(1)

Def. If $a \in K[[x]]$ has a mult. inverse a^{-1}, then we can define a^{-n} for each $n \in \mathbb{N}$ by setting $a^{-n} = (a^{-1})^n$.

So the powers a^k are defined $\forall k \in \mathbb{Z}$.

Thm. 7. The FPS $1-x$ has a mult. inverse, which is

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

Proof.

$$(1-x)(1+x+x^2+x^3+\cdots) = (1+\cancel{x}+\cancel{x^2}+\cancel{x^3}+\cdots) - (\cancel{x}+\cancel{x^2}+\cancel{x^3}+\cdots) = 1. \quad \blacksquare$$

Alternatively:

$$(1-x)(1+x+x^2+x^3+\cdots) = (1-\cancel{x}) + (\cancel{x}-\cancel{x^2}) + (\cancel{x^2}-\cancel{x^3}) + (\cancel{x^3}-\cancel{x^4}) + \cdots = 1. \quad \square$$

Thm. 8.

$$(1+x)^n = \underbrace{\sum_k \binom{n}{k} x^k}_{\text{actually an } \sim\sim \text{[illegible]} \sim\sim \text{ sum if } n<0} \qquad \forall n \in \mathbb{Z}.$$

("Newton's binomial theorem")

Proof idea: For $n \geq 0$, this follows from regular bin. theorem.

For $n = -1$, this boils down to Thm. 7 (with $-x$ repl by x).

Lem. 9. $(1-x)^{-n} = \sum_k \binom{n+k-1}{k} x^k \quad \forall n \in \mathbb{N}.$

Proof idea for Lem. 9: Induction over n.

Need to check: $$\left(\sum_k \binom{n+k-1}{k} x^k\right) \cdot \underbrace{\frac{1}{1-x}}_{\substack{=\sum_\ell x^\ell \\ \text{(by Thm. 7)}}} = \sum_k \binom{n+k}{k} x^k$$

This easily follow from Pascal's recurrence. $\square$

Now, we want to prove Thm. 8 using Lem. 9 by substituting $-x$ for x. What does "substituting" mean?
We need to define substitution.

~~Def~~ Prop. 9. Let f, g be FPS with $[x^0]g = 0$. (That is, $g = g_1 x + g_2 x^2 + \ldots$)
Then the FPS $f[g]$ is defined as follows:
Write f as $f = \sum_{n \geq 0} f_n \cdot x^n$, and set $f[g] = \sum_{n \geq 0} f_n g^n$.

This is well-defined, e.g. the family $(f_n g^n)_{n \in \mathbb{N}}$ is summable.

Ex2. We can substitute $x+x^2$ for x into $1+x+x^2+\dots$. The result is

$$1 + (x+x^2) + (x+x^2)^2 + (x+x^2)^3 + (x+x^2)^4 + \dots$$

$$= 1 + x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + \dots$$

$$= \sum_{n \geq 0} f_{n+1} x^n \quad \text{where } f_i = \text{Fibonacci \#s.}$$

This is because $1+x+x^2+\dots = \dfrac{1}{1-x}$, so

$$1 + (x+x^2) + (x+x^2)^2 + \dots = \frac{1}{1-(x+x^2)} = \frac{1}{1-x-x^2} \overset{\text{Ex.1}}{\underset{\text{Eq.1}}{=}} \sum_n f_{n+1} x^n .$$

Prop. 10. Substitution satisfies the rules you'd expect:

e.g. $f[g[h]] = (f[g])[h]$.

(We can restate it as $f \circ (g \circ h) = (f \circ g) \circ h$ where $f \circ g = f[g]$).

(see Loehr for details)

$\rightsquigarrow$ Thm. 8 (after a while).

This justifies Ex. 1.

Rmk. A polynomial is a FPS $(a_0, a_1, \dots)$ such that all but finitely many $i \in \mathbb{N}$ satisfy $a_i = 0$.

To justify Ex. 2, we need to make sense of $(1+x)^n$ for $n \notin \mathbb{Z}$.

Option 1: define $(1+x)^n$ as $\sum_k \binom{n}{k} x^k$.

But then, we would have ~~the~~ to prove all the rules of exponents:

$$(1+x)^n (1+x)^m = (1+x)^{n+m};$$

$$\left((1+x)^n\right)^m = (1+x)^{nm},$$

$$\dots$$

Option 2: define $(1+x)^n$ as $\exp\left[n \log(1+x)\right]$.

What are exp & log?

$$\exp = \sum_{n \geq 0} \frac{1}{n!} x^n, \qquad \log(1+x) = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n} x^n.$$

Same issues hold, but easier to deal with.

After some work, Ex. 2 becomes justified.

4.3. ONE MORE EXAMPLE FOR FPS

<u>Ex.</u> $(1-x)(1+x) = 1-x^2$. (Let $n \in \mathbb{N}$.)

$$\Longrightarrow \underbrace{(1-x)^n}_{=\sum_k (-1)^k \binom{n}{k} x^k} \underbrace{(1+x)^n}_{=\sum_k \binom{n}{k} x^k} = \underbrace{(1-x^2)^n}_{=\sum_k (-1)^k \binom{n}{k} x^{2k}}$$

$$\Longrightarrow \left(\sum_k (-1)^k \binom{n}{k} x^k\right)\left(\sum_k \binom{n}{k} x^k\right) = \sum_k (-1)^k \binom{n}{k} x^{2k}.$$

Now take the ~~coeff~~ x^m-coeff. for a given $m \in \mathbb{N}$,

$$\Longrightarrow \boxed{\sum_{a=0}^{m} (-1)^a \binom{n}{a}\binom{n}{m-a} = \begin{cases} (-1)^{m/2}\binom{n}{m/2} & \text{if } m \text{ is even} \\ 0 & \text{if } m \text{ is odd} \end{cases}}.$$

For example, if $m=n$, then this simplifies to

$$\sum_{a=0}^{n} (-1)^a \binom{n}{a}^2 = \begin{cases} (-1)^{n/2}\binom{n}{n/2} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}.$$

More in [Loehr], [Galvin], etc.

5. PERMUTATIONS

5.1. LENGTH & INVERSIONS

See [detnotes, Ch. 4].

Recall: · a perm. of a set X is a bij. $X \to X$.

Def. Given $n \in \mathbb{N}$. Let S_n denote ~~the~~ set of all perms of $[n]$.

This S_n is closed under composition & inverses.

Def. Let $n \in \mathbb{N}$ and $\sigma \in S_n$. The one-line notation of σ is the n-tuple $[\sigma(1), \sigma(2), \ldots, \sigma(n)]$.

The use of square brackets is a standard; it doesn't mean anything.

Often, the commas and the brackets are omitted.

E.g. the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} \in S_5$ is written in

one-line notation $[4, 2, 1, 3, 5]$, or, short, 42135.

Prop. 1. Let $n \in \mathbb{N}$. Then,

$$S_n \longrightarrow [n]^n_{dist} = \{n\text{-tuples of distinct elts of } [n]\},$$
$$\sigma \longmapsto [\sigma(1), \sigma(2), \ldots, \sigma(n)] = (\text{one-line notation of } \sigma)$$

is a bijection.

(**Proof.** Pigeonhole principle / box principle / doocot principle.)

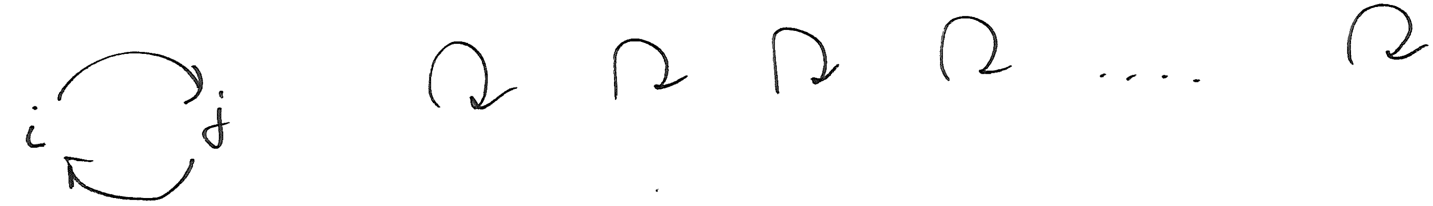
Def. (a) Let i, j be distinct elts of $[n]$. Then, the transposition $t_{i,j} \in S_n$ is the perm. that sends i to j, j to i, and leaves everything else in its place.

In one-line notation, $t_{i,j} = [1, 2, \ldots, i-1, j, i+1, \ldots, j-1, i, j+1, \ldots, n]$

for $i < j$.

(b) For each $i \in [n-1]$, set $s_i = t_{i,i+1}$.

Ex. The cycle digraph of $t_{i,j}$ is

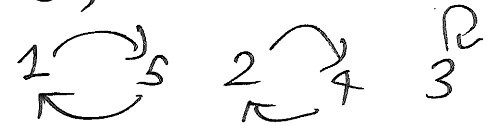


Prop. 2. (a) $s_i^2 = id \quad \forall i.$

(b) $s_i \circ s_j = s_j \circ s_i \quad \forall |i-j| > 1.$

(c) $\underbrace{s_i \circ s_{i+1} \circ s_i}_{t_{i,i+2}} = s_{i+1} \circ s_i \circ s_{i+1} \quad \forall i \in [n-1].$

Def. Let w_0 be the perm. in S_n sending each i to $n+1-i$.
In other words, it "reflects" numbers across the center of $[n]$.
It is the unique strictly decreasing perm. of $[n]$.

Example. If $n=5$, then $w_0 = [5,4,3,2,1]$,
with cycle digraph ,

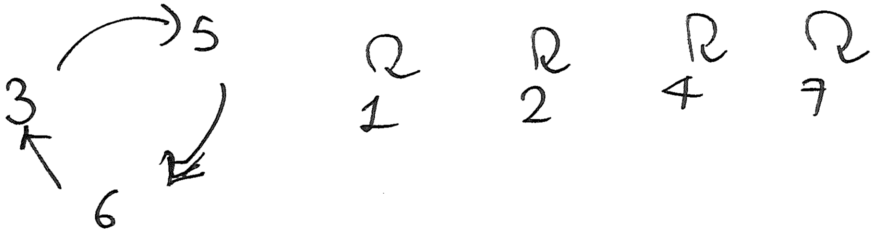
If $n=6$, then $w_0 = [6,5,4,3,2,1]$,
with cycle digraph .

Def. For any k distinct elements $i_1, \ldots, i_k$ of $[n]$
let $cyc_{i_1,\ldots,i_k}$ be the perm. in S_n sending

$i_1 \mapsto i_2, \quad i_2 \mapsto i_3, \quad \ldots, \quad i_{k-1} \mapsto i_k, \quad i_k \mapsto i_1,$

and leaving all other numbers in place.

Ex. (cycle digraph of) $\mathrm{cyc}_{3,5,6}$ =



Prop. 3. (a) $\mathrm{cyc}_{i_1,\dots,i_k} = t_{i_1,i_2} \circ t_{i_2,i_3} \circ \cdots \circ t_{i_{k-1},i_k}$.

(b) $\mathrm{cyc}_{i,i+1,\dots,i+k-1} = s_i \circ s_{i+1} \circ \cdots \circ s_{i+k-2}$.

(c) $w_0 = s_1 \circ (s_2 s_1) \circ (s_3 s_2 s_1) \circ \cdots \circ (s_{n-1} s_{n-2} \cdots s_1)$

$= (s_1 s_2 \cdots s_{n-1}) \circ (s_1 s_2 \cdots s_{n-2}) \circ \cdots \circ (s_1 s_2)\, s_1$.

[Here & in the following, $\alpha\beta$ means $\alpha \circ \beta$ when $\alpha, \beta \in S_n$].

(d) If $1 \leq i < j \leq n$, then $t_{i,j} = s_i s_{i+1} \cdots s_{j-1} \cdots s_{i+1} s_i$

$= s_j s_{j-1} \cdots s_{i+1} \cdots s_{j-1} s_j$

(e) $\sigma \, \mathrm{cyc}_{i_1,\ldots,i_k} \sigma^{-1} = \mathrm{cyc}_{\sigma(i_1),\ldots,\sigma(i_k)}$ $\forall \sigma \in S_n$.

(f) $t_{i,j} = \mathrm{cyc}_{i,j}$.

(g) $s_i = \mathrm{cyc}_{i,i+1}$.

<u>Def.</u> An <u>inversion</u> of a $\sigma \in S_n$ is a pair (i,j) ~~with~~ of elts of $[n]$ with $i<j$ and $\sigma(i) > \sigma(j)$.

<u>Example:</u> Let $\pi = [3,1,4,2] \in S_4$.
The inversions of π are $(1,2)$ (since $1<2$ but $\pi(1) = 3 > 1 = \pi(2)$), $(1,4)$, and $(3,4)$.

<u>Def.</u> The <u>length</u> of a $\sigma \in S_n$ is its number of inversions. (Also called the Coxeter length.) It is called $\ell(\sigma)$.

<u>Example:</u> $\ell(\pi) = 3$ for the π above.

<u>Remark:</u> If $\sigma \in S_n$, then $0 \leq \ell(\sigma) \leq \binom{n}{2}$.

The only σ with $\ell(\sigma) = 0$ is id.

The only σ with $\ell(\sigma) = \binom{n}{2}$ is w_0.

Inbetween, there are many.

S_3:

$$
\begin{tikzcd}
 & \underset{\displaystyle 321}{w_0 \,\shortparallel} \arrow[dl, dash, "s_1"'] \arrow[dr, dash, "s_2"] & & \ell = 3 \\
231 \arrow[d, dash, "s_2"'] & & 312 \arrow[d, dash, "s_1"] & \ell = 2 \\
213 \arrow[dr, dash, "s_1"'] & & 132 \arrow[dl, dash, "s_2"] & \ell = 1 \\
 & \overset{\displaystyle 123}{\shortparallel \, \mathrm{id}} & & \ell = 0
\end{tikzcd}
$$

(vertices = 6 perms. in S_3 in 1-line notation)

We draw an edge $\alpha \xrightarrow{s_i} \beta$ if $\alpha = \beta \circ s_i$ (or, equivalently, $\beta = \alpha \circ s_i$).