①

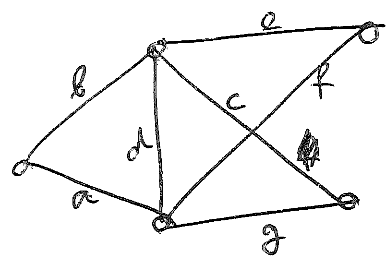
Prop. 10. Let $n \geq 2$. Let $d_1, d_2, \ldots, d_n$ be n nonneg. ints. with $d_1 + d_2 + \cdots + d_n = 2(n-1)$. Then, the # of trees with vertex set $[n]$ such that $\deg(i) = d_i$ is

$$\frac{(n-2)!}{(d_1-1)!\,(d_2-1)!\cdots(d_n-1)!}.$$

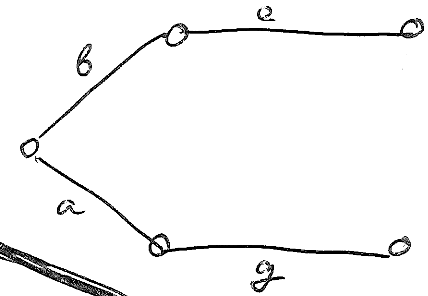
(See [Galvin].)

Def. Let G be a connected graph.
A spanning tree of G is (informally) a way to remove edges from G such that a tree is left behind.

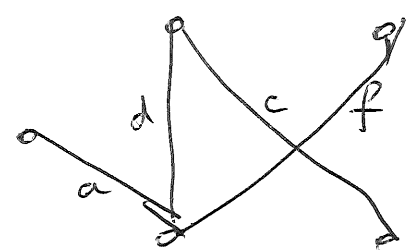
Example:



has spanning tree



but also



Prop. 11. Any connected graph G has a spanning tree.

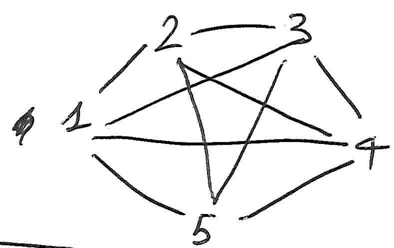
Thm. 12. (Matrix-tree theorem, Kirchhoff).
Let G be a connected graph with n vertices, $1, 2, \dots, n$.
The reduced Laplacian of G is defined as the $(n-1)\times(n-1)$-matrix L whose (i,j)-th entry is

$$\begin{cases} \deg i & \text{if } i=j; \\ -\#(\text{edges } i-j) & \text{if } i \neq j. \end{cases}$$

Then, the # of spanning trees of G is $\det L$.

Example: For each $n \geq 1$, consider the complete graph K_n:

$$K_n = ([n], \mathcal{P}_2([n])).$$

$K_5 =$ 

Then, the spanning trees of K_n are just the trees on vertex set $[n]$.

$\Rightarrow$ Thm. 12 says that the # of trees on ~~n~~ vertex set $[n]$

$\det L$, where

$$L = \underbrace{\begin{pmatrix} n-1 & -1 & -1 & \cdots & -1 \\ -1 & n-1 & -1 & & \vdots \\ -1 & -1 & n-1 & \ddots & \\ \vdots & & \ddots & \ddots & -1 \\ -1 & -1 & -1 & \cdots & n-1 \end{pmatrix}}_{n-1} \Big\} \, n-1$$

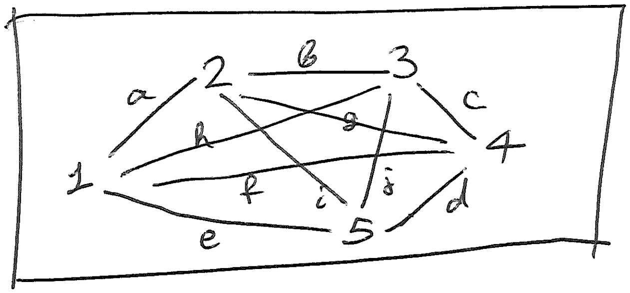
Elementary linear algebra says that $\det L = n^{n-2}$,

$\Rightarrow$ Thm. 9.

7.6. EULERIAN CIRCUITS

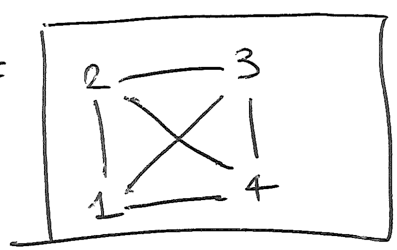
Given a graph $G = (V, E, \varphi)$, an <u>Eulerian circuit</u> of G means a circuit of G that contains each edge exactly once.

<u>Examples:</u> (a) If $G =$

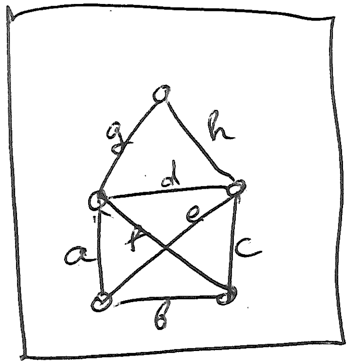


then $(1, a, 2, b, 3, c, 4, d, 5, e, 1, h, 3, j, 5, i, 2, g, 4, f, 1)$ is an Eulerian circuit.

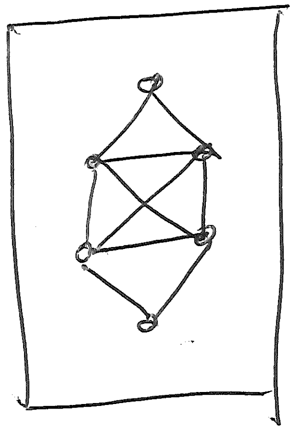
(b) If $G =$



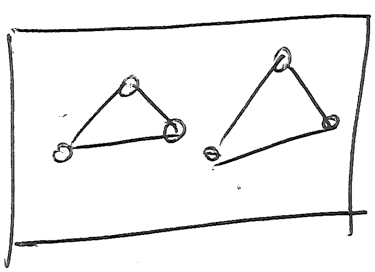
, then G has no Eulerian circuit because of the vertices with odd degrees.

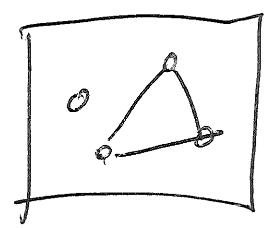
(c) If $G =$ 

, then G has no Eulerian circuit.

But  does.

<u>Thm. 13 (Euler & Hierholtzer).</u> Let G be a connected graph. Then, G has an Eulerian circuit if & only if each vertex of G has even degree.

<u>Ex:</u>  ← this has no Eul. circuit.

this one has. → 

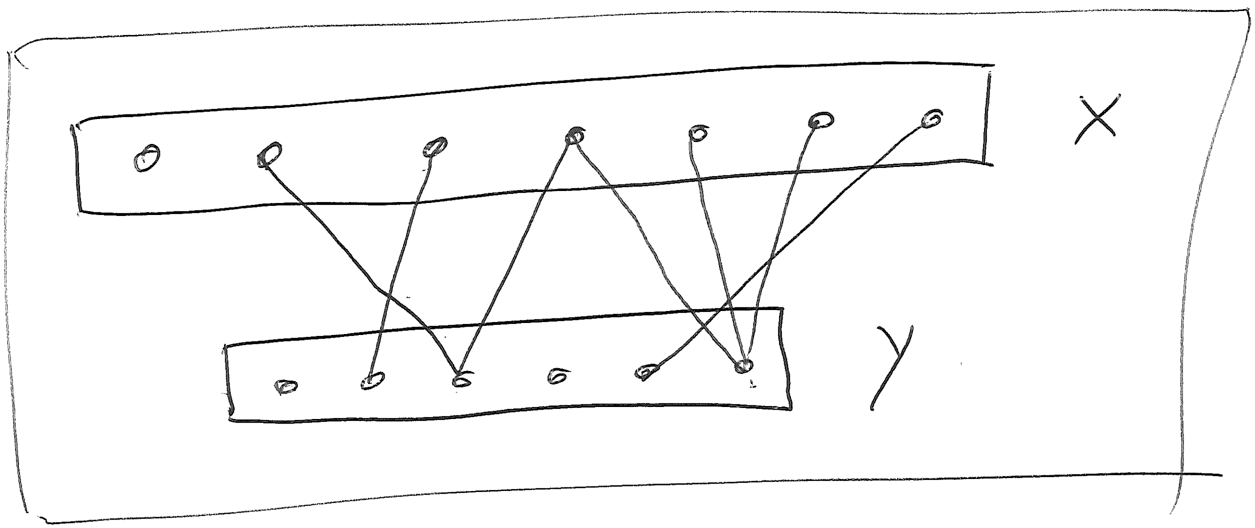
Proof: see [LeLeMe, Problem 12.43].

7.7. BIPARTITE GRAPHS & HALL'S THM.

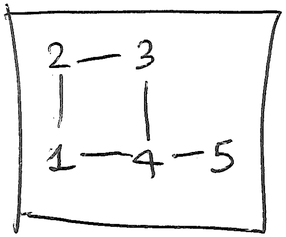
Def. A bipartite graph is a triple $(G; X, Y)$, where $G = (V, E)$ is a simple graph, X and Y are two subsets of V such that

- $X \cup Y = V$, $X \cap Y = \emptyset$;
- each edge $e \in E$ has exactly one endpoint in X & exactly one endpoint in Y.

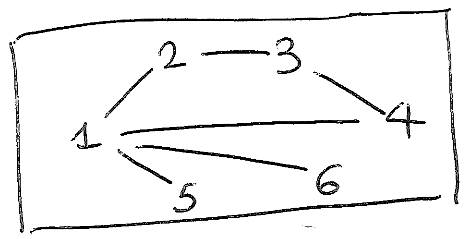
Ex:



Def. A **matching** in a graph G is a set of disjoint edges of G.

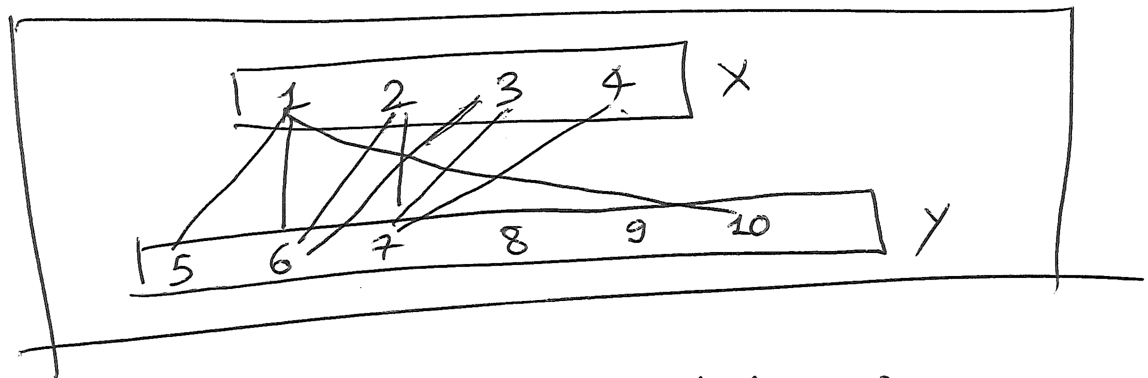
Examples: If $G =$  , then $\{\{1,2\},\{4,5\}\}$ is a matching in G.

But $\{\{1,2\},\{2,3\}\}$ is not.

If $G =$  , then $\{\{1,2\},\{3,4\}\}$ is a matching in G.

Def. Let S be a set of vertices of a graph G.
Let M be a matching in G.
We say that M is **S-complete** if each $v \in S$ is contained in an edge $e \in M$.

Ex: Let $G =$



Does G have an X-complete matching?

No, because the 3 vertices 2, 3, 4 have only 2 nbrs.

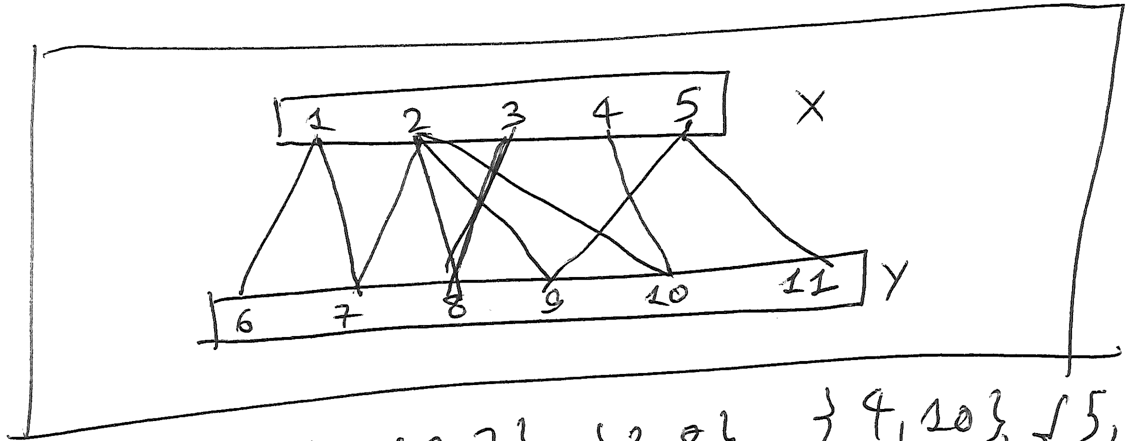
Thm. 14 ("Hall's marriage theorem"). Let $(G; X, Y)$ be a bipartite graph. Then, $\exists$ X-complete matching in G if & only if each subset U of X satisfies

$|N(U)| \geq |U|$, where $N(U) = \{v \in V \mid v \text{ has a neighbor in } U\}$,

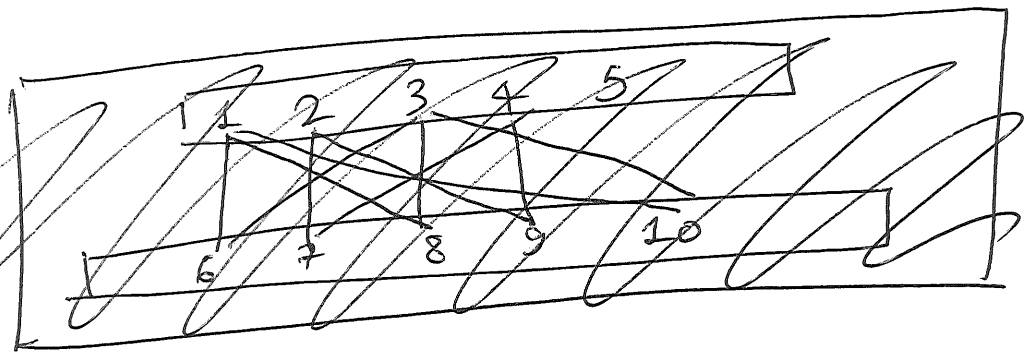
where $V = \{\text{vertices of } G\}$.

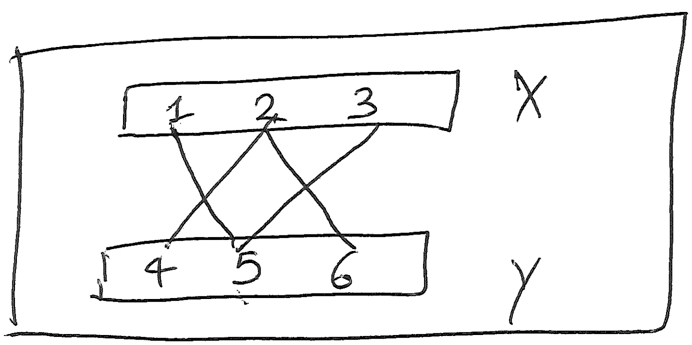
For an elementary proof, see [LeLe Me, Thm 12.5.2].

Ex:

$G =$ 

$\{\{1,6\}, \{2,7\}, \{3,8\}, \{4,10\}, \{5,11\}\}$

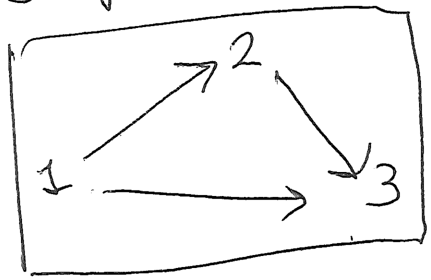
~~$G =$~~ 

$G =$ 

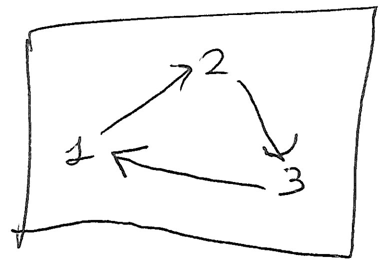
$\nexists$ X-complete matching

8. DIGRAPHS

Idea: digraphs = directed graphs



$\neq$



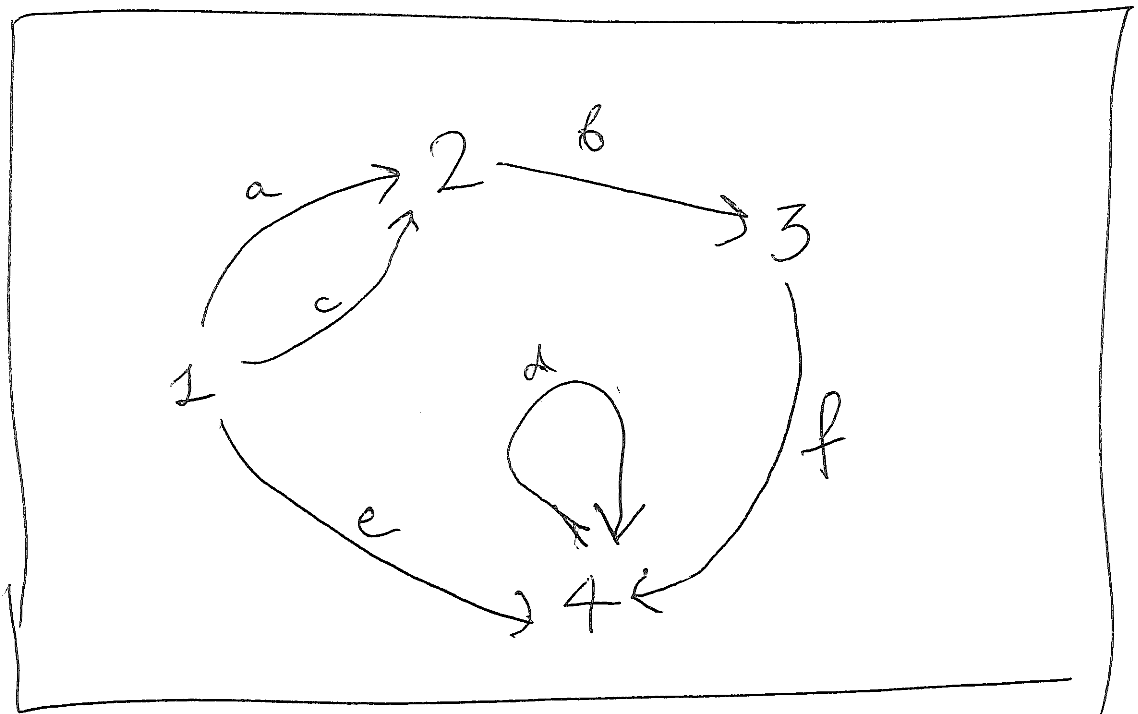
<u>Def.</u> A <u>multidigraph</u> is a triple (V, A, φ) where V and A are finite sets and $\varphi : A \to V \times V$.

The <u>vertices</u> of this multidigraph (V, A, φ) are the elts of V.

The <u>arcs</u> - - - - - - - - - - - - - - - - A.

If $a \in A$ with $\varphi(a) = (u, v)$, then u is the <u>source</u> of a, and v is the <u>target</u> of a. We draw a as $u \xrightarrow{a} v$.

<u>Ex.</u>



(Yes, we allow loops!)

"Simple ~~mult~~ digraphs" are similar (pairs (V, A) with $A \subseteq V \times V$).

<u>Def.</u> Let v be a vertex of a multidigraph D.

The <u>outdegree</u> $\deg^+ v$ of v is the # of arcs with source v.

The <u>indegree</u> $\deg^- v$ of v is the # of arcs with target v.

Analogue of Prop. 7.3:

Prop. 1. Let $D = (V, A, \varphi)$ be a multidigraph, then:

$$\sum_{v \in V} \deg^+ v = \sum_{v \in V} \deg^- v = |A|.$$

Def. Let $D = (V, A, \varphi)$ be a multidigraph. Let $p, q \in V$.

(a) A <u>walk</u> from p to q in D means a list $(v_0, a_1, v_1, a_2, v_2, \dots, v_{k-1}, a_k, v_k)$ where $v_i \in V$, $a_i \in A$, $\varphi(a_i) = (v_{i-1}, v_i)$, $v_0 = p$, $v_k = q$.

(b) This walk is called a <u>path</u> if $v_0, v_1, \dots, v_k$ are distinct.

(c) A walk from p to q is called a <u>circuit</u> if $p = q$.

(d) A circuit $(v_0, a_1, v_1, a_2, v_2, \dots, v_{k-1}, a_k, v_k)$ is a <u>cycle</u> if $v_0, v_1, \dots, v_{k-1}$ are distinct, $k \geq 1$.
(Note: The arcs $a_1, \dots, a_k$ are then distinct automatically.)

An <u>Eulerian circuit</u> in a multidigraph is a circuit that contains each arc exactly once. $\exists$ analogue to Thm 7.13:

<u>Thm. 2. (Euler – Hierholzer).</u> Let D be a strongly connected multidigraph. ("Strongly connected" means $\forall u \in V, v \in V$ $\exists$ walk $u \to v$.) Then, D has an Eulerian circuit if & only if each vertex v of D has $\deg^+ v = \deg^- v$.

See Math 5707 notes (lec. 16, Spring 2017) for network flow theory, which is an application of digraph to Hall's marriage theorem.

9. On colorings & chromatic polynomials

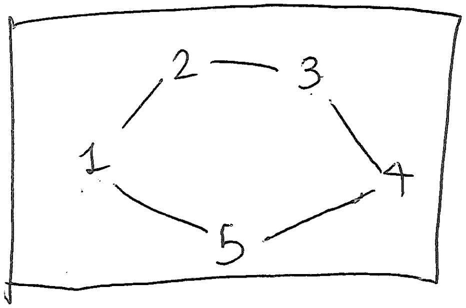
Def. Let $G=(V,E,\varphi)$ be a multigraph. Let $p\in\mathbb{N}$.

A *p-coloring* of G is a map $V\to[p]$.

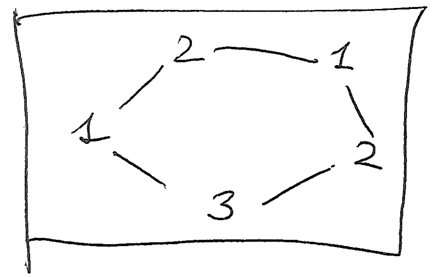
If f is a p-coloring & $v\in V$, then $f(v)$ is ~~also~~ called the *color* of v in f.

A *p-coloring* of G is called *proper* if no two adjacent vertices have the same color (i.e. if $\{u,v\}\in \varphi(E)$ we have $f(u)\neq f(v)$).

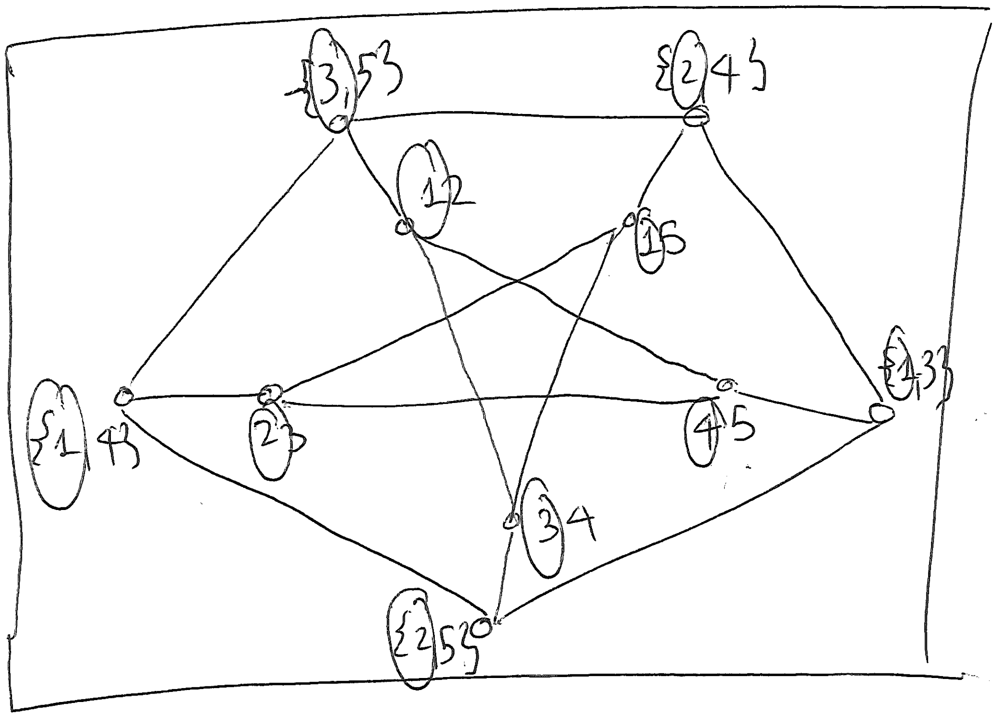
Examples: If $G=\boxed{\begin{array}{ccc}2&-&3\\ |&&|\\ 1&-&4\end{array}}$ then the 3-coloring $\boxed{\begin{array}{ccc}3&-&1\\ |&&|\\ 1&-&2\end{array}}$ is proper, but the 3-coloring $\boxed{\begin{array}{ccc}2&-&3\\ |&&|\\ 1&-&1\end{array}}$ is not. The 2-coloring $\boxed{\begin{array}{ccc}2&-&1\\ |&&|\\ 1&-&2\end{array}}$ is proper.

If $G =$ 

, then $\nexists$ proper 2-coloring,

but G has a proper 3-coloring 

.

If $G =$ 

"Petersen graph"

◯ ← proper 4-coloring

To get a proper 3-coloring, replace 4 by 3.

Prop. 1. Let $G = (V, E, \varphi)$ be a multigraph.

Let d = max. degree of a vertex of G.

Then, G has a proper $(d+1)$-coloring.

Proof. "Greedy algorithm" works: color vertices 1 by 1.

Prop. 2. Let $G = (V, E, \varphi)$ be a ~~simple graph~~ multigraph.

(a) Then, G has a proper 2-coloring

$\iff \exists\, X$ & Y such that $(G; X, Y)$ is a bipartite graph

$\iff$ G has no odd-length circuits

$\iff$ G has no odd-length cycles.

(b) If G has a proper 2-coloring, then

$\#$ of proper 2-colorings of $G = 2^{\#\text{ conn. comps of } G}$.

Thm. 3. Let $G = (V, E, \varphi)$ be a multigraph.

Then, there exists a unique polynomial $\chi_G \in \mathbb{Z}[x]$, monic of degree $|V|$, such that $\forall\, p \in \mathbb{N}$,

$$\#\text{ of proper } p\text{-colorings of } G = \chi_G(p).$$

Explicitly,

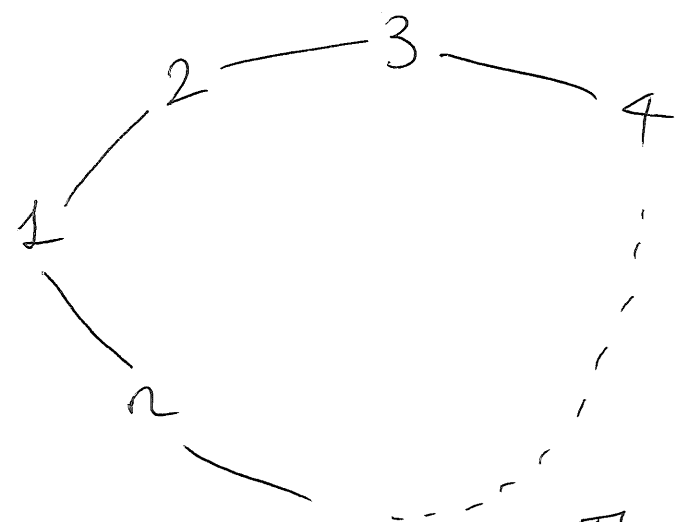
$$\chi_G = \sum_{F \subseteq E} (-1)^{|F|} x^{\overbrace{\text{conn}}^{\text{number of conn. comps.}} \underbrace{(V, F, \varphi|_F)}_{\text{sub-multigraph}}}.$$

This χ_G is called the <u>chromatic polynomial</u> of G.

(Proof: see Exe 4 on Spring 2017 Math 5707 MT 2.)

Remark: Fix $n \geq 1$.
Let P_n be the graph $1 — 2 — 3 — \cdots — n$.
What is $\chi_{P_n}(p)$? $p \cdot (p-1)^{n-1}$. The proper p-colorings of P_n are the n-smords ($=$ Smirnov words) with letters in $1, \ldots, p$.

Let C_n be the graph



What is $\chi_{C_n}(p)$? These are the rounded n-smords,

Thm. 4 (d/c recurrence). Let $G = (V, E, \varphi)$ be a multigraph.

Let $e \in E$.
Let $G \setminus e$ be G with e removed,
Let G/e be G with e "collapsed".
(a) If e is the only edge connecting the endpoints of e,
then $\chi_G = \chi_{G \setminus e} - \chi_{G/e}$.
(b) If e isn't, then $\chi_G = \chi_{G \setminus e}$.