Remarks on Krivine's "Lambda-calculus, types and models", Chapter 1, §2

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1. Introduction

The point of this note is to

1) add some lemmata to Chapter 1 §2 of [1] (lemmata that are used in [1] without mention, due to their intuitive obviousness);

2) show that the definition of α-equivalence given in [1] is equivalent to the definition of α-equivalence given in some other sources;

3) prove some rules for substitution (in order to answer a MathOverflow question of myself).

We are going to use the notations and the results of Chapter 1 of [1]. In particular, the sign $\equiv$ will stand for the α-equivalence defined in [1]. The different notion of α-equivalence that we consider will be denoted by $=^\alpha$ (in order not to confuse it with $\equiv$ as long as it is not yet proven that the two notions are equivalent).

2. Sidenotes to Chapter 1 §2 of [1]

Here come several facts silently used in some proofs in §1.2 of [1]. These facts are all pretty simple, intuitively clear and easy to prove, and I suspect this is why they have not been explicitly stated in [1]. I am making them explicit and proving them in detail in order to formalize the theory a little bit more.

We begin with some properties of bound variables (and their behaviour under substitution).

> **Definition:** If u is a term in L, let $\operatorname{BV} u$ denote the set of bounded variables of the term u.

Before we continue, let us give an inductive method to compute $\operatorname{BV} u$ for a term u:

If $u = x$ for a variable x, then $\operatorname{BV} u = \varnothing$.

If $u = (v)\, w$ for terms v and w, then $\operatorname{BV} u = (\operatorname{BV} v) \cup (\operatorname{BV} w)$.

If $u = \lambda x v$ for some variable x and some term v, then $\operatorname{BV} u = \{x\} \cup (\operatorname{BV} v)$.

> **Lemma 1.A.** Let t, t_1, ..., t_m be terms in L, and x_1, ..., x_m be distinct variables. Then, $\operatorname{BV}(t \langle t_1/x_1, ..., t_m/x_m\rangle) \subseteq (\operatorname{BV} t) \cup (\operatorname{BV} t_1) \cup ... \cup (\operatorname{BV} t_m)$.

Proof of Lemma 1.A. We proceed by induction over t:

If t is a variable or a term of the form $(u)\, v$, the induction step is clear.

Remains to consider the case when $t = \lambda x u$ for some variable x and some term u.

In this case, $\operatorname{BV} t = \{x\} \cup (\operatorname{BV} u)$. There are two subcases to consider: the subcase when $x \in \{x_1, ..., x_m\}$ and the subcase when $x \notin \{x_1, ..., x_m\}$.

First, let us consider the subcase when $x \in \{x_1, ..., x_m\}$. In this subcase, let us WLOG assume that $x = x_1$. Thus, $t = \lambda x_1 u$, so that $\operatorname{BV} t = \{x_1\} \cup (\operatorname{BV} u)$.

Now, $t = \lambda x_1 u$ and the definition of $t \langle t_1/x_1, ..., t_m/x_m \rangle$ result in $t \langle t_1/x_1, ..., t_m/x_m \rangle = \lambda x_1 (u \langle t_2/x_2, ..., t_m/x_m \rangle)$, so that

$$\operatorname{BV}(t \langle t_1/x_1, ..., t_m/x_m \rangle) = \{x_1\} \cup \operatorname{BV}(u \langle t_2/x_2, ..., t_m/x_m \rangle).$$

Since $\operatorname{BV}(u \langle t_2/x_2, ..., t_m/x_m \rangle) \subseteq (\operatorname{BV} u) \cup (\operatorname{BV} t_2) \cup ... \cup (\operatorname{BV} t_m)$ by the induction assumption, this becomes

$$\begin{aligned} &\operatorname{BV}(t \langle t_1/x_1, ..., t_m/x_m \rangle) \\ &\subseteq \underbrace{\{x_1\} \cup (\operatorname{BV} u)}_{= \operatorname{BV} t} \cup (\operatorname{BV} t_2) \cup ... \cup (\operatorname{BV} t_m) \\ &= (\operatorname{BV} t) \cup (\operatorname{BV} t_2) \cup ... \cup (\operatorname{BV} t_m) \subseteq (\operatorname{BV} t) \cup (\operatorname{BV} t_1) \cup ... \cup (\operatorname{BV} t_m). \end{aligned}$$

Now, let us consider the subcase when $x \notin \{x_1, ..., x_m\}$. In this subcase, $t = \lambda x u$ and the definition of $t \langle t_1/x_1, ..., t_m/x_m \rangle$ result in $t \langle t_1/x_1, ..., t_m/x_m \rangle = \lambda x (u \langle t_1/x_1, ..., t_m/x_m \rangle)$. Thus,

$$\operatorname{BV}(t \langle t_1/x_1, ..., t_m/x_m \rangle) = \{x\} \cup \operatorname{BV}(u \langle t_1/x_1, ..., t_m/x_m \rangle).$$

Since $\operatorname{BV}(u \langle t_1/x_1, ..., t_m/x_m \rangle) \subseteq (\operatorname{BV} u) \cup (\operatorname{BV} t_1) \cup ... \cup (\operatorname{BV} t_m)$ by the induction assumption, this becomes

$$\begin{aligned} \operatorname{BV}(t \langle t_1/x_1, ..., t_m/x_m \rangle) &\subseteq \underbrace{\{x\} \cup (\operatorname{BV} u)}_{= \operatorname{BV} t} \cup (\operatorname{BV} t_1) \cup ... \cup (\operatorname{BV} t_m) \\ &= (\operatorname{BV} t) \cup (\operatorname{BV} t_1) \cup ... \cup (\operatorname{BV} t_m). \end{aligned}$$

In both subcases, we have proven that $\operatorname{BV}(t \langle t_1/x_1, ..., t_m/x_m \rangle) \subseteq (\operatorname{BV} t) \cup (\operatorname{BV} t_1) \cup ... \cup (\operatorname{BV} t_m)$. This completes the induction and thus proves Lemma 1.A.

Lemma 1.A is used in the proof of Lemma 1.12 in [1]. (In fact, this proof claims that "no bound variable of this term is free in $\underline{u}_1, ..., \underline{u}_n$" [1]. The reason why this is true is the following: Lemma 1.A yields that $\operatorname{BV}(\underline{t} \langle \underline{t}_1/x_1, ..., \underline{t}_m/x_m \rangle) \subseteq (\operatorname{BV} \underline{t}) \cup (\operatorname{BV} \underline{t}_1) \cup ... \cup (\operatorname{BV} \underline{t}_m)$, and we know that no bound variable of any of the terms $\underline{t}, \underline{t}_1, ..., \underline{t}_m$ is free in $\underline{u}_1, ..., \underline{u}_n$.)

Lemma 1.B. Let u be a term in L, and let x and y be two variables. Then, $\operatorname{BV}(u \langle y/x \rangle) \subseteq \operatorname{BV} u$.

Proof of Lemma 1.B. Apply Lemma 1.A to $m = 1$, $t_1 = y$, $x_1 = x$ and $t = u$. This yields $\operatorname{BV}(u \langle y/x \rangle) \subseteq (\operatorname{BV} u) \cup \underbrace{(\operatorname{BV} y)}_{= \varnothing} = \operatorname{BV} u$, and thus Lemma 1.B is proven.

Lemma 1.B is used in the proof of Proposition 1.6 in [1]. (Namely, when this proof says "the induction hypothesis gives", it silently uses the fact that no free

[1] Here, "this term" refers to the term $\underline{t} \langle \underline{t}_1/x_1, ..., \underline{t}_m/x_m \rangle$.

variable in $t_1, ..., t_k$ is bound in $u\langle y/x\rangle$ or $u'\langle y/x'\rangle$ (this must be guaranteed, lest we could not apply the induction hypothesis!). This holds because Lemma 1.B yields $\operatorname{BV}(u\langle y/x\rangle) \subseteq \operatorname{BV} u \subseteq \{x\} \cup (\operatorname{BV} u) = \operatorname{BV} t$ (since $t = \lambda xu$) and $\operatorname{BV}(u'\langle y/x'\rangle) \subseteq \operatorname{BV} t'$ (for similar reasons), and because we know that no free variable in $t_1, ..., t_k$ is bound in t or t'.)

Next some lemmata about free variables:

Definition: If u is a term in L, let $\operatorname{FV} u$ denote the set of free variables of the term u.

Before we continue, let us give an inductive method to compute $\operatorname{FV} u$ for a term u:

If $u = x$ for a variable x, then $\operatorname{FV} u = \{x\}$.

If $u = (v)\, w$ for terms v and w, then $\operatorname{FV} u = (\operatorname{FV} v) \cup (\operatorname{FV} w)$.

If $u = \lambda xv$ for some variable x and some term v, then $\operatorname{FV} u = (\operatorname{FV} v) \setminus \{x\}$.

Lemma 1.C. Let u be a term in L, and let y be a variable which does not appear in u. Let x be a variable. Then, $\operatorname{FV}(u\langle y/x\rangle) = \operatorname{map}_{x,y}(\operatorname{FV} u)$. Here, $\operatorname{map}_{x,y}$ denotes the map $V \to V$ (where V is the set of variables) which maps x to y and maps v to v for every variable $v \neq x$.

Proof of Lemma 1.C. We proceed by induction over u:

If u is a variable, then everything is clear.

Consider the case when $u = (v)\, w$ for terms v and w. In this case, $u\langle y/x\rangle = (v\langle y/x\rangle)(w\langle y/x\rangle)$, so that

$$\operatorname{FV}(u\langle y/x\rangle) = (\operatorname{FV}(v\langle y/x\rangle)) \cup (\operatorname{FV}(w\langle y/x\rangle)). \tag{1}$$

By the induction assumption, $\operatorname{FV}(v\langle y/x\rangle) = \operatorname{map}_{x,y}(\operatorname{FV} v)$ and $\operatorname{FV}(w\langle y/x\rangle) = \operatorname{map}_{x,y}(\operatorname{FV} w)$. Thus (1) becomes

$$\operatorname{FV}(u\langle y/x\rangle) = \left(\operatorname{map}_{x,y}(\operatorname{FV} v)\right) \cup \left(\operatorname{map}_{x,y}(\operatorname{FV} w)\right) = \operatorname{map}_{x,y}((\operatorname{FV} v) \cup (\operatorname{FV} w)).$$

Since $(\operatorname{FV} v) \cup (\operatorname{FV} w) = \operatorname{FV} u$ (due to $u = (v)\, w$), this becomes $\operatorname{FV}(u\langle y/x\rangle) = \operatorname{map}_{x,y}(\operatorname{FV} u)$, completing the induction (in the case $u = (v)\, w$).

It remains to complete the induction step in the case when $u = \lambda zv$ for some variable z and some term $v \in L$.

Consider this case. Clearly, $\operatorname{FV} u = (\operatorname{FV} v) \setminus \{z\}$ in this case.

Two subcases are possible: the subcase $z = x$ and the subcase $z \neq x$.

Consider the subcase $z = x$. In this subcase, $u = \lambda zv = \lambda xv$, thus $u\langle y/x\rangle = \lambda xv = u$, so that $\operatorname{FV}(u\langle y/x\rangle) = \operatorname{FV} u$. But we want to prove that $\operatorname{FV}(u\langle y/x\rangle) = \operatorname{map}_{x,y}(\operatorname{FV} u)$. So we only need to check that $\operatorname{map}_{x,y}(\operatorname{FV} u) = \operatorname{FV} u$. But this is clear because $x \notin \operatorname{FV} u$ (since $u = \lambda xv$, so that $\operatorname{FV} u = (\operatorname{FV} v) \setminus \{x\}$) and because $\operatorname{map}_{x,y}$ leaves every variable except of x fixed.

Now consider the subcase $z \neq x$. In this subcase, $u = \lambda zv$ leads to $u \langle y/x \rangle = \lambda z (v \langle y/x \rangle)$, so that $\operatorname{FV}(u \langle y/x \rangle) = (\operatorname{FV}(v \langle y/x \rangle)) \setminus \{z\}$. By the induction hypothesis, $\operatorname{FV}(v \langle y/x \rangle) = \operatorname{map}_{x,y}(\operatorname{FV} v)$ (since y does not appear in v, which is because y does not appear in u). Thus, $\operatorname{FV}(u \langle y/x \rangle) = \underbrace{(\operatorname{FV}(v \langle y/x \rangle))}_{=\operatorname{map}_{x,y}(\operatorname{FV} v)} \setminus \{z\} = \left(\operatorname{map}_{x,y}(\operatorname{FV} v)\right) \setminus \{z\}$. Let us now show that $\left(\operatorname{map}_{x,y}(\operatorname{FV} v)\right) \setminus \{z\} = \operatorname{map}_{x,y}((\operatorname{FV} v) \setminus \{z\})$. In fact, $\operatorname{FV} v$ does not contain y (because y does not appear in u, and thus y does not appear in v either), so that $\operatorname{FV} v \subseteq V \diagdown \{y\}$ (where V denotes the set of all variables). Also, $\{z\} \subseteq V \diagdown \{y\}$ (since $z \neq y$ (which is because y does not appear in $u = \lambda zv$, while z does appear in λzv)). Now, $\operatorname{map}_{x,y}$ is injective on $V \diagdown \{y\}$ (because (x, y) is the only possible pair of distinct variables which have the same value under $\operatorname{map}_{x,y}$, and therefore there is no such pair inside $(V \diagdown \{y\}) \times (V \diagdown \{x\})$). Thus, $\operatorname{map}_{x,y}((\operatorname{FV} v) \setminus \{z\}) = \left(\operatorname{map}_{x,y}(\operatorname{FV} v)\right) \setminus \left(\operatorname{map}_{x,y}\{z\}\right)$ (since $\operatorname{FV} v$ and $\{z\}$ both are subsets of $V \diagdown \{y\}$). Since $\operatorname{map}_{x,y}\{z\} = \{z\}$ (because $z \neq x$, and thus $\operatorname{map}_{x,y} z = z$), this becomes $\operatorname{map}_{x,y}((\operatorname{FV} v) \setminus \{z\}) = \left(\operatorname{map}_{x,y}(\operatorname{FV} v)\right) \setminus \{z\}$. Now,

$$\operatorname{FV}(u \langle y/x \rangle) = \left(\operatorname{map}_{x,y}(\operatorname{FV} v)\right) \setminus \{z\} = \operatorname{map}_{x,y}\left(\underbrace{(\operatorname{FV} v) \setminus \{z\}}_{=\operatorname{FV} u}\right) = \operatorname{map}_{x,y}(\operatorname{FV} u).$$

Thus, $\operatorname{FV}(u \langle y/x \rangle) = \operatorname{map}_{x,y}(\operatorname{FV} u)$ is proven in every possible case and subcase. Lemma 1.C is proven.

> **Lemma 1.D.** Let x and x' be two variables, let u and u' be two terms in L, and let y be a variable which does not appear in any of the terms u and u'. Assume that $\operatorname{FV}(u \langle y/x \rangle) = \operatorname{FV}(u' \langle y/x' \rangle)$. Then, $\operatorname{FV}(\lambda xu) = \operatorname{FV}(\lambda x'u')$.

Proof of Lemma 1.D. Lemma 1.C yields $\operatorname{FV}(u \langle y/x \rangle) = \operatorname{map}_{x,y}(\operatorname{FV} u)$ (where $\operatorname{map}_{x,y}$ is defined as in Lemma 1.C). But $y \notin \operatorname{FV} u$ (because y does not appear in u). Now we will prove that $(\operatorname{FV} u) \setminus \{x\} = \left(\operatorname{map}_{x,y}(\operatorname{FV} u)\right) \setminus \{y\}$.

In fact, let z be an arbitrary element of $(\operatorname{FV} u) \setminus \{x\}$. Then, $z \neq x$, but also $z \in \operatorname{FV} u$, so that $z \neq y$ (since $z \in \operatorname{FV} u$ and $y \notin \operatorname{FV} u$). Now, due to $z \neq x$, we have $\operatorname{map}_{x,y} z = z$ (because $\operatorname{map}_{x,y} w = w$ for every variable $w \neq x$), and thus $z = \operatorname{map}_{x,y} z \in \operatorname{map}_{x,y}(\operatorname{FV} u)$ (since $z \in \operatorname{FV} u$). Together with $z \notin \{y\}$ (since $z \neq y$), this yields $z \in \left(\operatorname{map}_{x,y}(\operatorname{FV} u)\right) \setminus \{y\}$. Thus we have shown that every $z \in (\operatorname{FV} u) \setminus \{x\}$ satisfies $z \in \left(\operatorname{map}_{x,y}(\operatorname{FV} u)\right) \setminus \{y\}$. In other words, $(\operatorname{FV} u) \setminus \{x\} \subseteq \left(\operatorname{map}_{x,y}(\operatorname{FV} u)\right) \setminus \{y\}$.

Now, let z' be an arbitrary element of $\left(\operatorname{map}_{x,y}(\operatorname{FV} u)\right) \setminus \{y\}$. Then, $z' \in \operatorname{map}_{x,y}(\operatorname{FV} u)$, so that there exists some $w' \in \operatorname{FV} u$ such that $z' = \operatorname{map}_{x,y} w'$. Consider this w'. Clearly, $w' \neq x$ (since $w' = x$ would yield $z' = \operatorname{map}_{x,y} \underbrace{w'}_{=x} = \operatorname{map}_{x,y} x = y$, contradicting $z' \in \left(\operatorname{map}_{x,y}(\operatorname{FV} u)\right) \setminus \{y\}$). Thus, $\operatorname{map}_{x,y} w' = w'$

(since $\operatorname{map}_{x,y} w = w$ for every variable $w \neq x$). Thus, $z' = \operatorname{map}_{x,y} w' = w' \in \operatorname{FV} u$. Combined with $z' \notin \{x\}$ (since $z' = w' \neq x$), this yields $z' \in (\operatorname{FV} u) \setminus \{x\}$. Thus we have shown that every $z' \in \left(\operatorname{map}_{x,y}(\operatorname{FV} u)\right) \setminus \{y\}$ satisfies $z' \in (\operatorname{FV} u) \setminus \{x\}$. In other words, $\left(\operatorname{map}_{x,y}(\operatorname{FV} u)\right) \setminus \{y\} \subseteq (\operatorname{FV} u) \setminus \{x\}$. Combined with $(\operatorname{FV} u) \setminus \{x\} \subseteq \left(\operatorname{map}_{x,y}(\operatorname{FV} u)\right) \setminus \{y\}$, this yields $(\operatorname{FV} u) \setminus \{x\} = \left(\operatorname{map}_{x,y}(\operatorname{FV} u)\right) \setminus \{y\}$.

Thus,

$$\operatorname{FV}(\lambda x u) = (\operatorname{FV} u) \setminus \{x\} = \underbrace{\left(\operatorname{map}_{x,y}(\operatorname{FV} u)\right)}_{=\operatorname{FV}(u\langle y/x\rangle)} \setminus \{y\} = (\operatorname{FV}(u\langle y/x\rangle)) \setminus \{y\}.$$

Similarly, $\operatorname{FV}(\lambda x' u') = (\operatorname{FV}(u'\langle y/x'\rangle)) \setminus \{y\}$. Therefore, $\operatorname{FV}(u\langle y/x\rangle) = \operatorname{FV}(u'\langle y/x'\rangle)$ yields

$$\operatorname{FV}(\lambda x u) = \underbrace{(\operatorname{FV}(u\langle y/x\rangle))}_{=\operatorname{FV}(u'\langle y/x'\rangle)} \setminus \{y\} = (\operatorname{FV}(u'\langle y/x'\rangle)) \setminus \{y\} = \operatorname{FV}(\lambda x' u').$$

Lemma 1.D is proven.

Lemma 1.D is used in the proof that t and t' have the same free variables if $t \equiv t'$ (this fact is given without proof on page 12 of [1]).

Lemma 1.E. Let u be a term in L, and x be a variable. Then, $u\langle x/x\rangle = u$.

Proof of Lemma 1.E. This is a trivial induction proof (induction on u), so we omit it.

Lemma 1.E is used in the proof of Proposition 1.14 in [1] (in fact, it is the reason why $u'[x'/x'] = u'$).

3. Equivalent definitions of α-equivalence

Not everybody defines the notion of α-equivalence the same way as it is done in [1]. In some other texts, α-equivalence is defined in a different way, which, instead of the substitution $\langle t/x\rangle$ defined in [1], uses another notion of substitution:

Definition. For any term t in L and any variables x_1 and y_1, we define the term $t\{y_1/x_1\}$ as the result of the replacement of *every* occurence of x_1 in t by y_1 (where "every occurence" really means "every occurence", including bounded and free occurences and occurences in abstractions). The definition is by induction on t, as follows:

if $t = x_1$, then $t\{y_1/x_1\} = y_1$;

if t is a variable $\neq x_1$, then $t\{y_1/x_1\} = t$;

if $t = (u)\,v$ for some terms u and v, then $t\{y_1/x_1\} = (u\{y_1/x_1\})\,(v\{y_1/x_1\})$;

if $t = \lambda x u$ for some variable x and some term u, then $t\{y_1/x_1\} = \lambda\,(x\{y_1/x_1\})\,(u\{y_1/x_1\})$.

Intuitively, this $\{y_1/x_1\}$ substitution is a very low-level kind of substitution, best understood as a blind find-replace operation without regard to the meaning of the x_1's which are being replaced. Similarly one can define a substitution $\{y_1/x_1, ..., y_m/x_m\}$ for m variables $x_1, ..., x_m$ and m variables $y_1, ..., y_m$, but I will not use it.[2] Now here is the second definition of α-equivalence I am speaking about:

> **Definition.** Let us define a relation $=^\alpha$ on terms in L. [3] Namely, we define $t =^\alpha t'$ by induction on the length of t by the following clauses:
>
> if t is a variable, then $t =^\alpha t'$ if and only if $t = t'$;
>
> if $t = (u)\,v$ for some terms u and v, then $t =^\alpha t'$ if and only if $t' = (u')\,v'$ for some terms u' and v' with $u =^\alpha u'$ and $v =^\alpha v'$;
>
> if $t = \lambda x u$ for some variable x and some term u, then $t =^\alpha t'$ if and only if $t' = \lambda x' u'$ for some variable x' and some term u' such that all variables y except a finite number satisfy $u\,\{y/x\} =^\alpha u'\,\{y/x'\}$.

We claim that the relation $=^\alpha$ defined by this definition is the α-equivalence defined in [1]; i. e., we claim that the following theorem holds:

> **Theorem 1.F.** The relations $\equiv$ and $=^\alpha$ are identical.

We prove this using a lemma:

> **Lemma 1.G.** Let t be a term in L. Let x and y be two variables such that y does not occur in t. Then, $t\,\langle y/x\rangle \equiv t\,\{y/x\}$.

Proof of Lemma 1.G. We prove this by induction over t:

If t is a variable, then everything is clear because the definitions of $t\,\langle y/x\rangle$ and $t\,\{y/x\}$ for t being a variable are the same.

If $t = (u)\,v$ for some terms u and v, then everything is clear again because the definition of $t\,\langle y/x\rangle$ says

$$t\,\langle y/x\rangle = \underbrace{(u\,\langle y/x\rangle)}_{\substack{\equiv u\{y/x\} \\ \text{(by the induction} \\ \text{assumption)}}} \quad \underbrace{(v\,\langle y/x\rangle)}_{\substack{\equiv v\{y/x\} \\ \text{(by the induction} \\ \text{assumption)}}} \qquad (\text{since } t = (u)\,v)$$

$$\equiv (u\,\{y/x\})\,(v\,\{y/x\}) = t\,\{y/x\}$$

$$\left(\begin{array}{c} \text{since the definition of } t\,\{y/x\} \text{ says} \\ t\,\{y/x\} = (u\,\{y/x\})\,(v\,\{y/x\}) \ \ (\text{since } t = (u)\,v) \end{array}\right).$$

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[2]Note that $t\,\{s/x\}$ *cannot* be defined if s is just assumed to be an arbitrary term (rather than a single variable).

[3]We denote this relation by $=^\alpha$, but later (in Theorem 1.F) we will show that this relation is identical to the relation $\equiv$ from [1].

So it only remains to consider the case when $t = \lambda zu$ for some variable z and some term u. By the induction assumption, $u\langle y/x\rangle \equiv u\{y/x\}$.

Two subcases are possible: the subcase $z \neq x$ and the subcase $z = x$.

First consider the subcase $z \neq x$. In this subcase, $t\langle y/x\rangle = \lambda z\,(u\langle y/x\rangle) \equiv \lambda z\,(u\{y/x\})$ (by Corollary 1.7, since $u\langle y/x\rangle \equiv u\{y/x\}$) and $t\{y/x\} = \lambda\,(z\{y/x\})\,(u\{y/x\}) = \lambda z\,(u\{y/x\})$ (since $z \neq x$ and thus $z\{y/x\} = z$), so that $t\langle y/x\rangle \equiv \lambda z\,(u\{y/x\}) = t\{y/x\}$.

Now consider the subcase $z = x$. In this subcase, $t = \lambda zu = \lambda xu$, so that $t\langle y/x\rangle = \lambda xu$, but on the other hand $t = \lambda xu$ gives us $t\{y/x\} = \lambda\underbrace{(x\{y/x\})}_{=y}(u\{y/x\}) = \lambda y\,(u\{y/x\}) \equiv \lambda y\,(u\langle y/x\rangle)$ (by Corollary 1.7, since $u\{y/x\} \equiv u\langle y/x\rangle$). Since y does not occur in u (because y does not occur in t), we have $\lambda xu \equiv \lambda y\,(u\langle y/x\rangle)$ by Lemma 1.9, so that $t\langle y/x\rangle = \lambda xu \equiv \lambda y\,(u\langle y/x\rangle) \equiv t\{y/x\}$.

Hence, $t\langle y/x\rangle \equiv t\{y/x\}$ is proved in every case and every subcase. Lemma 1.G is thus proven.

Lemma 1.H. Let t and t' be two terms in L such that $t =^{\alpha} t'$. Then, $t \equiv t'$.

Proof of Lemma 1.H. We proceed by induction over the length of t.

There are three cases to consider: the case when t is a variable; the case when $t = (u)\,v$ for some terms u and v; the case when $t = \lambda xu$ for some variable x and some term u.

In the case when t is a variable, the relation $t =^{\alpha} t'$ yields that t' is the same variable as t. Thus, $t \equiv t'$.

In the case when $t = (u)\,v$ for some terms u and v, the relation $t =^{\alpha} t'$ yields that $t' = (u')\,v'$ for some terms u' and v' with $u =^{\alpha} u'$ and $v =^{\alpha} v'$. By the induction assumption, $u =^{\alpha} u'$ yields $u \equiv u'$, and $v =^{\alpha} v'$ yields $v \equiv v'$. Thus, $t' = (u')\,v'$ for some terms u' and v' with $u \equiv u'$ and $v \equiv v'$. This means that $t \equiv t'$.

Now let us consider the final remaining case: the case when $t = \lambda xu$ for some variable x and some term u. In this case, $t =^{\alpha} t'$ means that $t' = \lambda x'u'$ for some variable x' and some term u' such that all variables y except a finite number satisfy $u\{y/x\} =^{\alpha} u'\{y/x'\}$. By the induction assumption, this yields that all variables y except a finite number satisfy $u\{y/x\} \equiv u'\{y/x'\}$ (because the terms $u\{y/x\}$ and $u'\{y/x'\}$ are as long as u and u', respectively, and therefore shorter than t and t', respectively). Thus, all variables y except a finite number and except those which occur in u or u' satisfy $u\langle y/x\rangle \equiv u'\langle y/x'\rangle$ (because Lemma 1.G yields that these variables satisfy $u\langle y/x\rangle \equiv u\{y/x\}$ and $u'\langle y/x'\rangle \equiv u'\{y/x'\}$, so that they satisfy $u\langle y/x\rangle \equiv u\{y/x\} \equiv u'\{y/x'\} \equiv u'\langle y/x'\rangle$). But "all variables y except a finite number and except those which occur in u or u' " can be rewritten as "all variables y except a finite number", because only finitely many variables occur in u or u'. Thus, all variables y except a finite number satisfy $u\langle y/x\rangle \equiv u'\langle y/x'\rangle$. Hence, $t \equiv t'$ (by the definition of $\equiv$).

Thus we have proven that $t \equiv t'$ in all possible cases. The proof of Lemma 1.H is complete.

Lemma 1.I. Let t and t' be two terms in L such that $t \equiv t'$. Then, $t =^{\alpha} t'$.

Proof of Lemma 1.I. We proceed by induction over the length of t.

There are three cases to consider: the case when t is a variable; the case when $t = (u)\, v$ for some terms u and v; the case when $t = \lambda x u$ for some variable x and some term u.

In the case when t is a variable, the relation $t \equiv t'$ yields that t' is the same variable as t. Thus, $t =^{\alpha} t'$.

In the case when $t = (u)\, v$ for some terms u and v, the relation $t \equiv t'$ yields that $t' = (u')\, v'$ for some terms u' and v' with $u \equiv u'$ and $v \equiv v'$. By the induction assumption, $u \equiv u'$ yields $u =^{\alpha} u'$, and $v \equiv v'$ yields $v =^{\alpha} v'$. Thus, $t' = (u')\, v'$ for some terms u' and v' with $u =^{\alpha} u'$ and $v =^{\alpha} v'$. This means that $t =^{\alpha} t'$.

Now let us consider the final remaining case: the case when $t = \lambda x u$ for some variable x and some term u. In this case, $t \equiv t'$ means that $t' = \lambda x' u'$ for some variable x' and some term u' such that all variables y except a finite number satisfy $u \langle y/x \rangle \equiv u' \langle y/x' \rangle$. Thus, all variables y except a finite number and except those which occur in u or u' satisfy $u \{y/x\} \equiv u' \{y/x'\}$ (because Lemma 1.G yields that these variables satisfy $u \langle y/x \rangle \equiv u \{y/x\}$ and $u' \langle y/x' \rangle \equiv u' \{y/x'\}$, so that they satisfy $u \{y/x\} \equiv u \langle y/x \rangle \equiv u' \langle y/x' \rangle \equiv u' \{y/x'\}$). But "all variables y except a finite number and except those which occur in u or u' " can be rewritten as "all variables y except a finite number", because only finitely many variables occur in u or u'. Thus, all variables y except a finite number satisfy $u \{y/x\} \equiv u' \{y/x'\}$. By the induction assumption, this yields that all variables y except a finite number satisfy $u \{y/x\} =^{\alpha} u' \{y/x'\}$ (because the terms $u \{y/x\}$ and $u' \{y/x'\}$ are as long as u and u', respectively, and therefore shorter than t and t', respectively). Hence, $t =^{\alpha} t'$ (by the definition of $=^{\alpha}$).

Thus we have proven that $t =^{\alpha} t'$ in all possible cases. The proof of Lemma 1.I is complete.

Proof of Theorem 1.F. Theorem 1.F follows directly from Lemma 1.H and Lemma 1.I.

4. Some rules for substitution

Now we are going to prove the following properties of the substitution defined in Chapter 1 §2 of [1]:

Lemma 1.J. Any variable x and any $s \in \Lambda$ satisfy $x\,[s/x] = s$.

Lemma 1.K. Any two distinct variables x and y and any $s \in \Lambda$ satisfy $y\,[s/x] = y$.

Lemma 1.L. If $t_1 \in \Lambda$, $t_2 \in \Lambda$ and $s \in \Lambda$ are three equivalence classes and x is a variable, then $(t_1 t_2)\,[s/x] = (t_1\,[s/x])\,(t_2\,[s/x])$.

Lemma 1.M. If x and y are two distinct variables, and $s \in \Lambda$ and $r \in \Lambda$ are two equivalence classes, then $(\lambda y r)\,[s/x] = \lambda y' \,(r\,[y'/y]\,[s/x])$, where y' is any variable which is not free in x, s or r.

Lemma 1.N. If x and y are two distinct variables, and $s \in \Lambda$ and $r \in \Lambda$ are two equivalence classes such that y is not a free variable in s, then $(\lambda y r)[s/x] = \lambda y (r[s/x])$.

Lemma 1.O. If x is a variable, and $s \in \Lambda$ and $r \in \Lambda$ are two equivalence classes, then $(\lambda x r)[s/x] = \lambda x r$.

Proof of Lemma 1.J. Let $\underline{s}$ be a representative of the equivalence class s. Clearly, x is a representative of x, and no bound variable of x is free in $\underline{s}$ (since x has no bound variable). Therefore, by the definition of substitution, $x[s/x]$ is the equivalence class of $x\langle \underline{s}/x\rangle$. Since $x\langle \underline{s}/x\rangle = \underline{s}$, this means that $x[s/x]$ is the equivalence class of $\underline{s}$. In other words, $x[s/x] = s$ (because s is the equivalence class of $\underline{s}$). This proves Lemma 1.J.

Proof of Lemma 1.K. Let $\underline{s}$ be a representative of the equivalence class s. Clearly, y is a representative of y, and no bound variable of y is free in $\underline{s}$ (since y has no bound variable). Therefore, by the definition of substitution, $y[s/x]$ is the equivalence class of $y\langle \underline{s}/x\rangle$. Since $y\langle \underline{s}/x\rangle = y$, this means that $y[s/x]$ is the equivalence class of y. In other words, $y[s/x] = y$. This proves Lemma 1.K.

Proof of Lemma 1.L. Let $\underline{s}$ be a representative of the equivalence class s.

Let $\underline{t}_1$ be a representative of the equivalence class t_1 such that no bound variable of $\underline{t}_1$ is free in s. [4] Let $\underline{t}_2$ be a representative of the equivalence class t_2 such that no bound variable of $\underline{t}_2$ is free in s. [5] Then, clearly, no bound variable of $\underline{t}_1\underline{t}_2$ is free in s (since $\operatorname{BV}(\underline{t}_1\underline{t}_2) = (\operatorname{BV}\underline{t}_1) \cup (\operatorname{BV}\underline{t}_2)$), and we know that $\underline{t}_1\underline{t}_2$ is a representative of the equivalence class t_1t_2. Thus, the definition of $(t_1t_2)[s/x]$ says that $(t_1t_2)[s/x]$ is the equivalence class of $(\underline{t}_1\underline{t}_2)\langle \underline{s}/x\rangle$. On the other hand, the definition of $t_1[s/x]$ says that $t_1[s/x]$ is the equivalence class of $\underline{t}_1\langle \underline{s}/x\rangle$ (since no bound variable of $\underline{t}_1$ is free in s), and the definition of $t_2[s/x]$ says that $t_2[s/x]$ is the equivalence class of $\underline{t}_2\langle \underline{s}/x\rangle$ (since no bound variable of $\underline{t}_2$ is free in s). Since we know that $(\underline{t}_1\underline{t}_2)\langle \underline{s}/x\rangle = (\underline{t}_1\langle \underline{s}/x\rangle)(\underline{t}_2\langle \underline{s}/x\rangle)$, we therefore conclude that

$$\begin{aligned}
(t_1t_2)[s/x] &= \left(\text{equivalence class of } \underbrace{(\underline{t}_1\underline{t}_2)\langle \underline{s}/x\rangle}_{=(\underline{t}_1\langle \underline{s}/x\rangle)(\underline{t}_2\langle \underline{s}/x\rangle)}\right) \\
&= \left(\text{equivalence class of } (\underline{t}_1\langle \underline{s}/x\rangle)(\underline{t}_2\langle \underline{s}/x\rangle)\right) \\
&= \underbrace{\left(\text{equivalence class of } \underline{t}_1\langle \underline{s}/x\rangle\right)}_{=t_1[s/x]}\underbrace{\left(\text{equivalence class of } \underline{t}_2\langle \underline{s}/x\rangle\right)}_{=t_2[s/x]} \\
&= (t_1[s/x])(t_2[s/x]).
\end{aligned}$$

Lemma 1.L is proven.

Proof of Lemma 1.N. Let $\underline{s}$ be a representative of the equivalence class s. Let $\underline{r}$ be a representative of the equivalence class r such that no bound variable of $\underline{r}$

[4] Such a representative $\underline{t}_1$ exists due to Lemma 1.10.

[5] Such a representative $\underline{t}_2$ exists due to Lemma 1.10.

is free in s. [6] Then, $\lambda y \underline{r}$ is a representative of $\lambda y r$, and no bound variable of $\lambda y \underline{r}$ is free in s (because $\operatorname{BV}(\lambda y \underline{r}) = \{y\} \cup (\operatorname{BV} \underline{r})$, but neither y nor any bound variable of $\underline{r}$ is free in s). Therefore, by the definition of $(\lambda y r)[s/x]$, we know that $(\lambda y r)[s/x]$ is the equivalence class of $(\lambda y \underline{r}) \langle \underline{s}/x \rangle$. Since $(\lambda y \underline{r}) \langle \underline{s}/x \rangle = \lambda y (\underline{r} \langle \underline{s}/x \rangle)$ (because $x \neq y$), this rewrites as follows: $(\lambda y r)[s/x]$ is the equivalence class of $\lambda y (\underline{r} \langle \underline{s}/x \rangle)$. But since $r[s/x]$ is the equivalence class of $\underline{r} \langle \underline{s}/x \rangle$ (by the definition of $r[s/x]$, since no bound variable of $\underline{r}$ is free in s), the class $\lambda y (r[s/x])$ is the equivalence class of $\lambda y (\underline{r} \langle \underline{s}/x \rangle)$. So now we know that both $(\lambda y r)[s/x]$ and $\lambda y (r[s/x])$ are the equivalence class of $\lambda y (\underline{r} \langle \underline{s}/x \rangle)$. Thus, $(\lambda y r)[s/x] = \lambda y (r[s/x])$. This proves Lemma 1.N.

Proof of Lemma 1.M. Let y' be any variable which is not free in x, s or r. Then, y' is not free in $\lambda y r$ either. Proposition 1.14 (applied to $\lambda y r$, y, r and y' instead of t, x, u and x') yields $\lambda y r = \lambda y' (r[y'/y])$. Lemma 1.N (applied to y' and $r[y'/y]$ instead of y and r) yields $(\lambda y' (r[y'/y]))[s/x] = \lambda y' (r[y'/y][s/x])$ (here we use $y' \neq x$, which is because y' is not free in x). Thus, $\underbrace{(\lambda y r)}_{=\lambda y'(r[y'/y])} [s/x] = (\lambda y' (r[y'/y]))[s/x] = \lambda y' (r[y'/y][s/x])$. This proves Lemma 1.M.

Proof of Lemma 1.O. Let $\underline{s}$ be a representative of the equivalence class s. Let $\underline{p}$ be a representative of the equivalence class $\lambda x r$ such that no bound variable of $\underline{p}$ is free in s. [7] Then, the definition of $(\lambda x r)[s/x]$ yields that $(\lambda x r)[s/x]$ is the equivalence class of $\underline{p} \langle \underline{s}/x \rangle$. But x is not a free variable in $\underline{p}$ (because x is not a free variable in $\lambda x r$), and therefore $\underline{p} \langle \underline{s}/x \rangle = \underline{p}$ (by Lemma 1.1 in [1]). Hence, $(\lambda x r)[s/x]$ is the equivalence class of $\underline{p}$ (since $(\lambda x r)[s/x]$ is the equivalence class of $\underline{p} \langle \underline{s}/x \rangle$). In other words, $(\lambda x r)[s/x] = \lambda x r$ (since we know that the equivalence class of $\underline{p}$ is $\lambda x r$). This proves Lemma 1.O.

Appendix: Proof of Corollary 1.3 of [1], Chapter 1, §1

Below is a writeup of the proof of Corollary 1.3 of [1]. I made this writeup at a time when the proof given in [1] was wrong; now the proof in [1] was corrected, so there is no use in this writeup anymore except for the little bit of additional detail it gives.

Proof of Corollary 1.3: WLOG assume that $x_1, ..., x_u$ are those variables among the set $\{x_1, ..., x_m\}$ which don't occur in t. Then, $x_1, ..., x_u$ are not free in t, so that Lemma 1.1 yields $t \langle y_1/x_1, ..., y_m/x_m \rangle = t \langle y_{u+1}/x_{u+1}, ..., y_m/x_m \rangle$. Now, the sets $\{x_{u+1}, ..., x_m\}$ and $\{y_1, ..., y_m\}$ have no common elements (because every of the variables $x_{u+1}, ..., x_m$ occurs in t, while none of the variables $y_1, ..., y_m$ does). The hypothesis of Lemma 1.2 is satisfied (with $k = 0$), because none of the y_i is bound in t. Thus,

$$t \langle y_{u+1}/x_{u+1}, ..., y_m/x_m \rangle \langle t_1/y_1, ..., t_m/y_m \rangle = t \langle t_{u+1}/x_{u+1}, ..., t_m/x_m, t_1/y_1, ..., t_m/y_m \rangle .$$

But $y_1, ..., y_m$ are not free in t, and thus Lemma 1.1 yields

$$t \langle t_{u+1}/x_{u+1}, ..., t_m/x_m, t_1/y_1, ..., t_m/y_m \rangle = t \langle t_{u+1}/x_{u+1}, ..., t_m/x_m \rangle .$$

[6] Such a representative $\underline{r}$ exists due to Lemma 1.10.

[7] Such a representative $\underline{p}$ exists due to Lemma 1.10.

Finally, $x_1, ..., x_u$ are not free in t, so that Lemma 1.1 yields (again)

$$t \langle t_{u+1}/x_{u+1}, ..., t_m/x_m \rangle = t \langle t_1/x_1, ..., t_m/x_m \rangle .$$

Altogether,

$$\underbrace{t \langle y_1/x_1, ..., y_m/x_m \rangle}_{= t\langle y_{u+1}/x_{u+1}, ..., y_m/x_m \rangle} \langle t_1/y_1, ..., t_m/y_m \rangle$$
$$= t \langle y_{u+1}/x_{u+1}, ..., y_m/x_m \rangle \langle t_1/y_1, ..., t_m/y_m \rangle = t \langle t_{u+1}/x_{u+1}, ..., t_m/x_m, t_1/y_1, ..., t_m/y_m \rangle$$
$$= t \langle t_{u+1}/x_{u+1}, ..., t_m/x_m \rangle = t \langle t_1/x_1, ..., t_m/x_m \rangle ,$$

qed.

References

[1] Jean-Louis Krivine, *Lambda-calculus, types and models*, 22 January 2009, updated version of 5 June 2011.
`http://www.pps.jussieu.fr/~krivine/articles/Lambda.pdf`