# A hyperfactorial divisibility

*Darij Grinberg*

***brief version***

Let us define a function $H : \mathbb{N} \to \mathbb{N}$ by $H(n) = \prod_{k=0}^{n-1} k!$ for every $n \in \mathbb{N}$.

Our goal is to prove the following theorem:

**Theorem 0 (MacMahon).** We have

$$H(b+c) H(c+a) H(a+b) \mid H(a) H(b) H(c) H(a+b+c)$$

for every $a \in \mathbb{N}$, every $b \in \mathbb{N}$ and every $c \in \mathbb{N}$.

*Remark:* Here, we denote by $\mathbb{N}$ the set $\{0, 1, 2, ...\}$ (and not the set $\{1, 2, 3, ...\}$, as some authors do).

Before we come to the proof, first some definitions:

**Notations.**

- Let $R$ be a ring. Let $u \in \mathbb{N}$ and $v \in \mathbb{N}$, and let $a_{i,j}$ be an element of $R$ for every $(i,j) \in \{1, 2, ..., u\} \times \{1, 2, ..., v\}$. Then, we denote by $(a_{i,j})_{1 \leq i \leq u}^{1 \leq j \leq v}$ the $u \times v$ matrix $A \in R^{u \times v}$ whose entry in row $i$ and column $j$ is $a_{i,j}$ for every $(i,j) \in \{1, 2, ..., u\} \times \{1, 2, ..., v\}$.

- Let $R$ be a commutative ring with unity. Let $P \in R[X]$ be a polynomial. Let $j \in \mathbb{N}$. Then, we denote by $\operatorname{coeff}_j P$ the coefficient of the polynomial $P$ before $X^j$. (In particular, this implies $\operatorname{coeff}_j P = 0$ for every $j > \deg P$.)

We are first going to prove a known fact from linear algebra:

**Theorem 1 (Vandermonde determinant).** Let $R$ be a commutative ring with unity. Let $m \in \mathbb{N}$. Let $a_1, a_2, ..., a_m$ be $m$ elements of $R$. Then,

$$\det\left(\left(a_i^{j-1}\right)_{1 \leq i \leq m}^{1 \leq j \leq m}\right) = \prod_{\substack{(i,j) \in \{1,2,...,m\}^2; \\ i > j}} (a_i - a_j).$$

Actually we are more interested in a corollary - and generalization - of this fact:

**Theorem 2 (generalized Vandermonde determinant).** Let $R$ be a commutative ring with unity. Let $m \in \mathbb{N}$. For every $j \in \{1, 2, ..., m\}$, let $P_j \in R[X]$ be a polynomial such that $\deg(P_j) \leq j - 1$. Let $a_1, a_2, ..., a_m$ be $m$ elements of $R$. Then,

$$\det\left(\left(P_j(a_i)\right)_{1 \leq i \leq m}^{1 \leq j \leq m}\right) = \prod_{j=1}^{m} \operatorname{coeff}_{j-1}(P_j) \cdot \prod_{\substack{(i,j) \in \{1,2,...,m\}^2; \\ i > j}} (a_i - a_j).$$

Both Theorems 1 and 2 can be deduced from the following lemma:

**Lemma 3.** Let $R$ be a commutative ring with unity. Let $m \in \mathbb{N}$. For every $j \in \{1, 2, ..., m\}$, let $P_j \in R[X]$ be a polynomial such that $\deg(P_j) \leq j - 1$. Let $a_1$, $a_2$, ..., $a_m$ be $m$ elements of $R$. Then,

$$\det\left(\left(P_j(a_i)\right)_{1\leq i\leq m}^{1\leq j\leq m}\right) = \prod_{j=1}^{m} \operatorname{coeff}_{j-1}(P_j) \cdot \det\left(\left(a_i^{j-1}\right)_{1\leq i\leq m}^{1\leq j\leq m}\right).$$

*Proof of Lemma 3.* For every $j \in \{1, 2, ..., m\}$, we have $P_j(X) = \sum_{k=0}^{m-1} \operatorname{coeff}_k(P_j) \cdot X^k$ (since $\deg(P_j) \leq j - 1 \leq m - 1$). Thus, for every $i \in \{1, 2, ..., m\}$ and $j \in \{1, 2, ..., m\}$, we have

$$P_j(a_i) = \sum_{k=0}^{m-1} \operatorname{coeff}_k(P_j) \cdot a_i^k = \sum_{k=0}^{m-1} a_i^k \cdot \operatorname{coeff}_k(P_j) = \sum_{k=1}^{m} a_i^{k-1} \cdot \operatorname{coeff}_{k-1}(P_j)$$

(here we substituted $k - 1$ for $k$ in the sum). Hence,

$$\left(P_j(a_i)\right)_{1\leq i\leq m}^{1\leq j\leq m} = \left(a_i^{j-1}\right)_{1\leq i\leq m}^{1\leq j\leq m} \cdot \left(\operatorname{coeff}_{i-1}(P_j)\right)_{1\leq i\leq m}^{1\leq j\leq m}.$$

But the matrix $\left(\operatorname{coeff}_{i-1}(P_j)\right)_{1\leq i\leq m}^{1\leq j\leq m}$ is upper triangular (since $\operatorname{coeff}_{i-1}(P_j) = 0$ for every $i \in \{1, 2, ..., m\}$ and $j \in \{1, 2, ..., m\}$ satisfying $i > j$ [1]); hence, $\det\left(\left(\operatorname{coeff}_{i-1}(P_j)\right)_{1\leq i\leq m}^{1\leq j\leq m}\right) = \prod_{j=1}^{m} \operatorname{coeff}_{j-1}(P_j)$ (since the determinant of an upper triangular matrix equals the product of its diagonal entries).

Now,

$$\begin{aligned}\det\left(\left(P_j(a_i)\right)_{1\leq i\leq m}^{1\leq j\leq m}\right) &= \det\left(\left(a_i^{j-1}\right)_{1\leq i\leq m}^{1\leq j\leq m} \cdot \left(\operatorname{coeff}_{i-1}(P_j)\right)_{1\leq i\leq m}^{1\leq j\leq m}\right) \\ &= \det\left(\left(a_i^{j-1}\right)_{1\leq i\leq m}^{1\leq j\leq m}\right) \cdot \underbrace{\det\left(\left(\operatorname{coeff}_{i-1}(P_j)\right)_{1\leq i\leq m}^{1\leq j\leq m}\right)}_{=\prod_{j=1}^{m} \operatorname{coeff}_{j-1}(P_j)} \\ &= \prod_{j=1}^{m} \operatorname{coeff}_{j-1}(P_j) \cdot \det\left(\left(a_i^{j-1}\right)_{1\leq i\leq m}^{1\leq j\leq m}\right),\end{aligned}$$

and thus, Lemma 3 is proven.

*Proof of Theorem 1.* For every $j \in \{1, 2, ..., m\}$, define a polynomial $P_j \in R[X]$ by $P_j(X) = \prod_{k=1}^{j-1} (X - a_k)$. Then, $P_j$ is a monic polynomial of degree $j - 1$. In other words, $\deg(P_j) = j - 1$ and $\operatorname{coeff}_{j-1}(P_j) = 1$ for every $j \in \{1, 2, ..., m\}$. Thus, Lemma 3 yields

$$\det\left(\left(P_j(a_i)\right)_{1\leq i\leq m}^{1\leq j\leq m}\right) = \prod_{j=1}^{m} \operatorname{coeff}_{j-1}(P_j) \cdot \det\left(\left(a_i^{j-1}\right)_{1\leq i\leq m}^{1\leq j\leq m}\right). \quad (1)$$

[1] because $i > j$ yields $i - 1 > j - 1$, thus $i - 1 > \deg(P_j)$ (since $\deg(P_j) \leq j - 1$) and therefore $\operatorname{coeff}_{i-1}(P_j) = 0$

But the matrix $\left(P_j\left(a_i\right)\right)_{1\leq i\leq m}^{1\leq j\leq m}$ is lower triangular (since $P_j\left(a_i\right)=0$ for every $i\in\{1,2,...,m\}$ and $j\in\{1,2,...,m\}$ satisfying $i<j$, as follows quickly from the definition of $P_j$); hence, $\det\left(\left(P_j\left(a_i\right)\right)_{1\leq i\leq m}^{1\leq j\leq m}\right)=\prod\limits_{j=1}^{m}P_j\left(a_j\right)$ (since the determinant of a lower triangular matrix equals the product of its diagonal entries). Thus, (1) becomes

$$\prod_{j=1}^{m}P_j\left(a_j\right)=\prod_{j=1}^{m}\underbrace{\operatorname{coeff}_{j-1}\left(P_j\right)}_{=1}\cdot\det\left(\left(a_i^{j-1}\right)_{1\leq i\leq m}^{1\leq j\leq m}\right)=\det\left(\left(a_i^{j-1}\right)_{1\leq i\leq m}^{1\leq j\leq m}\right).$$

But $P_j\left(X\right)=\prod\limits_{k=1}^{j-1}\left(X-a_k\right)$ yields $P_j\left(a_j\right)=\prod\limits_{k=1}^{j-1}\left(a_j-a_k\right)$, so that

$$\det\left(\left(a_i^{j-1}\right)_{1\leq i\leq m}^{1\leq j\leq m}\right)=\prod_{j=1}^{m}P_j\left(a_j\right)=\prod_{j=1}^{m}\prod_{k=1}^{j-1}\left(a_j-a_k\right)=\prod_{\substack{(j,k)\in\{1,2,...,m\}^2;\\ k<j}}\left(a_j-a_k\right)$$
$$=\prod_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ j<i}}\left(a_i-a_j\right)=\prod_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ i>j}}\left(a_i-a_j\right).$$

Hence, Theorem 1 is proven.

Now, Theorem 2 immediately follows from Lemma 3 and Theorem 1.

A consequence of Theorem 2:

**Corollary 4.** Let $R$ be a commutative ring with unity. Let $m\in\mathbb{N}$. Let $a_1$, $a_2$, ..., $a_m$ be $m$ elements of $R$. Then,

$$\det\left(\left(\prod_{k=1}^{j-1}\left(a_i-k\right)\right)_{1\leq i\leq m}^{1\leq j\leq m}\right)=\prod_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ i>j}}\left(a_i-a_j\right).$$

*Proof of Corollary 4.* For every $j\in\{1,2,...,m\}$, define a polynomial $P_j\in R\left[X\right]$ by $P_j\left(X\right)=\prod\limits_{k=1}^{j-1}\left(X-k\right)$. Then, $P_j$ is a monic polynomial of degree $j-1$. In other words, $\deg\left(P_j\right)=j-1$ and $\operatorname{coeff}_{j-1}\left(P_j\right)=1$ for every $j\in\{1,2,...,m\}$. Thus, applying Theorem 2 to these polynomials $P_j$ yields the assertion of Corollary 4.

Also notice that:

**Lemma 5.** Let $m\in\mathbb{N}$. Then,

$$\prod_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ i>j}}\left(i-j\right)=H\left(m\right).$$

*Proof of Lemma 5.* We have

$$\prod_{\substack{(i,j)\in\{1,2,\ldots,m\}^2;\\ i>j}} (i-j) = \prod_{\substack{(i,j)\in\{0,1,\ldots,m-1\}^2;\\ i>j}} (i-j)$$

(here we shifted $i$ and $j$ by 1, which doesn't change anything since $i-j$ remains constant)

$$= \prod_{i\in\{0,1,\ldots,m-1\}} \prod_{\substack{j\in\{0,1,\ldots,m-1\};\\ i>j}} (i-j) = \prod_{i\in\{0,1,\ldots,m-1\}} \prod_{j=0}^{i-1} (i-j) = \prod_{i\in\{0,1,\ldots,m-1\}} \underbrace{\prod_{j=1}^{i} j}_{=i!}$$

(here, we substituted $j$ for $i-j$ in the second product)

$$= \prod_{i\in\{0,1,\ldots,m-1\}} i! = \prod_{k=0}^{m-1} k! = H(m).$$

Hence, Lemma 5 is proven.

Now, we notice that every $a \in \mathbb{N}$, every $b \in \mathbb{N}$ and every $c \in \mathbb{N}$ satisfy

$$H(a+b+c) = \prod_{k=0}^{a+b+c-1} k! = \underbrace{\prod_{k=0}^{a+b-1} k!}_{=H(a+b)} \cdot \prod_{k=a+b}^{a+b+c-1} k! = H(a+b) \cdot \prod_{k=a+b}^{a+b+c-1} k! = H(a+b) \cdot \prod_{i=1}^{c} (a+b+i-1)!$$

$$\text{(here we substituted } a+b+i-1 \text{ for } k \text{ in the product)}, \qquad (2)$$

$$H(b+c) = \prod_{k=0}^{b+c-1} k! = \underbrace{\prod_{k=0}^{b-1} k!}_{=H(b)} \cdot \prod_{k=b}^{b+c-1} k! = H(b) \cdot \prod_{k=b}^{b+c-1} k! = H(b) \cdot \prod_{i=1}^{c} (b+i-1)!$$

$$\text{(here we substituted } b+i-1 \text{ for } k \text{ in the product)}, \qquad (3)$$

$$H(c+a) = \prod_{k=0}^{c+a-1} k! = \underbrace{\prod_{k=0}^{a-1} k!}_{=H(a)} \cdot \prod_{k=a}^{c+a-1} k! = H(a) \cdot \prod_{k=a}^{c+a-1} k! = H(a) \cdot \prod_{i=1}^{c} (a+i-1)!$$

$$\text{(here we substituted } a+i-1 \text{ for } k \text{ in the product)}. \qquad (4)$$

Next, a technical lemma.

**Lemma 6.** For every $i \in \mathbb{N}$ and $j \in \mathbb{N}$ satisfying $i \geq 1$ and $j \geq 1$, we have

$$\binom{a+b+i-1}{a+i-j} = \frac{(a+b+i-1)!}{(a+i-1)! \cdot (b+j-1)!} \cdot \prod_{k=1}^{j-1} (a+i-k).$$

The *proof* of this lemma is completely straightforward: Either we have $a+i-j \geq 0$ and Lemma 6 follows from standard manipulations with binomial coefficients, or we have $a+i-j < 0$ and Lemma 6 follows from $\binom{a+b+i-1}{a+i-j} = 0$ and $\prod_{k=1}^{j-1} (a+i-k) = 0$.

Another trivial lemma:

**Lemma 7.** Let $R$ be a commutative ring with unity. Let $u \in \mathbb{N}$, and let $a_{i,j}$ be an element of $R$ for every $(i,j) \in \{1,2,...,u\}^2$.

Let $\alpha_1$, $\alpha_2$, ..., $\alpha_u$ be $u$ elements of $R$. Let $\beta_1$, $\beta_2$, ..., $\beta_u$ be $u$ elements of $R$. Then,

$$\det\left(\left(\alpha_i a_{i,j} \beta_j\right)_{1 \leq i \leq u}^{1 \leq j \leq u}\right) = \prod_{i=1}^{u} \alpha_i \cdot \prod_{i=1}^{u} \beta_i \cdot \det\left(\left(a_{i,j}\right)_{1 \leq i \leq u}^{1 \leq j \leq u}\right).$$

This is clear because the matrix $\left(\alpha_i a_{i,j} \beta_j\right)_{1 \leq i \leq u}^{1 \leq j \leq u}$ can be written as the product

$$\operatorname{diag}\left(\alpha_1, \alpha_2, ..., \alpha_u\right) \cdot \left(a_{i,j}\right)_{1 \leq i \leq u}^{1 \leq j \leq u} \cdot \operatorname{diag}\left(\beta_1, \beta_2, ..., \beta_u\right),$$

and thus

$$\begin{aligned} \det\left(\left(\alpha_i a_{i,j} \beta_j\right)_{1 \leq i \leq u}^{1 \leq j \leq u}\right) &= \det\left(\operatorname{diag}\left(\alpha_1, \alpha_2, ..., \alpha_u\right) \cdot \left(a_{i,j}\right)_{1 \leq i \leq u}^{1 \leq j \leq u} \cdot \operatorname{diag}\left(\beta_1, \beta_2, ..., \beta_u\right)\right) \\ &= \underbrace{\det\left(\operatorname{diag}\left(\alpha_1, \alpha_2, ..., \alpha_u\right)\right)}_{=\prod_{i=1}^{u} \alpha_i} \cdot \det\left(\left(a_{i,j}\right)_{1 \leq i \leq u}^{1 \leq j \leq u}\right) \cdot \underbrace{\det\left(\operatorname{diag}\left(\beta_1, \beta_2, ..., \beta_u\right)\right)}_{=\prod_{i=1}^{u} \beta_i}. \end{aligned}$$

Now, back to proving Theorem 0:

We have

$$\begin{aligned} &\det\left(\left(\binom{a+b+i-1}{a+i-j}\right)_{1 \leq i \leq c}^{1 \leq j \leq c}\right) \\ &= \det\left(\left(\frac{(a+b+i-1)!}{(a+i-1)! \cdot (b+j-1)!} \cdot \prod_{k=1}^{j-1} (a+i-k)\right)_{1 \leq i \leq c}^{1 \leq j \leq c}\right) \qquad \text{(by Lemma 6)} \\ &= \det\left(\left(\frac{(a+b+i-1)!}{(a+i-1)!} \cdot \prod_{k=1}^{j-1} (a+i-k) \cdot \frac{1}{(b+j-1)!}\right)_{1 \leq i \leq c}^{1 \leq j \leq c}\right) \\ &= \prod_{i=1}^{c} \frac{(a+b+i-1)!}{(a+i-1)!} \cdot \prod_{i=1}^{c} \frac{1}{(b+i-1)!} \cdot \det\left(\left(\prod_{k=1}^{j-1} (a+i-k)\right)_{1 \leq i \leq c}^{1 \leq j \leq c}\right) \end{aligned}$$

(by Lemma 7, applied to $R = \mathbb{Q}$, $u = c$, $a_{i,j} = \prod_{k=1}^{j-1} (a+i-k)$, $\alpha_i = \frac{(a+b+i-1)!}{(a+i-1)!}$ and $\beta_i = \frac{1}{(b+i-1)!}$). Since

$$\begin{aligned} \det\left(\left(\prod_{k=1}^{j-1} (a+i-k)\right)_{1 \leq i \leq c}^{1 \leq j \leq c}\right) &= \prod_{\substack{(i,j) \in \{1,2,...,c\}^2; \\ i > j}} \left(\underbrace{(a+i)-(a+j)}_{=i-j}\right) \\ &\qquad \text{(by Corollary 4, applied to } R = \mathbb{Z}\text{, } m = c \text{ and } a_i = a+i \text{ for every } i \in \{1,2,...,c\}\text{)} \\ &= \prod_{\substack{(i,j) \in \{1,2,...,c\}^2; \\ i > j}} (i-j) = H(c) \qquad \text{(by Lemma 5, applied to } m = c\text{)}, \end{aligned}$$

this becomes

$$\det\left(\left(\binom{a+b+i-1}{a+i-j}\right)_{1\leq i\leq c}^{1\leq j\leq c}\right)=\prod_{i=1}^{c}\frac{(a+b+i-1)!}{(a+i-1)!}\cdot\prod_{i=1}^{c}\frac{1}{(b+i-1)!}\cdot H(c). \qquad (5)$$

Now,

$$\frac{H(a)H(b)H(c)H(a+b+c)}{H(b+c)H(c+a)H(a+b)}=\frac{H(a)H(b)H(c)H(a+b)\cdot\prod_{i=1}^{c}(a+b+i-1)!}{H(b)\cdot\prod_{i=1}^{c}(b+i-1)!\cdot H(a)\cdot\prod_{i=1}^{c}(a+i-1)!\cdot H(a+b)}$$

(by (2), (3) and (4))

$$=\frac{\prod_{i=1}^{c}(a+b+i-1)!}{\prod_{i=1}^{c}(b+i-1)!\cdot\prod_{i=1}^{c}(a+i-1)!}\cdot H(c)=\underbrace{\frac{\prod_{i=1}^{c}(a+b+i-1)!}{\prod_{i=1}^{c}(a+i-1)!}}_{=\prod_{i=1}^{c}\frac{(a+b+i-1)!}{(a+i-1)!}}\cdot\underbrace{\frac{1}{\prod_{i=1}^{c}(b+i-1)!}}_{=\prod_{i=1}^{c}\frac{1}{(b+i-1)!}}\cdot H(c)$$

$$=\prod_{i=1}^{c}\frac{(a+b+i-1)!}{(a+i-1)!}\cdot\prod_{i=1}^{c}\frac{1}{(b+i-1)!}\cdot H(c)=\det\left(\left(\binom{a+b+i-1}{a+i-j}\right)_{1\leq i\leq c}^{1\leq j\leq c}\right) \quad \text{(by (5))} \qquad (6)$$

$$\in\mathbb{Z}$$

(since $\left(\binom{a+b+i-1}{a+i-j}\right)_{1\leq i\leq c}^{1\leq j\leq c}\in\mathbb{Z}^{c\times c}$). In other words,

$$H(b+c)H(c+a)H(a+b)\mid H(a)H(b)H(c)H(a+b+c).$$

Thus, Theorem 0 is finally proven.

**Remarks.**

**1.** Theorem 0 was briefly mentioned (with a combinatorial interpretation, but without proof) on the first page of [1]. It also follows from the formula (2.1) in [3] (since $\frac{H(a)H(b)H(c)H(a+b+c)}{H(b+c)H(c+a)H(a+b)}=\prod_{i=1}^{c}\frac{(a+b+i-1)!\,(i-1)!}{(a+i-1)!\,(b+i-1)!}$), or, equivalently, the formula (2.17) in [4]. It is also generalized in [2], Section 429 (where one has to consider the limit $x\to 1$).

**2.** We can prove more:

**Theorem 8.** For every $a\in\mathbb{N}$, every $b\in\mathbb{N}$ and every $c\in\mathbb{N}$, we have

$$\frac{H(a)H(b)H(c)H(a+b+c)}{H(b+c)H(c+a)H(a+b)}=\det\left(\left(\binom{a+b+i-1}{a+i-j}\right)_{1\leq i\leq c}^{1\leq j\leq c}\right)=\det\left(\left(\binom{a+b}{a+i-j}\right)_{1\leq i\leq c}^{1\leq j\leq c}\right).$$

We recall a useful fact to help us in the proof:

**Theorem 9, the Vandermonde convolution identity.** Let $x \in \mathbb{Z}$ and $y \in \mathbb{Z}$. Let $q \in \mathbb{Z}$. Then,

$$\binom{x+y}{q} = \sum_{k \in \mathbb{Z}} \binom{x}{k} \binom{y}{q-k}.$$

(The sum on the right hand side is an infinite sum, but only finitely many of its addends are nonzero.)

*Proof of Theorem 8.* For every $i \in \{1, 2, ..., c\}$ and every $j \in \{1, 2, ..., c\}$, we have

$$\binom{a+b+i-1}{a+i-j} = \sum_{k \in \mathbb{Z}} \binom{a+b}{k} \binom{i-1}{a+i-j-k}$$

(by Theorem 9, applied to $x = a+b$, $y = i-1$ and $q = a+i-j$)

$$= \sum_{\ell \in \mathbb{Z}} \binom{a+b}{a-j+\ell} \binom{i-1}{i-\ell} \qquad \text{(here we substituted } a-j+\ell \text{ for } k \text{ in the sum)}$$

$$= \sum_{\ell=1}^{c} \binom{a+b}{a-j+\ell} \binom{i-1}{i-\ell} \qquad \left( \begin{array}{c} \text{here, we restricted the summation from } \ell \in \mathbb{Z} \text{ to } \ell \in \{1,2,...,c\}, \\ \text{which doesn't change the sum because} \\ \binom{a+b}{a-j+\ell} \binom{i-1}{i-\ell} = 0 \text{ for all } \ell \in \mathbb{Z} \diagdown \{1,2,...,c\} \end{array} \right)$$

$$= \sum_{\ell=1}^{c} \binom{i-1}{i-\ell} \binom{a+b}{a-j+\ell}.$$

Thus,

$$\begin{aligned} \left( \binom{a+b+i-1}{a+i-j} \right)_{1 \le i \le c}^{1 \le j \le c} &= \left( \sum_{\ell=1}^{c} \binom{i-1}{i-\ell} \binom{a+b}{a-j+\ell} \right)_{1 \le i \le c}^{1 \le j \le c} \\ &= \left( \binom{i-1}{i-j} \right)_{1 \le i \le c}^{1 \le j \le c} \cdot \left( \binom{a+b}{a-j+i} \right)_{1 \le i \le c}^{1 \le j \le c} \\ &= \left( \binom{i-1}{i-j} \right)_{1 \le i \le c}^{1 \le j \le c} \cdot \left( \binom{a+b}{a+i-j} \right)_{1 \le i \le c}^{1 \le j \le c}. \end{aligned} \tag{7}$$

Now, the matrix $\left( \binom{i-1}{i-j} \right)_{1 \le i \le c}^{1 \le j \le c}$ is lower triangular (since $\binom{i-1}{i-j} = 0$ for every $i \in \{1, 2, ..., m\}$ and $j \in \{1, 2, ..., m\}$ satisfying $i < j$). Since the determinant of an lower triangular matrix equals the product of its diagonal entries, this yields

$$\det \left( \left( \binom{i-1}{i-j} \right)_{1 \le i \le c}^{1 \le j \le c} \right) = \prod_{j=1}^{m} \underbrace{\binom{j-1}{j-j}}_{= \binom{j-1}{0} = 1} = 1. \tag{8}$$

Now,

$$\det\left(\left(\binom{a+b+i-1}{a+i-j}\right)_{1\le i\le c}^{1\le j\le c}\right) = \det\left(\left(\binom{i-1}{i-j}\right)_{1\le i\le c}^{1\le j\le c} \cdot \left(\binom{a+b}{a+i-j}\right)_{1\le i\le c}^{1\le j\le c}\right)$$

(by (7))

$$= \underbrace{\det\left(\left(\binom{i-1}{i-j}\right)_{1\le i\le c}^{1\le j\le c}\right)}_{=1 \text{ by } (8)} \cdot \det\left(\left(\binom{a+b}{a+i-j}\right)_{1\le i\le c}^{1\le j\le c}\right)$$

$$= \det\left(\left(\binom{a+b}{a+i-j}\right)_{1\le i\le c}^{1\le j\le c}\right).$$

Combined with (6), this yields Theorem 8.

**3.** We notice a particularly known consequence of Corollary 4:

**Corollary 10.** Let $m \in \mathbb{N}$. Let $a_1$, $a_2$, ..., $a_m$ be $m$ integers. Then,

$$\det\left(\left(\binom{a_i-1}{j-1}\right)_{1\le i\le m}^{1\le j\le m}\right) \cdot H(m) = \prod_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ i>j}} (a_i - a_j).$$

In particular,

$$H(m) \mid \prod_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ i>j}} (a_i - a_j).$$

*Proof of Corollary 10.* For every $i \in \{1,2,...,m\}$ and $j \in \{1,2,...,m\}$, we have

$$\binom{a_i-1}{j-1} = \frac{\prod_{k=1}^{j-1}(a_i-k)}{(j-1)!} = 1 \cdot \prod_{k=1}^{j-1}(a_i-k)\cdot\frac{1}{(j-1)!}. \tag{9}$$

Therefore,

$$\det\left(\left(\binom{a_i-1}{j-1}\right)_{1\le i\le m}^{1\le j\le m}\right) = \det\left(\left(1\cdot\prod_{k=1}^{j-1}(a_i-k)\cdot\frac{1}{(j-1)!}\right)_{1\le i\le m}^{1\le j\le m}\right)$$

$$= \underbrace{\prod_{i=1}^{m} 1}_{=1} \cdot \underbrace{\prod_{i=1}^{m}\frac{1}{(i-1)!}}_{=\prod_{k=0}^{m-1}\frac{1}{k!}=\frac{1}{\prod_{i=1}^{m-1}k!}=\frac{1}{H(m)}} \cdot \underbrace{\det\left(\left(\prod_{k=1}^{j-1}(a_i-k)\right)_{1\le i\le m}^{1\le j\le m}\right)}_{=\prod_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ i>j}}(a_i-a_j) \text{ by Corollary 4}}$$

$$\left(\text{by Lemma 7, applied to } R=\mathbb{Q},\ u=m,\ a_{i,j}=\prod_{k=1}^{j-1}(a_i-k),\ \alpha_i=1 \text{ and } \beta_i=\frac{1}{(i-1)!}\right)$$

$$= \frac{1}{H(m)} \cdot \prod_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ i>j}} (a_i - a_j),$$

so that

$$\prod_{\substack{(i,j)\in\{1,2,\ldots,m\}^2;\\ i>j}} (a_i - a_j) = \det\left(\left(\binom{a_i - 1}{j-1}\right)_{1\leq i\leq m}^{1\leq j\leq m}\right) \cdot H(m).$$

Thus,

$$H(m) \mid \prod_{\substack{(i,j)\in\{1,2,\ldots,m\}^2;\\ i>j}} (a_i - a_j)$$

(since $\det\left(\left(\underbrace{\binom{a_i - 1}{j-1}}_{\in\mathbb{Z}}\right)_{1\leq i\leq m}^{1\leq j\leq m}\right) \in \mathbb{Z}$). Thus, Corollary 10 is proven.

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[2] See also
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for volume 1 and
`http://www.archive.org/details/introductiontoco00macmrich`
for an introduction.