \in SP_n \\ \text{of Type 2}}} (-1)^{\ell(\lambda)}.$$

7th step: 1st step yields

$$[x^n)(\mathrm{LHS}) = \sum_{\lambda \in SP_n} (-1)^{\ell(\lambda)} = \underbrace{\sum_{\substack{\lambda \in SP_n \\ \text{of Type 1}}} (-1)^{\ell(\lambda)} + \sum_{\substack{\lambda \in SP_n \\ \text{of Type 2}}} (-1)^{\ell(\lambda)}}_{\substack{=0 \\ \text{(by 6th step)}}} + \sum_{\substack{\lambda \in SP_n \\ \text{of Type 3}}} (-1)^{\ell(\lambda)}$$

$$= \sum_{\substack{\lambda \in SP_n \\ \text{of Type 3}}} (-1)^{\ell(\lambda)} = \begin{cases} (-1)^{\ell(\lambda)}, & \text{if } n = \omega_k \text{ for some } k \in \mathbb{Z} \\ 0, & \text{otherwise} \end{cases}$$

(where λ is the unique Type-3 partition)

$$= \begin{cases} (-1)^k, & \text{if } \ldots \\ 0, & \ldots \end{cases} = [x^n)(\mathrm{RHS}),$$

$\Rightarrow$ LHS = RHS,

$\square$