At least $|S|^{k-1} \cdot (|S| - 1)$ frontiers: a graph theory problem

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1. Problem

Let $S$ be a finite set. Let $k$ be a positive integer. Let $A$ be a subset of $S^k$ satisfying $|A| = |S|^{k-1}$. Let $B = S^k \setminus A$.

For every $v \in S^k$ and every $i \in \{1, 2, \ldots, k\}$, we denote by $v_i$ the $i$-th component of the $k$-tuple $v$ (remember that $v$ is an element of $S^k$, that is, a $k$-tuple of elements of $S$). Then, every $v \in S^k$ satisfies $v = (v_1, v_2, \ldots, v_k)$.

Let $F$ be the set of all pairs $(a, b) \in A \times B$ for which there exists an $i \in \{1, 2, \ldots, k\}$ satisfying $(a_j = b_j$ for all $j \neq i)$. (Speaking less formally, let $F$ be the set of all pairs $(a, b) \in A \times B$ for which the $k$-tuples $a$ and $b$ differ in at most one position.)

Prove that $|F| \geq |S|^{k-1} \cdot (|S| - 1)$.

2. Remark

In the case $|S| = 2$, this is an old problem (which appeared, for example, in a Moscow MO 1962 preparation booklet, and which is a particular case of Cheeger’s inequality for the hypercube).

I use to call the elements of $F$ “frontiers” between the sets $A$ and $B$.

3. Solution

Since $A \subseteq S^k$ and $B = S^k \setminus A$, we have $A \cap B = \emptyset$ and $A \cup B = S^k$. Hence, $S^k \setminus B = A$.

Define a map $\phi : A \times B \to F$ as follows:

\[\]

\[1\]Of course, “for all $j \neq i$” means “for all $j \in \{1, 2, \ldots, k\}$ satisfying $j \neq i$” here.
Let \((u, v) \in A \times B\) be a pair. Then, \(u \in A\) and \(v \in B\), so that \(u \notin B\) and \(v \notin A\) (since \(A \cap B = \emptyset\)).

We define a subset \(T\) of \(\{0, 1, 2, \ldots, k\}\) by

\[
T = \{i \in \{0, 1, 2, \ldots, k\} \mid (v_1, v_2, \ldots, v_i, u_{i+1}, u_{i+2}, \ldots, u_k) \in B\}
\]

Then, \(0 \notin T\) (since \((u_1, u_2, \ldots, u_k) = u \notin B\)) and \(k \in T\) (since \((v_1, v_2, \ldots, v_k) = v \in B\)). In particular, \(k \in T\) yields \(T \neq \emptyset\). Thus, the set \(T\) has a minimal element (since \(T\) is a finite set). Let \(a\) be this minimal element. Then, \(a \in T\) and \(a - 1 \notin T\). We have \(a \neq 0\) (since \(a \in T\) but \(0 \notin T\)). Thus, \(a - 1 \in \{0, 1, 2, \ldots, k\}\) (since \(a \in T \subseteq \{0, 1, 2, \ldots, k\}\)).

Now, \(a \in T\) yields \((v_1, v_2, \ldots, v_n, u_{n+1}, u_{n+2}, \ldots, u_k) \in B\), while \(a - 1 \notin T\) yields \((v_1, v_2, \ldots, v_{n+1}, u_{n+2}, \ldots, u_k) \notin B\), so that \((v_1, v_2, \ldots, v_{n+1}, u_{n+2}, \ldots, u_k) \in S^k \setminus B = A\). Set \(a = (v_1, v_2, \ldots, v_{n+1}, u_{n+2}, \ldots, u_k)\) and \(b = (v_1, v_2, \ldots, v_{n+1}, u_{n+2}, \ldots, u_k)\). Then, \(a = (v_1, v_2, \ldots, v_{n+1}, u_{n+2}, \ldots, u_k) \in A\) and \(b = (v_1, v_2, \ldots, v_{n+1}, u_{n+2}, \ldots, u_k) \in B\), so that \((a, b) \in A \times B\). Besides, there exists an \(i \in \{1, 2, \ldots, k\}\) satisfying \((a_i = b_j\) for all \(j \neq i\)) (namely, \(i = a\) \footnote{For \(i = 0\), the notation \((v_1, v_2, \ldots, v_i, u_{i+1}, u_{i+2}, \ldots, u_k)\) means \((u_1, u_2, \ldots, u_k)\). For \(i = k\), the notation \((v_1, v_2, \ldots, v_i, u_{i+1}, u_{i+2}, \ldots, u_k)\) means \((v_1, v_2, \ldots, v_k)\).}). Hence, \((a, b) \in F\) (by the definition of \(F\)).

Now set \(\phi (u, v) = (a, b)\). Thus we have defined a map \(\phi : A \times B \to F\).

Next, we will prove that \(|\phi^{-1} (\{(a, b)\})| \leq |S|^{k-1}\) for every \((a, b) \in F\). In fact, let \((a, b) \in F\). Since \((a, b) \in F\), we have \((a, b) \in A \times B\), so that \(a \in A\) and \(b \in B\), so that \(a \neq b\) (since \(A \cap B = \emptyset\)). But since \((a, b) \in F\),

there exists an \(i \in \{1, 2, \ldots, k\}\) satisfying \((a_j = b_j\) for all \(j \neq i\)). \footnote{In fact, \(a_j = b_j\) for all \(j \neq a\) (in fact, for any \(j\), we have \(a_j = \begin{cases} v_j, & \text{if } j < a; \\ u_j, & \text{if } j \geq a \end{cases}\) and \(b_j = \begin{cases} v_j, & \text{if } j \leq a; \\ u_j, & \text{if } j > a; \\ v_j, & \text{if } j > a \end{cases}\), so that \(a_j = b_j\) for all \(j \neq a\).} \footnote{In fact, otherwise, we would have \(a_j = b_j\), what, combined with \(a_j = b_j\) for all \(j \neq i\), would yield \(a_j = b_j\) for all \(j \in \{1, 2, \ldots, k\}\), so that \(a = b\), contradicting \(a \neq b\).} (1)

Consider this \(i\).

We must have \(a_i \neq b_i\) \footnote{In fact, the notation \((v_1, v_2, \ldots, v_i, u_{i+1}, u_{i+2}, \ldots, u_k)\) means \((u_1, u_2, \ldots, u_k)\). For \(i = k\), the notation \((v_1, v_2, \ldots, v_i, u_{i+1}, u_{i+2}, \ldots, u_k)\) means \((v_1, v_2, \ldots, v_k)\).}

Now, consider some \((u, v) \in \phi^{-1} (\{(a, b)\})\). Then, \(\phi (u, v) = (a, b)\). Thus, by the definition of \(\phi\), there exists an \(\alpha \in \{0, 1, 2, \ldots, k\}\) such that

\[
a = (v_1, v_2, \ldots, v_{n+1}, u_{n+2}, \ldots, u_k) \quad \text{and} \quad b = (v_1, v_2, \ldots, v_{n+1}, u_{n+2}, \ldots, u_k).
\] (2)

Consider this \(\alpha\).
We must have \( u_\alpha \neq v_\alpha \) \(^5\) Since \( a_\alpha = u_\alpha \) and \( b_\alpha = v_\alpha \), this yields \( a_\alpha \neq b_\alpha \). Hence, \( \alpha = i \) (since otherwise, we would have \( a \neq i \), so that \( a_\alpha = b_\alpha \) by \(^1\), contradicting \( a_\alpha \neq b_\alpha \)). Thus, \(^2\) becomes

\[
a = (v_1, v_2, \ldots, v_{i-1}, u_{i}, u_{i+1}, \ldots, u_k) \quad \text{and} \quad b = (v_1, v_2, \ldots, v_i, u_{i+1}, u_{i+2}, \ldots, u_k).
\]

Now, \( a = (v_1, v_2, \ldots, v_{i-1}, u_i, u_{i+1}, \ldots, u_k) \) yields \( a_j = u_j \) for all \( j \geq i \). Hence,

\[
u = (v_1, v_2, \ldots, v_{i-1}, u_i, u_{i+1}, \ldots, u_k) = (u_1, u_2, \ldots, u_{i-1}, a_i, a_{i+1}, \ldots, a_k)
\]

\( \in S^{i-1} \times \{a_i\} \times \{a_{i+1}\} \times \cdots \times \{a_k\} \). \(^3\)

Also, \( b = (v_1, v_2, \ldots, v_i, u_{i+1}, u_{i+2}, \ldots, u_k) \) yields \( b_j = v_j \) for all \( j \leq i \). Hence,

\[
v = (v_1, v_2, \ldots, v_i, v_{i+1}, v_{i+2}, \ldots, v_k) = (b_1, b_2, \ldots, b_i, v_{i+1}, v_{i+2}, \ldots, v_k)
\]

\( \in \{b_1\} \times \{b_2\} \times \cdots \times \{b_i\} \times S^{k-i}. \) \(^4\)

By \(^3\) and \(^4\), we have

\[
(u, v) \in \left( S^{i-1} \times \{a_i\} \times \{a_{i+1}\} \times \cdots \times \{a_k\} \right) \times \left( \{b_1\} \times \{b_2\} \times \cdots \times \{b_i\} \times S^{k-i} \right)
\]

for every \((u, v) \in \phi^{-1}(\{(a, b)\})\). Thus,

\[
\left| \phi^{-1}(\{(a, b)\}) \right| \leq \left| S^{i-1} \times \{a_i\} \times \{a_{i+1}\} \times \cdots \times \{a_k\} \right| \times \left| \{b_1\} \times \{b_2\} \times \cdots \times \{b_i\} \times S^{k-i} \right|
\]

\[
= S^{i-1} \times \{a_i\} \times \{a_{i+1}\} \times \cdots \times \{a_k\} \times \{b_1\} \times \{b_2\} \times \cdots \times \{b_i\} \times S^{k-i}
\]

\[
= |S|^{i-1} \cdot |S|^{k-i} = |S|^{k-1}
\] \(^5\)

\(^5\)because otherwise, we would have \( u_\alpha = v_\alpha \) and thus

\[
a = (v_1, v_2, \ldots, v_{i-1}, u_{\alpha}, u_{\alpha+1}, \ldots, u_k) = (v_1, v_2, \ldots, v_{i-1}, u_{\alpha}, u_{\alpha+1}, \ldots, u_k) = b,
\]

contradicting \( a \neq b \).

\(^6\)By abuse of notation, we are writing \( S^{i-1} \times \{a_i\} \times \{a_{i+1}\} \times \cdots \times \{a_k\} \) for \( S \times S \times \cdots \times S \times \{a_i\} \times \{a_{i+1}\} \times \cdots \times \{a_k\} \) here.

\(^7\)By abuse of notation, we are writing \( \{b_1\} \times \{b_2\} \times \cdots \times \{b_i\} \times S^{k-i} \) for \( \{b_1\} \times \{b_2\} \times \cdots \times \{b_i\} \times S \times S \times \cdots \times S \) here.
for every \((a, b) \in F\).

Thus,

\[
|A \times B| = \sum_{(a,b) \in F} |\{(u, v) \in A \times B \mid \phi(u, v) = (a, b)\}|
\]

\[
= \sum_{(a,b) \in F} \left| \phi^{-1}\left(\{(a, b)\}\right) \right| \leq \sum_{(a,b) \in F} |S|^{k-1}
\]

\[
= |F| \cdot |S|^{k-1}.
\]

But

\[
|A \times B| = |A| \cdot |B| = |A| \cdot \left| S^k \setminus A \right| = |A| \cdot \left( |S^k| - |A| \right)
\]

\[
= |S|^{k-1} \cdot \left( |S^k| - |S|^{k-1} \right) = |S|^{k-1} \cdot \left( |S|^k - |S|^{k-1} \right),
\]

so this becomes

\[
|S|^{k-1} \cdot \left( |S|^k - |S|^{k-1} \right) \leq |F| \cdot |S|^{k-1},
\]

thus \(|S|^k - |S|^{k-1} \leq |F|\), so that

\[
|F| \geq |S|^k - |S|^{k-1} = |S|^{k-1} \cdot (|S| - 1),
\]

qed.