

On binomial coefficients modulo squares of primes

Darij Grinberg

October 7, 2018

Abstract. We give elementary proofs for the Apagodu-Zeilberger-Stanton-Amdeberhan-Tauraso congruences

$$\begin{aligned} \sum_{n=0}^{p-1} \binom{2n}{n} &\equiv \eta_p \pmod{p^2}; \\ \sum_{n=0}^{rp-1} \binom{2n}{n} &\equiv \eta_p \sum_{n=0}^{r-1} \binom{2n}{n} \pmod{p^2}; \\ \sum_{n=0}^{rp-1} \sum_{m=0}^{sp-1} \binom{n+m}{m}^2 &\equiv \eta_p \sum_{m=0}^{r-1} \sum_{n=0}^{s-1} \binom{n+m}{m}^2 \pmod{p^2}, \end{aligned}$$

where p is an odd prime, r and s are nonnegative integers, and

$$\eta_p = \begin{cases} 0, & \text{if } p \equiv 0 \pmod{3}; \\ 1, & \text{if } p \equiv 1 \pmod{3}; \\ -1, & \text{if } p \equiv 2 \pmod{3} \end{cases}.$$

Contents

1. Introduction	2
1.1. Binomial coefficients	2
1.2. Classical congruences	3
1.3. The three modulo- p^2 congruences	4
2. The proofs	5
2.1. Identities and congruences from the literature	5
2.2. Variants and consequences of Vandermonde convolution	7
2.3. A congruence of Bailey's	11
2.4. Two congruences for polynomials	13

2.5. Proving Theorem 1.8 15
 2.6. Proving Theorem 1.9 18
 2.7. Proving Theorem 1.10 20
 2.8. Acknowledgments 25

1. Introduction

In this note, we prove that any odd prime p and any $r, s \in \mathbb{N}$ satisfy

$$\sum_{n=0}^{p-1} \binom{2n}{n} \equiv \eta_p \pmod{p^2} \quad (\text{Theorem 1.8});$$

$$\sum_{n=0}^{rp-1} \binom{2n}{n} \equiv \eta_p \sum_{n=0}^{r-1} \binom{2n}{n} \pmod{p^2} \quad (\text{Theorem 1.9});$$

$$\sum_{n=0}^{rp-1} \sum_{m=0}^{sp-1} \binom{n+m}{m}^2 \equiv \eta_p \sum_{m=0}^{r-1} \sum_{n=0}^{s-1} \binom{n+m}{m}^2 \pmod{p^2} \quad (\text{Theorem 1.10}),$$

where

$$\eta_p = \begin{cases} 0, & \text{if } p \equiv 0 \pmod{3}; \\ 1, & \text{if } p \equiv 1 \pmod{3}; \\ -1, & \text{if } p \equiv 2 \pmod{3} \end{cases}$$

These three congruences are (slightly extended versions of) three of the ‘‘Super-Conjectures’’ (namely, 1, 1’’ and 4’) stated by Apagodu and Zeilberger in [ApaZei16]¹. Our proofs are more elementary than previous proofs by Stanton [Stanto16] and Amdeberhan and Tauraso [AmdTau16].

1.1. Binomial coefficients

Let us first recall the definition of binomial coefficients:²

Definition 1.1. Let $n \in \mathbb{N}$ and $m \in \mathbb{Z}$. Then, the *binomial coefficient* $\binom{m}{n}$ is a rational number defined by

$$\binom{m}{n} = \frac{m(m-1)\cdots(m-n+1)}{n!}.$$

¹In the arXiv preprint version of [ApaZei16] (arXiv:1606.03351v2), these congruences appear as ‘‘Super-Conjectures’’ 1, 1’’ and 5’, respectively.

²We use the notation \mathbb{N} for the set $\{0, 1, 2, \dots\}$.

Definition 1.2. Let n be a negative integer. Let $m \in \mathbb{Z}$. Then, the *binomial coefficient* $\binom{m}{n}$ is a rational number defined by $\binom{m}{n} = 0$.

(This is the definition used in [GrKnPa94] and [Grinbe17b]. Some authors follow other conventions instead.)

The following proposition is well-known (see, e.g., [Grinbe17b, Proposition 1.9]):

Proposition 1.3. We have $\binom{m}{n} \in \mathbb{Z}$ for any $m \in \mathbb{Z}$ and $n \in \mathbb{Z}$.

Proposition 1.3 shows that $\binom{m}{n}$ is an integer whenever $m \in \mathbb{Z}$ and $n \in \mathbb{Z}$. We shall tacitly use this below, when we study congruences involving binomial coefficients.

One advantage of Definition 1.2 is that it makes the following hold:

Proposition 1.4. For any $n \in \mathbb{Z}$ and $m \in \mathbb{Z}$, the binomial coefficient $\binom{n}{m}$ is the coefficient of X^m in the formal power series $(1 + X)^n \in \mathbb{Z}[[X]]$. (Here, the coefficient of X^m in any formal power series is defined to be 0 when m is negative.)

1.2. Classical congruences

The behavior of binomial coefficients modulo primes and prime powers is a classical subject of research; see [Mestro14, §2.1] for a survey of much of it. Let us state two of the most basic results in this subject:

Theorem 1.5. Let p be a prime. Let a and b be two integers. Let c and d be two elements of $\{0, 1, \dots, p - 1\}$. Then,

$$\binom{ap + c}{bp + d} \equiv \binom{a}{b} \binom{c}{d} \pmod{p}.$$

Theorem 1.5 is known under the name of *Lucas's theorem*, and is proven in many places (e.g., [Mestro14, §2.1] or [Hausne83, Proof of §4] or [AnBeRo05, proof of Lucas's theorem] or [GrKnPa94, Exercise 5.61]) at least in the case when a and b are nonnegative integers. The standard proof of Theorem 1.5 in this case uses generating functions (specifically, Proposition 1.4); this proof applies (*mutatis mutandis*) in the general case as well. See [Grinbe17b, Theorem 1.11] for an elementary proof of Theorem 1.5.

Another fundamental result is the following:

Theorem 1.6. Let p be a prime. Let a and b be two integers. Then,

$$\binom{ap}{bp} \equiv \binom{a}{b} \pmod{p^2}.$$

Theorem 1.6 is a known result, perhaps due to Charles Babbage. It appears with proof in [Grinbe17b, Theorem 1.12]; again, many sources prove it for nonnegative a and b (for example [Stanle11, Exercise 1.14 c] or [GrKnPa94, Exercise 5.62]). Notice that if $p \geq 5$, then the modulus p^2 can be replaced by p^3 or (depending on a , b and p) by even higher powers of p ; see [Mestro14, (22) and (23)] for the details. See also [SunTau11, Lemma 2.1] for another strengthening of Theorem 1.6.

1.3. The three modulo- p^2 congruences

Definition 1.7. For any $p \in \mathbb{Z}$, we define an integer $\eta_p \in \{-1, 0, 1\}$ by

$$\eta_p = \begin{cases} 0, & \text{if } p \equiv 0 \pmod{3}; \\ 1, & \text{if } p \equiv 1 \pmod{3}; \\ -1, & \text{if } p \equiv 2 \pmod{3} \end{cases}.$$

Notice that η_p is the so-called *Legendre symbol* $\left(\frac{p}{3}\right)$ known from number theory.

We are now ready to state three conjectures by Apagodu and Zeilberger, which we shall prove in the sequel. The first one is [ApaZei16, Super-Conjecture 1]:³

Theorem 1.8. Let p be an odd prime. Then,

$$\sum_{n=0}^{p-1} \binom{2n}{n} \equiv \eta_p \pmod{p^2}.$$

The next one ([ApaZei16, Super-Conjecture 1'']) is a generalization:

Theorem 1.9. Let p be an odd prime. Let $r \in \mathbb{N}$. Set

$$\alpha_r = \sum_{n=0}^{r-1} \binom{2n}{n}.$$

³To be precise (and boastful), our Theorem 1.8 is somewhat stronger than [ApaZei16, Super-Conjecture 1], since we only require p to be odd (rather than $p \geq 5$). Of course, in the case of Theorem 1.8, this extra generality is insignificant, since it just adds the possibility of $p = 3$, in which case Theorem 1.8 can be checked by hand. However, for Theorems 1.9 and 1.10 further below, we gain somewhat more from this generality.

Then,

$$\sum_{n=0}^{rp-1} \binom{2n}{n} \equiv \eta_p \alpha_r \pmod{p^2}.$$

Theorem 1.8 and Theorem 1.9 both have been proven by Dennis Stanton [Stanto16] using Laurent series (in the case when $p \geq 5$), and by Liu [Liu16, (1.3)] using harmonic numbers. We shall reprove them elementarily.

The third conjecture that we shall prove is [ApaZei16, Super-Conjecture 5']:

Theorem 1.10. Let p be an odd prime. Let $r \in \mathbb{N}$ and $s \in \mathbb{N}$. Set

$$\epsilon_{r,s} = \sum_{m=0}^{r-1} \sum_{n=0}^{s-1} \binom{n+m}{m}^2.$$

Then,

$$\sum_{n=0}^{rp-1} \sum_{m=0}^{sp-1} \binom{n+m}{m}^2 \equiv \eta_p \epsilon_{r,s} \pmod{p^2}.$$

A proof of Theorem 1.10 has been found by Amdeberhan and Tauraso, and was outlined in [AmdTau16, §6]; we give a different, elementary proof.

2. The proofs

2.1. Identities and congruences from the literature

Before we come to the proofs of Theorems 1.8, 1.9 and 1.10, let us collect various well-known results that will prove useful.

The following properties of binomial coefficients are well-known (see, e.g., [Grinbe17, §3.1] and [Grinbe17b, §1]):

Proposition 2.1. We have $\binom{m}{0} = 1$ for every $m \in \mathbb{Z}$.

Proposition 2.2. We have $\binom{m}{n} = 0$ for every $m \in \mathbb{N}$ and $n \in \mathbb{N}$ satisfying $m < n$.

Proposition 2.3. We have $\binom{m}{n} = \binom{m}{m-n}$ for any $m \in \mathbb{N}$ and $n \in \mathbb{N}$ satisfying $m \geq n$.

Proposition 2.4. We have $\binom{m}{m} = 1$ for every $m \in \mathbb{N}$.

Proposition 2.5. We have

$$\binom{m}{n} = (-1)^n \binom{n-m-1}{n}$$

for any $m \in \mathbb{Z}$ and $n \in \mathbb{N}$.

Proposition 2.6. We have

$$\binom{m}{n} = \binom{m-1}{n-1} + \binom{m-1}{n}$$

for any $m \in \mathbb{Z}$ and $n \in \mathbb{Z}$.

Proposition 2.7. For every $x \in \mathbb{Z}$ and $y \in \mathbb{Z}$ and $n \in \mathbb{N}$, we have

$$\binom{x+y}{n} = \sum_{k=0}^n \binom{x}{k} \binom{y}{n-k}.$$

Proposition 2.7 is the so-called *Vandermonde convolution identity*, and is a particular case of [Grinbe17, Theorem 3.29].

Corollary 2.8. For each $n \in \mathbb{N}$, we have

$$\sum_{i=0}^{n-1} (-1)^i \binom{n-1-i}{i} = (-1)^n \cdot \begin{cases} 0, & \text{if } n \equiv 0 \pmod{3}; \\ -1, & \text{if } n \equiv 1 \pmod{3}; \\ 1, & \text{if } n \equiv 2 \pmod{3} \end{cases}$$

Corollary 2.8 is [Grinbe17, Corollary 8.63]. Apart from that, Corollary 2.8 can be easily derived from [GrKnPa94, §5.2, Problem 3], [BenQui03, Identity 172] or [BenQui08].

Another simple identity (sometimes known as the “absorption identity”) is the following:

Proposition 2.9. Let $n \in \mathbb{Z}$ and $k \in \mathbb{Z}$. Then, $k \binom{n}{k} = n \binom{n-1}{k-1}$.

Proposition 2.9 appears in [GrKnPa94, (5.6)], and is easily proven just from the definition of binomial coefficients.

Finally, we need the following result from elementary number theory:

Theorem 2.10. Let p be a prime. Let $k \in \mathbb{N}$. Assume that k is not a positive multiple of $p - 1$. Then,

$$\sum_{l=0}^{p-1} l^k \equiv 0 \pmod{p}.$$

Theorem 2.10 is proven, e.g., in [Grinbe17b, Theorem 3.1] and (in a slightly rewritten form) in [MacSon10, Theorem 1].

2.2. Variants and consequences of Vandermonde convolution

We are now going to state a number of identities that are restatements or particular cases of the Vandermonde convolution identity (Proposition 2.7). We begin with the following one:

Corollary 2.11. Let $u \in \mathbb{Z}$ and $l \in \mathbb{N}$ and $w \in \mathbb{N}$. Then,

$$\sum_{m=0}^l \binom{u}{w+m} \binom{l}{m} = \binom{u+l}{w+l}.$$

Proof of Corollary 2.11. Proposition 2.7 (applied to $x = u$, $y = l$ and $n = w + l$)

yields

$$\begin{aligned}
 \binom{u+l}{w+l} &= \sum_{k=0}^{w+l} \binom{u}{k} \binom{l}{w+l-k} \\
 &= \sum_{k=0}^{w-1} \binom{u}{k} \underbrace{\binom{l}{w+l-k}}_{=0} + \sum_{k=w}^{w+l} \binom{u}{k} \binom{l}{w+l-k} \\
 &\quad \text{(by Proposition 2.2} \\
 &\quad \text{(since } l < w+l-k \text{ (because } k < w)) \\
 &\quad \text{(here, we have split the sum at } k = w, \\
 &\quad \text{since } 0 \leq w \leq w+l \\
 &= \underbrace{\sum_{k=0}^{w-1} \binom{u}{k} 0}_{=0} + \sum_{k=w}^{w+l} \binom{u}{k} \binom{l}{w+l-k} = \sum_{k=w}^{w+l} \binom{u}{k} \binom{l}{w+l-k} \\
 &= \sum_{m=0}^l \binom{u}{w+m} \underbrace{\binom{l}{w+l-(w+m)}}_{= \binom{l}{l-m} = \binom{l}{m}} \\
 &\quad \text{(here, we have substituted } w+m \text{ for } k \text{ in the sum)} \\
 &= \sum_{m=0}^l \binom{u}{w+m} \binom{l}{m}.
 \end{aligned}$$

This proves Corollary 2.11. □

Let us also state another corollary of Proposition 2.7:

Corollary 2.12. Let $x \in \mathbb{Z}$ and $y \in \mathbb{N}$ and $n \in \mathbb{Z}$. Then,

$$\binom{x+y}{n} = \sum_{i=0}^y \binom{x}{n-i} \binom{y}{i}.$$

See [Grinbe17b, Corollary 2.2] for a proof of Corollary 2.12.

Lemma 2.13. Let $u \in \mathbb{Z}$ and $w \in \mathbb{N}$ and $l \in \mathbb{N}$. Then,

$$\binom{u+2l}{w+l} = \binom{u}{w} \binom{2l}{l} + \sum_{i=1}^l \left(\binom{u}{w+i} + \binom{u}{w-i} \right) \binom{2l}{l-i}.$$

Proof of Lemma 2.13. Corollary 2.12 (applied to $x = u, y = 2l$ and $n = w + l$) yields

$$\begin{aligned} \binom{u + 2l}{w + l} &= \sum_{i=0}^{2l} \binom{u}{w + l - i} \binom{2l}{i} = \sum_{i=-l}^l \binom{u}{w + i} \binom{2l}{l - i} \\ &\quad \text{(here, we have substituted } l - i \text{ for } i \text{ in the sum)} \\ &= \sum_{\substack{i \in \{-l, -l+1, \dots, l\}; \\ i \neq 0}} \binom{u}{w + i} \binom{2l}{l - i} + \binom{u}{w} \binom{2l}{l} \end{aligned}$$

(here, we have split off the addend for $i = 0$ from the sum). Hence,

$$\begin{aligned} \binom{u + 2l}{w + l} - \binom{u}{w} \binom{2l}{l} &= \sum_{\substack{i \in \{-l, -l+1, \dots, l\}; \\ i \neq 0}} \binom{u}{w + i} \binom{2l}{l - i} \\ &= \sum_{i=1}^l \binom{u}{w + i} \binom{2l}{l - i} + \sum_{i=-l}^{-1} \binom{u}{w + i} \binom{2l}{l - i} \\ &\quad \left(\begin{array}{l} \text{here, we have split the sum into two:} \\ \text{one for "positive } i" \text{ and one for "negative } i" \end{array} \right) \\ &= \sum_{i=1}^l \binom{u}{w + i} \binom{2l}{l - i} + \sum_{i=1}^l \binom{u}{w - i} \underbrace{\binom{2l}{l + i}}_{\substack{= \binom{2l}{l - i} \\ \text{(by Proposition 2.3)}}} \\ &\quad \left(\begin{array}{l} \text{here, we have substituted } -i \text{ for } i \\ \text{in the second sum} \end{array} \right) \\ &= \sum_{i=1}^l \binom{u}{w + i} \binom{2l}{l - i} + \sum_{i=1}^l \binom{u}{w - i} \binom{2l}{l - i} \\ &= \sum_{i=1}^l \left(\binom{u}{w + i} + \binom{u}{w - i} \right) \binom{2l}{l - i}. \end{aligned}$$

In other words,

$$\binom{u + 2l}{w + l} = \binom{u}{w} \binom{2l}{l} + \sum_{i=1}^l \left(\binom{u}{w + i} + \binom{u}{w - i} \right) \binom{2l}{l - i}.$$

This proves Lemma 2.13. □

Lemma 2.14. Let $p \in \mathbb{N}$. Let $c \in \mathbb{Z}$. Let $l \in \{0, 1, \dots, p - 1\}$. Then,

$$\binom{cp + 2l}{l} = \sum_{k=0}^{p-1} \binom{cp + l}{k} \binom{l}{k}.$$

Proof of Lemma 2.14. Corollary 2.12 (applied to $x = cp + l, y = l$ and $n = l$) yields

$$\begin{aligned} \binom{cp+l+l}{l} &= \sum_{i=0}^l \binom{cp+l}{l-i} \binom{l}{i} = \sum_{k=0}^l \binom{cp+l}{k} \underbrace{\binom{l}{l-k}}_{\substack{= \binom{l}{k} \\ \text{(by Proposition 2.3)}}} \\ &\text{(here, we have substituted } k \text{ for } l-i \text{ in the sum)} \\ &= \sum_{k=0}^l \binom{cp+l}{k} \binom{l}{k}. \end{aligned}$$

Comparing this with

$$\begin{aligned} \sum_{k=0}^{p-1} \binom{cp+l}{k} \binom{l}{k} &= \sum_{k=0}^l \binom{cp+l}{k} \binom{l}{k} + \underbrace{\sum_{k=l+1}^{p-1} \binom{cp+l}{k} \binom{l}{k}}_{\substack{=0 \\ \text{(by Proposition 2.2} \\ \text{(applied to } m=l \text{ and } n=k) \\ \text{(since } l < k))}} \\ &\text{(here, we have split the sum at } k=l, \text{ since } 0 \leq l \leq p-1) \\ &= \sum_{k=0}^l \binom{cp+l}{k} \binom{l}{k} + \underbrace{\sum_{k=l+1}^{p-1} \binom{cp+l}{k}}_{=0} \binom{l}{k} = \sum_{k=0}^l \binom{cp+l}{k} \binom{l}{k}, \end{aligned}$$

we obtain $\sum_{k=0}^{p-1} \binom{cp+l}{k} \binom{l}{k} = \binom{cp+l+l}{l} = \binom{cp+2l}{l}$. This proves Lemma 2.14. □

Lemma 2.15. Let $p \in \mathbb{N}$. Let $l \in \mathbb{N}$. Then,

$$\sum_{i=1}^l \binom{p}{i} \binom{2l}{l-i} = \binom{p+2l}{l} - \binom{2l}{l}.$$

Proof of Lemma 2.15. Proposition 2.7 (applied to $x = p, y = 2l$ and $n = l$) yields

$$\begin{aligned} \binom{p+2l}{l} &= \sum_{k=0}^l \binom{p}{k} \binom{2l}{l-k} = \sum_{i=0}^l \binom{p}{i} \binom{2l}{l-i} \\ &\text{(here, we have renamed the summation index } k \text{ as } i) \\ &= \underbrace{\binom{p}{0}}_{=1} \underbrace{\binom{2l}{l-0}}_{= \binom{2l}{l}} + \sum_{i=1}^l \binom{p}{i} \binom{2l}{l-i} = \binom{2l}{l} + \sum_{i=1}^l \binom{p}{i} \binom{2l}{l-i}. \end{aligned}$$

Thus,

$$\sum_{i=1}^l \binom{p}{i} \binom{2l}{l-i} = \binom{p+2l}{l} - \binom{2l}{l}.$$

This proves Lemma 2.15. □

2.3. A congruence of Bailey's

Next, we shall prove a modulo- p^2 congruence for certain binomial coefficients that can be regarded as a counterpart to Theorem 1.6:

Theorem 2.16. Let p be a prime. Let $N \in \mathbb{Z}$ and $K \in \mathbb{Z}$ and $i \in \{1, 2, \dots, p-1\}$. Then:

(a) We have

$$\binom{Np}{Kp+i} \equiv N \binom{N-1}{K} \binom{p}{i} \pmod{p^2}.$$

(b) We have

$$\binom{Np}{Kp-i} \equiv N \binom{N-1}{K-1} \binom{p}{i} \pmod{p^2}.$$

(c) We have

$$\binom{Np}{Kp+i} + \binom{Np}{Kp-i} \equiv N \binom{N}{K} \binom{p}{i} \pmod{p^2}.$$

Theorem 2.16 (a) is essentially the result [Bailey91, Theorem 4] by Bailey (see also [Mestro14, (26)]); in fact, it transforms into [Bailey91, Theorem 4] if we rewrite $N \binom{N-1}{K}$ as $(K+1) \binom{N}{K+1}$ (using Proposition 2.9). We shall nevertheless give our own proof.

Proof of Theorem 2.16. From $i \in \{1, 2, \dots, p-1\}$, we conclude that both $i-1$ and $p-i$ are elements of $\{0, 1, \dots, p-1\}$. Notice also that i is not divisible by p (since $i \in \{1, 2, \dots, p-1\}$); hence, i is coprime to p (since p is a prime). Therefore, i is also coprime to p^2 .

(a) Proposition 2.9 (applied to $n = Np$ and $k = Kp + i$) yields

$$\begin{aligned} (Kp+i) \binom{Np}{Kp+i} &= Np \binom{Np-1}{Kp+i-1} = Np \underbrace{\binom{(N-1)p + (p-1)}{Kp + (i-1)}}_{\equiv \binom{N-1}{K} \binom{p-1}{i-1} \pmod{p}} \\ &\equiv \binom{N-1}{K} \binom{p-1}{i-1} \pmod{p} \\ &\quad \text{(by Theorem 1.5, applied to } a=N-1, b=K, c=p-1 \text{ and } d=i-1) \\ &\equiv Np \binom{N-1}{K} \binom{p-1}{i-1} \pmod{p^2} \end{aligned} \tag{1}$$

(notice that the presence of the p factor has turned a congruence modulo p into a congruence modulo p^2). Thus,

$$(Kp + i) \binom{Np}{Kp + i} \equiv Np \binom{N-1}{K} \binom{p-1}{i-1} \equiv 0 \pmod{p},$$

so that $0 \equiv \underbrace{(Kp + i)}_{\equiv i \pmod{p}} \binom{Np}{Kp + i} \equiv i \binom{Np}{Kp + i} \pmod{p}$. We can cancel i from this

congruence (since i is coprime to p), and thus obtain $0 \equiv \binom{Np}{Kp + i} \pmod{p}$. Hence, $\binom{Np}{Kp + i}$ is divisible by p . Thus, $p \binom{Np}{Kp + i}$ is divisible by p^2 . In other words,

$$p \binom{Np}{Kp + i} \equiv 0 \pmod{p^2}. \tag{2}$$

Now,

$$(Kp + i) \binom{Np}{Kp + i} = \underbrace{Kp \binom{Np}{Kp + i}}_{\substack{\equiv 0 \pmod{p^2} \\ \text{(by (2))}}} + i \binom{Np}{Kp + i} \equiv i \binom{Np}{Kp + i} \pmod{p^2}.$$

Hence,

$$\begin{aligned} i \binom{Np}{Kp + i} &\equiv (Kp + i) \binom{Np}{Kp + i} \equiv Np \binom{N-1}{K} \binom{p-1}{i-1} && \text{(by (1))} \\ &= N \binom{N-1}{K} \underbrace{p \binom{p-1}{i-1}}_{=i \binom{p}{i}} = N \binom{N-1}{K} i \binom{p}{i} \pmod{p^2}. && \text{(by Proposition 2.9)} \end{aligned}$$

We can cancel i from this congruence (since i is coprime to p^2), and thus obtain

$$\binom{Np}{Kp + i} \equiv N \binom{N-1}{K} \binom{p}{i} \pmod{p^2}.$$

This proves Theorem 2.16 (a).

(b) We have $i \in \{1, 2, \dots, p-1\}$ and thus $p-i \in \{1, 2, \dots, p-1\}$. Hence, Theorem 2.16 (a) (applied to $K-1$ and $p-i$ instead of K and i) yields

$$\binom{Np}{(K-1)p + (p-i)} \equiv N \binom{N-1}{K-1} \underbrace{\binom{p}{p-i}}_{= \binom{p}{i}} = N \binom{N-1}{K-1} \binom{p}{i} \pmod{p^2}.$$

(by Proposition 2.3)

In view of $(K - 1)p + (p - i) = Kp - i$, this rewrites as

$$\binom{Np}{Kp - i} \equiv N \binom{N - 1}{K - 1} \binom{p}{i} \pmod{p^2}.$$

This proves Theorem 2.16 (b).

(c) We have

$$\begin{aligned} & \underbrace{\binom{Np}{Kp + i}} + \underbrace{\binom{Np}{Kp - i}} \\ & \equiv N \binom{N - 1}{K} \binom{p}{i} \pmod{p^2} \quad \equiv N \binom{N - 1}{K - 1} \binom{p}{i} \pmod{p^2} \\ & \quad \text{(by Theorem 2.16 (a))} \quad \text{(by Theorem 2.16 (b))} \\ & \equiv N \binom{N - 1}{K} \binom{p}{i} + N \binom{N - 1}{K - 1} \binom{p}{i} \\ & = N \underbrace{\left(\binom{N - 1}{K - 1} + \binom{N - 1}{K} \right)}_{= \binom{N}{K}} \binom{p}{i} = N \binom{N}{K} \binom{p}{i} \pmod{p^2}. \\ & \quad \text{(by Proposition 2.6)} \end{aligned}$$

This proves Theorem 2.16 (c). □

2.4. Two congruences for polynomials

Now, we recall that $\mathbb{Z}[X]$ is the ring of all polynomials in one indeterminate X with integer coefficients.

Lemma 2.17. Let p be a prime. Let $c \in \mathbb{Z}$. Let $P \in \mathbb{Z}[X]$ be a polynomial of degree $< 2p - 1$. Then, $\sum_{l=0}^{p-1} (P(cp + l) - P(l)) \equiv 0 \pmod{p^2}$.

Proof of Lemma 2.17. WLOG assume that $P = X^k$ for some $k \in \{0, 1, \dots, 2p - 2\}$ (since the congruence we are proving depends \mathbb{Z} -linearly on P). If $k = 0$, then Lemma 2.17 is easily checked (because in this case, P is constant). Thus, WLOG assume that $k \neq 0$. Hence, k is a positive integer (since $k \in \mathbb{N}$). Thus, $k - 1 \in \mathbb{N}$.

Each $l \in \{0, 1, \dots, p - 1\}$ satisfies

$$\begin{aligned}
 P(cp + l) &= (cp + l)^k && \left(\text{since } P = X^k\right) \\
 &= \sum_{i=0}^k \binom{k}{i} (cp)^i l^{k-i} && \left(\text{by the binomial formula}\right) \\
 &= \underbrace{(cp)^0 l^{k-0}}_{=l^k} + k \underbrace{(cp)^1 l^{k-1}}_{=cp} + \sum_{i=2}^k \binom{k}{i} \underbrace{(cp)^i l^{k-i}}_{\substack{\equiv 0 \pmod{p^2} \\ (\text{since } i \geq 2)}} \\
 &\equiv l^k + kcp l^{k-1} + \underbrace{\sum_{i=2}^k \binom{k}{i} 0 l^{k-i}}_{=0} \equiv l^k + kcp l^{k-1} \pmod{p^2}
 \end{aligned}$$

and $P(l) = l^k$ (since $P = X^k$). Thus,

$$\sum_{l=0}^{p-1} \left(\underbrace{P(cp + l)}_{\equiv l^k + kcp l^{k-1} \pmod{p^2}} - \underbrace{P(l)}_{=l^k} \right) \equiv \sum_{l=0}^{p-1} \underbrace{(l^k + kcp l^{k-1} - l^k)}_{=kcp l^{k-1}} = kcp \sum_{l=0}^{p-1} l^{k-1} \pmod{p^2}.$$

The claim of Lemma 2.17 now becomes obvious if $k = p$ (because if $k = p$, then kcp is already divisible by p^2); thus, we WLOG assume that $k \neq p$. Hence, $k - 1 \neq p - 1$.

If $k - 1$ was a positive multiple of $p - 1$, then we would have $k - 1 = p - 1$ (since $k \in \{0, 1, \dots, 2p - 2\}$), which would contradict $k - 1 \neq p - 1$. Hence, $k - 1$ is not a positive multiple of $p - 1$. Thus, Theorem 2.10 (applied to $k - 1$ instead of k) yields

$\sum_{l=0}^{p-1} l^{k-1} \equiv 0 \pmod{p}$. Thus, $p \sum_{l=0}^{p-1} l^{k-1} \equiv 0 \pmod{p^2}$, so that

$$\sum_{l=0}^{p-1} (P(cp + l) - P(l)) \equiv kcp \underbrace{\sum_{l=0}^{p-1} l^{k-1}}_{\equiv 0 \pmod{p^2}} \equiv 0 \pmod{p^2}.$$

This proves Lemma 2.17. □

Lemma 2.18. Let p, a and b be three integers such that $a - b$ is divisible by p . Then, $a^2 - b^2 \equiv 2(a - b)b \pmod{p^2}$.

Proof of Lemma 2.18. The difference $(a^2 - b^2) - 2(a - b)b = (a - b)^2$ is divisible by p^2 (since $a - b$ is divisible by p). In other words, $a^2 - b^2 \equiv 2(a - b)b \pmod{p^2}$. Lemma 2.18 is proven. □

Lemma 2.19. Let p be an odd prime. Let $c \in \mathbb{Z}$. Let $P \in \mathbb{Z}[X]$ be a polynomial of degree $\leq p - 1$. Then,

$$\sum_{l=0}^{p-1} (P(cp + l) - P(l)) P(l) \equiv 0 \pmod{p^2}.$$

Proof of Lemma 2.19. Fix $l \in \mathbb{Z}$. We have $P \in \mathbb{Z}[X]$. Thus, $P(u) - P(v)$ is divisible by $u - v$ whenever u and v are two integers⁴. Applying this to $u = cp + l$ and $v = l$, we conclude that $P(cp + l) - P(l)$ is divisible by $(cp + l) - l = cp$, and thus also divisible by p .

Hence, Lemma 2.18 (applied to $a = P(cp + l)$ and $b = P(l)$) shows that

$$(P(cp + l))^2 - (P(l))^2 \equiv 2(P(cp + l) - P(l)) P(l) \pmod{p^2}. \quad (3)$$

Now, forget that we fixed l . We thus have proven (3) for each $l \in \mathbb{Z}$.

The polynomial P has degree $\leq p - 1$. Hence, the polynomial P^2 has degree $\leq 2(p - 1) < 2p - 1$. Thus, Lemma 2.17 (applied to P^2 instead of P) shows that

$$\sum_{l=0}^{p-1} (P^2(cp + l) - P^2(l)) \equiv 0 \pmod{p^2}.$$

Thus,

$$\begin{aligned} 0 &\equiv \sum_{l=0}^{p-1} \underbrace{(P^2(cp + l) - P^2(l))}_{\substack{=(P(cp+l))^2 - (P(l))^2 \\ \equiv 2(P(cp+l) - P(l))P(l) \pmod{p^2} \\ \text{(by (3))}}} \equiv 2 \sum_{l=0}^{p-1} (P(cp + l) - P(l)) P(l) \pmod{p^2}. \end{aligned}$$

We can cancel 2 from this congruence (since p is odd), and conclude that

$$0 \equiv \sum_{l=0}^{p-1} (P(cp + l) - P(l)) P(l) \pmod{p^2}.$$

This proves Lemma 2.19. □

2.5. Proving Theorem 1.8

Now, let us prepare for the proofs of our results by showing several lemmas.

⁴This is a well-known fact. It can be proven as follows: WLOG assume that $P = X^k$ for some $k \in \mathbb{N}$ (this is a valid assumption, since the claim is \mathbb{Z} -linear in P); then, $P(u) - P(v) = u^k - v^k = (u - v) \sum_{i=0}^{k-1} u^i v^{k-i}$ is clearly divisible by $u - v$.

Lemma 2.20. Let p be an odd prime. Let $c \in \mathbb{Z}$. Let $k \in \{0, 1, \dots, p - 1\}$. Then,

$$\sum_{l=0}^{p-1} \left(\binom{cp+l}{k} - \binom{l}{k} \right) \binom{l}{k} \equiv 0 \pmod{p^2}.$$

Proof of Lemma 2.20. Notice that $k!$ is coprime to p (since $k \leq p - 1$), and thus $k!^2$ is coprime to p^2 .

Define a polynomial $P \in \mathbb{Z}[X]$ by

$$P = X(X - 1) \cdots (X - k + 1).$$

Then, P has degree $k \leq p - 1$. Thus, Lemma 2.19 yields

$$\sum_{l=0}^{p-1} (P(cp+l) - P(l)) P(l) \equiv 0 \pmod{p^2}.$$

Since each $n \in \mathbb{Z}$ satisfies $P(n) = n(n - 1) \cdots (n - k + 1) = k! \binom{n}{k}$, this rewrites as

$$\sum_{l=0}^{p-1} \left(k! \binom{cp+l}{k} - k! \binom{l}{k} \right) k! \binom{l}{k} \equiv 0 \pmod{p^2}.$$

We can cancel $k!^2$ from this congruence (since $k!^2$ is coprime to p^2), and thus obtain

$$\sum_{l=0}^{p-1} \left(\binom{cp+l}{k} - \binom{l}{k} \right) \binom{l}{k} \equiv 0 \pmod{p^2}.$$

This proves Lemma 2.20. □

Lemma 2.21. Let p be an odd prime. Let $c \in \mathbb{Z}$. Then,

$$\sum_{l=0}^{p-1} \left(\binom{cp+2l}{l} - \binom{2l}{l} \right) \equiv 0 \pmod{p^2}.$$

Proof of Lemma 2.21. For each $l \in \{0, 1, \dots, p - 1\}$, we have

$$\begin{aligned} & \binom{cp+2l}{l} - \binom{2l}{l} \\ &= \underbrace{\sum_{k=0}^{p-1} \binom{cp+l}{k} \binom{l}{k}}_{\text{(by Lemma 2.14)}} - \underbrace{\sum_{k=0}^{p-1} \binom{l}{k} \binom{l}{k}}_{\substack{\text{(by Lemma 2.14,} \\ \text{applied to 0 instead of } c)}} \\ &= \sum_{k=0}^{p-1} \binom{cp+l}{k} \binom{l}{k} - \sum_{k=0}^{p-1} \binom{l}{k} \binom{l}{k} = \sum_{k=0}^{p-1} \left(\binom{cp+l}{k} - \binom{l}{k} \right) \binom{l}{k}. \end{aligned}$$

Summing these equalities over all $l \in \{0, 1, \dots, p-1\}$, we find

$$\begin{aligned} \sum_{l=0}^{p-1} \left(\binom{cp+2l}{l} - \binom{2l}{l} \right) &= \sum_{l=0}^{p-1} \sum_{k=0}^{p-1} \left(\binom{cp+l}{k} - \binom{l}{k} \right) \binom{l}{k} \\ &= \sum_{k=0}^{p-1} \underbrace{\sum_{l=0}^{p-1} \left(\binom{cp+l}{k} - \binom{l}{k} \right) \binom{l}{k}}_{\substack{\equiv 0 \pmod{p^2} \\ \text{(by Lemma 2.20)}}} \equiv \sum_{k=0}^{p-1} 0 = 0 \pmod{p^2}. \end{aligned}$$

This proves Lemma 2.21. □

Proof of Theorem 1.8. Lemma 2.21 (applied to $c = -1$) yields

$$\sum_{l=0}^{p-1} \left(\binom{-p+2l}{l} - \binom{2l}{l} \right) \equiv 0 \pmod{p^2}.$$

Thus,

$$0 \equiv \sum_{l=0}^{p-1} \left(\binom{-p+2l}{l} - \binom{2l}{l} \right) = \sum_{l=0}^{p-1} \binom{-p+2l}{l} - \sum_{l=0}^{p-1} \binom{2l}{l} \pmod{p^2},$$

so that

$$\sum_{l=0}^{p-1} \binom{2l}{l} \equiv \sum_{l=0}^{p-1} \binom{-p+2l}{l} \pmod{p^2}. \tag{4}$$

Lemma 2.13 (applied to $u = Np$ and $w = Kp$) yields

$$\begin{aligned}
 \binom{Np+2l}{Kp+l} &= \underbrace{\binom{Np}{Kp}}_{\equiv \binom{N}{K} \pmod{p^2}} \binom{2l}{l} + \sum_{i=1}^l \underbrace{\left(\binom{Np}{Kp+i} + \binom{Np}{Kp-i} \right)}_{\equiv N \binom{N}{K} \binom{p}{i} \pmod{p^2}} \binom{2l}{l-i} \\
 &\equiv \binom{N}{K} \binom{2l}{l} + \sum_{i=1}^l N \binom{N}{K} \binom{p}{i} \binom{2l}{l-i} \\
 &= \binom{N}{K} \binom{2l}{l} + N \binom{N}{K} \underbrace{\sum_{i=1}^l \binom{p}{i} \binom{2l}{l-i}}_{= \binom{p+2l}{l} - \binom{2l}{l}} \\
 &= \binom{N}{K} \binom{2l}{l} + N \binom{N}{K} \left(\binom{p+2l}{l} - \binom{2l}{l} \right) \pmod{p^2}.
 \end{aligned}$$

Subtracting $\binom{N}{K} \binom{2l}{l}$ from both sides of this congruence, we obtain

$$\binom{Np+2l}{Kp+l} - \binom{N}{K} \binom{2l}{l} \equiv N \binom{N}{K} \left(\binom{p+2l}{l} - \binom{2l}{l} \right) \pmod{p^2}.$$

This proves Lemma 2.22. □

Lemma 2.23. Let p be an odd prime. Let $N \in \mathbb{Z}$ and $K \in \mathbb{N}$. Then,

$$\sum_{l=0}^{p-1} \binom{Np+2l}{Kp+l} \equiv \binom{N}{K} \eta_p \pmod{p^2}.$$

Proof of Lemma 2.23. For any $l \in \{0, 1, \dots, p-1\}$, we have

$$\binom{Np+2l}{Kp+l} \equiv \binom{N}{K} \binom{2l}{l} + N \binom{N}{K} \left(\binom{p+2l}{l} - \binom{2l}{l} \right) \pmod{p^2}$$

(by Lemma 2.22). Summing these congruences over all $l \in \{0, 1, \dots, p-1\}$, we find

$$\begin{aligned} \sum_{l=0}^{p-1} \binom{Np+2l}{Kp+l} &\equiv \sum_{l=0}^{p-1} \left(\binom{N}{K} \binom{2l}{l} + N \binom{N}{K} \left(\binom{p+2l}{l} - \binom{2l}{l} \right) \right) \\ &= \binom{N}{K} \sum_{l=0}^{p-1} \binom{2l}{l} + N \binom{N}{K} \underbrace{\sum_{l=0}^{p-1} \left(\binom{p+2l}{l} - \binom{2l}{l} \right)}_{\equiv 0 \pmod{p^2}} \\ &\quad \text{(by Lemma 2.21, applied to } c=1) \\ &\equiv \binom{N}{K} \underbrace{\sum_{l=0}^{p-1} \binom{2l}{l}}_{= \sum_{n=0}^{p-1} \binom{2n}{n} \equiv \eta_p \pmod{p^2}} \equiv \binom{N}{K} \eta_p \pmod{p^2}. \\ &\quad \text{(by Theorem 1.8)} \end{aligned}$$

This proves Lemma 2.23. □

Proof of Theorem 1.9. The map

$$\begin{aligned} \{0, 1, \dots, p-1\} \times \{0, 1, \dots, r-1\} &\rightarrow \{0, 1, \dots, rp-1\}, \\ (l, K) &\mapsto Kp+l \end{aligned}$$

is a bijection (since each element of $\{0, 1, \dots, rp-1\}$ can be uniquely divided by p with remainder, and said remainder will belong to $\{0, 1, \dots, r-1\}$). Thus, we can substitute $Kp+l$ for n in the sum $\sum_{n=0}^{rp-1} \binom{2n}{n}$. This sum thus rewrites as follows:

$$\begin{aligned} \sum_{n=0}^{rp-1} \binom{2n}{n} &= \underbrace{\sum_{(l,K) \in \{0,1,\dots,p-1\} \times \{0,1,\dots,r-1\}} \binom{2(Kp+l)}{Kp+l}}_{= \sum_{K=0}^{r-1} \sum_{l=0}^{p-1}} = \sum_{K=0}^{r-1} \underbrace{\sum_{l=0}^{p-1} \binom{2Kp+2l}{Kp+l}}_{\equiv \binom{2K}{K} \eta_p \pmod{p^2}} \\ &\quad \text{(by Lemma 2.23, applied to } N=2K) \\ &\equiv \underbrace{\sum_{K=0}^{r-1} \binom{2K}{K}}_{= \sum_{n=0}^{r-1} \binom{2n}{n} = \alpha_r} \eta_p = \alpha_r \eta_p = \eta_p \alpha_r \pmod{p^2}. \end{aligned}$$

This proves Theorem 1.9. □

2.7. Proving Theorem 1.10

Lemma 2.24. Let p be an odd prime. Let $N \in \mathbb{Z}$ and $K \in \mathbb{N}$. Then,

$$\sum_{l=0}^{p-1} \sum_{m=0}^l \left(\binom{Np+l}{Kp+m} - \binom{N}{K} \binom{l}{m} \right) \binom{l}{m} \equiv 0 \pmod{p^2}.$$

Proof of Lemma 2.24. We have

$$\begin{aligned} & \sum_{l=0}^{p-1} \sum_{m=0}^l \left(\binom{Np+l}{Kp+m} - \binom{N}{K} \binom{l}{m} \right) \binom{l}{m} \\ &= \sum_{l=0}^{p-1} \underbrace{\sum_{m=0}^l \binom{Np+l}{Kp+m} \binom{l}{m}}_{\substack{= \binom{Np+2l}{Kp+l} \\ \text{(by Corollary 2.11,} \\ \text{applied to } u=Np+l \text{ and } w=Kp)}} - \binom{N}{K} \sum_{l=0}^{p-1} \underbrace{\sum_{m=0}^l \binom{l}{m} \binom{l}{m}}_{\substack{= \binom{2l}{l} \\ \text{(by Corollary 2.11,} \\ \text{applied to } u=l \text{ and } w=0)}} \\ &= \sum_{l=0}^{p-1} \binom{Np+2l}{Kp+l} - \binom{N}{K} \sum_{l=0}^{p-1} \binom{2l}{l} = \sum_{l=0}^{p-1} \underbrace{\left(\binom{Np+2l}{Kp+l} - \binom{N}{K} \binom{2l}{l} \right)}_{\substack{\equiv N \binom{N}{K} \left(\binom{p+2l}{l} - \binom{2l}{l} \right) \pmod{p^2} \\ \text{(by Lemma 2.22)}}} \\ &\equiv N \binom{N}{K} \underbrace{\sum_{l=0}^{p-1} \left(\binom{p+2l}{l} - \binom{2l}{l} \right)}_{\substack{\equiv 0 \pmod{p^2} \\ \text{(by Lemma 2.21, applied to } c=1)}} \equiv 0 \pmod{p^2}. \end{aligned}$$

This proves Lemma 2.24. □

Lemma 2.25. Let p be an odd prime. Let $N \in \mathbb{Z}$ and $K \in \mathbb{N}$. Then,

$$\sum_{l=0}^{p-1} \sum_{m=0}^l \binom{Np+l}{Kp+m}^2 \equiv \binom{N}{K}^2 \eta_p \pmod{p^2}.$$

Proof of Lemma 2.25. Fix $l \in \{0, 1, \dots, p-1\}$ and $m \in \{0, 1, \dots, p-1\}$. Then, Theorem 1.5 (applied to $a = N$, $b = K$, $c = l$ and $d = m$) yields that $\binom{Np+l}{Kp+m} \equiv \binom{N}{K} \binom{l}{m} \pmod{p}$. In other words, $\binom{Np+l}{Kp+m} - \binom{N}{K} \binom{l}{m}$ is divisible by p . Hence,

Lemma 2.18 (applied to $a = \binom{Np+l}{Kp+m}$ and $b = \binom{N}{K} \binom{l}{m}$) shows that

$$\begin{aligned} & \binom{Np+l}{Kp+m}^2 - \left(\binom{N}{K} \binom{l}{m} \right)^2 \\ & \equiv 2 \left(\binom{Np+l}{Kp+m} - \binom{N}{K} \binom{l}{m} \right) \binom{N}{K} \binom{l}{m} \pmod{p^2}. \end{aligned} \tag{5}$$

Now, forget that we fixed l and m . We thus have proven (5) for all $l \in \{0, 1, \dots, p-1\}$ and $m \in \{0, 1, \dots, p-1\}$. Now,

$$\begin{aligned} & \sum_{l=0}^{p-1} \sum_{m=0}^l \binom{Np+l}{Kp+m}^2 - \sum_{l=0}^{p-1} \sum_{m=0}^l \left(\binom{N}{K} \binom{l}{m} \right)^2 \\ & = \sum_{l=0}^{p-1} \sum_{m=0}^l \underbrace{\left(\binom{Np+l}{Kp+m}^2 - \left(\binom{N}{K} \binom{l}{m} \right)^2 \right)}_{\substack{\equiv 2 \left(\binom{Np+l}{Kp+m} - \binom{N}{K} \binom{l}{m} \right) \binom{N}{K} \binom{l}{m} \pmod{p^2} \\ \text{(by (5))}}} \\ & \equiv 2 \binom{N}{K} \underbrace{\sum_{l=0}^{p-1} \sum_{m=0}^l \left(\binom{Np+l}{Kp+m} - \binom{N}{K} \binom{l}{m} \right) \binom{l}{m}}_{\substack{\equiv 0 \pmod{p^2} \\ \text{(by Lemma 2.24)}}} \equiv 0 \pmod{p^2}. \end{aligned}$$

Thus,

$$\begin{aligned} \sum_{l=0}^{p-1} \sum_{m=0}^l \binom{Np+l}{Kp+m}^2 & \equiv \sum_{l=0}^{p-1} \sum_{m=0}^l \left(\binom{N}{K} \binom{l}{m} \right)^2 = \binom{N}{K}^2 \sum_{l=0}^{p-1} \underbrace{\sum_{m=0}^l \binom{l}{m}^2}_{\substack{= \sum_{m=0}^l \binom{l}{m} \binom{l}{m} = \binom{2l}{l} \\ \text{(by Corollary 2.11,} \\ \text{applied to } u=l \text{ and } w=0)}} \\ & = \binom{N}{K}^2 \underbrace{\sum_{l=0}^{p-1} \binom{2l}{l}}_{\substack{= \sum_{n=0}^{p-1} \binom{2n}{n} \equiv \eta_p \pmod{p^2} \\ \text{(by Theorem 1.8)}}} \equiv \binom{N}{K}^2 \eta_p \pmod{p^2}. \end{aligned}$$

This proves Lemma 2.25. □

Lemma 2.26. Let p be a prime. Let $N \in \mathbb{Z}$ and $K \in \mathbb{Z}$. Let u and v be two elements of $\{0, 1, \dots, p-1\}$ satisfying $u+v \geq p$. Then, $p \mid \binom{Np+u+v}{Kp+u}$.

Proof of Lemma 2.26. We have $u+v \geq p$. Thus, $u+v = p+c$ for some $c \in \mathbb{N}$. Consider this c . From $v \in \{0, 1, \dots, p-1\}$, we obtain $v < p$. Thus, $c+p = p+c = u + \underbrace{v}_{< p} < u+p$, so that $c < u \leq p-1$ (since $u \in \{0, 1, \dots, p-1\}$). Thus, $c \in \{0, 1, \dots, p-1\}$ (since $c \in \mathbb{N}$). Also, $c < u$. Hence, Proposition 2.2 (applied to $m=c$ and $n=u$) yields $\binom{c}{u} = 0$.

Now, $u+v = p+c$, so that $Np+u+v = Np+p+c = (N+1)p+c$. Hence,

$$\begin{aligned} \binom{Np+u+v}{Kp+u} &= \binom{(N+1)p+c}{Kp+u} \equiv \binom{N+1}{K} \underbrace{\binom{c}{u}}_{=0} \\ &\quad \text{(by Theorem 1.5, applied to } a=N+1, b=K \text{ and } d=u) \\ &= 0 \pmod{p}. \end{aligned}$$

In other words, $p \mid \binom{Np+u+v}{Kp+u}$. This proves Lemma 2.26. □

Lemma 2.27. Let p be an odd prime. Let $N \in \mathbb{Z}$ and $K \in \mathbb{N}$. Then,

$$\sum_{u=0}^{p-1} \sum_{v=0}^{p-1} \binom{Np+u+v}{Kp+u}^2 \equiv \binom{N}{K}^2 \eta_p \pmod{p^2}.$$

Proof of Lemma 2.27. If u and v are two elements of $\{0, 1, \dots, p-1\}$ satisfying $v \geq p-u$, then

$$\binom{Np+u+v}{Kp+u}^2 \equiv 0 \pmod{p^2} \tag{6}$$

5.

⁵*Proof of (6):* Let u and v be two elements of $\{0, 1, \dots, p-1\}$ satisfying $v \geq p-u$. From $v \geq p-u$, we obtain $u+v \geq p$. Thus, Lemma 2.26 yields $p \mid \binom{Np+u+v}{Kp+u}$. Hence, $p^2 \mid \binom{Np+u+v}{Kp+u}^2$. This proves (6).

Hence, any $u \in \{0, 1, \dots, p-1\}$ satisfies

$$\begin{aligned} \sum_{v=0}^{p-1} \binom{Np+u+v}{Kp+u}^2 &= \sum_{v=0}^{p-u-1} \binom{Np+u+v}{Kp+u}^2 + \underbrace{\sum_{v=p-u}^{p-1} \binom{Np+u+v}{Kp+u}^2}_{\equiv 0 \pmod{p^2} \text{ (by (6))}} \\ &\quad \text{(here, we have split the sum at } v = p - u) \\ &\equiv \sum_{v=0}^{p-u-1} \binom{Np+u+v}{Kp+u}^2 = \sum_{l=u}^{p-1} \binom{Np+l}{Kp+u}^2 \pmod{p^2} \end{aligned}$$

(here, we have substituted l for $u+v$ in the sum). Summing up these congruences for all $u \in \{0, 1, \dots, p-1\}$, we obtain

$$\begin{aligned} &\sum_{u=0}^{p-1} \sum_{v=0}^{p-1} \binom{Np+u+v}{Kp+u}^2 \\ &\equiv \underbrace{\sum_{u=0}^{p-1} \sum_{l=u}^{p-1} \binom{Np+l}{Kp+u}^2}_{= \sum_{l=0}^{p-1} \sum_{u=0}^l} = \sum_{l=0}^{p-1} \sum_{u=0}^l \binom{Np+l}{Kp+u}^2 = \sum_{l=0}^{p-1} \sum_{m=0}^l \binom{Np+l}{Kp+m}^2 \\ &\quad \text{(here, we have renamed the index } u \text{ as } m \text{ in the second sum)} \\ &\equiv \binom{N}{K}^2 \eta_p \pmod{p^2} \end{aligned}$$

(by Lemma 2.25). This proves Lemma 2.27. □

Proof of Theorem 1.10. First, let us observe that

$$\begin{aligned} \epsilon_{r,s} &= \sum_{m=0}^{r-1} \sum_{n=0}^{s-1} \binom{n+m}{m}^2 = \sum_{n=0}^{s-1} \sum_{m=0}^{r-1} \binom{n+m}{m}^2 = \sum_{K=0}^{s-1} \sum_{L=0}^{r-1} \binom{K+L}{L}^2 \\ &= \sum_{K=0}^{s-1} \sum_{L=0}^{r-1} \binom{K+L}{K}^2 \end{aligned} \tag{7}$$

(since Proposition 2.3 yields $\binom{K+L}{L} = \binom{K+L}{K}$ for all $K \in \mathbb{N}$ and $L \in \mathbb{N}$).

Each $n \in \mathbb{N}$ satisfies

$$\sum_{m=0}^{sp-1} \binom{n+m}{m}^2 = \sum_{u=0}^{p-1} \sum_{K=0}^{s-1} \binom{n+Kp+u}{Kp+u}^2$$

(here, we have substituted $Kp+u$ for m in the sum, since the map

$$\begin{aligned} \{0, 1, \dots, p-1\} \times \{0, 1, \dots, s-1\} &\rightarrow \{0, 1, \dots, sp-1\}, \\ (u, K) &\mapsto Kp+u \end{aligned}$$

is a bijection). Summing up this equality over all $n \in \{0, 1, \dots, rp - 1\}$, we obtain

$$\begin{aligned} \sum_{n=0}^{rp-1} \sum_{m=0}^{sp-1} \binom{n+m}{m}^2 &= \sum_{n=0}^{rp-1} \sum_{u=0}^{p-1} \sum_{K=0}^{s-1} \binom{n+Kp+u}{Kp+u}^2 \\ &= \sum_{v=0}^{p-1} \sum_{L=0}^{r-1} \sum_{u=0}^{p-1} \sum_{K=0}^{s-1} \binom{Lp+v+Kp+u}{Kp+u}^2 \end{aligned}$$

(here, we have substituted $Lp + v$ for n in the sum, since the map

$$\begin{aligned} \{0, 1, \dots, p - 1\} \times \{0, 1, \dots, r - 1\} &\rightarrow \{0, 1, \dots, rp - 1\}, \\ (v, L) &\mapsto Lp + v \end{aligned}$$

is a bijection).

Thus,

$$\begin{aligned} \sum_{n=0}^{rp-1} \sum_{m=0}^{sp-1} \binom{n+m}{m}^2 &= \sum_{v=0}^{p-1} \sum_{L=0}^{r-1} \sum_{u=0}^{p-1} \sum_{K=0}^{s-1} \underbrace{\binom{Lp+v+Kp+u}{Kp+u}^2}_{\substack{= \binom{(K+L)p+u+v}{Kp+u}^2 \\ = \binom{(K+L)p+u+v}{Kp+u}^2}} \\ &= \sum_{K=0}^{s-1} \sum_{L=0}^{r-1} \sum_{u=0}^{p-1} \sum_{v=0}^{p-1} \underbrace{\binom{(K+L)p+u+v}{Kp+u}^2}_{\substack{\equiv \binom{K+L}{K}^2 \eta_p \pmod{p^2} \\ \text{(by Lemma 2.27, applied to } N=K+L)}} \\ &\equiv \underbrace{\sum_{K=0}^{s-1} \sum_{L=0}^{r-1} \binom{K+L}{K}^2}_{\substack{= \epsilon_{r,s} \\ \text{(by (7))}}} \eta_p = \epsilon_{r,s} \eta_p = \eta_p \epsilon_{r,s} \pmod{p^2}. \end{aligned}$$

This proves Theorem 1.10. □

2.8. Acknowledgments

Thanks to Doron Zeilberger and Roberto Tauraso for alerting me to [AmdTau16] and [SunTau11].

References

- [AmdTau16] Tewodros Amdeberhan, Roberto Tauraso, *Two triple binomial sum supercongruences*, Journal of Number Theory **175** (2017), pp. 140–157. A preprint is arXiv:1607.02483v1.

- [AnBeRo05] Peter G. Anderson, Arthur T. Benjamin and Jeremy A. Rouse, *Combinatorial Proofs of Fermat's, Lucas's, and Wilson's Theorems*, The American Mathematical Monthly, Vol. **112**, No. 3 (Mar., 2005), pp. 266–268.
- [ApaZei16] Moa Apagodu, Doron Zeilberger, *Using the "Freshman's Dream" to Prove Combinatorial Congruences*, The American Mathematical Monthly, Vol. **124**, No. 7 (August-September 2017), pp. 597–608.
(A preprint can be found at arXiv:1606.03351v2, but is less up-to-date and uses a different numbering of the conjectures.)
- [Bailey91] D. F. Bailey, *Some Binomial Coefficient Congruences*, Applied Mathematics Letters, Volume **4**, Issue 4, 1991, pp. 1–5.
[https://doi.org/10.1016/0893-9659\(91\)90043-U](https://doi.org/10.1016/0893-9659(91)90043-U)
- [BenQui03] Arthur T. Benjamin and Jennifer J. Quinn, *Proofs that Really Count: The Art of Combinatorial Proof*, The Mathematical Association of America, 2003.
- [BenQui08] Arthur T. Benjamin and Jennifer J. Quinn, *An Alternate Approach to Alternating Sums: A Method to DIE for*, The College Mathematics Journal, Volume 39, Number 3, May 2008, pp. 191-202(12).
- [GrKnPa94] Ronald L. Graham, Donald E. Knuth, Oren Patashnik, *Concrete Mathematics, Second Edition*, Addison-Wesley 1994.
- [Grinbe17] Darij Grinberg, *Notes on the combinatorial fundamentals of algebra*, 4 October 2018.
<https://github.com/darijgr/detnotes/releases/tag/2018-10-04>
See also <http://www.cip.ifi.lmu.de/~grinberg/primes2015/sols.pdf> for a version that is getting updates.
- [Grinbe17b] Darij Grinberg, *The Lucas and Babbage congruences*, 3 October 2018.
<http://www.cip.ifi.lmu.de/~grinberg/lucascong.pdf>
- [Hausne83] Melvin Hausner, *Applications of a Simple Counting Technique*, The American Mathematical Monthly, Vol. 90, No. 2 (Feb., 1983), pp. 127–129.
- [Liu16] Ji-Cai Liu, *On two conjectural supercongruences of Apagodu and Zeilberger*, arXiv:1606.08432v3.
- [MacSon10] Kieren MacMillan and Jonathan Sondow, *Proofs of Power Sum and Binomial Coefficient Congruences Via Pascal's Identity*, arXiv:1011.0076v1.
Published in: The American Mathematical Monthly, Vol. **118** (2011), pp. 549–551.
- [Mestro14] Romeo Meštrović, *Lucas' theorem: its generalizations, extensions and applications (1878–2014)*, arXiv:1409.3820v1.
-

- [Stanle11] Richard Stanley, *Enumerative Combinatorics, volume 1*, Second edition, version of 15 July 2011. Available at <http://math.mit.edu/~rstan/ec/>.
- [Stanto16] Dennis Stanton, *Addendum to "Using the "Freshman's Dream" to Prove Combinatorial Congruences"*, <http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/freshmanDennisStanton.pdf>
- [SunTau11] Zhi-Wei Sun, Roberto Tauraso, *On some new congruences for binomial coefficients*, *International Journal of Number Theory*, Vol. 7, No. 3 (2011), pp. 645–662. A preprint is arXiv:0709.1665v10.
-