Proof of a CWMO problem generalized

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The following result is due to Dan Schwarz. It was proposed as problem 4 **(c)** for the 9th grade of the Romanian Mathematical Olympiad 2004. It was discussed in [1] (where it was posted by tanlsth), in [2] and in [3].

Theorem 1. Let X be a set. Let n and $m \geq 1$ be two nonnegative integers such that $|X| \geq m(n-1)+1$. Let B_1, B_2, ..., B_n be n subsets of X such that $|B_i| \leq m$ for every $i \in \{1, 2, ..., n\}$. Then, there exists a subset Y of X such that $|Y| = n$ and $|Y \cap B_i| \leq 1$ for every $i \in \{1, 2, ..., n\}$.

Proof of Theorem 1. We will prove Theorem 1 by induction over n.

Induction base: If $n = 0$, then Theorem 1 is trivially true (just set $Y = \varnothing$; then, $|Y| = 0 = n$ and $|Y \cap B_i| = |\varnothing \cap B_i| = |\varnothing| = 0 \leq 1$ for every $i \in \{1, 2, ..., n\}$). This completes the induction base.

Induction step: Let N be a nonnegative integer. Assume that Theorem 1 holds for $n = N$. We have to show that Theorem 1 also holds for $n = N + 1$.

We assumed that Theorem 1 holds for $n = N$. In other words, we assumed the following assertion:

Assertion $\mathcal{A}$: Let X be a set. Let $m \geq 1$ be a nonnegative integer such that $|X| \geq m(N-1)+1$. Let B_1, B_2, ..., B_N be N subsets of X such that $|B_i| \leq m$ for every $i \in \{1, 2, ..., N\}$. Then, there exists a subset Y of X such that $|Y| = N$ and $|Y \cap B_i| \leq 1$ for every $i \in \{1, 2, ..., N\}$.

Upon renaming X, Y and B_i into X', Y' and B'_i, respectively, this assertion rewrites as:

Assertion $\mathcal{A}'$: Let X' be a set. Let $m \geq 1$ be a nonnegative integer such that $|X'| \geq m(N-1)+1$. Let B'_1, B'_2, ..., B'_N be N subsets of X' such that $|B'_i| \leq m$ for every $i \in \{1, 2, ..., N\}$. Then, there exists a subset Y' of X' such that $|Y'| = N$ and $|Y' \cap B'_i| \leq 1$ for every $i \in \{1, 2, ..., N\}$.

Now, we have to show that Theorem 1 also holds for $n = N + 1$. In other words, we have to prove the following assertion:

Assertion $\mathcal{B}$: Let X be a set. Let $m \geq 1$ be a nonnegative integer such that $|X| \geq m((N+1)-1)+1$. Let B_1, B_2, ..., B_{N+1} be $N+1$ subsets of X such that $|B_i| \leq m$ for every $i \in \{1, 2, ..., N+1\}$. Then, there exists a subset Y of X such that $|Y| = N+1$ and $|Y \cap B_i| \leq 1$ for every $i \in \{1, 2, ..., N+1\}$.

Proof of Assertion $\mathcal{B}$. For every choice of X, m and B_1, B_2, ..., B_{N+1}, one of the following two cases must hold:

Case 1: We have $X = \bigcup\limits_{j \in \{1,2,...,N+1\}} B_j$.

Case 2: We have $X \neq \bigcup\limits_{j \in \{1,2,...,N+1\}} B_j$.

Let us consider Case 1. In this case, let $k \in \{1, 2, ..., N+1\}$. Then,

$$\left| \bigcup_{j \in \{1,2,...,N+1\} \setminus \{k\}} B_j \right| \leq \sum_{j \in \{1,2,...,N+1\} \setminus \{k\}} \underbrace{|B_j|}_{\leq m} \leq \sum_{j \in \{1,2,...,N+1\} \setminus \{k\}} m = Nm$$

$$= mN < mN + 1 = m\left((N+1) - 1\right) + 1 \leq |X| = \left| \bigcup_{j \in \{1,2,...,N+1\}} B_j \right|,$$

so that $\bigcup_{j \in \{1,2,...,N+1\} \setminus \{k\}} B_j \neq \bigcup_{j \in \{1,2,...,N+1\}} B_j$. Since $\bigcup_{j \in \{1,2,...,N+1\}} B_j = B_k \cup \left(\bigcup_{j \in \{1,2,...,N+1\} \setminus \{k\}} B_j \right)$, this becomes $\bigcup_{j \in \{1,2,...,N+1\} \setminus \{k\}} B_j \neq B_k \cup \left(\bigcup_{j \in \{1,2,...,N+1\} \setminus \{k\}} B_j \right)$. Thus, $B_k \not\subseteq \bigcup_{j \in \{1,2,...,N+1\} \setminus \{k\}} B_j$ (since $B_k \subseteq \bigcup_{j \in \{1,2,...,N+1\} \setminus \{k\}} B_j$ would yield $\bigcup_{j \in \{1,2,...,N+1\} \setminus \{k\}} B_j = B_k \cup \left(\bigcup_{j \in \{1,2,...,N+1\} \setminus \{k\}} B_j \right)$). Hence, we have shown that

$$B_k \not\subseteq \bigcup_{j \in \{1,2,...,N+1\} \setminus \{k\}} B_j \qquad \text{for every } k \in \{1, 2, ..., N+1\}.$$

For every $k \in \{1, 2, ..., N+1\}$, let x_k be an element of B_k satisfying $x_k \notin \bigcup_{j \in \{1,2,...,N+1\} \setminus \{k\}} B_j$ (such an x_k exists, since $B_k \not\subseteq \bigcup_{j \in \{1,2,...,N+1\} \setminus \{k\}} B_j$). Then, for every $k \in \{1, 2, ..., N+1\}$ and for every $i \in \{1, 2, ..., N+1\}$ satisfying $i \neq k$, we have $x_k \notin B_i$ (since $x_k \notin \bigcup_{j \in \{1,2,...,N+1\} \setminus \{k\}} B_j$ and $B_i \subseteq \bigcup_{j \in \{1,2,...,N+1\} \setminus \{k\}} B_j$). Hence, for every $k \in \{1, 2, ..., N+1\}$ and for every $i \in \{1, 2, ..., N+1\}$ satisfying $i \neq k$, we have $x_k \neq x_i$ (since $x_k \notin B_i$ while $x_i \in B_i$). Thus, the $N+1$ elements x_1, x_2, ..., x_{N+1} are pairwise distinct. Set $Y = \{x_1, x_2, ..., x_{N+1}\}$. Then, $|Y| = N+1$ (since the $N+1$ elements x_1, x_2, ..., x_{N+1} are pairwise distinct). Besides, for every $i \in \{1, 2, ..., N+1\}$, we have $\{x_1, x_2, ..., x_{N+1}\} \cap B_i = \{x_i\}$ (since $x_i \in B_i$, but $x_k \notin B_i$ for every $k \in \{1, 2, ..., N+1\}$ satisfying $i \neq k$), and thus

$$|Y \cap B_i| = |\{x_1, x_2, ..., x_{N+1}\} \cap B_i| = |\{x_i\}| = 1 \leq 1.$$

Thus, Assertion $\mathcal{B}$ is proven in Case 1.

Now, let us consider Case 2. In this case, $X \supseteq \bigcup_{j \in \{1,2,...,N+1\}} B_j$, but $X \neq \bigcup_{j \in \{1,2,...,N+1\}} B_j$. Hence, $X \not\subseteq \bigcup_{j \in \{1,2,...,N+1\}} B_j$, so that there exists some $x \in X$ such that $x \notin \bigcup_{j \in \{1,2,...,N+1\}} B_j$. Thus, $x \notin B_i$ for every $i \in \{1, 2, ..., N+1\}$.

We want to prove Assertion $\mathcal{B}$. If every $i \in \{1, 2, ..., N+1\}$ satisfies $B_i = \varnothing$, then Assertion $\mathcal{B}$ is trivial (just let Y be any subset of X satisfying $|Y| = N+1$ [1]; then, for every $i \in \{1, 2, ..., N+1\}$, we have $|Y \cap B_i| = |Y \cap \varnothing| = |\varnothing| = 0 \leq 1$, so that Assertion $\mathcal{B}$ is fulfilled). Hence, for the rest of the proof of Assertion $\mathcal{B}$, we

[1] Such a subset Y exists, since $|X| \geq m\left((N+1) - 1\right) + 1 = \underbrace{m}_{\geq 1} N + 1 \geq N + 1$.

may assume that not every $i \in \{1,2,...,N+1\}$ satisfies $B_i = \varnothing$. So assume that not every $i \in \{1,2,...,N+1\}$ satisfies $B_i = \varnothing$. In other words, there exists some $k \in \{1,2,...,N+1\}$ such that $B_k \neq \varnothing$. WLOG assume that $B_{N+1} \neq \varnothing$. Let u be an element of B_{N+1}.

Set $X' = X \setminus ((B_{N+1} \setminus \{u\}) \cup \{x\})$ and $B_i' = B_i \cap X'$ for every $i \in \{1,2,...,N+1\}$. Then, B_1', B_2', ..., B_N' are N subsets of X', and we have

$$|B_{N+1} \setminus \{u\}| = |B_{N+1}| - 1 \qquad (\text{since } u \in B_{N+1})$$
$$\leq m - 1 \qquad (\text{since } |B_{N+1}| \leq m),$$

thus

$$|(B_{N+1} \setminus \{u\}) \cup \{x\}| = |B_{N+1} \setminus \{u\}| + 1 \qquad (\text{since } x \notin B_{N+1} \text{ yields } x \notin B_{N+1} \setminus \{u\})$$
$$\leq (m-1) + 1 = m,$$

hence

$$|X'| = |X \setminus ((B_{N+1} \setminus \{u\}) \cup \{x\})| = |X| - |(B_{N+1} \setminus \{u\}) \cup \{x\}| \geq m((N+1)-1) + 1 - m$$
$$(\text{since } |X| \geq m((N+1)-1)+1 \text{ and } |(B_{N+1} \setminus \{u\}) \cup \{x\}| \leq m)$$
$$= mN + 1 - m = m(N-1) + 1$$

and $|B_i'| = |B_i \cap X'| \leq |B_i| \leq m$ for every $i \in \{1,2,...,N\}$. Hence, by Assertion $\mathcal{A}'$, there exists a subset Y' of X' such that $|Y'| = N$ and $|Y' \cap B_i'| \leq 1$ for every $i \in \{1,2,...,N\}$. Note that $x \notin Y'$, since $Y' \subseteq X' = X \setminus ((B_{N+1} \setminus \{u\}) \cup \{x\})$ and $x \notin X \setminus ((B_{N+1} \setminus \{u\}) \cup \{x\})$.

Notice that

$$B_{N+1}' = B_{N+1} \cap X' = B_{N+1} \cap \underbrace{(X \setminus ((B_{N+1} \setminus \{u\}) \cup \{x\}))}_{\substack{=(X \setminus (B_{N+1} \setminus \{u\})) \setminus \{x\} \\ \subseteq X \setminus (B_{N+1} \setminus \{u\})}}$$
$$\subseteq B_{N+1} \cap (X \setminus (B_{N+1} \setminus \{u\})) = (B_{N+1} \cap X) \setminus (B_{N+1} \setminus \{u\})$$
$$= B_{N+1} \setminus (B_{N+1} \setminus \{u\}) \qquad (\text{since } B_{N+1} \subseteq X \text{ yields } B_{N+1} \cap X = B_{N+1})$$
$$= \{u\} \qquad (\text{since } u \in B_{N+1}),$$

so that $Y' \cap B_{N+1}' \subseteq B_{N+1}' \subseteq \{u\}$ and thus $\left|Y' \cap B_{N+1}'\right| \leq |\{u\}| = 1$.

Altogether, we have seen that $|Y' \cap B_i'| \leq 1$ for every $i \in \{1,2,...,N\}$ and that $\left|Y' \cap B_{N+1}'\right| \leq 1$. Combining these two facts, we conclude that $|Y' \cap B_i'| \leq 1$ for every $i \in \{1,2,...,N+1\}$.

Now, let $Y = Y' \cup \{x\}$. Then,

$$|Y| = |Y' \cup \{x\}| = |Y'| + 1 \qquad (\text{since } x \notin Y')$$
$$= N + 1.$$

Besides, for every $i \in \{1, 2, ..., N+1\}$, we have

$$\left|Y \cap B_i\right| = \left|(Y' \cup \{x\}) \cap B_i\right| = \left|(Y' \cap B_i) \cup \underbrace{(\{x\} \cap B_i)}_{\substack{=\varnothing, \text{ since} \\ x \notin B_i}}\right| = \left|(Y' \cap B_i) \cup \varnothing\right| = \left|Y' \cap B_i\right| = \left|(Y' \cap X') \cap B_i\right|$$

$$\left(\text{since } Y' \subseteq X' \text{ yields } Y' = Y' \cap X'\right)$$

$$= \left|Y' \cap \underbrace{(B_i \cap X')}_{=B_i'}\right| = \left|Y' \cap B_i'\right| \leq 1.$$

Thus, Assertion $\mathcal{B}$ is proven in Case 2.

Altogether, we have now verified Assertion $\mathcal{B}$ in both Cases 1 and 2. But we know that for every choice of X, m and B_1, B_2, ..., B_{N+1}, either Case 1 or Case 2 is satisfied. Thus, Assertion $\mathcal{B}$ is proven in every possible case. In other words, Theorem 1 holds for $n = N + 1$. This completes the induction step.

Therefore, the induction proof of Theorem 1 is complete.

References

[1] tanlsth et al., *MathLinks topic #118091 ("subset conditions")*, posts #3-#5.
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[2] Dan Schwarz (alias mavropnevma), *MathLinks topic #299325 ("Renewal of an old problem")*.
http://www.mathlinks.ro/viewtopic.php?t=299325

[3] perfect_radio, grobber et al., *MathLinks topic #76786 ("S-property of a set")*.
http://www.mathlinks.ro/viewtopic.php?t=76786