

A hyperfactorial divisibility

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long version

Let us define a function $H : \mathbb{N} \rightarrow \mathbb{N}$ by $H(n) = \prod_{k=0}^{n-1} k!$ for every $n \in \mathbb{N}$.

Our goal is to prove the following theorem:

Theorem 0 (MacMahon). We have

$$H(b+c)H(c+a)H(a+b) \mid H(a)H(b)H(c)H(a+b+c)$$

for every $a \in \mathbb{N}$, every $b \in \mathbb{N}$ and every $c \in \mathbb{N}$.

Remark: Here, we denote by \mathbb{N} the set $\{0, 1, 2, \dots\}$ (and not the set $\{1, 2, 3, \dots\}$, as some authors do).

Before we come to the proof, first some definitions:

Notations.

- For any matrix A , we denote by $A \begin{bmatrix} j \\ i \end{bmatrix}$ the entry in the j -th column and the i -th row of A . [This is usually denoted by A_{ij} or by $A_{i,j}$.]
- Let R be a ring. Let $u \in \mathbb{N}$ and $v \in \mathbb{N}$, and let $a_{i,j}$ be an element of R for every $(i, j) \in \{1, 2, \dots, u\} \times \{1, 2, \dots, v\}$. Then, we denote by $(a_{i,j})_{\substack{1 \leq j \leq v \\ 1 \leq i \leq u}}$ the $u \times v$ matrix $A \in R^{u \times v}$ which satisfies $A \begin{bmatrix} j \\ i \end{bmatrix} = a_{i,j}$ for every $(i, j) \in \{1, 2, \dots, u\} \times \{1, 2, \dots, v\}$.
- Let R be a commutative ring with unity. Let $P \in R[X]$ be a polynomial. Let $j \in \mathbb{N}$. Then, we denote by $\text{coeff}_j P$ the coefficient of the polynomial P before X^j . (In particular, this implies $\text{coeff}_j P = 0$ for every $j > \deg P$.)

We recall a fact from linear algebra:

Theorem 1 (Vandermonde determinant). Let R be a commutative ring with unity. Let $m \in \mathbb{N}$. Let a_1, a_2, \dots, a_m be m elements of R . Then,

$$\det \left((a_i^{j-1})_{\substack{1 \leq j \leq m \\ 1 \leq i \leq m}} \right) = \prod_{\substack{(i,j) \in \{1,2,\dots,m\}^2; \\ i > j}} (a_i - a_j).$$

We are more interested in a corollary - and generalization - of this fact:

Theorem 2 (generalized Vandermonde determinant). Let R be a commutative ring with unity. Let $m \in \mathbb{N}$. For every $j \in \{1, 2, \dots, m\}$, let $P_j \in R[X]$ be a polynomial such that $\deg(P_j) \leq j - 1$. Let a_1, a_2, \dots, a_m be m elements of R . Then,

$$\det \left((P_j(a_i))_{\substack{1 \leq j \leq m \\ 1 \leq i \leq m}} \right) = \prod_{j=1}^m \text{coeff}_{j-1}(P_j) \cdot \prod_{\substack{(i,j) \in \{1,2,\dots,m\}^2; \\ i > j}} (a_i - a_j).$$

Both Theorems 1 and 2 can be deduced from the following lemma:

Lemma 3. Let R be a commutative ring with unity. Let $m \in \mathbb{N}$. For every $j \in \{1, 2, \dots, m\}$, let $P_j \in R[X]$ be a polynomial such that $\deg(P_j) \leq j - 1$. Let a_1, a_2, \dots, a_m be m elements of R . Then,

$$\det \left((P_j(a_i))_{\substack{1 \leq j \leq m \\ 1 \leq i \leq m}} \right) = \prod_{j=1}^m \text{coeff}_{j-1}(P_j) \cdot \det \left((a_i^{j-1})_{\substack{1 \leq j \leq m \\ 1 \leq i \leq m}} \right).$$

Proof of Lemma 3. For every $j \in \{1, 2, \dots, m\}$, we have $P_j(X) = \sum_{k=0}^{m-1} \text{coeff}_k(P_j) \cdot X^k$ (since $\deg(P_j) \leq j - 1 \leq m - 1$, since $j \leq m$). Thus, for every $i \in \{1, 2, \dots, m\}$ and $j \in \{1, 2, \dots, m\}$, we have

$$P_j(a_i) = \sum_{k=0}^{m-1} \text{coeff}_k(P_j) \cdot a_i^k = \sum_{k=0}^{m-1} a_i^k \cdot \text{coeff}_k(P_j) = \sum_{k=1}^m a_i^{k-1} \cdot \text{coeff}_{k-1}(P_j)$$

(here we substituted $k - 1$ for k in the sum).

Hence,

$$(P_j(a_i))_{\substack{1 \leq j \leq m \\ 1 \leq i \leq m}} = (a_i^{j-1})_{\substack{1 \leq j \leq m \\ 1 \leq i \leq m}} \cdot (\text{coeff}_{i-1}(P_j))_{\substack{1 \leq j \leq m \\ 1 \leq i \leq m}} \quad (1)$$

(according to the definition of the product of two matrices).

But the matrix $(\text{coeff}_{i-1}(P_j))_{\substack{1 \leq j \leq m \\ 1 \leq i \leq m}}$ is upper triangular (since $\text{coeff}_{i-1}(P_j) = 0$ for every $i \in \{1, 2, \dots, m\}$ and $j \in \{1, 2, \dots, m\}$ satisfying $i > j - 1$); hence, $\det \left((\text{coeff}_{i-1}(P_j))_{\substack{1 \leq j \leq m \\ 1 \leq i \leq m}} \right) = \prod_{j=1}^m \text{coeff}_{j-1}(P_j)$ (since the determinant of an upper triangular matrix equals the product of its diagonal entries).

Now, (1) yields

$$\begin{aligned} \det \left((P_j(a_i))_{\substack{1 \leq j \leq m \\ 1 \leq i \leq m}} \right) &= \det \left((a_i^{j-1})_{\substack{1 \leq j \leq m \\ 1 \leq i \leq m}} \cdot (\text{coeff}_{i-1}(P_j))_{\substack{1 \leq j \leq m \\ 1 \leq i \leq m}} \right) \\ &= \det \left((a_i^{j-1})_{\substack{1 \leq j \leq m \\ 1 \leq i \leq m}} \right) \cdot \underbrace{\det \left((\text{coeff}_{i-1}(P_j))_{\substack{1 \leq j \leq m \\ 1 \leq i \leq m}} \right)}_{= \prod_{j=1}^m \text{coeff}_{j-1}(P_j)} \\ &= \det \left((a_i^{j-1})_{\substack{1 \leq j \leq m \\ 1 \leq i \leq m}} \right) \cdot \prod_{j=1}^m \text{coeff}_{j-1}(P_j) = \prod_{j=1}^m \text{coeff}_{j-1}(P_j) \cdot \det \left((a_i^{j-1})_{\substack{1 \leq j \leq m \\ 1 \leq i \leq m}} \right), \end{aligned}$$

and thus, Lemma 3 is proven.

Proof of Theorem 1. For every $j \in \{1, 2, \dots, m\}$, define a polynomial $P_j \in R[X]$ by $P_j(X) = \prod_{k=1}^{j-1} (X - a_k)$. Then, P_j is a monic polynomial of degree $j - 1$ (since P_j is a product of $j - 1$ monic polynomials of degree 1 each²). In other words, $\deg(P_j) = j - 1$

¹because $i > j$ yields $i - 1 > j - 1$, thus $i - 1 > \deg(P_j)$ (since $\deg(P_j) \leq j - 1$) and therefore $\text{coeff}_{i-1}(P_j) = 0$

²because $X - a_k$ is a monic polynomial of degree 1 for every $k \in \{1, 2, \dots, j - 1\}$, and we have $P_j(X) = \prod_{k=1}^{j-1} (X - a_k)$

and $\text{coeff}_{j-1}(P_j) = 1$ for every $j \in \{1, 2, \dots, m\}$. Obviously, $\deg(P_j) = j - 1$ yields $\deg(P_j) \leq j - 1$ for every $j \in \{1, 2, \dots, m\}$. Thus, Lemma 3 yields

$$\det \left((P_j(a_i))_{\substack{1 \leq j \leq m \\ 1 \leq i \leq m}} \right) = \prod_{j=1}^m \text{coeff}_{j-1}(P_j) \cdot \det \left((a_i^{j-1})_{\substack{1 \leq j \leq m \\ 1 \leq i \leq m}} \right). \quad (2)$$

But the matrix $(P_j(a_i))_{\substack{1 \leq j \leq m \\ 1 \leq i \leq m}}$ is lower triangular (since $P_j(a_i) = 0$ for every $i \in \{1, 2, \dots, m\}$ and $j \in \{1, 2, \dots, m\}$ satisfying $i < j$ ³); hence, $\det \left((P_j(a_i))_{\substack{1 \leq j \leq m \\ 1 \leq i \leq m}} \right) = \prod_{j=1}^m P_j(a_j)$ (since the determinant of a lower triangular matrix equals the product of its diagonal entries). Thus, (2) becomes

$$\prod_{j=1}^m P_j(a_j) = \prod_{j=1}^m \text{coeff}_{j-1}(P_j) \cdot \det \left((a_i^{j-1})_{\substack{1 \leq j \leq m \\ 1 \leq i \leq m}} \right).$$

Since $\prod_{j=1}^m \underbrace{\text{coeff}_{j-1}(P_j)}_{=1} = \prod_{j=1}^m 1 = 1$, this simplifies to

$$\prod_{j=1}^m P_j(a_j) = \det \left((a_i^{j-1})_{\substack{1 \leq j \leq m \\ 1 \leq i \leq m}} \right).$$

Thus,

$$\begin{aligned} \det \left((a_i^{j-1})_{\substack{1 \leq j \leq m \\ 1 \leq i \leq m}} \right) &= \prod_{j=1}^m P_j(a_j) = \prod_{j=1}^m \prod_{k=1}^{j-1} (a_j - a_k) \\ &= \prod_{j \in \{1, 2, \dots, m\}} \prod_{k \in \{1, 2, \dots, j-1\}} (a_j - a_k) = \prod_{\substack{k \in \{1, 2, \dots, m\} \\ k < j}} (a_j - a_k); \\ &\left(\text{since } P_j(X) = \prod_{k=1}^{j-1} (X - a_k) \text{ yields } P_j(a_j) = \prod_{k=1}^{j-1} (a_j - a_k) \right) \\ &= \prod_{j \in \{1, 2, \dots, m\}} \prod_{\substack{k \in \{1, 2, \dots, m\} \\ k < j}} (a_j - a_k) = \prod_{\substack{(j, k) \in \{1, 2, \dots, m\}^2 \\ k < j}} (a_j - a_k) = \prod_{\substack{(i, j) \in \{1, 2, \dots, m\}^2 \\ j < i}} (a_i - a_j) \\ &\text{(here we renamed } j \text{ and } k \text{ as } i \text{ and } j \text{ in the product)} \\ &= \prod_{\substack{(i, j) \in \{1, 2, \dots, m\}^2 \\ i > j}} (a_i - a_j). \end{aligned}$$

³because $i < j$ yields $i \leq j - 1$ (since i and j are integers) and thus

$$\begin{aligned} P_j(a_i) &= \prod_{k=1}^{j-1} (a_i - a_k) \quad \left(\text{since } P_j(X) = \prod_{k=1}^{j-1} (X - a_k) \right) \\ &= 0 \end{aligned}$$

(since the factor of the product $\prod_{k=1}^{j-1} (a_i - a_k)$ for $k = i$ equals $a_i - a_i = 0$)

Hence, Theorem 1 is proven.

Proof of Theorem 2. Lemma 3 yields

$$\begin{aligned} \det \left((P_j(a_i))_{\substack{1 \leq j \leq m \\ 1 \leq i \leq m}} \right) &= \prod_{j=1}^m \text{coeff}_{j-1}(P_j) \cdot \underbrace{\det \left((a_i^{j-1})_{\substack{1 \leq j \leq m \\ 1 \leq i \leq m}} \right)}_{\substack{\prod_{\substack{(i,j) \in \{1,2,\dots,m\}^2; \\ i > j}} (a_i - a_j) \\ \text{by Theorem 1}}} \\ &= \prod_{j=1}^m \text{coeff}_{j-1}(P_j) \cdot \prod_{\substack{(i,j) \in \{1,2,\dots,m\}^2; \\ i > j}} (a_i - a_j). \end{aligned}$$

Hence, Theorem 2 is proven.

A consequence of Theorem 2:

Corollary 4. Let R be a commutative ring with unity. Let $m \in \mathbb{N}$. Let a_1, a_2, \dots, a_m be m elements of R . Then,

$$\det \left(\left(\prod_{k=1}^{j-1} (a_i - k) \right)_{\substack{1 \leq j \leq m \\ 1 \leq i \leq m}} \right) = \prod_{\substack{(i,j) \in \{1,2,\dots,m\}^2; \\ i > j}} (a_i - a_j).$$

Proof of Corollary 4. For every $j \in \{1, 2, \dots, m\}$, define a polynomial $P_j \in R[X]$ by $P_j(X) = \prod_{k=1}^{j-1} (X - k)$. Then, P_j is a monic polynomial of degree $j - 1$ (since P_j is a product of $j - 1$ monic polynomials of degree 1 each⁴). In other words, $\deg(P_j) = j - 1$ and $\text{coeff}_{j-1}(P_j) = 1$ for every $j \in \{1, 2, \dots, m\}$. Obviously, $\deg(P_j) = j - 1$ yields $\deg(P_j) \leq j - 1$ for every $j \in \{1, 2, \dots, m\}$. Thus, Theorem 2 yields

$$\det \left((P_j(a_i))_{\substack{1 \leq j \leq m \\ 1 \leq i \leq m}} \right) = \prod_{j=1}^m \text{coeff}_{j-1}(P_j) \cdot \prod_{\substack{(i,j) \in \{1,2,\dots,m\}^2; \\ i > j}} (a_i - a_j).$$

Since $P_j(a_i) = \prod_{k=1}^{j-1} (a_i - k)$ for every $i \in \{1, 2, \dots, m\}$ and $j \in \{1, 2, \dots, m\}$ (since

$P_j(X) = \prod_{k=1}^{j-1} (X - k)$), this becomes

$$\det \left(\left(\prod_{k=1}^{j-1} (a_i - k) \right)_{\substack{1 \leq j \leq m \\ 1 \leq i \leq m}} \right) = \prod_{j=1}^m \text{coeff}_{j-1}(P_j) \cdot \prod_{\substack{(i,j) \in \{1,2,\dots,m\}^2; \\ i > j}} (a_i - a_j).$$

⁴because $X - k$ is a monic polynomial of degree 1 for every $k \in \{1, 2, \dots, j - 1\}$, and we have

$$P_j(X) = \prod_{k=1}^{j-1} (X - k)$$

Since $\prod_{j=1}^m \underbrace{\text{coeff}_{j-1}(P_j)}_{=1} = \prod_{j=1}^m 1 = 1$, this simplifies to

$$\det \left(\left(\prod_{k=1}^{j-1} (a_i - k) \right)_{\substack{1 \leq j \leq m \\ 1 \leq i \leq m}} \right) = \prod_{\substack{(i,j) \in \{1,2,\dots,m\}^2; \\ i > j}} (a_i - a_j).$$

Hence, Corollary 4 is proven.

Also notice that:

Lemma 5. Let $m \in \mathbb{N}$. Then,

$$\prod_{\substack{(i,j) \in \{1,2,\dots,m\}^2; \\ i > j}} (i - j) = H(m).$$

Proof of Lemma 5. We have

$$\begin{aligned}
& \prod_{\substack{(i,j) \in \{1,2,\dots,m\}^2; \\ i > j}} (i-j) = \prod_{\substack{(i,j) \in \{0,1,\dots,m-1\}^2; \\ i+1 > j+1}} ((i+1) - (j+1)) \\
& \quad \text{(here we substituted } i+1 \text{ and } j+1 \text{ for } i \text{ and } j \text{ in the product)} \\
& = \prod_{\substack{(i,j) \in \{0,1,\dots,m-1\}^2; \\ i > j}} (i-j) \\
& \quad \left(\begin{array}{l} \text{since } (i+1) - (j+1) = i-j, \text{ and since} \\ i+1 > j+1 \text{ is equivalent to } i > j \end{array} \right) \\
& = \prod_{i \in \{0,1,\dots,m-1\}} \prod_{\substack{j \in \{0,1,\dots,m-1\}; \\ i > j}} (i-j) = \prod_{i \in \{0,1,\dots,m-1\}} \prod_{\substack{j \in \mathbb{N}; \\ j \leq m-1 \text{ and } i > j}} (i-j) \\
& \quad \text{(since } j \in \{0,1,\dots,m-1\} \text{ is equivalent to } (j \in \mathbb{N} \text{ and } j \leq m-1)) \\
& = \prod_{i \in \{0,1,\dots,m-1\}} \underbrace{\prod_{\substack{j \in \mathbb{N}; \\ i > j}} (i-j)}_{= \prod_{\substack{j \in \mathbb{N}; \\ j < i}}^{i-1}} \\
& \quad \left(\begin{array}{l} \text{since for every } i \in \{0,1,\dots,m-1\}, \text{ the assertion} \\ (j \leq m-1 \text{ and } i > j) \text{ is equivalent to } (i > j), \text{ because} \\ \text{if } i > j, \text{ then } j \leq m-1 \text{ (since } i \in \{0,1,\dots,m-1\} \text{ yields } i \leq m-1) \end{array} \right) \\
& = \prod_{i \in \{0,1,\dots,m-1\}} \prod_{j=0}^{i-1} (i-j) = \prod_{i \in \{0,1,\dots,m-1\}} \underbrace{\prod_{j=1}^i j}_{=i!} \\
& \quad \text{(here we substituted } i-j \text{ for } j \text{ in the second product)} \\
& = \underbrace{\prod_{i \in \{0,1,\dots,m-1\}} i!}_{= \prod_{i=0}^{m-1}} = \prod_{i=0}^{m-1} i! = \prod_{k=0}^{m-1} k! \quad \text{(here we renamed } i \text{ as } k \text{ in the product)} \\
& = H(m).
\end{aligned}$$

Hence, Lemma 5 is proven.

Now let us prove Theorem 0: Let $a \in \mathbb{N}$, let $b \in \mathbb{N}$ and let $c \in \mathbb{N}$.

We have

$$\begin{aligned}
H(a+b+c) &= \prod_{k=0}^{a+b+c-1} k! = \underbrace{\prod_{k=0}^{a+b-1} k!}_{=H(a+b)} \cdot \prod_{k=a+b}^{a+b+c-1} k! = H(a+b) \cdot \prod_{k=a+b}^{a+b+c-1} k! = H(a+b) \cdot \prod_{i=1}^c (a+b+i-1)! \\
& \quad \text{(here we substituted } a+b+i-1 \text{ for } k \text{ in the product),} \quad (3)
\end{aligned}$$

$$H(b+c) = \prod_{k=0}^{b+c-1} k! = \underbrace{\prod_{k=0}^{b-1} k!}_{=H(b)} \cdot \prod_{k=b}^{b+c-1} k! = H(b) \cdot \prod_{k=b}^{b+c-1} k! = H(b) \cdot \prod_{i=1}^c (b+i-1)! \quad (4)$$

(here we substituted $b+i-1$ for k in the product),

$$H(c+a) = \prod_{k=0}^{c+a-1} k! = \underbrace{\prod_{k=0}^{a-1} k!}_{=H(a)} \cdot \prod_{k=a}^{c+a-1} k! = H(a) \cdot \prod_{k=a}^{c+a-1} k! = H(a) \cdot \prod_{i=1}^c (a+i-1)! \quad (5)$$

(here we substituted $a+i-1$ for k in the product).

Next, we show a lemma:

Lemma 6. For every $i \in \mathbb{N}$ and $j \in \mathbb{N}$ satisfying $i \geq 1$ and $j \geq 1$, we have

$$\binom{a+b+i-1}{a+i-j} = \frac{(a+b+i-1)!}{(a+i-1)! \cdot (b+j-1)!} \cdot \prod_{k=1}^{j-1} (a+i-k).$$

Proof of Lemma 6. One of the following two cases must hold:

Case 1: We have $a+i-j \geq 0$.

Case 2: We have $a+i-j < 0$.

In Case 1, we have

$$\begin{aligned} (a+i-1)! &= \prod_{k=1}^{a+i-1} k = \underbrace{\prod_{k=1}^{a+i-j} k}_{=(a+i-j)!} \cdot \prod_{k=a+i-j+1}^{a+i-1} k = (a+i-j)! \cdot \prod_{k=a+i-j+1}^{a+i-1} k \\ &= (a+i-j)! \cdot \prod_{k=1}^{j-1} (a+i-k) \quad (\text{here we substituted } a+i-k \text{ for } k \text{ in the product}), \end{aligned}$$

so that

$$\frac{(a+i-1)!}{(a+i-j)!} = \prod_{k=1}^{j-1} (a+i-k). \quad (6)$$

Now,

$$\begin{aligned} \binom{a+b+i-1}{a+i-j} &= \frac{(a+b+i-1)!}{(a+i-j)! \cdot ((a+b+i-1) - (a+i-j))!} = \frac{(a+b+i-1)!}{(a+i-j)! \cdot (b+j-1)!} \\ &\quad (\text{since } (a+b+i-1) - (a+i-j) = b+j-1) \\ &= \frac{(a+b+i-1)!}{(a+i-1)! \cdot (b+j-1)!} \cdot \frac{(a+i-1)!}{(a+i-j)!} \\ &= \frac{(a+b+i-1)!}{(a+i-1)! \cdot (b+j-1)!} \cdot \prod_{k=1}^{j-1} (a+i-k) \quad (\text{by (6)}). \end{aligned}$$

Hence, Lemma 6 holds in Case 1.

In Case 2, we have

$$\begin{aligned} \prod_{k=1}^{j-1} (a+i-k) &= \prod_{k=1}^{a+i-1} (a+i-k) \cdot \underbrace{\prod_{k=a+i}^{a+i} (a+i-k)}_{=a+i-(a+i)=0} \cdot \prod_{k=a+i+1}^{j-1} (a+i-k) \\ & \quad (\text{since } a+i-j < 0 \text{ yields } a+i < j) \\ &= 0, \end{aligned}$$

so that

$$\begin{aligned} \binom{a+b+i-1}{a+i-j} &= 0 \quad (\text{since } a+i-j < 0) \\ &= \frac{(a+b+i-1)!}{(a+i-1)! \cdot (b+j-1)!} \cdot \underbrace{0}_{= \prod_{k=1}^{j-1} (a+i-k)} \\ &= \frac{(a+b+i-1)!}{(a+i-1)! \cdot (b+j-1)!} \cdot \prod_{k=1}^{j-1} (a+i-k). \end{aligned}$$

Hence, Lemma 6 holds in Case 2.

Hence, in both cases, Lemma 6 holds. Thus, Lemma 6 always holds, and this completes the proof of Lemma 6.

Another trivial lemma:

Lemma 7. Let R be a commutative ring with unity. Let $u \in \mathbb{N}$ and $v \in \mathbb{N}$, and let $a_{i,j}$ be an element of R for every $(i, j) \in \{1, 2, \dots, u\} \times \{1, 2, \dots, v\}$.

(a) Let $\alpha_1, \alpha_2, \dots, \alpha_u$ be u elements of R . Then,

$$\left(\begin{array}{c} \alpha_i, \text{ if } j = i; \\ 0, \text{ if } j \neq i \end{array} \right)_{1 \leq i \leq u}^{1 \leq j \leq u} \cdot (a_{i,j})_{1 \leq i \leq u}^{1 \leq j \leq v} = (\alpha_i a_{i,j})_{1 \leq i \leq u}^{1 \leq j \leq v}.$$

(b) Let $\beta_1, \beta_2, \dots, \beta_v$ be v elements of R . Then,

$$(a_{i,j})_{1 \leq i \leq u}^{1 \leq j \leq v} \cdot \left(\begin{array}{c} \beta_i, \text{ if } j = i; \\ 0, \text{ if } j \neq i \end{array} \right)_{1 \leq i \leq v}^{1 \leq j \leq v} = (a_{i,j} \beta_j)_{1 \leq i \leq u}^{1 \leq j \leq v}.$$

(c) Let $\alpha_1, \alpha_2, \dots, \alpha_u$ be u elements of R . Let $\beta_1, \beta_2, \dots, \beta_v$ be v elements of R . Then,

$$\left(\begin{array}{c} \alpha_i, \text{ if } j = i; \\ 0, \text{ if } j \neq i \end{array} \right)_{1 \leq i \leq u}^{1 \leq j \leq u} \cdot (a_{i,j})_{1 \leq i \leq u}^{1 \leq j \leq v} \cdot \left(\begin{array}{c} \beta_i, \text{ if } j = i; \\ 0, \text{ if } j \neq i \end{array} \right)_{1 \leq i \leq v}^{1 \leq j \leq v} = (\alpha_i a_{i,j} \beta_j)_{1 \leq i \leq u}^{1 \leq j \leq v}.$$

(d) Let $\alpha_1, \alpha_2, \dots, \alpha_u$ be u elements of R . Let $\beta_1, \beta_2, \dots, \beta_v$ be v elements of R . If $u = v$, then

$$\det \left((\alpha_i a_{i,j} \beta_j)_{1 \leq i \leq u}^{1 \leq j \leq v} \right) = \prod_{i=1}^u \alpha_i \cdot \prod_{i=1}^v \beta_i \cdot \det \left((a_{i,j})_{1 \leq i \leq u}^{1 \leq j \leq v} \right).$$

Proof of Lemma 7. (a) For every $i \in \{1, 2, \dots, u\}$ and $j \in \{1, 2, \dots, v\}$, we have

$$\begin{aligned}
& \sum_{\ell=1}^u \begin{cases} \alpha_i, & \text{if } \ell = i; \\ 0, & \text{if } \ell \neq i \end{cases} \cdot a_{\ell,j} = \sum_{\ell \in \{1,2,\dots,u\}} \begin{cases} \alpha_i, & \text{if } \ell = i; \\ 0, & \text{if } \ell \neq i \end{cases} \cdot a_{\ell,j} \\
&= \sum_{\ell \in \{1,2,\dots,u\}} \begin{cases} \alpha_i, & \text{if } \ell = i; \\ 0, & \text{if } \ell \neq i \end{cases} \cdot a_{\ell,j} + \sum_{\ell \in \{1,2,\dots,u\}; \ell \neq i} \begin{cases} \alpha_i, & \text{if } \ell = i; \\ 0, & \text{if } \ell \neq i \end{cases} \cdot a_{\ell,j} \\
&= \sum_{\ell \in \{1,2,\dots,u\}; \ell=i} \alpha_i a_{\ell,j} + \underbrace{\sum_{\ell \in \{1,2,\dots,u\}; \ell \neq i} 0 \cdot a_{\ell,j}}_{=0} = \sum_{\ell \in \{1,2,\dots,u\}; \ell=i} \alpha_i a_{\ell,j} = \sum_{\ell \in \{i\}} \alpha_i a_{\ell,j} \\
&\quad (\text{since } i \in \{1, 2, \dots, u\} \text{ yields } \{\ell \in \{1, 2, \dots, u\} \mid \ell = i\} = \{i\}) \\
&= \alpha_i a_{i,j}.
\end{aligned}$$

Thus,

$$\left(\begin{matrix} \alpha_i, & \text{if } j = i; \\ 0, & \text{if } j \neq i \end{matrix} \right)_{1 \leq i \leq u}^{1 \leq j \leq v} \cdot (a_{i,j})_{1 \leq i \leq u}^{1 \leq j \leq v} = \left(\sum_{\ell=1}^u \begin{cases} \alpha_i, & \text{if } \ell = i; \\ 0, & \text{if } \ell \neq i \end{cases} \cdot a_{\ell,j} \right)_{1 \leq i \leq u}^{1 \leq j \leq v} = (\alpha_i a_{i,j})_{1 \leq i \leq u}^{1 \leq j \leq v},$$

and thus, Lemma 7 (a) is proven.

(b) For every $i \in \{1, 2, \dots, u\}$ and $j \in \{1, 2, \dots, v\}$, we have

$$\begin{aligned}
& \sum_{\ell=1}^v a_{i,\ell} \cdot \begin{cases} \beta_\ell, & \text{if } j = \ell; \\ 0, & \text{if } j \neq \ell \end{cases} = \sum_{\ell \in \{1,2,\dots,v\}} a_{i,\ell} \cdot \begin{cases} \beta_\ell, & \text{if } j = \ell; \\ 0, & \text{if } j \neq \ell \end{cases} \\
&= \sum_{\ell \in \{1,2,\dots,v\}} a_{i,\ell} \cdot \begin{cases} \beta_\ell, & \text{if } j = \ell; \\ 0, & \text{if } j \neq \ell \end{cases} + \sum_{\ell \in \{1,2,\dots,v\}; \ell \neq j} a_{i,\ell} \cdot \begin{cases} \beta_\ell, & \text{if } j = \ell; \\ 0, & \text{if } j \neq \ell \end{cases} \\
&= \sum_{\ell \in \{1,2,\dots,v\}; \ell=j} a_{i,\ell} \beta_\ell + \underbrace{\sum_{\ell \in \{1,2,\dots,v\}; \ell \neq j} a_{i,\ell} \cdot 0}_{=0} = \sum_{\ell \in \{1,2,\dots,v\}; \ell=j} a_{i,\ell} \beta_\ell = \sum_{\ell \in \{j\}} a_{i,\ell} \beta_\ell \\
&\quad (\text{since } j \in \{1, 2, \dots, v\} \text{ yields } \{\ell \in \{1, 2, \dots, v\} \mid \ell = j\} = \{j\}) \\
&= a_{i,j} \beta_j.
\end{aligned}$$

Thus,

$$(a_{i,j})_{1 \leq i \leq u}^{1 \leq j \leq v} \cdot \left(\begin{matrix} \beta_i, & \text{if } j = i; \\ 0, & \text{if } j \neq i \end{matrix} \right)_{1 \leq i \leq v}^{1 \leq j \leq v} = \left(\sum_{\ell=1}^v a_{i,\ell} \cdot \begin{cases} \beta_\ell, & \text{if } j = \ell; \\ 0, & \text{if } j \neq \ell \end{cases} \right)_{1 \leq i \leq u}^{1 \leq j \leq v} = (a_{i,j} \beta_j)_{1 \leq i \leq u}^{1 \leq j \leq v},$$

and thus, Lemma 7 (b) is proven.

(c) We have

$$\begin{aligned} & \underbrace{\left(\begin{array}{c} \alpha_i, \text{ if } j = i; \\ 0, \text{ if } j \neq i \end{array} \right)_{1 \leq i \leq u}^{1 \leq j \leq u}}_{=(\alpha_i a_{i,j})_{1 \leq i \leq u}^{1 \leq j \leq v} \text{ by Lemma 7 (a)}} \cdot (a_{i,j})_{1 \leq i \leq u}^{1 \leq j \leq v} \cdot \left(\begin{array}{c} \beta_i, \text{ if } j = i; \\ 0, \text{ if } j \neq i \end{array} \right)_{1 \leq i \leq v}^{1 \leq j \leq v} \\ & = (\alpha_i a_{i,j})_{1 \leq i \leq u}^{1 \leq j \leq v} \cdot \left(\begin{array}{c} \beta_i, \text{ if } j = i; \\ 0, \text{ if } j \neq i \end{array} \right)_{1 \leq i \leq v}^{1 \leq j \leq v} = (\alpha_i a_{i,j} \beta_j)_{1 \leq i \leq u}^{1 \leq j \leq v} \end{aligned}$$

by Lemma 7 (b) (applied to $\alpha_i a_{i,j}$ instead of $a_{i,j}$).

Thus, Lemma 7 (c) is proven.

(d) The matrix $\left(\begin{array}{c} \alpha_i, \text{ if } j = i; \\ 0, \text{ if } j \neq i \end{array} \right)_{1 \leq i \leq u}^{1 \leq j \leq u}$ is diagonal (since $\begin{cases} \alpha_i, & \text{if } j = i; \\ 0, & \text{if } j \neq i \end{cases} = 0$ for every $i \in \{1, 2, \dots, u\}$ and $j \in \{1, 2, \dots, u\}$ satisfying $j \neq i$). Since the determinant of a diagonal matrix equals the product of its diagonal entries, this yields

$$\det \left(\left(\begin{array}{c} \alpha_i, \text{ if } j = i; \\ 0, \text{ if } j \neq i \end{array} \right)_{1 \leq i \leq u}^{1 \leq j \leq u} \right) = \prod_{i=1}^u \underbrace{\begin{cases} \alpha_i, & \text{if } i = i; \\ 0, & \text{if } i \neq i \end{cases}}_{=\alpha_i, \text{ since } i=i} = \prod_{i=1}^u \alpha_i.$$

Similarly,

$$\det \left(\left(\begin{array}{c} \beta_i, \text{ if } j = i; \\ 0, \text{ if } j \neq i \end{array} \right)_{1 \leq i \leq v}^{1 \leq j \leq v} \right) = \prod_{i=1}^v \beta_i.$$

Lemma 7 (c) yields

$$(\alpha_i a_{i,j} \beta_j)_{1 \leq i \leq u}^{1 \leq j \leq v} = \left(\begin{array}{c} \alpha_i, \text{ if } j = i; \\ 0, \text{ if } j \neq i \end{array} \right)_{1 \leq i \leq u}^{1 \leq j \leq u} \cdot (a_{i,j})_{1 \leq i \leq u}^{1 \leq j \leq v} \cdot \left(\begin{array}{c} \beta_i, \text{ if } j = i; \\ 0, \text{ if } j \neq i \end{array} \right)_{1 \leq i \leq v}^{1 \leq j \leq v}.$$

Thus, if $u = v$, then

$$\begin{aligned} \det \left((\alpha_i a_{i,j} \beta_j)_{1 \leq i \leq u}^{1 \leq j \leq v} \right) &= \det \left(\left(\begin{array}{c} \alpha_i, \text{ if } j = i; \\ 0, \text{ if } j \neq i \end{array} \right)_{1 \leq i \leq u}^{1 \leq j \leq u} \cdot (a_{i,j})_{1 \leq i \leq u}^{1 \leq j \leq v} \cdot \left(\begin{array}{c} \beta_i, \text{ if } j = i; \\ 0, \text{ if } j \neq i \end{array} \right)_{1 \leq i \leq v}^{1 \leq j \leq v} \right) \\ &= \underbrace{\det \left(\left(\begin{array}{c} \alpha_i, \text{ if } j = i; \\ 0, \text{ if } j \neq i \end{array} \right)_{1 \leq i \leq u}^{1 \leq j \leq u} \right)}_{=\prod_{i=1}^u \alpha_i} \cdot \det \left((a_{i,j})_{1 \leq i \leq u}^{1 \leq j \leq v} \right) \cdot \underbrace{\det \left(\left(\begin{array}{c} \beta_i, \text{ if } j = i; \\ 0, \text{ if } j \neq i \end{array} \right)_{1 \leq i \leq v}^{1 \leq j \leq v} \right)}_{=\prod_{i=1}^v \beta_i} \\ &= \prod_{i=1}^u \alpha_i \cdot \prod_{i=1}^v \beta_i \cdot \det \left((a_{i,j})_{1 \leq i \leq u}^{1 \leq j \leq v} \right). \end{aligned}$$

Thus, Lemma 7 (d) is proven.

Now, back to Theorem 0:

We have

$$\begin{aligned}
& \det \left(\left(\binom{a+b+i-1}{a+i-j} \right)_{\substack{1 \leq j \leq c \\ 1 \leq i \leq c}} \right) \\
&= \det \left(\left(\frac{(a+b+i-1)!}{(a+i-1)! \cdot (b+j-1)!} \cdot \prod_{k=1}^{j-1} (a+i-k) \right)_{\substack{1 \leq j \leq c \\ 1 \leq i \leq c}} \right) \quad (\text{by Lemma 6}) \\
&= \det \left(\left(\frac{(a+b+i-1)!}{(a+i-1)!} \cdot \prod_{k=1}^{j-1} (a+i-k) \cdot \frac{1}{(b+j-1)!} \right)_{\substack{1 \leq j \leq c \\ 1 \leq i \leq c}} \right) \\
&= \prod_{i=1}^c \frac{(a+b+i-1)!}{(a+i-1)!} \cdot \prod_{i=1}^c \frac{1}{(b+i-1)!} \cdot \det \left(\left(\prod_{k=1}^{j-1} (a+i-k) \right)_{\substack{1 \leq j \leq c \\ 1 \leq i \leq c}} \right)
\end{aligned}$$

(by Lemma 7 **(d)**, applied to $R = \mathbb{Q}$, $u = c$, $v = c$, $a_{i,j} = \prod_{k=1}^{j-1} (a+i-k)$, $\alpha_i = \frac{(a+b+i-1)!}{(a+i-1)!}$ and $\beta_i = \frac{1}{(b+i-1)!}$). Since

$$\begin{aligned}
& \det \left(\left(\prod_{k=1}^{j-1} (a+i-k) \right)_{\substack{1 \leq j \leq c \\ 1 \leq i \leq c}} \right) = \prod_{\substack{(i,j) \in \{1,2,\dots,c\}^2; \\ i > j}} \left(\underbrace{(a+i) - (a+j)}_{=i-j} \right) \\
& \quad (\text{by Corollary 4, applied to } R = \mathbb{Z}, m = c \text{ and } a_i = a+i \text{ for every } i \in \{1, 2, \dots, c\}) \\
&= \prod_{\substack{(i,j) \in \{1,2,\dots,c\}^2; \\ i > j}} (i-j) = H(c) \quad (\text{by Lemma 5, applied to } m = c),
\end{aligned}$$

this becomes

$$\det \left(\left(\binom{a+b+i-1}{a+i-j} \right)_{\substack{1 \leq j \leq c \\ 1 \leq i \leq c}} \right) = \prod_{i=1}^c \frac{(a+b+i-1)!}{(a+i-1)!} \cdot \prod_{i=1}^c \frac{1}{(b+i-1)!} \cdot H(c). \quad (7)$$

Now,

$$\begin{aligned}
\frac{H(a) H(b) H(c) H(a+b+c)}{H(b+c) H(c+a) H(a+b)} &= \frac{H(a) H(b) H(c) H(a+b) \cdot \prod_{i=1}^c (a+b+i-1)!}{H(b) \cdot \prod_{i=1}^c (b+i-1)! \cdot H(a) \cdot \prod_{i=1}^c (a+i-1)! \cdot H(a+b)} \\
&\text{(by (3), (4) and (5))} \\
&= \frac{\prod_{i=1}^c (a+b+i-1)!}{\prod_{i=1}^c (b+i-1)! \cdot \prod_{i=1}^c (a+i-1)!} \cdot H(c) = \frac{\prod_{i=1}^c (a+b+i-1)!}{\underbrace{\prod_{i=1}^c (a+i-1)!}_{= \prod_{i=1}^c \frac{(a+b+i-1)!}{(a+i-1)!}} \cdot \underbrace{\prod_{i=1}^c (b+i-1)!}_{= \prod_{i=1}^c \frac{1}{(b+i-1)!}}} \cdot H(c) \\
&= \prod_{i=1}^c \frac{(a+b+i-1)!}{(a+i-1)!} \cdot \prod_{i=1}^c \frac{1}{(b+i-1)!} \cdot H(c) = \det \left(\left(\binom{a+b+i-1}{a+i-j} \right)_{\substack{1 \leq j \leq c \\ 1 \leq i \leq c}} \right) \quad \text{(by (7))} \\
&\tag{8}
\end{aligned}$$

$\in \mathbb{Z}$

(since $\left(\binom{a+b+i-1}{a+i-j} \right)_{\substack{1 \leq j \leq c \\ 1 \leq i \leq c}} \in \mathbb{Z}^{c \times c}$). In other words,

$$H(b+c) H(c+a) H(a+b) \mid H(a) H(b) H(c) H(a+b+c).$$

Thus, Theorem 0 is proven.

Remarks.

1. Theorem 0 was briefly mentioned (with a combinatorial interpretation, but without proof) on the first page of [1]. It also follows from the formula (2.1) in [3] (since $\frac{H(a) H(b) H(c) H(a+b+c)}{H(b+c) H(c+a) H(a+b)} = \prod_{i=1}^c \frac{(a+b+i-1)! (i-1)!}{(a+i-1)! (b+i-1)!}$), or, equivalently, the formula (2.17) in [4]. It is also generalized in [2], Section 429 (where one has to consider the limit $x \rightarrow 1$).

2. We can prove more:

Theorem 8. For every $a \in \mathbb{N}$, every $b \in \mathbb{N}$ and every $c \in \mathbb{N}$, we have

$$\frac{H(a) H(b) H(c) H(a+b+c)}{H(b+c) H(c+a) H(a+b)} = \det \left(\left(\binom{a+b+i-1}{a+i-j} \right)_{\substack{1 \leq j \leq c \\ 1 \leq i \leq c}} \right) = \det \left(\left(\binom{a+b}{a+i-j} \right)_{\substack{1 \leq j \leq c \\ 1 \leq i \leq c}} \right).$$

We recall a useful fact to help us in the proof:

Theorem 9, the Vandermonde convolution identity. Let $x \in \mathbb{Z}$ and $y \in \mathbb{Z}$. Let $q \in \mathbb{Z}$. Then,

$$\binom{x+y}{q} = \sum_{k \in \mathbb{Z}} \binom{x}{k} \binom{y}{q-k}.$$

(The sum on the right hand side is an infinite sum, but only finitely many of its addends are nonzero.)

Proof of Theorem 8. For every $i \in \{1, 2, \dots, c\}$ and every $j \in \{1, 2, \dots, c\}$, we have

$$\begin{aligned}
\binom{a+b+i-1}{a+i-j} &= \sum_{k \in \mathbb{Z}} \binom{a+b}{k} \binom{i-1}{a+i-j-k} \\
&\quad (\text{by Theorem 9, applied to } x = a+b, y = i-1 \text{ and } q = a+i-j) \\
&= \sum_{\ell \in \mathbb{Z}} \binom{a+b}{a-j+\ell} \binom{i-1}{a+i-j-(a-j+\ell)} \\
&\quad (\text{here we substituted } a-j+\ell \text{ for } k \text{ in the sum}) \\
&= \sum_{\ell \in \mathbb{Z}} \binom{a+b}{a-j+\ell} \binom{i-1}{i-\ell} \\
&= \underbrace{\sum_{\substack{\ell \in \mathbb{Z}; \\ (0 \leq i-\ell \leq i-1 \text{ is true})}} \binom{a+b}{a-j+\ell} \binom{i-1}{i-\ell}}_{= \sum_{\substack{\ell \in \mathbb{Z}; \\ 0 \leq i-\ell \leq i-1 \\ (\text{since } 0 \leq i-\ell \leq i-1 \text{ is} \\ \text{equivalent to } 1 \leq \ell \leq i)}}} + \underbrace{\sum_{\substack{\ell \in \mathbb{Z}; \\ (0 \leq i-\ell \leq i-1 \text{ is false})}} \binom{a+b}{a-j+\ell} \binom{i-1}{i-\ell}}_{=0, \text{ since } i-1 \geq 0 \text{ and } (0 \leq i-\ell \leq i-1 \text{ is false})} \\
&= \sum_{\substack{\ell \in \mathbb{Z}; \\ 1 \leq \ell \leq i}} \binom{a+b}{a-j+\ell} \binom{i-1}{i-\ell} + \underbrace{\sum_{\substack{\ell \in \mathbb{Z}; \\ (0 \leq i-\ell \leq i-1 \text{ is false})}} \binom{a+b}{a-j+\ell}}_{=0} \cdot 0 = \sum_{\substack{\ell \in \mathbb{Z}; \\ 1 \leq \ell \leq i}} \underbrace{\binom{a+b}{a-j+\ell} \binom{i-1}{i-\ell}}_{= \sum_{\ell=1}^i \binom{i-1}{i-\ell} \binom{a+b}{a-j+\ell}} \\
&= \sum_{\ell=1}^i \binom{i-1}{i-\ell} \binom{a+b}{a-j+\ell} = \sum_{\ell=1}^c \binom{i-1}{i-\ell} \binom{a+b}{a-j+\ell} \\
&\quad \left(\begin{array}{l} \text{here we replaced the } \sum_{\ell=1}^i \text{ sign by an } \sum_{\ell=1}^c \text{ sign, since all addends} \\ \text{for } \ell > i \text{ are zero (as } \binom{i-1}{i-\ell} = 0 \text{ for } \ell > i, \text{ since } i-\ell < 0 \text{ for } \ell > i) \text{ and since } c \geq i \end{array} \right).
\end{aligned}$$

Thus,

$$\begin{aligned}
\left(\binom{a+b+i-1}{a+i-j} \right)_{1 \leq i \leq c}^{1 \leq j \leq c} &= \left(\sum_{\ell=1}^c \binom{i-1}{i-\ell} \binom{a+b}{a-j+\ell} \right)_{1 \leq i \leq c}^{1 \leq j \leq c} \\
&= \left(\binom{i-1}{i-j} \right)_{1 \leq i \leq c}^{1 \leq j \leq c} \cdot \left(\underbrace{\binom{a+b}{a-j+i}}_{= \binom{a+b}{a+i-j}} \right)_{1 \leq i \leq c}^{1 \leq j \leq c} \\
&= \left(\binom{i-1}{i-j} \right)_{1 \leq i \leq c}^{1 \leq j \leq c} \cdot \left(\binom{a+b}{a+i-j} \right)_{1 \leq i \leq c}^{1 \leq j \leq c}. \tag{9}
\end{aligned}$$

Now, the matrix $\left(\binom{i-1}{i-j}\right)_{\substack{1 \leq j \leq c \\ 1 \leq i \leq c}}$ is lower triangular (since $\binom{i-1}{i-j} = 0$ for every $i \in \{1, 2, \dots, m\}$ and $j \in \{1, 2, \dots, m\}$ satisfying $i < j$ ⁵). Since the determinant of an lower triangular matrix equals the product of its diagonal entries, this yields

$$\det \left(\left(\binom{i-1}{i-j} \right)_{\substack{1 \leq j \leq c \\ 1 \leq i \leq c}} \right) = \prod_{j=1}^m \underbrace{\binom{j-1}{j-j}}_{=\binom{j-1}{0}=1} = \prod_{j=1}^m 1 = 1. \quad (10)$$

Now,

$$\begin{aligned} \det \left(\left(\binom{a+b+i-1}{a+i-j} \right)_{\substack{1 \leq j \leq c \\ 1 \leq i \leq c}} \right) &= \det \left(\left(\binom{i-1}{i-j} \right)_{\substack{1 \leq j \leq c \\ 1 \leq i \leq c}} \cdot \left(\binom{a+b}{a+i-j} \right)_{\substack{1 \leq j \leq c \\ 1 \leq i \leq c}} \right) \\ &\quad \text{(by (9))} \\ &= \underbrace{\det \left(\left(\binom{i-1}{i-j} \right)_{\substack{1 \leq j \leq c \\ 1 \leq i \leq c}} \right)}_{=1 \text{ by (10)}} \cdot \det \left(\left(\binom{a+b}{a+i-j} \right)_{\substack{1 \leq j \leq c \\ 1 \leq i \leq c}} \right) \\ &= \det \left(\left(\binom{a+b}{a+i-j} \right)_{\substack{1 \leq j \leq c \\ 1 \leq i \leq c}} \right). \end{aligned}$$

Combined with (8), this yields

$$\frac{H(a)H(b)H(c)H(a+b+c)}{H(b+c)H(c+a)H(a+b)} = \det \left(\left(\binom{a+b+i-1}{a+i-j} \right)_{\substack{1 \leq j \leq c \\ 1 \leq i \leq c}} \right) = \det \left(\left(\binom{a+b}{a+i-j} \right)_{\substack{1 \leq j \leq c \\ 1 \leq i \leq c}} \right).$$

Thus, Theorem 8 is proven.

3. We notice a particularly known consequence of Corollary 4:

Corollary 10. Let $m \in \mathbb{N}$. Let a_1, a_2, \dots, a_m be m integers. Then,

$$\det \left(\left(\binom{a_i-1}{j-1} \right)_{\substack{1 \leq j \leq m \\ 1 \leq i \leq m}} \right) \cdot H(m) = \prod_{\substack{(i,j) \in \{1,2,\dots,m\}^2; \\ i > j}} (a_i - a_j).$$

In particular,

$$H(m) \mid \prod_{\substack{(i,j) \in \{1,2,\dots,m\}^2; \\ i > j}} (a_i - a_j).$$

Proof of Corollary 10. Corollary 4 (applied to $R = \mathbb{Z}$) yields

$$\det \left(\left(\prod_{k=1}^{j-1} (a_i - k) \right)_{\substack{1 \leq j \leq m \\ 1 \leq i \leq m}} \right) = \prod_{\substack{(i,j) \in \{1,2,\dots,m\}^2; \\ i > j}} (a_i - a_j). \quad (11)$$

⁵because $i < j$ yields $i - j < 0$ and thus $\binom{i-1}{i-j} = 0$

Now, for every $i \in \{1, 2, \dots, m\}$ and $j \in \{1, 2, \dots, m\}$, we have

$$\begin{aligned}
\binom{a_i - 1}{j - 1} &= \frac{\prod_{k=0}^{(j-1)-1} ((a_i - 1) - k)}{(j - 1)!} = \frac{1}{(j - 1)!} \prod_{k=0}^{(j-1)-1} ((a_i - 1) - k) = \frac{1}{(j - 1)!} \prod_{k=1}^{j-1} \underbrace{\left((a_i - 1) - (k - 1) \right)}_{=a_i - k} \\
&\quad \text{(here we substituted } k - 1 \text{ for } k \text{ in the product)} \\
&= \frac{1}{(j - 1)!} \prod_{k=1}^{j-1} (a_i - k) = 1 \cdot \prod_{k=1}^{j-1} (a_i - k) \cdot \frac{1}{(j - 1)!}. \tag{12}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\det \left(\left(\binom{a_i - 1}{j - 1} \right)_{\substack{1 \leq j \leq m \\ 1 \leq i \leq m}} \right) &= \det \left(\left(1 \cdot \prod_{k=1}^{j-1} (a_i - k) \cdot \frac{1}{(j - 1)!} \right)_{\substack{1 \leq j \leq m \\ 1 \leq i \leq m}} \right) \\
&= \underbrace{\prod_{i=1}^m 1}_{=1} \cdot \underbrace{\prod_{i=1}^m \frac{1}{(i - 1)!}}_1 \cdot \det \left(\left(\prod_{k=1}^{j-1} (a_i - k) \right)_{\substack{1 \leq j \leq m \\ 1 \leq i \leq m}} \right) \\
&= \frac{1}{\prod_{i=1}^m (i - 1)!} = \prod_{\substack{(i,j) \in \{1,2,\dots,m\}^2; \\ i > j}} (a_i - a_j) \quad \text{(by Lemma 7 (d), applied to } R = \mathbb{Q}, u = m, v = m, a_{i,j} = \prod_{k=1}^{j-1} (a_i - k), \alpha_i = 1 \text{ and } \beta_i = \frac{1}{(i - 1)!}) \\
&= \frac{1}{\prod_{i=1}^m (i - 1)!} \cdot \prod_{\substack{(i,j) \in \{1,2,\dots,m\}^2; \\ i > j}} (a_i - a_j),
\end{aligned}$$

so that

$$\begin{aligned}
\prod_{\substack{(i,j) \in \{1,2,\dots,m\}^2; \\ i > j}} (a_i - a_j) &= \det \left(\left(\binom{a_i - 1}{j - 1} \right)_{\substack{1 \leq j \leq m \\ 1 \leq i \leq m}} \right) \cdot \prod_{i=1}^m (i - 1)! \\
&= \det \left(\left(\binom{a_i - 1}{j - 1} \right)_{\substack{1 \leq j \leq m \\ 1 \leq i \leq m}} \right) \cdot \underbrace{\prod_{k=0}^{m-1} k!}_{=H(m)} \\
&\quad \text{(here we substituted } k \text{ for } i - 1 \text{ in the product)} \\
&= \det \left(\left(\binom{a_i - 1}{j - 1} \right)_{\substack{1 \leq j \leq m \\ 1 \leq i \leq m}} \right) \cdot H(m).
\end{aligned}$$

Thus,

$$H(m) \mid \prod_{\substack{(i,j) \in \{1,2,\dots,m\}^2; \\ i > j}} (a_i - a_j)$$

(since $\det \left(\left(\binom{a_i - 1}{j - 1} \right)_{\substack{1 \leq j \leq m \\ 1 \leq i \leq m}} \right) \in \mathbb{Z}$). Thus, Corollary 10 is proven.

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⁶See also

<http://www.archive.org/details/combinatoryanal01macmuoft>
for volume 1 and
<http://www.archive.org/details/introductiontoco00macmrich>
for an introduction.