Iterative properties of birational rowmotion

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version 6.0 (March 9, 2016).
This is the arXiv version, not the published version. The published version has been abridged in several places and split into two papers.
Mathematics Subject Classifications: 06A07, 05E99

Abstract

We study a birational map associated to any finite poset P. This map is a far-reaching generalization (found by Einstein and Propp) of classical rowmotion, which is a certain permutation of the set of order ideals of P. Classical rowmotion has been studied by various authors (Fon-der-Flaass, Cameron, Brouwer, Schrijver, Striker, Williams and many more) under different guises (Striker-Williams promotion and Panyushev complementation are two examples of maps equivalent to it). In contrast, birational rowmotion is new and has yet to reveal several of its mysteries. In this paper, we prove that birational rowmotion has order $p + q$ on the (p, q)-rectangle poset (i.e., on the product of a p-element chain with a q-element chain); we furthermore compute its orders on some triangle-shaped posets and on a class of posets which we call "skeletal" (this class includes all graded forests). In all cases mentioned, birational rowmotion turns out to have a finite (and explicitly computable) order, a property it does not exhibit for general finite posets (unlike classical rowmotion, which is a permutation of a finite set). Our proof in the case of the rectangle poset uses an idea introduced by Volkov (arXiv:hep-th/0606094) to prove the AA case of the Zamolodchikov periodicity conjecture; in fact, the finite order of birational rowmotion on many posets can be considered an analogue to Zamolodchikov periodicity. We comment on suspected, but so far enigmatic, connections to the theory of root posets. We also make a digression to study classical rowmotion on skeletal posets, since this case has seemingly been overlooked so far.

Keywords: rowmotion; posets; order ideals; Zamolodchikov periodicity; root systems; promotion; trees; graded posets; Grassmannian; tropicalization.

*Supported by NSF grant 1001905.
†Supported by NSF grant 1001905.

Contents

Introduction

The present paper had originally been intended as a companion paper to David Einstein's and James Propp's work [EiPr13], which introduced piecewise-linear and birational rowmotion as extensions of the classical concept of rowmotion on order ideals. While the present paper is mathematically self-contained (and indeed gives some proofs on which [EiPr13] relies), it provides only a modicum of motivation and applications for the results it discusses. For the latter, the reader may consult [EiPr13].

Let P be a finite poset, and $J(P)$ the set of the order ideals[1] of P. *Rowmotion* is a classical map $J(P) \to J(P)$ which can be defined in various ways, one of which is as follows: For every $v \in P$, let $\mathbf{t}_v : J(P) \to J(P)$ be the map sending every order ideal $S \in J(P)$ to $\begin{cases} S \cup \{v\}, \text{ if } v \notin S \text{ and } S \cup \{v\} \in J(P); \\ S \setminus \{v\}, \text{ if } v \in S \text{ and } S \setminus \{v\} \in J(P); \\ S, \text{ otherwise} \end{cases}$. These maps $\mathbf{t}_v$ are called *classical toggles*[2], since all they do is "toggle" an element into or out of an order ideal. Let $(v_1, v_2, ..., v_m)$ be a linear extension of P (see Definition 1.3 for the meaning of this). Then, (classical) rowmotion is defined as the composition $\mathbf{t}_{v_1} \circ \mathbf{t}_{v_2} \circ ... \circ \mathbf{t}_{v_m}$ (which, as can be seen, does not depend on the choice of the particular linear extension $(v_1, v_2, ..., v_m)$). This rowmotion map has been studied from various perspectives; in particular, it is isomorphic[3] to the map f of Fon-der-Flaass [Flaa93][4], the map F^{-1} of Brouwer and Schrijver [BrSchr74], and the map f^{-1} of Cameron and Fon-der-Flaass [CaFl95][5]. More recently, it has been studied (and christened "rowmotion") in Striker and Williams [StWi11], where further sources and context are also given. Since so much has already been said about this rowmotion map, we will only briefly touch on its properties in Section 10, while most of this paper will be spent studying a much more general construction.

Among the questions that have been posed about rowmotion, the most prevalent was probably that of its order: While it clearly has finite order (being a bijective map from the finite set $J(P)$ to itself), it turns out to have a much smaller order than what one would naively expect when the poset P has certain "special" forms (e.g., a rectangle, a root poset, a product of a rectangle with a 2-chain, or – apparently first considered in this paper – a forest). Most strikingly, when P is the rectangle $[p] \times [q]$ (denoted $\operatorname{Rect}(p, q)$ in Definition 11.1), then the $(p+q)$-th power of the rowmotion operator is the identity map. This is proven in [BrSchr74, Theorem 3.6] and [Flaa93, Theorem 2][6]. We will (in

[1] An *order ideal* of a poset P is a subset S of P such that every $s \in S$ and $p \in P$ with $p \leqslant s$ satisfy $p \in S$.

[2] or just *toggles* in literature which doesn't occupy itself with birational rowmotion

[3] By this, we mean that there exists a bijection ϕ from $J(P)$ to the set of all antichains of P such that rowmotion is $\phi^{-1} \circ f \circ \phi$.

[4] Indeed, let $\mathcal{A}(P)$ denote the set of all antichains of P. Then, the map $J(P) \to \mathcal{A}(P)$ which sends every order ideal $I \in J(P)$ to the antichain of the maximal elements of I is a bijection which intertwines rowmotion and Fon-der-Flaass' map f.

[5] This time, the intertwining bijection from rowmotion to the map f^{-1} of [CaFl95] is given by mapping every order ideal I to its indicator function. This is a bijection from $J(P)$ to the set of Boolean monotonic functions $P \to \{0, 1\}$.

[6] Another proof follows from two observations made in [PrRo14]: first, that the rowmotion operator on

Section 10) give a simple algorithm to find the order of rowmotion on graded forests and similar posets.

In [EiPr13], David Einstein and James Propp have lifted the rowmotion map from the set $J(P)$ of order ideals to the progressively more general setups of:

(a) the order polytope $\mathcal{O}(P)$ of the poset P (as defined in [Stan11, Example 4.6.17] or [Stan86, Definition 1.1]), and

(b) even more generally, the affine variety of $\mathbb{K}$-labellings of P for $\mathbb{K}$ an arbitrary infinite field.

In case **(a)**, order ideals of P are replaced by points in the order polytope $\mathcal{O}(P)$, and the role of the map $\mathbf{t}_v$ (for a given $v \in P$) is assumed by the map which reflects the v-coordinate of a point in $\mathcal{O}(P)$ around the midpoint of the interval of all values it could take without the point leaving $\mathcal{O}(P)$ (while all other coordinates are considered fixed). The operation of "piecewise linear" rowmotion (inspired by work of Arkady Berenstein) is still defined as the composition of these reflection maps in the same way as rowmotion is the composition of the toggles $\mathbf{t}_v$. This "piecewise linear" rowmotion extends (interpolates, even) classical rowmotion, as order ideals correspond to the vertices of the order polytope $\mathcal{O}(P)$ (see [Stan86, Corollary 1.3]). We will not study case **(a)** here, since all of the results we could find in this case can be obtained by tropicalization from similar results for case **(b)**.

In case **(b)**, instead of order ideals of P one considers maps from the poset $\widehat{P} := \{0\} \oplus P \oplus \{1\}$ (where $\oplus$ stands for the ordinal sum[7]) to a given infinite field $\mathbb{K}$ (or, to speak more graphically, of all labellings of the elements of P by elements of $\mathbb{K}$, along with two additional labels "at the very bottom" and "at the very top"). The maps $\mathbf{t}_v$ are then replaced by certain birational maps which we call *birational v-toggles* (Definition 2.6); the resulting composition is called *birational rowmotion* and denoted by R. By a careful limiting procedure (the tropical limit), we can "degenerate" R to the "piecewise linear" rowmotion of case **(a)**, and thus it can be seen as an even higher generalization of classical rowmotion. We refer to the body of this paper for precise definitions of these maps. Note that birational v-toggles (but not birational rowmotion) in the case of a rectangle poset have also appeared in [OSZ13, (3.5)], but (apparently) have not been composed there in a way that yields birational rowmotion.

As in the case of classical rowmotion on $J(P)$, the most interesting question is the order of this map R, which in general no longer has an obvious reason to be finite (since the affine variety of $\mathbb{K}$-labellings is not a finite set like $J(P)$). Indeed, for some posets P this order is infinite. In this paper we will prove the following facts:

- Birational rowmotion (i.e., the map R) on any graded poset (in the meaning of this

the order ideals of the rectangle $[p] \times [q]$ is equivalent to the operator named Φ_A in [PrRo14] (i.e., there is a bijection between order ideals and antichains of $[p] \times [q]$ which intertwines these two operators), and second, that the $(p+q)$-th power of this latter operator Φ_A is the identity map (this is proven in [PrRo14, right after Proposition 26]). This argument can also be constructed from ideas given in [PrRo13, §3.3.1].

[7] More explicitly, $\widehat{P}$ is the poset obtained by adding a new element 0 to P, which is set to be lower than every element of P, and adding a new element 1 to P, which is set to be higher than every element of P (and 0). We shall repeat this definition in more formal terms in Definition 2.1.

word introduced in Definition 3.3) has a very simple effect (namely, cyclic shifting) on the so-called "w-tuple" of a labelling (a rather simple fingerprint of the labelling). This does not mean R itself has finite order (but turns out to be crucial in proving this in several cases).

- Birational rowmotion on graded forests and, slightly more generally, skeletal posets (Definition 9.5) has finite order (which can be bounded from above by an iterative lcm, and also easily computed algorithmically). Moreover, its order in these cases coincides with the order of classical rowmotion (Section 10).

- Birational rowmotion on a $p \times q$-rectangle has order $p + q$ and satisfies a further symmetry property (Theorem 11.7). These results have originally been conjectured by James Propp and the second author, and can be used as an alternative route to certain properties of (Schützenberger's) promotion map on semistandard Young tableaux.

- Birational rowmotion on certain triangle-shaped posets (this is made precise in Sections 17, 18, 19) also has finite order (computed explicitly below). We show this for three kinds of triangle-shaped posets (obtained by cutting the $p \times p$-square in two along either of its two diagonals) and conjecture it for a fourth (a quarter of a $p \times p$-square obtained by cutting it along both diagonals).

The proof of the most difficult and fundamental case – that of a $p \times q$-rectangle – is inspired by Volkov's proof of the "rectangular" (type-AA) Zamolodchikov conjecture [Volk06], which uses a similar idea of parametrizing (generic) $\mathbb{K}$-labellings by matrices (or tuples of points in projective space). There is, of course, a striking similarity between the fact itself and the Zamolodchikov conjecture; yet, we were not able to reduce either result to the other.

Applications of the results of this paper (specifically Theorems 11.5 and 11.7) are found in [EiPr13]. Further directions currently under study of the authors are relations to the totally positive Grassmannian and generalizations to further classes of posets.

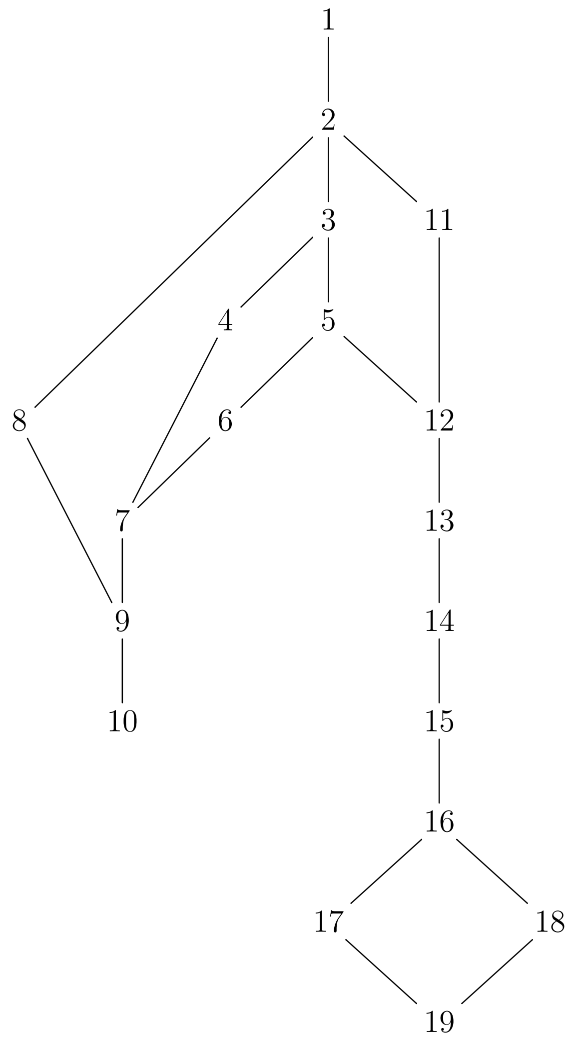
An extended (12-page) abstract [GrRo13] of this paper has been published in the proceedings of the FPSAC 2014 conference.

For publication, this preprint has been split into two papers: "*Iterative properties of birational rowmotion I: generalities and skeletal posets*" (published in Volume 23, Issue 1 (2016) of the Electronic Journal of Combinatorics), and "*Iterative properties of birational rowmotion II: rectangles and triangles*") (published in Volume 22, Issue 3 (2015) of the Electronic Journal of Combinatorics). These two papers are somewhat less detailed than the preprint that you are currently reading; they also (unlike this preprint) have undergone some stylistic changes during the refereeing process.[8]

[8]However, all errors found by the referees have been corrected both in the two papers and in this preprint.

0.1 Leitfaden

The following Hasse diagram shows how the sections of this paper depend upon each other.



A section n depends substantially on a section m if and only if $m > n$ in the poset whose Hasse diagram is depicted above. Only substantial dependencies are shown; dependencies upon definitions do not count as substantial (e.g., many sections depend on Definition 7.1, but this does not make them substantially dependent on Section 7), and dependencies which are only used in proving inessential claims do not count (e.g., the proof of Theorem 11.5 relies on Proposition 7.3 in order to show that $\operatorname{ord}\left(R_{\operatorname{Rect}(p,q)}\right) = p + q$ rather than just $\operatorname{ord}\left(R_{\operatorname{Rect}(p,q)}\right) \mid p+q$, but since the $\operatorname{ord}\left(R_{\operatorname{Rect}(p,q)}\right) \mid p+q$ statement is in our opinion the only important part of the theorem, we do not count this as a dependency on Section 7). Sections 20 and 21 are not shown.

No section of this paper depends on the Introduction.

0.2 Acknowledgments

When confronted with the (then open) problem of proving what is Theorem 11.5 in this paper, Pavlo Pylyavskyy and Gregg Musiker suggested reading [Volk06]. This suggestion proved highly useful and built the cornerstone of this paper, without which the latter would have ended at its "Skeletal posets" section.

The notion of birational rowmotion is due to James Propp and Arkady Berenstein. This paper owes James Propp also for a constant flow of inspiration and useful suggestions.

David Einstein found errors in our computations, and Hugh Thomas corrected slips in the writing including an abuse of Zariski topology and some accidental alternative history.

Nathan Williams noticed typos, too, and suggested a path connecting this subject to the theory of minuscule posets (which we will not explore in this paper).

The first author came to know birational rowmotion in Alexander Postnikov's combinatorics pre-seminar at MIT. Postnikov also suggested veins of further study.

Jessica Striker helped the first author understand some of the past work on this subject, in particular the labyrinthine connections between the various operators (rowmotion, Panyushev complementation, Striker-Williams promotion, Schützenberger promotion, etc.). The present paper explores merely one corner of this labyrinth (the rowmotion corner).

We thank Dan Bump, Anne Schilling and the two referees of our FPSAC abstract [GrRo13] for further helpful comments. We also owe a number of improvements in this paper to the suggestions of two anonymous EJC referees.

Both authors were partially supported by NSF grant #1001905, and have utilized the open-source CAS Sage ([S$^+$09], [Sage08]) to perform laborious computations. We thank Travis Scrimshaw, Frédéric Chapoton, Viviane Pons and Nathann Cohen for reviewing Sage patches relevant to this project.

1 Linear extensions of posets

This first section serves to introduce some general notions concerning posets and their linear extensions. In particular, we highlight that the set of linear extensions of any finite poset is non-empty and connected by a simple equivalence relation (Proposition 1.7). This will be used in subsequent sections for defining the basic maps that we consider throughout the paper.

Let us first get a basic convention out of the way:

Convention 1.1. We let $\mathbb{N}$ denote the set $\{0, 1, 2, ...\}$.

We start by defining general notations related to posets:

Definition 1.2. Let P be a poset. Let $u \in P$ and $v \in P$. In this definition, we will use $\leqslant$, $<$, $\geqslant$ and $>$ to denote the lesser-or-equal relation, the lesser relation, the greater-or-equal relation and the greater relation, respectively, of the poset P.

(a) The elements u and v of P are said to be *incomparable* if we have neither $u \leqslant v$ nor $u \geqslant v$.

(b) We write $u \lessdot v$ if we have $u < v$ and there is no $w \in P$ such that $u < w < v$. One often says that "u is covered by v" to signify that $u \lessdot v$.

(c) We write $u \gtrdot v$ if we have $u > v$ and there is no $w \in P$ such that $u > w > v$. (Thus, $u \gtrdot v$ holds if and only if $v \lessdot u$.) One often says that "u covers v" to signify that $u \gtrdot v$.

(d) An element u of P is called *maximal* if every $v \in P$ satisfying $v \geqslant u$ satisfies $v = u$. It is easy to see that every nonempty finite poset has at least one maximal element.

(e) An element u of P is called *minimal* if every $v \in P$ satisfying $v \leqslant u$ satisfies $v = u$. It is easy to see that every nonempty finite poset has at least one minimal element.

When any of these notations becomes ambiguous because the elements involved belong to several different posets simultaneously, we will disambiguate it by adding the words "in P" (where P is the poset which we want to use).[9]

Definition 1.3. Let P be a finite poset. A *linear extension* of P will mean a list $(v_1, v_2, ..., v_m)$ of the elements of P such that every element of P occurs exactly once in this list, and such that any $i \in \{1, 2, ..., m\}$ and $j \in \{1, 2, ..., m\}$ satisfying $v_i < v_j$ (where $<$ is the smaller relation of P) must satisfy $i < j$.

A brief remark on this definition is in order. Stanley, in [Stan11, one paragraph below the proof of Proposition 3.5.2], defines a linear extension of a poset P as an order-preserving bijection from P to the chain $\{1, 2, ..., |P|\}$; this is equivalent to our definition (indeed, our linear extension $(v_1, v_2, ..., v_m)$, whose length obviously is $m = |P|$, corresponds to the bijection $P \to \{1, 2, ..., |P|\}$ which sends each v_i to i). Another widespread definition of a linear extension of P is as a total order on P compatible with the given order of the poset P; this is equivalent to our definition as well (the total order is the one defined by $v_i < v_j$ whenever $i < j$).

Notice that if $(v_1, v_2, ..., v_m)$ is a linear extension of a nonempty finite poset P, then v_1 is a minimal element of P and v_m is a maximal element of P. The only linear extension of the empty poset $\varnothing$ is the empty list $()$.

Theorem 1.4. Let P be a finite poset. Then, there exists a linear extension of P.

Theorem 1.4 is a well-known fact, and can be proven, e.g., by induction over $|P|$ (with the induction step consisting of splitting off a maximal element u of P and appending it to a linear extension of the residual poset $P \setminus \{u\}$).

The following proposition can be easily checked by the reader:

[9] For instance, if R denotes the poset $\mathbb{Z}$ endowed with the reverse of its usual order, then we say (for instance) that "$1 \lessdot 0$ in R" rather than just "$1 \lessdot 0$".

Proposition 1.5. Let P be a finite poset. Let $(v_1, v_2, ..., v_m)$ be a linear extension of P. Let $i \in \{1, 2, ..., m-1\}$ be such that the elements v_i and v_{i+1} of P are incomparable. Then, $(v_1, v_2, ..., v_{i-1}, v_{i+1}, v_i, v_{i+2}, v_{i+3}, ..., v_m)$ (this is the tuple obtained from the tuple $(v_1, v_2, ..., v_m)$ by interchanging the adjacent entries v_i and v_{i+1}) is a linear extension of P as well.

Definition 1.6. Let P be a finite poset. The set of all linear extensions of P will be called $\mathcal{L}(P)$. Thus, $\mathcal{L}(P) \neq \varnothing$ (by Theorem 1.4).

In our approach to birational rowmotion, we will use the following fact (which is folklore and has applications in various contexts, including Young tableau theory):

Proposition 1.7. Let P be a finite poset. Let $\sim$ denote the equivalence relation on $\mathcal{L}(P)$ generated by the following requirement: For any linear extension $(v_1, v_2, ..., v_m)$ of P and any $i \in \{1, 2, ..., m-1\}$ such that the elements v_i and v_{i+1} of P are incomparable, we set $(v_1, v_2, ..., v_m) \sim (v_1, v_2, ..., v_{i-1}, v_{i+1}, v_i, v_{i+2}, v_{i+3}, ..., v_m)$ (noting that $(v_1, v_2, ..., v_{i-1}, v_{i+1}, v_i, v_{i+2}, v_{i+3}, ..., v_m)$ is also a linear extension of P, because of Proposition 1.5). [10] Then, any two elements of $\mathcal{L}(P)$ are equivalent under the relation $\sim$.

This proposition is very basic (it generalizes the fact that the symmetric group S_n is generated by the adjacent-element transpositions) and is classical, and proofs can be found in the literature. One proof is in [AKSch12, Proposition 4.1 (for the $\pi' = \pi\tau_j$ case)]; another is sketched in [Rusk92, p. 79] and presented in more detail in [Etienn84, Lemma 1].[11] In order to keep our paper self-contained, we will prove it too. Our proof is based on the following lemma (which is more or less a simple particular case of Proposition 1.7):

[10] Here is a more formal way to restate this definition of $\sim$:

We first introduce a binary relation $\equiv$ on the set $\mathcal{L}(P)$ as follows: If $\mathbf{v}$ and $\mathbf{w}$ are two linear extensions of P, then we set $\mathbf{v} \equiv \mathbf{w}$ if and only if the list $\mathbf{w}$ can be obtained from the list $\mathbf{v}$ by interchanging two adjacent entries v and v' which are incomparable in P. It is clear that this binary relation $\equiv$ is symmetric. It is also clear that for any linear extension $(v_1, v_2, ..., v_m)$ of P and any $i \in \{1, 2, ..., m-1\}$ such that the elements v_i and v_{i+1} of P are incomparable, the list $(v_1, v_2, ..., v_{i-1}, v_{i+1}, v_i, v_{i+2}, v_{i+3}, ..., v_m)$ is also a linear extension of P (according to Proposition 1.5) and satisfies $(v_1, v_2, ..., v_m) \equiv (v_1, v_2, ..., v_{i-1}, v_{i+1}, v_i, v_{i+2}, v_{i+3}, ..., v_m)$. Now, we define $\sim$ as the reflexive and transitive closure of the binary relation $\equiv$. Then, $\sim$ is an equivalence relation on $\mathcal{L}(P)$.

[11] For the sake of intellectual enrichment, let us outline yet another proof, which has been suggested by Thomas McConville. This proof is geometric (it uses the theory of hyperplane arrangements), and making it fully precise would require certain topological technicalities which we shall not delve into; but the idea is instructive and provides intuition. Namely, assume WLOG that $P = \{1, 2, \ldots, n\}$ as sets. A *listing* shall mean an n-tuple of distinct elements of P. Thus, listings are in bijection with the permutations of $\{1, 2, \ldots, n\}$.

Let $\mathcal{B}$ be the braid arrangement in $\mathbb{R}^n$ (that is, the hyperplane arrangement formed by the hyperplanes $x_i = x_j$ for all $1 \leqslant i < j \leqslant n$). The chambers of $\mathcal{B}$ are known to be in a 1-to-1 correspondence with the listings: Namely, to any listing $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ corresponds the chamber given by the inequalities $x_{p_1} < x_{p_2} < \cdots < x_{p_n}$; we denote the latter chamber by $C(\mathbf{p})$.

On the other hand, let K_P be the open cone in $\mathbb{R}^n$ defined by the inequalities $x_i < x_j$ for all pairs $(i, j) \in P^2$ satisfying $i < j$ in P. Then, a linear extension of P is precisely a listing $\mathbf{p}$ satisfying $C(\mathbf{p}) \subseteq K_P$.

Lemma 1.8. Let P be a finite poset. Define the equivalence relation $\sim$ on $\mathcal{L}(P)$ as in Proposition 1.7. Let a_1, a_2, ..., a_k be some elements of P. Let b_1, b_2, ..., b_ℓ be some further elements of P. Let u be a maximal element of P. Assume that $(a_1, a_2, ..., a_k, u, b_1, b_2, ..., b_\ell)$ is a linear extension of P. Then, $(a_1, a_2, ..., a_k, b_1, b_2, ..., b_\ell, u)$ is a linear extension of P satisfying $(a_1, a_2, ..., a_k, u, b_1, b_2, ..., b_\ell) \sim (a_1, a_2, ..., a_k, b_1, b_2, ..., b_\ell, u)$.

Proof of Lemma 1.8 (sketched). We will show that every $i \in \{0, 1, ..., \ell\}$ satisfies the following assertion:

$$\left(\begin{array}{c} \text{The tuple } (a_1, a_2, ..., a_k, b_1, b_2, ..., b_i, u, b_{i+1}, b_{i+2}, ..., b_\ell) \text{ is a} \\ \text{linear extension of } P \text{ satisfying} \\ (a_1, a_2, ..., a_k, u, b_1, b_2, ..., b_\ell) \sim (a_1, a_2, ..., a_k, b_1, b_2, ..., b_i, u, b_{i+1}, b_{i+2}, ..., b_\ell) \end{array} \right). \qquad (1)$$

Proof of (1): We will prove (1) by induction over i:

Induction base: If $i = 0$, then $(a_1, a_2, ..., a_k, b_1, b_2, ..., b_i, u, b_{i+1}, b_{i+2}, ..., b_\ell) = (a_1, a_2, ..., a_k, u, b_1, b_2, ..., b_\ell)$. Hence, (1) is a tautology for $i = 0$, and the induction base is done.

Induction step: Let $I \in \{1, 2, ..., \ell\}$. Assume that (1) holds for $i = I - 1$. We need to prove that (1) holds for $i = I$.

We have assumed that (1) holds for $i = I - 1$. In other words, the tuple $(a_1, a_2, ..., a_k, b_1, b_2, ..., b_{I-1}, u, b_{I-1+1}, b_{I-1+2}, ..., b_\ell)$ is a linear extension of P satisfying $(a_1, a_2, ..., a_k, u, b_1, b_2, ..., b_\ell) \sim (a_1, a_2, ..., a_k, b_1, b_2, ..., b_{I-1}, u, b_{I-1+1}, b_{I-1+2}, ..., b_\ell)$.

Denote the smaller relation of P by $<$. Since the tuple $(a_1, a_2, ..., a_k, u, b_1, b_2, ..., b_\ell)$ is a linear extension of P, we cannot have $u \geqslant b_I$ (because u appears strictly to the left of b_I in this tuple). But we cannot have $u < b_I$ either (since u is a maximal element of P). Thus, u and b_I are incomparable.

Let now $\mathbf{u}$ and $\mathbf{v}$ be two elements of $\mathcal{L}(P)$, that is, two linear extensions of P. Thus, $C(\mathbf{u}) \subseteq K_P$ and $C(\mathbf{v}) \subseteq K_P$. Pick any two points $y \in C(\mathbf{u})$ and $z \in C(\mathbf{v})$. Then, both y and z lie in K_P; therefore, so does every point on the segment joining y with z (since K_P is convex). Thus, there exists a continuous path from y to z staying entirely inside K_P. By slightly deforming this path, we can ensure that it never intersects more than one hyperplane of $\mathcal{B}$ at the same point (at the expense of no longer being a straight line); if we do this with care, then it still will remain inside K_P (since K_P is an open set; here is where we are using some topology). Consider this latter path. Let $C(\mathbf{u}_1), C(\mathbf{u}_2), \ldots, C(\mathbf{u}_k)$ be the chambers of $\mathcal{B}$ it traverses. Thus, all the listings $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ are linear extensions of P (since all the chambers $C(\mathbf{u}_1), C(\mathbf{u}_2), \ldots, C(\mathbf{u}_k)$ are contained in K_P), thus belong to $\mathcal{L}(P)$. Moreover, $C(\mathbf{u}_1) = C(\mathbf{u})$ (since the path starts at $y \in C(\mathbf{u})$), so that $\mathbf{u}_1 = \mathbf{u}$. Similarly, $\mathbf{u}_k = \mathbf{v}$. Moreover, for every $i \in \{1, 2, \ldots, k-1\}$, the chambers $C(\mathbf{u}_i)$ and $C(\mathbf{u}_{i+1})$ are separated by precisely one hyperplane. This can easily be translated as follows: For every $i \in \{1, 2, \ldots, k-1\}$, the listing $\mathbf{u}_{i+1}$ is obtained from $\mathbf{u}_i$ by interchanging two adjacent entries. Hence, for every $i \in \{1, 2, \ldots, k-1\}$, we have $\mathbf{u}_{i+1} \sim \mathbf{u}_i$. Since $\sim$ is an equivalence relation, this shows that $\mathbf{u}_k \sim \mathbf{u}_1$. In other words, $\mathbf{v} \sim \mathbf{u}$ (since $\mathbf{u}_1 = \mathbf{u}$ and $\mathbf{u}_k = \mathbf{v}$), qed.

Now,

$$\begin{aligned}
&(a_1, a_2, ..., a_k, u, b_1, b_2, ..., b_\ell) \\
&\sim (a_1, a_2, ..., a_k, b_1, b_2, ..., b_{I-1}, u, b_{I-1+1}, b_{I-1+2}, ..., b_\ell) \\
&= (a_1, a_2, ..., a_k, b_1, b_2, ..., b_{I-1}, u, b_I, b_{I+1}, b_{I+2}, ..., b_\ell) \\
&\sim (a_1, a_2, ..., a_k, b_1, b_2, ..., b_{I-1}, b_I, u, b_{I+1}, b_{I+2}, ..., b_\ell) \\
&\qquad\qquad \text{(by the definition of the relation } \sim \text{, since } u \text{ and } b_I \text{ are incomparable)} \\
&= (a_1, a_2, ..., a_k, b_1, b_2, ..., b_I, u, b_{I+1}, b_{I+2}, ..., b_\ell) .
\end{aligned}$$

The proof of this equivalence also shows that its right hand side is a linear extension of P. Thus, (1) holds for $i = I$. This completes the induction step, whence (1) is proven.

Lemma 1.8 now follows by applying (1) to $i = \ell$.

□

Proof of Proposition 1.7 (sketched). We prove Proposition 1.7 by induction over $|P|$. The induction base $|P| = 0$ is trivial. For the induction step, let N be a positive integer. Assume that Proposition 1.7 is proven for all posets P with $|P| = N - 1$. Now, let P be a poset with $|P| = N$.

Let $(v_1, v_2, ..., v_N)$ and $(w_1, w_2, ..., w_N)$ be two elements of $\mathcal{L}(P)$. We are going to prove that $(v_1, v_2, ..., v_N) \sim (w_1, w_2, ..., w_N)$.

Let $u = v_N$. Then, u is a maximal element of P (since it comes last in the linear extension $(v_1, v_2, ..., v_N)$). Let i be the index satisfying $w_i = u$.

Consider the poset $P \setminus \{u\}$. This poset has size $|P \setminus \{u\}| = \underbrace{|P|}_{=N} - 1 = N - 1$. Define a relation $\sim$ on $\mathcal{L}(P \setminus \{u\})$ in the same way as the relation $\sim$ on $\mathcal{L}(P)$ was defined. Recall that u is a maximal element of P. Hence,

$$\left(\begin{array}{c} \text{if } (a_1, a_2, ..., a_{N-1}) \text{ is a linear extension of } P \setminus \{u\} \text{, then} \\ (a_1, a_2, ..., a_{N-1}, u) \text{ is a linear extension of } P \end{array} \right) . \tag{2}$$

Moreover, just by recalling how the relations $\sim$ were defined, we can easily see that

$$\left(\begin{array}{c} \text{if two linear extensions } (a_1, a_2, ..., a_{N-1}) \text{ and } (b_1, b_2, ..., b_{N-1}) \text{ of } P \setminus \{u\} \\ \text{satisfy } (a_1, a_2, ..., a_{N-1}) \sim (b_1, b_2, ..., b_{N-1}) \text{ in } \mathcal{L}(P \setminus \{u\}) \text{, then} \\ (a_1, a_2, ..., a_{N-1}, u) \text{ and } (b_1, b_2, ..., b_{N-1}, u) \text{ are two linear extensions} \\ \text{of } P \text{ satisfying } (a_1, a_2, ..., a_{N-1}, u) \sim (b_1, b_2, ..., b_{N-1}, u) \text{ in } \mathcal{L}(P) \end{array} \right) \tag{3}$$

(here, the fact that $(a_1, a_2, ..., a_{N-1}, u)$ and $(b_1, b_2, ..., b_{N-1}, u)$ are linear extensions of P follows from (2)).

It is rather clear that $(v_1, v_2, ..., v_{N-1})$ and $(w_1, w_2, ..., w_{i-1}, w_{i+1}, w_{i+2}, ..., w_N)$ are two linear extensions of the poset $P \setminus \{u\}$ (since they are obtained from the linear extensions $(v_1, v_2, ..., v_N)$ and $(w_1, w_2, ..., w_N)$ of P by removing u). Since we can apply Proposition 1.7 to this poset $P \setminus \{u\}$ in lieu of P (by the induction hypothesis, since $|P \setminus \{u\}| = N - 1$), we thus see that

$$(v_1, v_2, ..., v_{N-1}) \sim (w_1, w_2, ..., w_{i-1}, w_{i+1}, w_{i+2}, ..., w_N)$$

in $\mathcal{L}(P \setminus \{u\})$. By (3), this yields that $(v_1, v_2, ..., v_{N-1}, u)$ and $(w_1, w_2, ..., w_{i-1}, w_{i+1}, w_{i+2}, ..., w_N, u)$ are two linear extensions of P satisfying

$$(v_1, v_2, ..., v_{N-1}, u) \sim (w_1, w_2, ..., w_{i-1}, w_{i+1}, w_{i+2}, ..., w_N, u)$$

in $\mathcal{L}(P)$.

Now, we know that the tuple $(w_1, w_2, ..., w_N)$ is a linear extension of P. Since

$$\begin{aligned}&(w_1, w_2, ..., w_N)\\&= \left(w_1, w_2, ..., w_{i-1}, \underbrace{w_i}_{=u}, w_{i+1}, w_{i+2}, ..., w_N\right) = (w_1, w_2, ..., w_{i-1}, u, w_{i+1}, w_{i+2}, ..., w_N),\end{aligned}$$

this rewrites as follows: The tuple $(w_1, w_2, ..., w_{i-1}, u, w_{i+1}, w_{i+2}, ..., w_N)$ is a linear extension of P. Hence, we can apply Lemma 1.8 to $k = i-1$, $\ell = N-i$, $a_j = w_j$ and $b_j = w_{i+j}$. As a result, we see that $(w_1, w_2, ..., w_{i-1}, w_{i+1}, w_{i+2}, ..., w_N, u)$ is a linear extension of P satisfying $(w_1, w_2, ..., w_{i-1}, u, w_{i+1}, w_{i+2}, ..., w_N) \sim (w_1, w_2, ..., w_{i-1}, w_{i+1}, w_{i+2}, ..., w_N, u)$. Since the relation $\sim$ is symmetric (because $\sim$ is an equivalence relation), this yields

$$(w_1, w_2, ..., w_{i-1}, w_{i+1}, w_{i+2}, ..., w_N, u) \sim (w_1, w_2, ..., w_{i-1}, u, w_{i+1}, w_{i+2}, ..., w_N).$$

Altogether,

$$\begin{aligned}(v_1, v_2, ..., v_N) &= \left(v_1, v_2, ..., v_{N-1}, \underbrace{v_N}_{=u}\right) = (v_1, v_2, ..., v_{N-1}, u)\\&\sim (w_1, w_2, ..., w_{i-1}, w_{i+1}, w_{i+2}, ..., w_N, u)\\&\sim \left(w_1, w_2, ..., w_{i-1}, \underbrace{u}_{=w_i}, w_{i+1}, w_{i+2}, ..., w_N\right)\\&= (w_1, w_2, ..., w_{i-1}, w_i, w_{i+1}, w_{i+2}, ..., w_N) = (w_1, w_2, ..., w_N).\end{aligned}$$

We thus have shown that any two elements $(v_1, v_2, ..., v_N)$ and $(w_1, w_2, ..., w_N)$ of $\mathcal{L}(P)$ satisfy $(v_1, v_2, ..., v_N) \sim (w_1, w_2, ..., w_N)$. In other words, Proposition 1.7 is proven for $|P| = N$, so the induction step is complete, and Proposition 1.7 is proven. $\square$

2 Birational rowmotion

In this section, we introduce the basic objects whose nature we will investigate: labellings of a finite poset P (by elements of a field) and a birational map between them called "birational rowmotion". This map generalizes (in a certain sense) the notion of ordinary rowmotion on the set $J(P)$ of order ideals of P to the vastly more general setting of field-valued labellings. We will discuss the technical concerns raised by the definitions, and provide two examples and an alternative description of birational rowmotion. A deeper study of birational rowmotion is deferred to the following sections.

The concepts which we are going to define now go back to [EiPr13] and earlier sources, and are often motivated there. The reader should be warned that the notations used in [EiPr13] are not identical with those used in the present paper (not to mention that [EiPr13] is working over $\mathbb{R}_+$ rather than over fields as we do).

Definition 2.1. Let P be a poset. Then, $\widehat{P}$ will denote the poset defined as follows: As a set, let $\widehat{P}$ be the disjoint union of the set P with the two-element set $\{0, 1\}$. The smaller-or-equal relation $\leqslant$ on $\widehat{P}$ will be given by

$$(a \leqslant b) \iff (\text{either } (a \in P \text{ and } b \in P \text{ and } a \leqslant b \text{ in } P) \text{ or } a = 0 \text{ or } b = 1)$$

[12]. Here and in the following, we regard the canonical injection of the set P into the disjoint union $\widehat{P}$ as an inclusion; thus, P becomes a subposet of $\widehat{P}$. In the terminology of Stanley's [Stan11, section 3.2], this poset $\widehat{P}$ is the ordinal sum $\{0\} \oplus P \oplus \{1\}$.

Convention 2.2. Let P is a finite poset, and let u and v be two elements of P. Then, u and v are also elements of $\widehat{P}$ (since we are regarding P as a subposet of $\widehat{P}$). Thus, strictly speaking, statements like "$u < v$" or "$u \lessdot v$" are ambiguous because it is not clear whether they are referring to the poset P or to the poset $\widehat{P}$. However, this ambiguity is irrelevant, because it is easily seen that the truth of each of the statements "$u < v$", "$u \leqslant v$", "$u > v$", "$u \geqslant v$", "$u \lessdot v$", "$u \gtrdot v$" and "u and v are incomparable" is independent on whether it refers to the poset P or to the poset $\widehat{P}$. We are going to therefore omit mentioning the poset in these statements, unless there are other reasons for us to do so.

Definition 2.3. Let P be a poset. Let $\mathbb{K}$ be a field. A $\mathbb{K}$-*labelling of* P will mean a map $f : \widehat{P} \to \mathbb{K}$. Thus, $\mathbb{K}^{\widehat{P}}$ is the set of all $\mathbb{K}$-labellings of P. If f is a $\mathbb{K}$-labelling of P and v is an element of $\widehat{P}$, then $f(v)$ will be called the *label of* f *at* v.

Definition 2.4. In the following, whenever we are working with a field $\mathbb{K}$, we are going to tacitly assume that $\mathbb{K}$ is either infinite or at least can be enlarged when necessity arises. This assumption is needed in order to clarify the notions of rational maps and generic elements of algebraic varieties over $\mathbb{K}$. (We will not require $\mathbb{K}$ to be algebraically closed.)

We will use the terminology of algebraic varieties and rational maps between them, although the only algebraic varieties that we will be considering are products of affine and projective spaces, as well as their open subsets. We use the punctured arrow $\dashrightarrow$ to signify rational maps (i.e., a rational map from a variety U to a variety V is called

[12] Here and in the following, the expression "either/or" always has a non-exclusive meaning. (Thus, in particular, $0 \leqslant 1$ in $\widehat{P}$.)

a rational map $U \dashrightarrow V$). A rational map $U \dashrightarrow V$ is said to be *dominant* if its image is dense in V (with respect to the Zariski topology).

The words "generic" and "almost" will always refer to the Zariski topology. For example, if U is a finite set, then an assertion saying that some statement holds "for almost every point $p \in \mathbb{K}^U$" is supposed to mean that there is a Zariski-dense open subset D of $\mathbb{K}^U$ such that this statement holds for every point $p \in D$. A "generic" point on an algebraic variety V (for example, this can be a "generic matrix" when V is a space of matrices, or a "generic $\mathbb{K}$-labelling of a poset P" when V is the space of all $\mathbb{K}$-labellings of P) means a point lying in some fixed Zariski-dense open subset S of V; the concrete definition of S can usually be inferred from the context (often, it will be the subset of V on which everything we want to do with our point is well-defined), but of course should never depend on the actual point. (Note that one often has to read the whole proof in order to be able to tell what this S is. This is similar to the use of the "for ϵ small enough" wording in analysis, where it is often not clear until the end of the proof how small exactly the ϵ needs to be.) We are sometimes going to abuse notation and say that an equality holds "for every point" instead of "for almost every point" when it is really clear what the S is. (For example, if we say that "the equality $\dfrac{x^3 - y^3}{x - y} = x^2 + xy + y^2$ holds for every $x \in \mathbb{K}$ and $y \in \mathbb{K}$", it is clear that S has to be the set $\mathbb{K}^2 \setminus \{(x, y) \in \mathbb{K}^2 \mid x = y\}$, because the left hand side of the equality makes no sense when (x, y) is outside of this set.)

Remark 2.5. Most statements that we make below work not only for fields, but also more generally for semifields[13] such as the semifield $\mathbb{Q}_+$ of positive rationals or the tropical semiring. Some (but not all!) statements actually simplify when the underlying field is replaced by a semifield in which no two nonzero elements add to zero (because in such cases, e.g., the denominators in (4) cannot become zero unless some labels of f are 0). Thus, working with such semifields instead of fields would save us the trouble of having things defined "almost everywhere". Moreover, applying our results to the tropical semifield would yield some of the statements about order polytopes made in [EiPr13]. Nevertheless, we prefer to work with fields, for the following reasons:

– While most of our results can be formulated for semifields, not all of them can (and sometimes, even when a result holds over semifields, its proof might not work over semifields). In particular, Proposition 13.13 makes no sense over semifields, because determinants involve subtraction. Also, if we were to work in semifields which **do** contain two nonzero elements summing up to zero, then we would still have the issue of zero denominators, but we are not aware of a theoretical framework in the spirit of Zariski topology for fields to reassure us in this case that these issues are negligible.

– If an identity between subtraction-free rational expressions (such as $\dfrac{x^3 + y^3}{x + y} + 3xy = (x + y)^2$) holds over every field (as long as the denominators involved are nonzero), then it must hold over every semifield as well (again as long as the denominators involved are nonzero), even if the identity has only been proven with the help of subtraction (e.g., a proof of $\dfrac{x^3+y^3}{x+y}+3xy=(x+y)^2$ over a field can begin by simplifying $\dfrac{x^3+y^3}{x+y}$ to x^2-xy+y^2, a technique not available over a semifield). This is simply because every true identity between subtraction-free rational expressions can be verified by multiplying by a common denominator (an operation which does not introduce any subtractions) and comparing coefficients. Since our main results (such as Theorem 11.7, or the $p+q \mid \operatorname{ord}\left(R_{\operatorname{Rect}(p,q)}\right)$ part of Theorem 11.5) can be construed as identities between subtraction-free rational expressions, this yields that all these results hold over any semifield (provided the denominators are nonzero) if they hold over every field. So we are not losing any generality by restricting ourselves to considering only fields.

Definition 2.6. Let P be a finite poset. Let $\mathbb{K}$ be a field. Let $v \in P$. We define a rational map $T_v : \mathbb{K}^{\widehat{P}} \dashrightarrow \mathbb{K}^{\widehat{P}}$ by

$$(T_v f)(w) = \begin{cases} f(w), & \text{if } w \neq v; \\ \dfrac{1}{f(v)} \cdot \dfrac{\sum\limits_{\substack{u \in \widehat{P}; \\ u \lessdot v}} f(u)}{\sum\limits_{\substack{u \in \widehat{P}; \\ u \gtrdot v}} \dfrac{1}{f(u)}}, & \text{if } w = v \end{cases} \qquad \text{for all } w \in \widehat{P} \tag{4}$$

for all $f \in \mathbb{K}^{\widehat{P}}$. Note that this rational map T_v is well-defined, because the right-hand side of (4) is well-defined on a Zariski-dense open subset of $\mathbb{K}^{\widehat{P}}$. (This follows from the fact that for every $v \in P$, there is at least one $u \in \widehat{P}$ such that $u \gtrdot v$ [14].)

This rational map T_v is called the *v-toggle*.

The map T_v that we have just introduced (although defined over the semifield $\mathbb{R}_+$ instead of our field $\mathbb{K}$) is called a "birational toggle operation" in [EiPr13] (where it is denoted by ϕ_i with i being a number indexing the elements v of P; however, the same notation is used for the "tropicalized" version of T_v). As is clear from its definition, it only changes the label at the element v.

Note also the following almost trivial fact:

[13] The word "*semifield*" here means a commutative semiring in which each element other than 0 has a multiplicative inverse. (In contrast to other authors' conventions, our semifields do have zeroes.) A *semiring* is defined as a set with two binary operations called "addition" and "multiplication" and two elements 0 and 1 which satisfies all axioms of a ring (in particular, it must be associative and satisfy $0 \cdot a = a \cdot 0 = 0$ and $1 \cdot a = a \cdot 1 = a$ for all a) except for having additive inverses.

[14] Indeed, either there is at least one $u \in P$ such that $u \gtrdot v$ in P (and therefore also $u \gtrdot v$ in $\widehat{P}$), or else v is maximal in P and then we have $1 \gtrdot v$ in $\widehat{P}$.

Proposition 2.7. Let P be a finite poset. Let $\mathbb{K}$ be a field. Let $v \in P$. Then, the rational map T_v is an involution, i.e., the map T_v^2 is well-defined on a Zariski-dense open subset of $\mathbb{K}^{\widehat{P}}$ and satisfies $T_v^2 = \mathrm{id}$ on this subset.

We are calling this "almost trivial" because one subtlety is easily overlooked: We have to check that the map T_v^2 is well-defined on a Zariski-dense open subset of $\mathbb{K}^{\widehat{P}}$; this requires observing that for every $v \in P$, there exists at least one $u \in \widehat{P}$ such that $u \lessdot v$.

Proposition 2.7 yields the following:

Corollary 2.8. Let P be a finite poset. Let $\mathbb{K}$ be a field. Let $v \in P$. Then, the map T_v is a dominant rational map.

The reader should remember that dominant rational maps (unlike general rational maps) can be composed, and their compositions are still dominant rational maps. Of course, we are brushing aside subtleties like the fact that dominant rational maps are defined only over infinite fields (unless we are considering them in a sufficiently formal sense); as far as this paper is concerned, it never hurts to extend the field $\mathbb{K}$ (say, by introducing a new indeterminate), so when in doubt the reader can assume that the field $\mathbb{K}$ is infinite.

The following proposition is trivially obtained by rewriting (4); we are merely stating it for easier reference in proofs:

Proposition 2.9. Let P be a finite poset. Let $\mathbb{K}$ be a field. Let $v \in P$. For every $f \in \mathbb{K}^{\widehat{P}}$ for which $T_v f$ is well-defined, we have:

(a) Every $w \in \widehat{P}$ such that $w \neq v$ satisfies $(T_v f)(w) = f(w)$.

(b) We have

$$(T_v f)(v) = \frac{1}{f(v)} \cdot \frac{\sum\limits_{\substack{u \in \widehat{P};\\ u \lessdot v}} f(u)}{\sum\limits_{\substack{u \in \widehat{P};\\ u \gtrdot v}} \frac{1}{f(u)}}.$$

It is very easy to check the following "locality principle":

Proposition 2.10. Let P be a finite poset. Let $\mathbb{K}$ be a field. Let $v \in P$ and $w \in P$. Then, $T_v \circ T_w = T_w \circ T_v$, unless we have either $v \lessdot w$ or $w \lessdot v$.

Proof of Proposition 2.10 (sketched). Assume that neither $v \lessdot w$ nor $w \lessdot v$. Also, WLOG, assume that $v \neq w$, lest the claim of the proposition be obvious.

The action of T_v on a labelling of P merely changes the label at v. The new value depends on the label at v, on the labels at the elements $u \in \widehat{P}$ satisfying $u \lessdot v$, and on the labels at the elements $u \in \widehat{P}$ satisfying $u \gtrdot v$. A similar thing can be said about the action of T_w. Since we have neither $v \lessdot w$ nor $w \lessdot v$ nor $v = w$, it thus becomes clear that the actions of T_v and T_w don't interfere with each other, in the sense that the changes made by either of them are the same no matter whether the other has been applied before it or not. That is, $T_v \circ T_w = T_w \circ T_v$, so that Proposition 2.10 is proven. $\square$

Corollary 2.11. Let P be a finite poset. Let $\mathbb{K}$ be a field. Let v and w be two elements of P which are incomparable. Then, $T_v \circ T_w = T_w \circ T_v$.

This follows from Proposition 2.10 because incomparable elements never cover each other. $\square$

Combining Corollary 2.11 with Proposition 1.7, we obtain:

Corollary 2.12. Let P be a finite poset. Let $\mathbb{K}$ be a field. Let $(v_1, v_2, ..., v_m)$ be a linear extension of P. Then, the dominant rational map $T_{v_1} \circ T_{v_2} \circ ... \circ T_{v_m} : \mathbb{K}^{\widehat{P}} \dashrightarrow \mathbb{K}^{\widehat{P}}$ is well-defined and independent of the choice of the linear extension $(v_1, v_2, ..., v_m)$.

Definition 2.13. Let P be a finite poset. Let $\mathbb{K}$ be a field. *Birational rowmotion* is defined as the dominant rational map $T_{v_1} \circ T_{v_2} \circ ... \circ T_{v_m} : \mathbb{K}^{\widehat{P}} \dashrightarrow \mathbb{K}^{\widehat{P}}$, where $(v_1, v_2, ..., v_m)$ is a linear extension of P. This rational map is well-defined (in particular, it does not depend on the linear extension $(v_1, v_2, ..., v_m)$ chosen) because of Corollary 2.12 (and also because a linear extension of P always exists; this is Theorem 1.4). This rational map will be denoted by R.

The reason for the names "birational toggle" and "birational rowmotion" is explained in the paper [EiPr13], in which birational rowmotion (again, defined over $\mathbb{R}_+$ rather than over $\mathbb{K}$) is denoted (serendipitously from the standpoint of the second author of this paper) by $\rho_{\mathcal{B}}$.

Example 2.14. Let us demonstrate the effect of birational toggles and birational rowmotion on a rather simple 4-element poset. Namely, for this example, we let P be the poset $\{p, q_1, q_2, q_3\}$ with order relation defined by setting $p < q_i$ for each $i \in \{1, 2, 3\}$. This poset has Hasse diagram

$$\begin{tikzcd} q_1 \arrow[dr, dash] & q_2 \arrow[d, dash] & q_3 \arrow[dl, dash] \\ & p & \end{tikzcd} .$$

The extended poset $\widehat{P}$ has Hasse diagram

$$\begin{tikzcd} & 1 \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\ q_1 \arrow[dr, dash] & q_2 \arrow[d, dash] & q_3 \arrow[dl, dash] \\ & p \arrow[d, dash] & \\ & 0 & \end{tikzcd} .$$

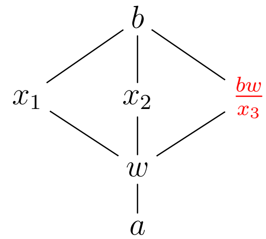
We can visualize a $\mathbb{K}$-labelling f of P by replacing, in the Hasse diagram of $\widehat{P}$, each element $v \in \widehat{P}$ by the label $f(v)$. Let f be a $\mathbb{K}$-labelling sending 0, p, q_1, q_2, q_3, and 1 to a, w, x_1, x_2, x_3, and b, respectively (for some elements a, b, w, x_1, x_2, x_3 of $\mathbb{K}$); this f is then visualized as follows:

$$\begin{tikzcd} & b \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\ x_1 \arrow[dr, dash] & x_2 \arrow[d, dash] & x_3 \arrow[dl, dash] \\ & w \arrow[d, dash] & \\ & a & \end{tikzcd} .$$

Now, recall the definition of birational rowmotion R on our poset P. Since (p, q_1, q_2, q_3) is a linear extension of P, we have $R = T_p \circ T_{q_1} \circ T_{q_2} \circ T_{q_3}$. Let us track how this transforms our labelling f:

We first apply T_{q_3}, obtaining

$T_{q_3} f =$

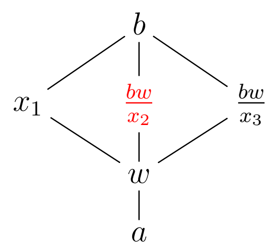


(where we colored the label at q_3 red to signify that it is the label at the element which got toggled). Indeed, the only label that changes under T_{q_3} is the one at q_3, and this label becomes

$$\left(T_{q_3} f\right)(q_3) = \frac{1}{f(q_3)} \cdot \frac{\sum\limits_{\substack{u \in \widehat{P}; \\ u \lessdot q_3}} f(u)}{\sum\limits_{\substack{u \in \widehat{P}; \\ u \gtrdot q_3}} \frac{1}{f(u)}} = \frac{1}{f(q_3)} \cdot \frac{f(p)}{\left(\frac{1}{f(1)}\right)} = \frac{1}{x_3} \cdot \frac{w}{\left(\frac{1}{b}\right)} = \frac{bw}{x_3}.$$

Having applied T_{q_3}, we next apply T_{q_2}, obtaining

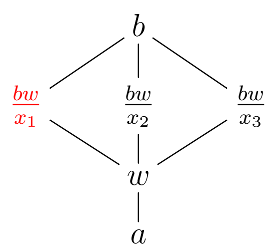
$T_{q_2} T_{q_3} f =$



.

Next, we apply T_{q_1}, obtaining

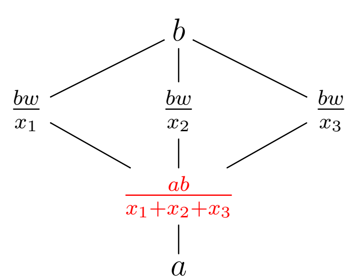
$T_{q_1} T_{q_2} T_{q_3} f =$



.

Finally, we apply T_p, resulting in

$T_p T_{q_1} T_{q_2} T_{q_3} f =$



,

since the birational p-toggle T_p has changed the label at p to

$$\begin{aligned}
&\left(T_p T_{q_1} T_{q_2} T_{q_3} f\right)(p) \\
&= \frac{1}{\left(T_{q_1} T_{q_2} T_{q_3} f\right)(p)} \cdot \frac{\sum\limits_{\substack{u \in \widehat{P}; \\ u \lessdot p}} \left(T_{q_1} T_{q_2} T_{q_3} f\right)(u)}{\sum\limits_{\substack{u \in \widehat{P}; \\ u \gtrdot p}} \frac{1}{\left(T_{q_1} T_{q_2} T_{q_3} f\right)(u)}} \\
&= \frac{1}{\left(T_{q_1} T_{q_2} T_{q_3} f\right)(p)} \cdot \frac{\left(T_{q_1} T_{q_2} T_{q_3} f\right)(0)}{\frac{1}{\left(T_{q_1} T_{q_2} T_{q_3} f\right)(q_1)} + \frac{1}{\left(T_{q_1} T_{q_2} T_{q_3} f\right)(q_2)} + \frac{1}{\left(T_{q_1} T_{q_2} T_{q_3} f\right)(q_3)}} \\
&= \frac{1}{w} \cdot \frac{a}{\frac{1}{bw/x_1} + \frac{1}{bw/x_2} + \frac{1}{bw/x_3}} = \frac{ab}{x_1 + x_2 + x_3}.
\end{aligned}$$

We thus have computed Rf (since $R = T_p T_{q_1} T_{q_2} T_{q_3}$). By repeating this procedure (or just substituting the labels of Rf obtained as variables), we can compute $R^2 f$, $R^3 f$ etc.. Specifically, we obtain

$$Rf = \begin{tikzcd}
& b \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\frac{bw}{x_1} \arrow[dr, dash] & \frac{bw}{x_2} \arrow[d, dash] & \frac{bw}{x_3} \arrow[dl, dash] \\
& \frac{ab}{x_1+x_2+x_3} \arrow[d, dash] & \\
& a &
\end{tikzcd},$$

$$R^2 f = \begin{tikzcd}
& b \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\frac{abx_1}{w(x_1+x_2+x_3)} \arrow[dr, dash] & \frac{abx_2}{w(x_1+x_2+x_3)} \arrow[d, dash] & \frac{abx_3}{w(x_1+x_2+x_3)} \arrow[dl, dash] \\
& \frac{ax_1x_2x_3}{w(x_2x_3+x_3x_1+x_1x_2)} \arrow[d, dash] & \\
& a &
\end{tikzcd},$$

$$R^3 f = \begin{tikzcd}
& b \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\frac{x_2x_3(x_1+x_2+x_3)}{x_2x_3+x_3x_1+x_1x_2} \arrow[dr, dash] & \frac{x_3x_1(x_1+x_2+x_3)}{x_2x_3+x_3x_1+x_1x_2} \arrow[d, dash] & \frac{x_1x_2(x_1+x_2+x_3)}{x_2x_3+x_3x_1+x_1x_2} \arrow[dl, dash] \\
& w \arrow[d, dash] & \\
& a &
\end{tikzcd},$$

$$R^4 f = \begin{tikzcd}[column sep=small] & b \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\ \frac{bw(x_2x_3+x_3x_1+x_1x_2)}{x_2x_3(x_1+x_2+x_3)} & \frac{bw(x_2x_3+x_3x_1+x_1x_2)}{x_3x_1(x_1+x_2+x_3)} & \frac{bw(x_2x_3+x_3x_1+x_1x_2)}{x_1x_2(x_1+x_2+x_3)} \\ & \frac{ab}{x_1+x_2+x_3} \arrow[ul, dash] \arrow[u, dash] \arrow[ur, dash] & \\ & a \arrow[u, dash] & \end{tikzcd},$$

$$R^5 f = \begin{tikzcd}[column sep=small] & b \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\ \frac{abx_2x_3}{w(x_2x_3+x_3x_1+x_1x_2)} & \frac{abx_3x_1}{w(x_2x_3+x_3x_1+x_1x_2)} & \frac{abx_1x_2}{w(x_2x_3+x_3x_1+x_1x_2)} \\ & \frac{ax_1x_2x_3}{w(x_2x_3+x_3x_1+x_1x_2)} \arrow[ul, dash] \arrow[u, dash] \arrow[ur, dash] & \\ & a \arrow[u, dash] & \end{tikzcd},$$

$$R^6 f = \begin{tikzcd} & b \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\ x_1 & x_2 & x_3 \\ & w \arrow[ul, dash] \arrow[u, dash] \arrow[ur, dash] & \\ & a \arrow[u, dash] & \end{tikzcd}.$$

There are several patterns here that catch the eye, some of which are related to the very simple structure of P and don't seem to generalize well. However, the most striking observation here is that $R^n f = f$ for some positive integer n (namely, $n = 6$ for this particular P). We will see in Proposition 9.7 that this generalizes to a rather wide class of posets, which we call "skeletal posets" (defined in Definition 9.5), a class of posets which contain (in particular) all graded forests such as our poset P here. (See Definition 9.5 for the definitions of the concepts involved here.)

Example 2.15. Let us demonstrate the effect of birational toggles and birational rowmotion on another 4-element poset. Namely, for this example, we let P be the poset $\{1, 2\} \times \{1, 2\}$ with order relation defined by setting $(i, k) \leqslant (i', k')$ if and only if ($i \leqslant i'$ and $k \leqslant k'$). This poset will later be called the "2×2-rectangle" in Definition 11.1. It has Hasse diagram

$$\begin{tikzcd}[column sep=small] & (2,2) \arrow[dl, dash] \arrow[dr, dash] & \\ (2,1) & & (1,2) \\ & (1,1) \arrow[ul, dash] \arrow[ur, dash] & \end{tikzcd}.$$

The extended poset $\widehat{P}$ has Hasse diagram

$$\begin{tikzcd}[column sep=small]
& 1 \arrow[d, dash] & \\
& (2,2) \arrow[dl, dash] \arrow[dr, dash] & \\
(2,1) \arrow[dr, dash] & & (1,2) \arrow[dl, dash] \\
& (1,1) \arrow[d, dash] & \\
& 0 &
\end{tikzcd} .$$

We can visualize a $\mathbb{K}$-labelling f of P by replacing, in the Hasse diagram of $\widehat{P}$, each element $v \in \widehat{P}$ by the label $f(v)$. Let f be a $\mathbb{K}$-labelling sending 0, $(1,1)$, $(1,2)$, $(2,1)$, $(2,2)$, and 1 to a, w, y, x, z, and b, respectively (for some elements a, b, x, y, z, w of $\mathbb{K}$); this f is then visualized as follows:

$$f = \begin{tikzcd}[column sep=small]
& b \arrow[d, dash] & \\
& z \arrow[dl, dash] \arrow[dr, dash] & \\
x \arrow[dr, dash] & & y \arrow[dl, dash] \\
& w \arrow[d, dash] & \\
& a &
\end{tikzcd} .$$

Now, recall the definition of birational rowmotion R on our poset P. Since $((1,1),(1,2),(2,1),(2,2))$ is a linear extension of P, we have $R = T_{(1,1)} \circ T_{(1,2)} \circ T_{(2,1)} \circ T_{(2,2)}$. Let us track how this transforms our labelling f:

We first apply $T_{(2,2)}$, obtaining

$$T_{(2,2)} f = \begin{tikzcd}[column sep=small, row sep=small] & b \arrow[d, dash] & \\ & \frac{b(x+y)}{z} \arrow[dl, dash] \arrow[dr, dash] & \\ x \arrow[dr, dash] & & y \arrow[dl, dash] \\ & w \arrow[d, dash] & \\ & a & \end{tikzcd}$$

(where we colored the label at $(2,2)$ red to signify that it is the label at the element which got toggled). Indeed, the only label that changes under $T_{(2,2)}$ is the one at $(2,2)$, and this label becomes

$$\left(T_{(2,2)} f\right)(2,2) = \frac{1}{f((2,2))} \cdot \frac{\sum\limits_{\substack{u \in \widehat{P};\\ u \lessdot (2,2)}} f(u)}{\sum\limits_{\substack{u \in \widehat{P};\\ u \gtrdot (2,2)}} \frac{1}{f(u)}} = \frac{1}{f((2,2))} \cdot \frac{f((1,2)) + f((2,1))}{\left(\frac{1}{f(1)}\right)}$$
$$= \frac{1}{z} \cdot \frac{y+x}{\left(\frac{1}{b}\right)} = \frac{b(x+y)}{z}.$$

Having applied $T_{(2,2)}$, we next apply $T_{(2,1)}$, obtaining

$$T_{(2,1)} T_{(2,2)} f = \begin{tikzcd}[column sep=small, row sep=small] & b \arrow[d, dash] & \\ & \frac{b(x+y)}{z} \arrow[dl, dash] \arrow[dr, dash] & \\ x \arrow[dr, dash] & & \frac{b(x+y)w}{yz} \arrow[dl, dash] \\ & w \arrow[d, dash] & \\ & a & \end{tikzcd}.$$

Next, we apply $T_{(1,2)}$, obtaining

$$T_{(1,2)} T_{(2,1)} T_{(2,2)} f = \begin{tikzcd}[column sep=small, row sep=small] & b \arrow[d, dash] & \\ & \frac{b(x+y)}{z} \arrow[dl, dash] \arrow[dr, dash] & \\ \frac{b(x+y)w}{xz} \arrow[dr, dash] & & \frac{b(x+y)w}{yz} \arrow[dl, dash] \\ & w \arrow[d, dash] & \\ & a & \end{tikzcd}.$$

Finally, we apply $T_{(1,1)}$, resulting in

$$Rf = T_{(1,1)}T_{(1,2)}T_{(2,1)}T_{(2,2)}f = \begin{tikzcd}[arrows=-] & b \arrow[d] & \\ & \frac{b(x+y)}{z} \arrow[dl] \arrow[dr] & \\ \frac{b(x+y)w}{xz} \arrow[dr] & & \frac{b(x+y)w}{yz} \arrow[dl] \\ & \color{red}{\frac{ab}{z}} \arrow[d] & \\ & a & \end{tikzcd}$$

(after cancelling terms). We thus have computed Rf. By repeating this procedure (or just substituting the labels of Rf obtained as variables), we can compute $R^2 f$, $R^3 f$ etc.. Specifically, we obtain

$$Rf = \begin{tikzcd}[arrows=-] & b \arrow[d] & \\ & \frac{b(x+y)}{z} \arrow[dl] \arrow[dr] & \\ \frac{b(x+y)w}{xz} \arrow[dr] & & \frac{b(x+y)w}{yz} \arrow[dl] \\ & \frac{ab}{z} \arrow[d] & \\ & a & \end{tikzcd}, \qquad R^2 f = \begin{tikzcd}[arrows=-] & b \arrow[d] & \\ & \frac{b(x+y)w}{xy} \arrow[dl] \arrow[dr] & \\ \frac{ab}{y} \arrow[dr] & & \frac{ab}{x} \arrow[dl] \\ & \frac{az}{x+y} \arrow[d] & \\ & a & \end{tikzcd},$$

$$R^3 f = \begin{tikzcd}[arrows=-] & b \arrow[d] & \\ & \frac{ab}{w} \arrow[dl] \arrow[dr] & \\ \frac{ayz}{(x+y)w} \arrow[dr] & & \frac{axz}{(x+y)w} \arrow[dl] \\ & \frac{axy}{(x+y)w} \arrow[d] & \\ & a & \end{tikzcd}, \qquad R^4 f = \begin{tikzcd}[arrows=-] & b \arrow[d] & \\ & z \arrow[dl] \arrow[dr] & \\ x \arrow[dr] & & y \arrow[dl] \\ & w \arrow[d] & \\ & a & \end{tikzcd}.$$

There are two surprises here. First, it turns out that $R^4 f = f$. This is not obvious, but generalizes in at least two ways: On the one hand, our poset P is a particular case of what we call a "skeletal poset" (Definition 9.5), a class of posets which all share the property (Proposition 9.7) that $R^n = \mathrm{id}$ for some sufficiently high positive integer n (which can be explicitly computed). On the other hand, our poset P is a particular case of rectangle posets, which turn out (Theorem 11.5) to satisfy $R^{p+q} = \mathrm{id}$ with p and q being the side lengths (here, 2 and 2) of the rectangle. Second, on a more subtle level, the rational functions appearing as labels in Rf, $R^2 f$ and $R^3 f$ are not as "wild" as one might expect. The values $(Rf)((1,1))$, $(R^2 f)((1,2))$, $(R^2 f)((2,1))$ and $(R^3 f)((2,2))$ each have the form $\dfrac{ab}{f(v)}$ for some $v \in P$. This is a "reciprocity" phenomenon which turns out to generalize to arbitrary rectangles (Theorem 11.7).

In the above calculation, we used the linear extension $((1,1),(1,2),(2,1),(2,2))$ of P to compute R as $T_{(1,1)} \circ T_{(1,2)} \circ T_{(2,1)} \circ T_{(2,2)}$. We could have just as well used the linear extension $((1,1),(2,1),(1,2),(2,2))$, obtaining the same result. But we could not have used the list $((1,1),(1,2),(2,2),(2,1))$ (for example), since it is not a linear extension (and indeed, the order of $T_{(1,1)} \circ T_{(1,2)} \circ T_{(2,2)} \circ T_{(2,1)}$ is infinite, as follows from the results of [EiPr13, §12.2]).

Let us state another proposition, which describes birational rowmotion implicitly:

Proposition 2.16. Let P be a finite poset. Let $\mathbb{K}$ be a field. Let $v \in P$. Let $f \in \mathbb{K}^{\widehat{P}}$. Then,

$$(Rf)(v) = \frac{1}{f(v)} \cdot \frac{\sum\limits_{\substack{u \in \widehat{P};\\ u \lessdot v}} f(u)}{\sum\limits_{\substack{u \in \widehat{P};\\ u \gtrdot v}} \dfrac{1}{(Rf)(u)}}. \tag{5}$$

Here (and in statements further down this paper), we are taking the liberty to leave assumptions such as "Assume that Rf is well-defined" unsaid (for instance, such an assumption is needed in Proposition 2.16) because these assumptions are satisfied when the parameters belong to some Zariski-dense open subset of their domains.

Proof of Proposition 2.16 (sketched). Fix a linear extension $(v_1, v_2, ..., v_m)$ of P. Recall that R has been defined as the composition $T_{v_1} \circ T_{v_2} \circ ... \circ T_{v_m}$. Hence, Rf can be obtained from f by traversing the linear extension $(v_1, v_2, ..., v_m)$ from right to left (thus starting with the largest element v_m, then proceeding to v_{m-1}, etc.), and at every step toggling the element being traversed. When an element v is being toggled, the elements $u \in \widehat{P}$ satisfying $u \lessdot v$ have not yet been toggled (they are further left than v in the linear extension), whereas those satisfying $u \gtrdot v$ have been toggled already. Denoting the state of the $\mathbb{K}$-labelling before the v-toggle by g, we see that the state *after* the v-toggle will be $T_v g$ with

$$(T_v g)(w) = \begin{cases} g(w), & \text{if } w \neq v; \\ \dfrac{1}{g(v)} \cdot \dfrac{\sum\limits_{\substack{u \in \widehat{P};\\ u \lessdot v}} g(u)}{\sum\limits_{\substack{u \in \widehat{P};\\ u \gtrdot v}} \dfrac{1}{g(u)}}, & \text{if } w = v \end{cases} \qquad \text{for all } w \in \widehat{P}. \tag{6}$$

But $g(v) = f(v)$ (since v has not yet been toggled at the time of g) and $(T_v g)(v) = (Rf)(v)$ (since v has been toggled at the time of $T_v g$, and is not going to be toggled ever again during the process of computing Rf); moreover, all $u \in \widehat{P}$ satisfying $u \lessdot v$ satisfy $g(u) = f(u)$ (since these u have not yet been toggled), whereas all $u \in \widehat{P}$ satisfying $u \gtrdot v$ satisfy $g(u) = (Rf)(u)$ (since these u have already been toggled and will not be toggled ever again). Thus, (6) (applied to $w = v$) transforms into (5). Proposition 2.16 is proven. □

Here is a little triviality to complete the picture of Proposition 2.16:

Proposition 2.17. Let P be a finite poset. Let $\mathbb{K}$ be a field. Let $f \in \mathbb{K}^{\widehat{P}}$. Then, $(Rf)(0) = f(0)$ and $(Rf)(1) = f(1)$.

This is clear since no toggle changes the labels at 0 and 1. □

We will often use Proposition 2.17 tacitly. A trivial corollary of Proposition 2.17 is:

Corollary 2.18. Let P be a finite poset. Let $\mathbb{K}$ be a field. Let $f \in \mathbb{K}^{\widehat{P}}$ and $\ell \in \mathbb{N}$. Then, $\left(R^{\ell} f\right)(0) = f(0)$ and $\left(R^{\ell} f\right)(1) = f(1)$.

(Recall that $\mathbb{N}$ denotes the set $\{0, 1, 2, ...\}$ in this paper.)

We will also need a converse of Propositions 2.16 and 2.17:

Proposition 2.19. Let P be a finite poset. Let $\mathbb{K}$ be a field. Let $f \in \mathbb{K}^{\widehat{P}}$ and $g \in \mathbb{K}^{\widehat{P}}$ be such that $f(0) = g(0)$ and $f(1) = g(1)$. Assume that

$$g(v) = \frac{1}{f(v)} \cdot \frac{\sum\limits_{\substack{u \in \widehat{P};\\ u \lessdot v}} f(u)}{\sum\limits_{\substack{u \in \widehat{P};\\ u \gtrdot v}} \frac{1}{g(u)}} \qquad \text{for every } v \in P. \tag{7}$$

(This means, in particular, that we assume that all denominators in (7) are nonzero.) Then, $g = Rf$. [15]

Proof of Proposition 2.19 (sketched). It is clearly enough to show that $g(v) = (Rf)(v)$ for every $v \in \widehat{P}$. Since this is clear for $v = 0$ (since $g(0) = f(0) = (Rf)(0)$), we only need to consider the case when $v \in \{1\} \cup P$. In this case, we can prove $g(v) = (Rf)(v)$ by descending induction over v – that is, we assume as an induction hypothesis that $g(u) = (Rf)(u)$ holds for all elements $u \in \{1\} \cup P$ which are greater than v in $\widehat{P}$. The induction base ($v = 1$) is clear (just like $v = 0$), and the induction step follows by comparing (5) with (7). We leave the details (including a check that Rf is well-defined, which piggybacks on the induction) to the reader. □

[15] More precisely, Rf **is well-defined** and equal to g.

As an aside, at this point we could give an alternative proof of Corollary 2.12, foregoing the use of Proposition 1.7. In fact, the proofs of Propositions 2.16, 2.17 and 2.19 only used that R is a composition $T_{v_1} \circ T_{v_2} \circ ... \circ T_{v_m}$ for **some** linear extension $(v_1, v_2, ..., v_m)$ of P. Thus, starting with **any** linear extension $(v_1, v_2, ..., v_m)$ of P, we could have defined R as the composition $T_{v_1} \circ T_{v_2} \circ ... \circ T_{v_m}$, and then used Propositions 2.16, 2.17 and 2.19 to characterize the image Rf of a $\mathbb{K}$-labelling f under this map R in a unique way without reference to $(v_1, v_2, ..., v_m)$, and thus concluded that R does not depend on $(v_1, v_2, ..., v_m)$. The details of this derivation are left to the reader.

On a related note, Proposition 2.16, Proposition 2.17 and Proposition 2.19 combined can be used as an alternative definition of birational rowmotion R, which works even when the poset P fails to be finite, as long as for every $v \in P$, there exist only finitely many $u \in P$ satisfying $u > v$ and there exist only finitely many $u \in P$ satisfying $u \lessdot v$ (provided that some technicalities arising from Zariski topology on infinite-dimensional spaces are dealt with).[16] We will not dwell on this.

Another general property of birational rowmotion concerns the question of what happens if the birational toggles are composed not in the "from top to bottom" order as in the definition of birational rowmotion, but the other way round. It turns out that the result is the inverse of birational rowmotion:

Proposition 2.20. Let P be a finite poset. Let $\mathbb{K}$ be a field. Then, birational rowmotion R is invertible (as a rational map). Its inverse R^{-1} is $T_{v_m} \circ T_{v_{m-1}} \circ ... \circ T_{v_1} : \mathbb{K}^{\widehat{P}} \dashrightarrow \mathbb{K}^{\widehat{P}}$, where $(v_1, v_2, ..., v_m)$ is a linear extension of P.

Proof of Proposition 2.20 (sketched). We know that T_w is an involution for every $w \in P$. Thus, in particular, for every $w \in P$, the map T_w is invertible and satisfies $T_w^{-1} = T_w$.

Let $(v_1, v_2, ..., v_m)$ be a linear extension of P. Then, $R = T_{v_1} \circ T_{v_2} \circ ... \circ T_{v_m}$ (by the definition of R), so that $R^{-1} = T_{v_m}^{-1} \circ T_{v_{m-1}}^{-1} \circ ... \circ T_{v_1}^{-1}$ (this makes sense since the map T_w is invertible for every $w \in P$). Since $T_w^{-1} = T_w$ for every $w \in P$, this simplifies to $R^{-1} = T_{v_m} \circ T_{v_{m-1}} \circ ... \circ T_{v_1}$. This proves Proposition 2.20. $\square$

3 Graded posets

In this section, we restrict our attention to what we call *graded posets* (a notion that encompasses most of the posets we are interested in; see Definition 3.1), and define (for this kind of posets) a family of "refined rowmotion" operators R_i which toggle only the labels of the i-th degree of the poset. These each turn out to be involutions, and their composition from top to bottom degree is R on the entire poset. We will later on use these R_i to get a better understanding of R on graded posets.

Let us first introduce our notion of a graded poset:

[16] The asymmetry between the $>$ and $\lessdot$ signs in this requirement is intentional. For instance, birational rowmotion can be defined (but will not be invertible) for the poset $\{0, -1, -2, -3, \ldots\}$ (with the usual order relation), but not for the poset $\{0, 1, 2, 3, \ldots\}$ (again with the usual order relation).

Definition 3.1. Let P be a finite poset. Let n be a nonnegative integer. We say that the poset P is *n-graded* if there exists a surjective map $\deg : P \to \{1, 2, ..., n\}$ such that the following three assertions hold:

Assertion 1: Any two elements u and v of P such that $u \gtrdot v$ satisfy $\deg u = \deg v + 1$.

Assertion 2: We have $\deg u = 1$ for every minimal element u of P.

Assertion 3: We have $\deg v = n$ for every maximal element v of P.

Note that the word "surjective" in Definition 3.1 is almost superfluous: Indeed, whenever $P \neq \varnothing$, then any map $\deg : P \to \{1, 2, ..., n\}$ satisfying the Assertions 1, 2 and 3 of Definition 3.1 is automatically surjective (this is easy to prove). But if $P = \varnothing$, such a map exists (vacuously) for every n, whereas requiring surjectivity forced $n = 0$.

Example 3.2. The poset $\{1, 2\} \times \{1, 2\}$ studied in Example 2.15 is 3-graded. The poset P studied in Example 2.14 is 2-graded. The empty poset is 0-graded, but not n-graded for any positive n. A chain with k elements is k-graded.

Definition 3.3. Let P be a finite poset. We say that the poset P is *graded* if there exists an $n \in \mathbb{N}$ such that P is n-graded. This n is then called the *height* of P.

The reader should be warned that the notion of a "graded poset" is not standard across literature; we have found at least four non-equivalent definitions of this notion in different sources.

Definition 3.4. Let $n \in \mathbb{N}$. Let P be an n-graded poset. Then, there exists a surjective map $\deg : P \to \{1, 2, ..., n\}$ that satisfies the Assertions 1, 2 and 3 of Definition 3.1. A moment of thought reveals that such a map deg is also uniquely determined by P [17]. Thus, we will call deg the *degree map* of P.

Moreover, we extend this map deg to a map $\widehat{P} \to \{0, 1, ..., n+1\}$ by letting it map 0 to 0 and 1 to $n + 1$. This extended map will also be denoted by deg and called the degree map. Notice that this extended map deg still satisfies Assertion 1 of Definition 3.1 if P is replaced by $\widehat{P}$ in that assertion.

For every $i \in \{0, 1, ..., n+1\}$, we will denote by $\widehat{P}_i$ the subset $\deg^{-1}(\{i\})$ of $\widehat{P}$. For every $v \in \widehat{P}$, the number $\deg v$ is called the *degree* of v.

The notion of an "n-graded poset" we just defined is identical with the notion of a "graded finite poset of rank $n - 1$" as defined in [Stan11, §3.1]. The degree of an element v of P as defined in Definition 3.4 is off by 1 from the rank of v in P in the sense of [Stan11, §3.1], but the degree $\deg v$ of an element v of $\widehat{P}$ equals its rank in $\widehat{P}$ in the sense of [Stan11, §3.1].

The way we extended the map $\deg : P \to \{1, 2, ..., n\}$ to a map $\deg : \widehat{P} \to \{0, 1, ..., n+1\}$ in Definition 3.4, of course, was not arbitrary. In fact, it was tailored to make the following true:

[17] In fact, if $v \in P$, then it is easy to see that $\deg v$ equals the number of elements of any maximal chain in P with highest element v. This clearly determines $\deg v$ uniquely.

Proposition 3.5. Let $n \in \mathbb{N}$. Let P be an n-graded poset. Let $u \in \widehat{P}$ and $v \in \widehat{P}$. Consider the map $\deg : \widehat{P} \to \{0, 1, ..., n+1\}$ defined in Definition 3.4.

(a) If $u \lessdot v$ in $\widehat{P}$, then $\deg u = \deg v - 1$.

(b) If $u < v$ in $\widehat{P}$, then $\deg u < \deg v$.

(c) If $u < v$ in $\widehat{P}$ and $\deg u = \deg v - 1$, then $u \lessdot v$ in $\widehat{P}$.

(d) If $u \neq v$ and $\deg u = \deg v$, then u and v are incomparable in $\widehat{P}$.

The rather simple proofs of these facts are left to the reader. (Note that part **(a)** incorporates all three Assertions 1, 2 and 3 of Definition 3.1.) $\square$

In words, Proposition 3.5 **(d)** states that any two distinct elements of $\widehat{P}$ having the same degree are incomparable. We will use this several times below.

One important observation is that any two distinct elements of a graded poset having the same degree are incomparable. Hence:

Corollary 3.6. Let $n \in \mathbb{N}$. Let $\mathbb{K}$ be a field. Let P be an n-graded poset. Let $i \in \{1, 2, ..., n\}$. Let $(u_1, u_2, ..., u_k)$ be any list of the elements of $\widehat{P}_i$ with every element of $\widehat{P}_i$ appearing exactly once in the list. Then, the dominant rational map $T_{u_1} \circ T_{u_2} \circ ... \circ T_{u_k} : \mathbb{K}^{\widehat{P}} \dashrightarrow \mathbb{K}^{\widehat{P}}$ is well-defined and independent of the choice of the list $(u_1, u_2, ..., u_k)$.

Proof of Corollary 3.6 (sketched). This is analogous to the proof of Corollary 2.12, because any two distinct elements of $\widehat{P}_i$ are incomparable. (In place of the set $\mathcal{L}(P)$ now serves the set of all lists of elements of $\widehat{P}_i$ (with every element of $\widehat{P}_i$ appearing exactly once in the list). Any two elements of this latter set are equivalent under the relation $\sim$, because any two adjacent elements in such a list of elements of $\widehat{P}_i$ are incomparable and can thus be switched.) $\square$

Definition 3.7. Let $n \in \mathbb{N}$. Let $\mathbb{K}$ be a field. Let P be an n-graded poset. Let $i \in \{1, 2, ..., n\}$. Then, let R_i denote the dominant rational map $T_{u_1} \circ T_{u_2} \circ ... \circ T_{u_k} : \mathbb{K}^{\widehat{P}} \dashrightarrow \mathbb{K}^{\widehat{P}}$, where $(u_1, u_2, ..., u_k)$ is any list of the elements of $\widehat{P}_i$ with every element of $\widehat{P}_i$ appearing exactly once in the list. This map $T_{u_1} \circ T_{u_2} \circ ... \circ T_{u_k}$ is well-defined (in particular, it does not depend on the list $(u_1, u_2, ..., u_k)$) because of Corollary 3.6.

Proposition 3.8. Let $n \in \mathbb{N}$. Let $\mathbb{K}$ be a field. Let P be an n-graded poset. Then,

$$R = R_1 \circ R_2 \circ ... \circ R_n. \tag{8}$$

Proof of Proposition 3.8 (sketched). For every $i \in \{1, 2, ..., n\}$, let $\left(u_1^{[i]}, u_2^{[i]}, ..., u_{k_i}^{[i]}\right)$ be a list of the elements of $\widehat{P}_i$ with every element of $\widehat{P}_i$ appearing exactly once in the list. Then, every $i \in \{1, 2, ..., n\}$ satisfies $R_i = T_{u_1^{[i]}} \circ T_{u_2^{[i]}} \circ ... \circ T_{u_{k_i}^{[i]}}$.

But any listing of the elements of P in order of increasing degree is a linear extension of P (because any two distinct elements of a graded poset having the same degree are incomparable). Thus,

$$\left(u_1^{[1]}, u_2^{[1]}, ..., u_{k_1}^{[1]}, \quad u_1^{[2]}, u_2^{[2]}, ..., u_{k_2}^{[2]}, \quad ..., \quad u_1^{[n]}, u_2^{[n]}, ..., u_{k_n}^{[n]}\right)$$

is a linear extension of P. Thus, by the definition of R, we have

$$\begin{aligned} R &= \left(T_{u_1^{[1]}} \circ T_{u_2^{[1]}} \circ ... \circ T_{u_{k_1}^{[1]}}\right) \circ \left(T_{u_1^{[2]}} \circ T_{u_2^{[2]}} \circ ... \circ T_{u_{k_2}^{[2]}}\right) \circ ... \circ \left(T_{u_1^{[n]}} \circ T_{u_2^{[n]}} \circ ... \circ T_{u_{k_n}^{[n]}}\right) \\ &= R_1 \circ R_2 \circ ... \circ R_n \end{aligned}$$

(since every $i \in \{1, 2, ..., n\}$ satisfies $T_{u_1^{[i]}} \circ T_{u_2^{[i]}} \circ ... \circ T_{u_{k_i}^{[i]}} = R_i$). This proves Proposition 3.8. □

We recall that birational rowmotion is a composition of toggle maps. As Proposition 3.8 shows, the operators R_i are an "intermediate" step between these toggle maps and birational rowmotion as a whole, though they are defined only when the poset P is graded. They will be rather useful for us in our understanding of birational rowmotion (and the condition on P to be graded doesn't prevent us from using them, since most of our results concern only graded posets anyway).

Proposition 3.9. Let $n \in \mathbb{N}$. Let $\mathbb{K}$ be a field. Let P be an n-graded poset. Let $i \in \{1, 2, ..., n\}$. Then, R_i is an involution (that is, $R_i^2 = \mathrm{id}$ on the set where R_i is defined).

Proof of Proposition 3.9 (sketched). We defined R_i as the composition $T_{u_1} \circ T_{u_2} \circ ... \circ T_{u_k}$ of the toggles T_{u_i} where $(u_1, u_2, ..., u_k)$ is any list of the elements of $\widehat{P}_i$ with every element of $\widehat{P}_i$ appearing exactly once in the list. These toggles are involutions and commute (the latter because any two distinct elements of $\widehat{P}_i$ are incomparable, having the same degree in P). Since a composition of commuting involutions is always an involution, this shows that R_i is an involution, qed. □

Similarly to Proposition 2.16, we have:

Proposition 3.10. Let $n \in \mathbb{N}$. Let P be an n-graded poset. Let $i \in \{1, 2, ..., n\}$. Let $\mathbb{K}$ be a field. Let $v \in \widehat{P}$. Let $f \in \mathbb{K}^{\widehat{P}}$.

(a) If $\deg v \neq i$, then $(R_i f)(v) = f(v)$.

(b) If $\deg v = i$, then

$$(R_i f)(v) = \frac{1}{f(v)} \cdot \frac{\sum\limits_{\substack{u \in \widehat{P};\\ u \lessdot v}} f(u)}{\sum\limits_{\substack{u \in \widehat{P};\\ u \gtrdot v}} \frac{1}{f(u)}}. \tag{9}$$

The proof of this proposition is very similar to that of Proposition 2.16 and therefore left to the reader. $\square$

Notice that using the proof of Proposition 3.10, it is easy to give an alternative proof of Corollary 3.6 (in the same way as we saw that an alternative proof of Corollary 2.12 could be given using the proofs of Propositions 2.16, 2.17 and 2.19).

4 w-tuples

This section continues the study of birational rowmotion on graded posets by introducing a "fingerprint" or "checksum" of a $\mathbb{K}$-labelling called the w-tuple, defined by summing ratios of elements between successive degrees (i.e., rows in the Hasse diagram). This w-tuple serves to extract some information from a $\mathbb{K}$-labelling; we will later see how to make the "rest" of the labelling more manageable.

Definition 4.1. Let $n \in \mathbb{N}$. Let $\mathbb{K}$ be a field. Let P be an n-graded poset. Let $f \in \mathbb{K}^{\widehat{P}}$. Let $i \in \{0, 1, ..., n\}$. Then, $\mathbf{w}_i(f)$ will denote the element of $\mathbb{K}$ defined by

$$\mathbf{w}_i(f) = \sum_{\substack{x \in \widehat{P}_i;\ y \in \widehat{P}_{i+1}; \\ y \gtrdot x}} \frac{f(x)}{f(y)}.$$

(This element is not always defined, but is defined in the "generic" case when $0 \notin f\left(\widehat{P}\right)$.)

Intuitively, one could think of $\mathbf{w}_i(f)$ as a kind of "checksum" for the labelling f which displays how much its labels at degree $i+1$ differ from those at degree i. Of course, in general, the knowledge of $\mathbf{w}_i(f)$ for all $i \in \{0, 1, ..., n\}$ is far from sufficient to reconstruct the whole labelling f; however, in Definition 6.2, we will introduce the so-called homogenization of f, which will provide "complementary data" to these $\mathbf{w}_i(f)$. As for now, let us show that the $\mathbf{w}_i(f)$ behave in a rather simple way under the maps R and R_j.

Definition 4.2. Let $n \in \mathbb{N}$. Let $\mathbb{K}$ be a field. Let P be an n-graded poset. Let $f \in \mathbb{K}^{\widehat{P}}$. The $(n+1)$-tuple $(\mathbf{w}_0(f), \mathbf{w}_1(f), ..., \mathbf{w}_n(f))$ will be called the *w-tuple* of the $\mathbb{K}$-labelling f.

It is easy to see:

Proposition 4.3. Let $n \in \mathbb{N}$. Let $\mathbb{K}$ be a field. Let P be an n-graded poset. Let $i \in \{1, 2, ..., n\}$. Then, every $f \in \mathbb{K}^{\widehat{P}}$ satisfies

$$\begin{aligned} &(\mathbf{w}_0(R_i f), \mathbf{w}_1(R_i f), ..., \mathbf{w}_n(R_i f)) \\ &= (\mathbf{w}_0(f), \mathbf{w}_1(f), ..., \mathbf{w}_{i-2}(f), \mathbf{w}_i(f), \mathbf{w}_{i-1}(f), \mathbf{w}_{i+1}(f), \mathbf{w}_{i+2}(f), ..., \mathbf{w}_n(f)). \end{aligned}$$

In other words, the map R_i changes the w-tuple of a $\mathbb{K}$-labelling by interchanging its $(i-1)$-st entry with its i-th entry (where the entries are labelled starting at 0).

Proof of Proposition 4.3 (sketched). Let $f \in \mathbb{K}^{\widehat{P}}$. We need to show that every $j \in \{0, 1, ..., n\}$ satisfies

$$\mathbf{w}_j (R_i f) = \mathbf{w}_{\tau_i(j)} (f), \tag{10}$$

where τ_i is the permutation of the set $\{0, 1, ..., n\}$ which transposes $i-1$ with i (while leaving all other elements of this set invariant).

Proof of (10): Let $j \in \{0, 1, ..., n\}$. We distinguish between three cases:

Case 1: We have $j = i$.

Case 2: We have $j = i - 1$.

Case 3: We have $j \notin \{i-1, i\}$.

Let us first consider Case 1. In this case, we have $j = i$. By the definition of $\mathbf{w}_i (R_i f)$, we have

$$\begin{aligned}\mathbf{w}_i (R_i f) &= \sum_{\substack{x \in \widehat{P}_i;\ y \in \widehat{P}_{i+1}; \\ y \gtrdot x}} \frac{(R_i f)(x)}{(R_i f)(y)} = \sum_{x \in \widehat{P}_i} (R_i f)(x) \sum_{\substack{y \in \widehat{P}_{i+1}; \\ y \gtrdot x}} \Bigg(\underbrace{(R_i f)(y)}_{\substack{= f(y) \\ \text{(by Proposition 3.10 } \textbf{(a)}\text{)}}} \Bigg)^{-1} \\ &= \sum_{x \in \widehat{P}_i} (R_i f)(x) \sum_{\substack{y \in \widehat{P}_{i+1}; \\ y \gtrdot x}} (f(y))^{-1}. \end{aligned} \tag{11}$$

But every $x \in \widehat{P}_i$ satisfies

$$\begin{aligned}(R_i f)(x) &= \frac{1}{f(x)} \cdot \frac{\sum\limits_{\substack{u \in \widehat{P}; \\ u \lessdot x}} f(u)}{\sum\limits_{\substack{u \in \widehat{P}; \\ u \gtrdot x}} \frac{1}{f(u)}} \qquad \text{(by Proposition 3.10 } \textbf{(b)}\text{)} \\ &= \frac{1}{f(x)} \cdot \sum_{\substack{u \in \widehat{P}; \\ u \lessdot x}} f(u) \cdot \left(\sum_{\substack{u \in \widehat{P}; \\ u \gtrdot x}} (f(u))^{-1} \right)^{-1} = \frac{1}{f(x)} \cdot \sum_{\substack{y \in \widehat{P}; \\ y \lessdot x}} f(y) \cdot \left(\sum_{\substack{y \in \widehat{P}; \\ y \gtrdot x}} (f(y))^{-1} \right)^{-1} \\ &= \frac{1}{f(x)} \cdot \sum_{\substack{y \in \widehat{P}_{i-1}; \\ y \lessdot x}} f(y) \cdot \left(\sum_{\substack{y \in \widehat{P}_{i+1}; \\ y \gtrdot x}} (f(y))^{-1} \right)^{-1}\end{aligned}$$

(here, we replaced $y \in \widehat{P}$ by $y \in \widehat{P}_{i-1}$ in the first sum (because every $y \in \widehat{P}$ satisfying $y \lessdot x$ must belong to $\widehat{P}_{i-1}$ [18]) and we replaced $y \in \widehat{P}$ by $y \in \widehat{P}_{i+1}$ in the second sum

[18] since $x \in \widehat{P}_i$

(for similar reasons)) and thus

$$(R_i f)(x) \sum_{\substack{y \in \widehat{P}_{i+1};\\ y \gtrdot x}} (f(y))^{-1} = \frac{1}{f(x)} \cdot \sum_{\substack{y \in \widehat{P}_{i-1};\\ y \lessdot x}} f(y) = \sum_{\substack{y \in \widehat{P}_{i-1};\\ y \lessdot x}} \frac{f(y)}{f(x)}.$$

Hence, (11) becomes

$$\begin{aligned}
\mathbf{w}_i(R_i f) &= \sum_{x \in \widehat{P}_i} \underbrace{(R_i f)(x) \sum_{\substack{y \in \widehat{P}_{i+1};\\ y \gtrdot x}} (f(y))^{-1}}_{= \sum_{\substack{y \in \widehat{P}_{i-1};\\ y \lessdot x}} \frac{f(y)}{f(x)}} = \sum_{x \in \widehat{P}_i} \sum_{\substack{y \in \widehat{P}_{i-1};\\ y \lessdot x}} \frac{f(y)}{f(x)} = \sum_{\substack{y \in \widehat{P}_{i-1};\ x \in \widehat{P}_i;\\ x \gtrdot y}} \frac{f(y)}{f(x)} \\
&= \sum_{\substack{x \in \widehat{P}_{i-1};\ y \in \widehat{P}_i\\ y \gtrdot x}} \frac{f(x)}{f(y)} \qquad \text{(here, we switched the indices in the sum)} \\
&= \mathbf{w}_{i-1}(f) \qquad \text{(by the definition of } \mathbf{w}_{i-1}(f)\text{)} \qquad (12) \\
&= \mathbf{w}_{\tau_i(i)}(f).
\end{aligned}$$

In other words, (10) holds for $j = i$. Thus, (10) is proven in Case 1.

Let us now consider Case 2. In this case, $j = i - 1$. Now, it can be shown that $\mathbf{w}_{i-1}(R_i f) = \mathbf{w}_i(f)$. This can be proven either in a similar way to how we proved $\mathbf{w}_i(R_i f) = \mathbf{w}_{i-1}(f)$ (the details of this are left to the reader), or by noticing that

$$\begin{aligned}
\mathbf{w}_i(f) &= \mathbf{w}_i\left(R_i^2 f\right) \qquad \left(\begin{array}{c}\text{since Proposition 3.9 yields that } R_i^2 = \mathrm{id}, \\ \text{hence } \mathbf{w}_i(R_i^2 f) = \mathbf{w}_i(\mathrm{id}\, f) = \mathbf{w}_i(f)\end{array}\right) \\
&= \mathbf{w}_i(R_i(R_i f)) = \mathbf{w}_{i-1}(R_i f) \qquad \text{(by (12), applied to } R_i f \text{ instead of } f\text{)}.
\end{aligned}$$

Either way, we end up knowing that $\mathbf{w}_{i-1}(R_i f) = \mathbf{w}_i(f)$. Thus, $\mathbf{w}_{i-1}(R_i f) = \mathbf{w}_i(f) = \mathbf{w}_{\tau_i(i-1)}(f)$. In other words, (10) holds for $j = i - 1$. Thus, (10) is proven in Case 2.

Let us finally consider Case 3. In this case, $j \notin \{i-1, i\}$. Hence, $\tau_i(j) = j$. On the other hand, by the definition of $\mathbf{w}_j(R_i f)$, we have

$$\begin{aligned}
\mathbf{w}_j(R_i f) &= \sum_{\substack{x \in \widehat{P}_j;\ y \in \widehat{P}_{j+1};\\ y \gtrdot x}} \frac{(R_i f)(x)}{(R_i f)(y)} = \sum_{\substack{x \in \widehat{P}_j;\ y \in \widehat{P}_{j+1};\\ y \gtrdot x}} \Bigg(\underbrace{(R_i f)(y)}_{\substack{= f(y) \\ \text{(by Proposition 3.10 (a))}}} \Bigg)^{-1} \cdot \underbrace{(R_i f)(x)}_{\substack{= f(x) \\ \text{(by Proposition 3.10 (a))}}} \\
&= \sum_{\substack{x \in \widehat{P}_j;\ y \in \widehat{P}_{j+1};\\ y \gtrdot x}} (f(y))^{-1} \cdot f(x) = \sum_{\substack{x \in \widehat{P}_j;\ y \in \widehat{P}_{j+1};\\ y \gtrdot x}} \frac{f(x)}{f(y)}.
\end{aligned}$$

Compared with $\mathbf{w}_j(f) = \sum_{\substack{x\in\widehat{P}_j;\ y\in\widehat{P}_{j+1};\\ y\gtrdot x}} \frac{f(x)}{f(y)}$ (by the definition of $\mathbf{w}_j(f)$), this yields $\mathbf{w}_j(R_i f) = \mathbf{w}_j(f)$. Since $j = \tau_i(j)$, this becomes $\mathbf{w}_j(R_i f) = \mathbf{w}_{\tau_i(j)}(f)$. Hence, (10) is proven in Case 3.

We have thus proven (10) in each of the three possible cases 1, 2 and 3. This completes the proof of (10) and thus of Proposition 4.3. $\square$

From Proposition 4.3, and (8), we conclude:

Proposition 4.4. Let $n \in \mathbb{N}$. Let $\mathbb{K}$ be a field. Let P be an n-graded poset. Then, every $f \in \mathbb{K}^{\widehat{P}}$ satisfies

$$(\mathbf{w}_0(Rf), \mathbf{w}_1(Rf), ..., \mathbf{w}_n(Rf)) = (\mathbf{w}_n(f), \mathbf{w}_0(f), \mathbf{w}_1(f), ..., \mathbf{w}_{n-1}(f)).$$

In other words, the map R changes the w-tuple of a $\mathbb{K}$-labelling by shifting it cyclically.

Proof of Proposition 4.4 (sketched). Proposition 3.8 yields $R = R_1 \circ R_2 \circ ... \circ R_n$. But for every $i \in \{1, 2, ..., n\}$, recall from Proposition 4.3 that the map R_i changes the w-tuple of a $\mathbb{K}$-labelling by interchanging its $(i-1)$-st entry with its i-th entry (where the entries are labelled starting at 0). Hence, the effect of the compound map $R = R_1 \circ R_2 \circ ... \circ R_n$ on the w-tuple is that of first interchanging the $(n-1)$-st entry with the n-th entry, then interchanging the $(n-2)$-st entry with the $(n-1)$-st entry, and so on, through to finally interchanging the 0-th entry with the 1-st entry. But this latter sequence of interchanges is equivalent to a cyclic shift of the w-tuple[19]. Hence, the map R changes the w-tuple of a $\mathbb{K}$-labelling by shifting it cyclically, qed. $\square$

As a consequence of Proposition 4.4, the map R^{n+1} (for an n-graded poset P) leaves the w-tuple of a $\mathbb{K}$-labelling fixed.

5 Graded rescaling of labellings

In general, birational rowmotion R has something that one might call an "avalanche effect": If f and g are two $\mathbb{K}$-labellings of a poset P which differ from each other only in their labels at one single element v, then the labellings Rf and Rg (in general) differ at all elements covering v and all elements beneath v, and further applications of R make the labellings even more different. Thus, a change of just one label in a labelling will often "spread" through a large part of the poset when R is repeatedly applied; the effect of such a change is hard to track in general. Thus, knowing the behavior of one particular $\mathbb{K}$-labelling f under R does not help us at understanding the behaviors of $\mathbb{K}$-labellings obtained from f by changing labels at particular elements. However, if P is a **graded**

[19] Indeed, the composition $(0,1) \circ (1,2) \circ ... \circ (n-1, n)$ of transpositions in the symmetric group on the set $\{0, 1, ..., n\}$ is the $(n+1)$-cycle $(0, 1, ..., n)$.

poset and we simultaneously multiply the labels at **all elements of a given degree** in a given labelling of P with a given scalar, then the changes this causes to the behavior of the labelling under R are rather predictable. We are going to formalize this observation in this section, proving some explicit formulas for how birational rowmotion R and its iterates react to such rescalings. These explicit formulas will be subsumed into slick conclusions in Section 6, where we will introduce a notion of *homogeneous equivalence* which formalizes the idea of a "labelling modulo scalar factors at each degree".

Definition 5.1. Let $\mathbb{K}$ be a field. Then, $\mathbb{K}^\times$ denotes the multiplicative group of nonzero elements of $\mathbb{K}$.

The following definition formalizes the idea of multiplying the labels at all elements of a certain degree with one and the same scalar factor:

Definition 5.2. Let $n \in \mathbb{N}$. Let $\mathbb{K}$ be a field. Let P be an n-graded poset. For every $\mathbb{K}$-labelling $f \in \mathbb{K}^{\widehat{P}}$ and any $(n+2)$-tuple $(a_0, a_1, ..., a_{n+1}) \in (\mathbb{K}^\times)^{n+2}$, we define a $\mathbb{K}$-labelling $(a_0, a_1, ..., a_{n+1}) \flat f \in \mathbb{K}^{\widehat{P}}$ by

$$((a_0, a_1, ..., a_{n+1}) \flat f)(v) = a_{\deg v} \cdot f(v) \qquad \text{for every } v \in \widehat{P}.$$

Straightforward application of this definition and that of R_i shows:

Proposition 5.3. Let $n \in \mathbb{N}$. Let $\mathbb{K}$ be a field. Let P be an n-graded poset. Let us use the notation introduced in Definition 5.2.

Let $f \in \mathbb{K}^{\widehat{P}}$ be a $\mathbb{K}$-labelling. Let $(a_0, a_1, ..., a_{n+1}) \in (\mathbb{K}^\times)^{n+2}$. Let $i \in \{1, 2, ..., n\}$. Then,

$$\begin{aligned} &R_i((a_0, a_1, ..., a_{n+1}) \flat f) \\ &= \left(a_0, a_1, ..., a_{i-1}, \frac{a_{i+1}a_{i-1}}{a_i}, a_{i+1}, a_{i+2}, ..., a_{n+1}\right) \flat (R_i f) \end{aligned}$$

(provided that $R_i f$ is well-defined).

A similar result can be obtained for R instead of R_i:

Proposition 5.4. Let $n \in \mathbb{N}$. Let $\mathbb{K}$ be a field. Let P be an n-graded poset. For every $\mathbb{K}$-labelling $f \in \mathbb{K}^{\widehat{P}}$ and any $(n+2)$-tuple $(a_0, a_1, ..., a_{n+1}) \in (\mathbb{K}^\times)^{n+2}$, we define a $\mathbb{K}$-labelling $(a_0, a_1, ..., a_{n+1}) \flat f \in \mathbb{K}^{\widehat{P}}$ as in Definition 5.2.

Let $f \in \mathbb{K}^{\widehat{P}}$ be a $\mathbb{K}$-labelling. Let $(a_0, a_1, ..., a_{n+1}) \in (\mathbb{K}^\times)^{n+2}$. Then,

$$R((a_0, a_1, ..., a_{n+1}) \flat f) = (a_0, ga_0, ga_1, ..., ga_{n-1}, a_{n+1}) \flat (Rf),$$

where $g = \frac{a_{n+1}}{a_n}$ (provided that Rf is well-defined).

Proof of Proposition 5.4 (sketched). Let $g = \dfrac{a_{n+1}}{a_n}$. We claim that every $j \in \{1, 2, ..., n+1\}$ satisfies

$$\begin{aligned}&(R_j \circ R_{j+1} \circ ... \circ R_n) ((a_0, a_1, ..., a_{n+1}) \flat f) \\ &= (a_0, a_1, a_2, ..., a_{j-1}, ga_{j-1}, ga_j, ..., ga_{n-1}, a_{n+1}) \flat ((R_j \circ R_{j+1} \circ ... \circ R_n) f). \qquad (13)\end{aligned}$$

Indeed, (13) is easily verified by reverse induction over j (that is, induction over $n+1-j$), using Proposition 5.3 in the step. Now, applying (13) to $j = 1$ and recalling that $R = R_1 \circ R_2 \circ ... \circ R_n$, we obtain the claim of Proposition 5.4. □

We can go further and generalize Proposition 5.4 to iterated birational rowmotion:

Proposition 5.5. Let $n \in \mathbb{N}$. Let $\mathbb{K}$ be a field. Let P be an n-graded poset. For every $\mathbb{K}$-labelling $f \in \mathbb{K}^{\widehat{P}}$ and any $(n+2)$-tuple $(a_0, a_1, ..., a_{n+1}) \in (\mathbb{K}^\times)^{n+2}$, we define a $\mathbb{K}$-labelling $(a_0, a_1, ..., a_{n+1}) \flat f \in \mathbb{K}^{\widehat{P}}$ as in Definition 5.2.

Let $(a_0, a_1, ..., a_{n+1}) \in (\mathbb{K}^\times)^{n+2}$. For every $\ell \in \{0, 1, ..., n+1\}$ and $k \in \{0, 1, ..., n+1\}$, define an element $\widehat{a}_k^{(\ell)} \in \mathbb{K}^\times$ by

$$\widehat{a}_k^{(\ell)} = \begin{cases} \dfrac{a_{n+1} a_{k-\ell}}{a_{n+1-\ell}}, & \text{if } k \geqslant \ell; \\ \dfrac{a_{n+1+k-\ell} a_0}{a_{n+1-\ell}}, & \text{if } k < \ell \end{cases}.$$

Let $f \in \mathbb{K}^{\widehat{P}}$ be a $\mathbb{K}$-labelling. Then, every $\ell \in \{0, 1, ..., n+1\}$ satisfies

$$R^\ell ((a_0, a_1, ..., a_{n+1}) \flat f) = \left(\widehat{a}_0^{(\ell)}, \widehat{a}_1^{(\ell)}, ..., \widehat{a}_{n+1}^{(\ell)}\right) \flat \left(R^\ell f\right)$$

(provided that $R^\ell f$ is well-defined).

Example 5.6. For this example, let $n = 3$, and let P be a 3-graded poset. Then,

$$\begin{aligned}
\left(\widehat{a}_0^{(0)}, \widehat{a}_1^{(0)}, \widehat{a}_2^{(0)}, \widehat{a}_3^{(0)}, \widehat{a}_4^{(0)}\right) &= (a_0, a_1, a_2, a_3, a_4); \\
\left(\widehat{a}_0^{(1)}, \widehat{a}_1^{(1)}, \widehat{a}_2^{(1)}, \widehat{a}_3^{(1)}, \widehat{a}_4^{(1)}\right) &= \left(a_0, \frac{a_4 a_0}{a_3}, \frac{a_4 a_1}{a_3}, \frac{a_4 a_2}{a_3}, a_4\right); \\
\left(\widehat{a}_0^{(2)}, \widehat{a}_1^{(2)}, \widehat{a}_2^{(2)}, \widehat{a}_3^{(2)}, \widehat{a}_4^{(2)}\right) &= \left(a_0, \frac{a_3 a_0}{a_2}, \frac{a_4 a_0}{a_2}, \frac{a_4 a_1}{a_2}, a_4\right); \\
\left(\widehat{a}_0^{(3)}, \widehat{a}_1^{(3)}, \widehat{a}_2^{(3)}, \widehat{a}_3^{(3)}, \widehat{a}_4^{(3)}\right) &= \left(a_0, \frac{a_2 a_0}{a_1}, \frac{a_3 a_0}{a_1}, \frac{a_4 a_0}{a_1}, a_4\right); \\
\left(\widehat{a}_0^{(4)}, \widehat{a}_1^{(4)}, \widehat{a}_2^{(4)}, \widehat{a}_3^{(4)}, \widehat{a}_4^{(4)}\right) &= (a_0, a_1, a_2, a_3, a_4).
\end{aligned}$$

More generally, we always have $\left(\widehat{a}_0^{(0)}, \widehat{a}_1^{(0)}, ..., \widehat{a}_{n+1}^{(0)}\right) = (a_0, a_1, ..., a_{n+1})$ and $\left(\widehat{a}_0^{(n+1)}, \widehat{a}_1^{(n+1)}, ..., \widehat{a}_{n+1}^{(n+1)}\right) = (a_0, a_1, ..., a_{n+1})$ (as can be verified directly).

Proof of Proposition 5.5 (sketched). This proof is a completely straightforward induction over ℓ, with the base case being trivial and the induction step relying on Proposition 5.4. It is useful to notice that every $\ell \in \{0, 1, ..., n+1\}$ and $k \in \{0, 1, ..., n+1\}$ satisfy

$$\widehat{a}_k^{(\ell)} = \frac{a_{n+1+k-\ell} a_0}{a_{n+1-\ell}} \qquad \text{if } k \leqslant \ell$$

to simplify the computations (this identity follows from the definition when $k < \ell$ and can be easily checked for $k = \ell$). □

As a consequence of Proposition 5.5, we notice a very simple behavior of rescaled labellings under R^{n+1} for an n-graded poset P:

Corollary 5.7. Let $n \in \mathbb{N}$. Let $\mathbb{K}$ be a field. Let P be an n-graded poset. For every $\mathbb{K}$-labelling $f \in \mathbb{K}^{\widehat{P}}$ and any $(n+2)$-tuple $(a_0, a_1, ..., a_{n+1}) \in (\mathbb{K}^{\times})^{n+2}$, we define a $\mathbb{K}$-labelling $(a_0, a_1, ..., a_{n+1}) \flat f \in \mathbb{K}^{\widehat{P}}$ as in Definition 5.2.

Let $(a_0, a_1, ..., a_{n+1}) \in (\mathbb{K}^{\times})^{n+2}$. Let $f \in \mathbb{K}^{\widehat{P}}$ be a $\mathbb{K}$-labelling. Then,

$$R^{n+1} \left((a_0, a_1, ..., a_{n+1}) \flat f \right) = (a_0, a_1, ..., a_{n+1}) \flat \left(R^{n+1} f \right)$$

(provided that $R^{n+1} f$ is well-defined).

We leave deriving Corollary 5.7 from Proposition 5.5 to the reader. □

Let us furthermore record how the rescaling of labels according to their degree affects their w-tuples (as defined in Definition 4.1):

Proposition 5.8. Let $n \in \mathbb{N}$. Let $\mathbb{K}$ be a field. Let P be an n-graded poset. For every $\mathbb{K}$-labelling $f \in \mathbb{K}^{\widehat{P}}$ and any $(n+2)$-tuple $(a_0, a_1, ..., a_{n+1}) \in (\mathbb{K}^{\times})^{n+2}$, we define a $\mathbb{K}$-labelling $(a_0, a_1, ..., a_{n+1}) \flat f \in \mathbb{K}^{\widehat{P}}$ as in Definition 5.2.

Let $f \in \mathbb{K}^{\widehat{P}}$ be a $\mathbb{K}$-labelling of P. Let $(a_0, a_1, ..., a_{n+1}) \in (\mathbb{K}^{\times})^{n+2}$. Then, the w-tuple of the $\mathbb{K}$-labelling $(a_0, a_1, ..., a_{n+1}) \flat f$ is

$$\left(\frac{a_0}{a_1} \mathbf{w}_0 (f), \frac{a_1}{a_2} \mathbf{w}_1 (f), ..., \frac{a_n}{a_{n+1}} \mathbf{w}_n (f) \right).$$

Proposition 5.8 follows by computation using just the definitions of the notions involved. □

6 Homogeneous labellings

In the previous section, we have quantified how the rescaling of all labels at a given degree affects a labelling (of a graded poset) under birational rowmotion. In this section, we will introduce a notion of "homogeneous labellings" which (roughly speaking) are "labellings up to rescaling at a given degree" in the same way as a point in a projective space can be regarded as (roughly speaking) "a point in the affine space up to rescaling the coordinates". To be precise, we will need to restrict ourselves to considering only "zero-free" labellings (a Zariski-dense open subset of all labellings) for the same reason as we need to exclude 0 when defining a projective space. Once done with the definitions, we will see that birational rowmotion (and the maps R_i) can be defined on homogeneous labellings (it is here that we will make use of the results of the previous section).

Let us begin with the definitions:

Definition 6.1. Let $\mathbb{K}$ be a field.

(a) For every $\mathbb{K}$-vector space V, let $\mathbb{P}(V)$ denote the projective space of V (that is, the set of equivalence classes of vectors in $V \setminus \{0\}$ modulo proportionality).

(b) For every $n \in \mathbb{N}$, we let $\mathbb{P}^n(\mathbb{K})$ denote the projective space $\mathbb{P}(\mathbb{K}^{n+1})$.

Definition 6.2. Let $n \in \mathbb{N}$. Let $\mathbb{K}$ be a field. Let P be an n-graded poset.

(a) Denote by $\overline{\mathbb{K}^{\widehat{P}}}$ the product $\prod\limits_{i=1}^{n} \mathbb{P}\left(\mathbb{K}^{\widehat{P}_i}\right)$ of projective spaces. Notice that the product is just a Cartesian product of algebraic varieties, and a reader unfamiliar with algebraic geometry can just regard it as a Cartesian product of sets.[20]

We have $\overline{\mathbb{K}^{\widehat{P}}} = \prod\limits_{i=1}^{n} \mathbb{P}\left(\mathbb{K}^{\widehat{P}_i}\right) \cong \prod\limits_{i=1}^{n} \mathbb{P}^{\left|\widehat{P}_i\right|-1}(\mathbb{K})$ (since every $i \in \{1, 2, ..., n\}$ satisfies $\mathbb{P}\left(\mathbb{K}^{\widehat{P}_i}\right) \cong \mathbb{P}^{\left|\widehat{P}_i\right|-1}(\mathbb{K})$). We denote the elements of $\overline{\mathbb{K}^{\widehat{P}}}$ as *homogeneous labellings*.

Notice that $\overline{\mathbb{K}^{\widehat{P}}} = \prod\limits_{i=1}^{n} \mathbb{P}\left(\mathbb{K}^{\widehat{P}_i}\right) \cong \prod\limits_{i=0}^{n+1} \mathbb{P}\left(\mathbb{K}^{\widehat{P}_i}\right)$ (as algebraic varieties). This is because $\mathbb{K}^{\widehat{P}_0}$ and $\mathbb{K}^{\widehat{P}_{n+1}}$ are 1-dimensional vector spaces (since $\left|\widehat{P}_0\right| = 1$ and $\left|\widehat{P}_{n+1}\right| = 1$), and thus the projective spaces $\mathbb{P}\left(\mathbb{K}^{\widehat{P}_0}\right)$ and $\mathbb{P}\left(\mathbb{K}^{\widehat{P}_{n+1}}\right)$ each consist of a single point.

(b) A $\mathbb{K}$-labelling $f \in \mathbb{K}^{\widehat{P}}$ is said to be *zero-free* if for every $i \in \{0, 1, ..., n+1\}$, there exists some $v \in \widehat{P}_i$ satisfying $f(v) \neq 0$. (In other words, a $\mathbb{K}$-labelling $f \in \mathbb{K}^{\widehat{P}}$ is said to be zero-free if there exists no $i \in \{0, 1, ..., n+1\}$ such that f is identically 0 on all elements of $\widehat{P}$ having degree i.) Let $\mathbb{K}_{\neq 0}^{\widehat{P}}$ be the set of all zero-free $\mathbb{K}$-labellings. Clearly, this set $\mathbb{K}_{\neq 0}^{\widehat{P}}$ is a Zariski-dense open subset of $\mathbb{K}^{\widehat{P}}$.

(c) Identify the set $\mathbb{K}^{\widehat{P}}$ with $\prod\limits_{i=0}^{n+1} \mathbb{K}^{\widehat{P}_i}$ in the obvious way (since $\widehat{P}$, regarded as a set, is the disjoint union of the sets $\widehat{P}_i$ over all $i \in \{0, 1, ..., n+1\}$).

Using the identifications $\mathbb{K}^{\widehat{P}} \cong \prod\limits_{i=0}^{n+1} \mathbb{K}^{\widehat{P}_i}$ and $\overline{\mathbb{K}^{\widehat{P}}} \cong \prod\limits_{i=0}^{n+1} \mathbb{P}\left(\mathbb{K}^{\widehat{P}_i}\right)$, we now define a rational map $\pi : \mathbb{K}^{\widehat{P}} \dashrightarrow \overline{\mathbb{K}^{\widehat{P}}}$ as the product of the canonical projections $\mathbb{K}^{\widehat{P}_i} \dashrightarrow \mathbb{P}\left(\mathbb{K}^{\widehat{P}_i}\right)$ (which are defined everywhere outside of the $\{0\}$ subsets) over all $i \in \{0, 1, ..., n+1\}$. Notice that the domain of definition of this rational map π is precisely $\mathbb{K}_{\neq 0}^{\widehat{P}}$. For every $f \in \mathbb{K}^{\widehat{P}}$, we denote $\pi(f)$ as the *homogenization* of the $\mathbb{K}$-labelling f.

(d) Two zero-free $\mathbb{K}$-labellings $f \in \mathbb{K}^{\widehat{P}}$ and $g \in \mathbb{K}^{\widehat{P}}$ are said to be *homogeneously equivalent* if and only if they satisfy one of the following equivalent conditions:

Condition 1: For every $i \in \{0, 1, ..., n+1\}$ and any two elements x and y of $\widehat{P}_i$, we have $\dfrac{f(x)}{f(y)} = \dfrac{g(x)}{g(y)}$.

Condition 2: There exists an $(n+2)$-tuple $(a_0, a_1, ..., a_{n+1}) \in (\mathbb{K}^\times)^{n+2}$ such that every $x \in \widehat{P}$ satisfies $g(x) = a_{\deg x} \cdot f(x)$.

Condition 3: We have $\pi(f) = \pi(g)$.

(The equivalence between these three conditions is very easy to check. We will never actually use Condition 1.)

Remark 6.3. Clearly, homogeneous equivalence is an equivalence relation on the set $\mathbb{K}^{\widehat{P}}_{\neq 0}$ of all zero-free $\mathbb{K}$-labellings. We can identify $\overline{\mathbb{K}^{\widehat{P}}}$ with the quotient of the set $\mathbb{K}^{\widehat{P}}_{\neq 0}$ modulo this relation. Then, π becomes the canonical projection map $\mathbb{K}^{\widehat{P}} \dashrightarrow \overline{\mathbb{K}^{\widehat{P}}}$.

One remark about the notion "zero-free": Being zero-free is a very weak condition on a $\mathbb{K}$-labelling (indeed the zero-free $\mathbb{K}$-labellings form a Zariski-dense open subset of the space of all $\mathbb{K}$-labellings), and the $\mathbb{K}$-labellings which don't satisfy this condition are rather useless for us (if f is a $\mathbb{K}$-labelling which is not zero-free, then $R^2 f$ is not well-defined, and usually not even Rf is well-defined). We are almost never giving up any generality if we require a labelling to be zero-free.

Remark 6.4. Let $n \in \mathbb{N}$. Let $\mathbb{K}$ be a field. Let P be an n-graded poset. For every $\mathbb{K}$-labelling $f \in \mathbb{K}^{\widehat{P}}$ and any $(n+2)$-tuple $(a_0, a_1, ..., a_{n+1}) \in (\mathbb{K}^\times)^{n+2}$, we define a $\mathbb{K}$-labelling $(a_0, a_1, ..., a_{n+1}) \flat f \in \mathbb{K}^{\widehat{P}}$ as in Definition 5.2.

Let $f \in \mathbb{K}^{\widehat{P}}$ be a zero-free $\mathbb{K}$-labelling of P. Let $(a_0, a_1, ..., a_{n+1}) \in (\mathbb{K}^\times)^{n+2}$. Then, the $\mathbb{K}$-labelling $(a_0, a_1, ..., a_{n+1}) \flat f$ is also zero-free. (This follows immediately from the definitions.)

Definition 6.5. Let $n \in \mathbb{N}$. Let $\mathbb{K}$ be a field. Let P be an n-graded poset. For every zero-free $f \in \mathbb{K}^{\widehat{P}}$ and every $i \in \{1, 2, ..., n\}$, the image of the restriction of $f : \widehat{P} \to \mathbb{K}$ to $\widehat{P}_i$ under the canonical projection $\mathbb{K}^{\widehat{P}_i} \dashrightarrow \mathbb{P}\left(\mathbb{K}^{\widehat{P}_i}\right)$ will be denoted by $\pi_i(f)$. This image $\pi_i(f)$ encodes the values of f at the elements of $\widehat{P}$ of degree i up to multiplying

[20] The structure of algebraic variety will only be needed to define the Zariski topology on $\overline{\mathbb{K}^{\widehat{P}}}$, which is more or less obvious already (e.g., when we say that something holds "for almost every element x of $\prod_{i=1}^{n} \mathbb{P}\left(\mathbb{K}^{\widehat{P}_i}\right)$", we could equivalently say that it holds "for $x = \operatorname{proj}(X)$ for almost every element X of $\prod_{i=1}^{n} \left(\mathbb{K}^{\widehat{P}_i} \setminus \{0\}\right)$", where proj is the canonical map $\prod_{i=1}^{n} \left(\mathbb{K}^{\widehat{P}_i} \setminus \{0\}\right) \to \prod_{i=1}^{n} \mathbb{P}\left(\mathbb{K}^{\widehat{P}_i}\right)$ defined as the product of the projections $\mathbb{K}^{\widehat{P}_i} \setminus \{0\} \to \mathbb{P}\left(\mathbb{K}^{\widehat{P}_i}\right)$).

all these values by a common nonzero scalar. Notice that

$$\pi(f) = (\pi_1(f), \pi_2(f), ..., \pi_n(f)) \tag{14}$$

for every $f \in \mathbb{K}^{\widehat{P}}$. (Here, the right hand side of (14) is regarded as an element of $\overline{\mathbb{K}^{\widehat{P}}}$ because it belongs to $\prod_{i=1}^{n} \mathbb{P}\left(\mathbb{K}^{\widehat{P}_i}\right) = \overline{\mathbb{K}^{\widehat{P}}}$.)

We are next going to see:

Corollary 6.6. Let $n \in \mathbb{N}$. Let $\mathbb{K}$ be a field. Let P be an n-graded poset. Let $i \in \{1, 2, ..., n\}$. If $f \in \mathbb{K}^{\widehat{P}}$ and $g \in \mathbb{K}^{\widehat{P}}$ are two homogeneously equivalent zero-free $\mathbb{K}$-labellings, then $R_i f$ is homogeneously equivalent to $R_i g$ (as long as $R_i f$ and $R_i g$ are zero-free).

Corollary 6.7. Let $n \in \mathbb{N}$. Let $\mathbb{K}$ be a field. Let P be an n-graded poset. If $f \in \mathbb{K}^{\widehat{P}}$ and $g \in \mathbb{K}^{\widehat{P}}$ are two homogeneously equivalent zero-free $\mathbb{K}$-labellings, then Rf is homogeneously equivalent to Rg (as long as Rf and Rg are zero-free).

Notice that Corollary 6.6 would not be valid if we were to replace R_i by a single toggle T_v! So the operators R_i in some sense combine the nice properties of T_v (like being an involution, cf. Proposition 3.9) with the nice properties of R (like having an easily describable action on w-tuples, cf. Proposition 4.3, and respecting homogeneous equivalence, cf. Corollary 6.6).

Proof of Corollary 6.6 (sketched). Let $f \in \mathbb{K}^{\widehat{P}}$ and $g \in \mathbb{K}^{\widehat{P}}$ be two homogeneously equivalent zero-free $\mathbb{K}$-labellings.

We know that f and g are homogeneously equivalent. By Condition 2 in Definition 6.2 **(d)**, this means that there exists an $(n+2)$-tuple $(a_0, a_1, ..., a_{n+1}) \in (\mathbb{K}^\times)^{n+2}$ such that every $x \in \widehat{P}$ satisfies $g(x) = a_{\deg x} \cdot f(x)$. In other words, there exists an $(n+2)$-tuple $(a_0, a_1, ..., a_{n+1}) \in (\mathbb{K}^\times)^{n+2}$ such that

$$g = (a_0, a_1, ..., a_{n+1}) \flat f.$$

Consider this $(n+2)$-tuple $(a_0, a_1, ..., a_{n+1})$. Since $g = (a_0, a_1, ..., a_{n+1}) \flat f$, we have

$$\begin{aligned} R_i g &= R_i((a_0, a_1, ..., a_{n+1}) \flat f) \\ &= \left(a_0, a_1, ..., a_{i-1}, \frac{a_{i+1}a_{i-1}}{a_i}, a_{i+1}, a_{i+2}, ..., a_{n+1}\right) \flat (R_i f) \end{aligned}$$

(by Proposition 5.3). Hence, there exists an $(n+2)$-tuple $(b_0, b_1, ..., b_{n+1}) \in (\mathbb{K}^\times)^{n+2}$ such that

$$R_i g = (b_0, b_1, ..., b_{n+1}) \flat (R_i f)$$

(namely, $(b_0, b_1, ..., b_{n+1}) = \left(a_0, a_1, ..., a_{i-1}, \dfrac{a_{i+1}a_{i-1}}{a_i}, a_{i+1}, a_{i+2}, ..., a_{n+1}\right)$). In other words, there exists an $(n+2)$-tuple $(b_0, b_1, ..., b_{n+1}) \in (\mathbb{K}^\times)^{n+2}$ such that every $x \in \widehat{P}$ satisfies $(R_i g)(x) = b_{\deg x} \cdot (R_i f)(x)$. But this is precisely Condition 2 in Definition 6.2 **(d)**, stated for the labellings $R_i f$ and $R_i g$ instead of f and g. Hence, $R_i f$ and $R_i g$ are homogeneously equivalent. This proves Corollary 6.6. □

Proof of Corollary 6.7 (sketched). Let $f \in \mathbb{K}^{\widehat{P}}$ and $g \in \mathbb{K}^{\widehat{P}}$ be two homogeneously equivalent zero-free $\mathbb{K}$-labellings. By iterative application of Corollary 6.6, we then conclude that the $\mathbb{K}$-labellings $(R_1 \circ R_2 \circ ... \circ R_n) f$ and $(R_1 \circ R_2 \circ ... \circ R_n) g$ are homogeneously equivalent (if they are well-defined). Since $R_1 \circ R_2 \circ ... \circ R_n = R$ (by Proposition 3.8), this rewrites as follows: The $\mathbb{K}$-labellings Rf and Rg are homogeneously equivalent. This proves Corollary 6.7. □

Let us introduce a general piece of notation:

Definition 6.8. Let S and T be two sets. Let $\sim_S$ be an equivalence relation on the set S, and let $\sim_T$ be an equivalence relation on the set T. Let $\overline{S}$ be the quotient of the set S modulo the equivalence relation $\sim_S$, and let $\overline{T}$ be the quotient of the set T modulo the equivalence relation $\sim_T$. Let $\pi_S : S \to \overline{S}$ and $\pi_T : T \to \overline{T}$ be the canonical projections of a set on its quotient. Let $f : S \to T$ be a map. If $\overline{f} : \overline{S} \to \overline{T}$ is a map for which the diagram

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, "\pi_S"'] & T \arrow[d, "\pi_T"] \\ \overline{S} \arrow[r, "\overline{f}"'] & \overline{T} \end{tikzcd}$$

is commutative, then we say that "the map f *descends* to the map $\overline{f}$". It is easy to see that there exists **at most one** map $\overline{f} : \overline{S} \to \overline{T}$ such that the map f descends to the map $\overline{f}$ (for given S, T, $\sim_S$, $\sim_T$ and f). Moreover, the existence of a map $\overline{f} : \overline{S} \to \overline{T}$ such that the map f descends to the map $\overline{f}$ is equivalent to the statement that every two elements x and y of S satisfying $x \sim_S y$ satisfy $f(x) \sim_T f(y)$.

The above statements are not literally true if we replace the map $f : S \to T$ by a partial map $f : S \dashrightarrow T$. However, when S and T are two algebraic varieties and $\sim_S$ and $\sim_T$ are algebraic equivalences (i.e., equivalence relations defined by polynomial relations between coordinates of points) and $f : S \dashrightarrow T$ is a **rational map**, then the above statements still are true (of course, with $\overline{f}$ being a partial map).

Definition 6.9. Let $n \in \mathbb{N}$. Let $\mathbb{K}$ be a field. Let P be an n-graded poset. Let $i \in \{1, 2, ..., n\}$. Because of Corollary 6.6, the rational map $R_i : \mathbb{K}^{\widehat{P}} \dashrightarrow \mathbb{K}^{\widehat{P}}$ descends (through the projection $\pi : \mathbb{K}^{\widehat{P}} \dashrightarrow \overline{\mathbb{K}^{\widehat{P}}}$) to a partial map $\overline{\mathbb{K}^{\widehat{P}}} \dashrightarrow \overline{\mathbb{K}^{\widehat{P}}}$. We denote this partial map $\overline{\mathbb{K}^{\widehat{P}}} \dashrightarrow \overline{\mathbb{K}^{\widehat{P}}}$ by $\overline{R_i}$. Thus, the diagram

$$\begin{tikzcd} \mathbb{K}^{\widehat{P}} \arrow[r, dashed, "R_i"] \arrow[d, dashed, "\pi"'] & \mathbb{K}^{\widehat{P}} \arrow[d, dashed, "\pi"] \\ \overline{\mathbb{K}^{\widehat{P}}} \arrow[r, dashed, "\overline{R_i}"'] & \overline{\mathbb{K}^{\widehat{P}}} \end{tikzcd} \qquad (15)$$

is commutative.

Definition 6.10. Let $n \in \mathbb{N}$. Let $\mathbb{K}$ be a field. Let P be an n-graded poset. We define the partial map $\overline{R} : \overline{\mathbb{K}^{\widehat{P}}} \dashrightarrow \overline{\mathbb{K}^{\widehat{P}}}$ by

$$\overline{R} = \overline{R_1} \circ \overline{R_2} \circ ... \circ \overline{R_n}.$$

Then, the diagram

$$\begin{tikzcd} \mathbb{K}^{\widehat{P}} \arrow[r, dashed, "R"] \arrow[d, dashed, "\pi"'] & \mathbb{K}^{\widehat{P}} \arrow[d, dashed, "\pi"] \\ \overline{\mathbb{K}^{\widehat{P}}} \arrow[r, dashed, "\overline{R}"'] & \overline{\mathbb{K}^{\widehat{P}}} \end{tikzcd} \qquad (16)$$

is commutative[21]. In other words, $\overline{R}$ is the partial map $\overline{\mathbb{K}^{\widehat{P}}} \dashrightarrow \overline{\mathbb{K}^{\widehat{P}}}$ to which the partial map $R : \mathbb{K}^{\widehat{P}} \dashrightarrow \mathbb{K}^{\widehat{P}}$ descends (through the projection $\pi : \mathbb{K}^{\widehat{P}} \dashrightarrow \overline{\mathbb{K}^{\widehat{P}}}$).

Next, we formulate a result which says something to the extent of "a zero-free $\mathbb{K}$-labelling $f \in \mathbb{K}^{\widehat{P}}$ is almost always uniquely determined by its w-tuple $(\mathbf{w}_0(f), \mathbf{w}_1(f), ..., \mathbf{w}_n(f))$, its homogenization $\pi(f)$ and the value $f(0)$". The words "almost always" are required here because otherwise the statement would be wrong; but they have to be made precise. Here is the exact statement we want to make:

Proposition 6.11. Let $n \in \mathbb{N}$. Let $\mathbb{K}$ be a field. Let P be an n-graded poset. Let f and g be two zero-free $\mathbb{K}$-labellings in $\mathbb{K}^{\widehat{P}}$ such that $(\mathbf{w}_0(f), \mathbf{w}_1(f), ..., \mathbf{w}_n(f)) = (\mathbf{w}_0(g), \mathbf{w}_1(g), ..., \mathbf{w}_n(g))$ and such that no $i \in \{0, 1, ..., n\}$ satisfies $\mathbf{w}_i(f) = 0$. Also assume that $\pi(f) = \pi(g)$ and $f(0) = g(0)$. Then, $f = g$.

Proposition 6.11 is easily proven by reconstructing f and g "bottom-up" along $\widehat{P}$. Alternatively, we can prove Proposition 6.11 directly using Proposition 5.8, as follows:

[21] *Proof.* We have $R = R_1 \circ R_2 \circ ... \circ R_n$ and $\overline{R} = \overline{R_1} \circ \overline{R_2} \circ ... \circ \overline{R_n}$. Hence, the diagram (16) can be obtained by stringing together the diagrams (15) for all $i \in \{1, 2, ..., n\}$ and then removing the "interior edges". Therefore, the diagram (16) is commutative (since the diagrams (15) are commutative for all i), qed.

Proof of Proposition 6.11 (sketched). Since $\pi(f) = \pi(g)$, we know that f and g are homogeneously equivalent. By Condition 2 in Definition 6.2 **(d)**, this means that there exists an $(n+2)$-tuple $(a_0, a_1, ..., a_{n+1}) \in (\mathbb{K}^\times)^{n+2}$ such that every $x \in \widehat{P}$ satisfies $g(x) = a_{\deg x} \cdot f(x)$. In other words, there exists an $(n+2)$-tuple $(a_0, a_1, ..., a_{n+1}) \in (\mathbb{K}^\times)^{n+2}$ such that

$$g = (a_0, a_1, ..., a_{n+1}) \flat f$$

(where $(a_0, a_1, ..., a_{n+1}) \flat f \in \mathbb{K}^{\widehat{P}}$ is defined as in Definition 5.2). Consider this $(n+2)$-tuple $(a_0, a_1, ..., a_{n+1})$.

Since $g = (a_0, a_1, ..., a_{n+1}) \flat f$, we know that

$$\begin{aligned}(\text{the w-tuple of } g) &= (\text{the w-tuple of } (a_0, a_1, ..., a_{n+1}) \flat f) \\ &= \left(\frac{a_0}{a_1}\mathbf{w}_0(f), \frac{a_1}{a_2}\mathbf{w}_1(f), ..., \frac{a_n}{a_{n+1}}\mathbf{w}_n(f)\right)\end{aligned}$$

(by Proposition 5.8). Compared with

$$(\text{the w-tuple of } g) = (\mathbf{w}_0(g), \mathbf{w}_1(g), ..., \mathbf{w}_n(g)) = (\mathbf{w}_0(f), \mathbf{w}_1(f), ..., \mathbf{w}_n(f)),$$

this yields

$$\left(\frac{a_0}{a_1}\mathbf{w}_0(f), \frac{a_1}{a_2}\mathbf{w}_1(f), ..., \frac{a_n}{a_{n+1}}\mathbf{w}_n(f)\right) = (\mathbf{w}_0(f), \mathbf{w}_1(f), ..., \mathbf{w}_n(f)).$$

In other words, $\frac{a_i}{a_{i+1}}\mathbf{w}_i(f) = \mathbf{w}_i(f)$ for every $i \in \{0, 1, ..., n\}$. Hence, $\frac{a_i}{a_{i+1}} = 1$ for every $i \in \{0, 1, ..., n\}$ (here, we cancelled out $\mathbf{w}_i(f)$, because by assumption we don't have $\mathbf{w}_i(f) = 0$). In other words, $a_i = a_{i+1}$ for every $i \in \{0, 1, ..., n\}$. Thus, $a_0 = a_1 = ... = a_{n+1}$.

But since $g = (a_0, a_1, ..., a_{n+1}) \flat f$, we have $g(0) = ((a_0, a_1, ..., a_{n+1}) \flat f)(0) = a_{\deg 0} \cdot f(0) = a_0 \cdot f(0)$ (since $\deg 0 = 0$), so that $f(0) = g(0) = a_0 \cdot f(0)$. Since $f(0) \neq 0$ (because f is zero-free, and the only element of $\widehat{P}_0$ is 0), we can cancel $f(0)$ here and obtain $1 = a_0$. In view of this, $a_0 = a_1 = ... = a_{n+1}$ becomes $a_0 = a_1 = ... = a_{n+1} = 1$. Thus, $(a_0, a_1, ..., a_{n+1}) = \left(\underbrace{1, 1, ..., 1}_{n+2 \text{ times}}\right)$, so that $g = (a_0, a_1, ..., a_{n+1}) \flat f = \left(\underbrace{1, 1, ..., 1}_{n+2 \text{ times}}\right) \flat f = f$, proving Proposition 6.11.

□

Definition 6.12. Let $\mathbb{K}$ be a field. In the following, if S is a finite set, and q is an element of a projective space $\mathbb{P}(\mathbb{K}^S)$ of the free vector space with basis S, and k is an integer, then q^k will denote the element of $\mathbb{P}(\mathbb{K}^S)$ obtained by replacing every homogeneous coordinate of q by its k-th power. This is well-defined (and will mostly be used for $k = -1$). In particular, this definition applies to $S = \{1, 2, ..., n\}$ for $n \in \mathbb{N}$ (in which case $\mathbb{K}^S = \mathbb{K}^n$).

We can explicitly describe the action of the $\overline{R_i}$ when the "structure of the poset P between degrees $i-1$, i and $i+1$" is particularly simple:

Proposition 6.13. Let $n \in \mathbb{N}$. Let $\mathbb{K}$ be a field. Let P be an n-graded poset. Fix $i \in \{1, 2, ..., n\}$. Assume that every $u \in \widehat{P}_i$ and every $v \in \widehat{P}_{i+1}$ satisfy $u \lessdot v$. Assume further that every $u \in \widehat{P}_{i-1}$ and every $v \in \widehat{P}_i$ satisfy $u \lessdot v$. Let $f \in \mathbb{K}^{\widehat{P}}$. Then,

$$\begin{aligned}&\left(\pi_1\left(R_i f\right), \pi_2\left(R_i f\right), ..., \pi_n\left(R_i f\right)\right)\\&= \left(\pi_1\left(f\right), \pi_2\left(f\right), ..., \pi_{i-1}\left(f\right), \left(\pi_i\left(f\right)\right)^{-1}, \pi_{i+1}\left(f\right), \pi_{i+2}\left(f\right), ..., \pi_n\left(f\right)\right).\end{aligned}$$

From this proposition, we obtain two corollaries:

Corollary 6.14. Let $n \in \mathbb{N}$. Let $\mathbb{K}$ be a field. Let P be an n-graded poset. Fix $i \in \{1, 2, ..., n\}$. Assume that every $u \in \widehat{P}_i$ and every $v \in \widehat{P}_{i+1}$ satisfy $u \lessdot v$. Assume further that every $u \in \widehat{P}_{i-1}$ and every $v \in \widehat{P}_i$ satisfy $u \lessdot v$. Let $\widetilde{f} = \left(\widetilde{f}_1, \widetilde{f}_2, ..., \widetilde{f}_n\right) \in \overline{\mathbb{K}^{\widehat{P}}}$. Then,

$$\overline{R_i}\left(\widetilde{f}\right) = \left(\widetilde{f}_1, \widetilde{f}_2, ..., \widetilde{f}_{i-1}, \widetilde{f}_i^{-1}, \widetilde{f}_{i+1}, \widetilde{f}_{i+2}, ..., \widetilde{f}_n\right).$$

Corollary 6.15. Let $n \in \mathbb{N}$. Let $\mathbb{K}$ be a field. Let P be an n-graded poset. Assume that, for every $i \in \{1, 2, ..., n-1\}$, every $u \in \widehat{P}_i$ and every $v \in \widehat{P}_{i+1}$ satisfy $u \lessdot v$. Let $f \in \mathbb{K}^{\widehat{P}}$ be zero-free. Then,

$$\left(\pi_1\left(Rf\right), \pi_2\left(Rf\right), ..., \pi_n\left(Rf\right)\right) = \left(\left(\pi_1\left(f\right)\right)^{-1}, \left(\pi_2\left(f\right)\right)^{-1}, ..., \left(\pi_n\left(f\right)\right)^{-1}\right).$$

7 Order

In this short section, we will relate the orders of the maps R and $\overline{R}$ for a graded poset P. The relation will later be used to gain knowledge on both of these orders.

We begin by defining the order of a partial map:

Definition 7.1. Let S be a set.

(a) If α and β are two partial maps from the set S, then we write "$\alpha = \beta$" if and only if every $s \in S$ for which both $\alpha(s)$ and $\beta(s)$ are well-defined satisfies $\alpha(s) = \beta(s)$. This is, per se, not a well-behaved notation (e.g., it is possible that three partial maps α, β and γ satisfy $\alpha = \beta$ and $\beta = \gamma$ but not $\alpha = \gamma$). However, we are going to use this notation for rational maps and their quotients (and, of course, total maps) only; in all of these cases, the notation **is** well-behaved (e.g., if α, β and γ are three rational maps satisfying $\alpha = \beta$ and $\beta = \gamma$, then $\alpha = \gamma$, because the intersection of two Zariski-dense open subsets is Zariski-dense and open).

(b) The *order* of a partial map $\varphi : S \dashrightarrow S$ is defined to be the smallest positive integer k satisfying $\varphi^k = \mathrm{id}_S$, if such a positive integer k exists, and ∞ otherwise. Here, we are disregarding the fact that φ is only a partial map; we will be working only with dominant rational maps and their quotients (and total maps), so nothing will go wrong.

We denote the order of a partial map $\varphi : S \dashrightarrow S$ as $\operatorname{ord} \varphi$.

Convention 7.2. In the following, we are going to occasionally make arithmetical statements involving the symbol ∞. We declare that 0 and ∞ are divisible by ∞, but no positive integer is divisible by ∞. We further declare that every positive integer (but not 0) divides ∞. We set $\operatorname{lcm}(a, \infty)$ and $\operatorname{lcm}(\infty, a)$ to mean ∞ whenever a is a positive integer.

As a consequence of Proposition 6.11, we have:

Proposition 7.3. Let $n \in \mathbb{N}$. Let $\mathbb{K}$ be a field. Let P be an n-graded poset. Then, $\operatorname{ord} R = \operatorname{lcm}\left(n+1, \operatorname{ord} \overline{R}\right)$. (Recall that $\operatorname{lcm}(n+1, \infty)$ is to be understood as ∞.)

The proof of this boils down to considering the effect of R on the w-tuple $(\mathbf{w}_0(f), \mathbf{w}_1(f), ..., \mathbf{w}_n(f))$ and on the homogenization $\pi(f)$ of a $\mathbb{K}$-labelling f. The effect on the w-tuple is a cyclic shift (by Proposition 4.4), which has order $n+1$. The effect on the homogenization is $\overline{R}$. It is now easy to see (invoking Proposition 6.11) that the order of R is the lcm of the orders of these two actions. Here are the details of this derivation:

Proof of Proposition 7.3 (sketched). 1st step: The commutativity of the diagram (16) yields $\overline{R} \circ \pi = \pi \circ R$. Hence,

$$\text{every } \ell \in \mathbb{N} \text{ satisfies } \overline{R}^{\ell} \circ \pi = \pi \circ R^{\ell} \tag{17}$$

(this is clear by induction over ℓ). Thus, if some $\ell \in \mathbb{N}$ satisfies $R^{\ell} = \mathrm{id}$, then it satisfies $\overline{R}^{\ell} = \mathrm{id}$ as well[22]. Hence, $\operatorname{ord} \overline{R} \mid \operatorname{ord} R$ (recall that every positive integer divides ∞, but only 0 and ∞ are divisible by ∞). In particular, if $\operatorname{ord} \overline{R} = \infty$, then $\operatorname{ord} R = \infty$. Thus, Proposition 7.3 is obvious in the case when $\operatorname{ord} \overline{R} = \infty$. Hence, for the rest of the proof of Proposition 7.3, we can WLOG assume that $\operatorname{ord} \overline{R} \neq \infty$. Assume this.

2nd step: Since $\operatorname{ord} \overline{R} \neq \infty$, we know that $\operatorname{ord} \overline{R}$ is a positive integer. Let m be this positive integer. Then, $m = \operatorname{ord} \overline{R}$, so that $\overline{R}^{m} = \mathrm{id}$.

Let $\ell = \operatorname{lcm}(n+1, m)$. Then, $n+1 \mid \ell$ and $m \mid \ell$. Since $\operatorname{ord} \overline{R} = m \mid \ell$, we have $\overline{R}^{\ell} = \mathrm{id}$. But from (17), we have $\pi \circ R^{\ell} = \underbrace{\overline{R}^{\ell}}_{=\mathrm{id}} \circ \pi = \pi$.

We are now going to prove that $R^{\ell} = \mathrm{id}$. In order to prove this, it is clearly enough to show that almost every (in the sense of Zariski topology) zero-free $\mathbb{K}$-labelling f of P

[22] *Proof.* Let $\ell \in \mathbb{N}$ be such that $R^{\ell} = \mathrm{id}$. Then, $\overline{R}^{\ell} \circ \pi = \pi \circ \underbrace{R^{\ell}}_{=\mathrm{id}} = \pi$. Since π is right-cancellable (since π is surjective), this yields $\overline{R}^{\ell} = \mathrm{id}$, qed.

satisfies $R^\ell f = \operatorname{id} f$ (because $R^\ell f = \operatorname{id} f$ is a polynomial identity in the labels of f). But it is easily shown that for almost every (in the sense of Zariski topology) zero-free $\mathbb{K}$-labelling f of P, the w-tuple $(\mathbf{w}_0(f), \mathbf{w}_1(f), ..., \mathbf{w}_n(f))$ of f consists of nonzero elements of $\mathbb{K}$. [23]

Hence, in order to prove $R^\ell = \operatorname{id}$, it is enough to show that every zero-free $\mathbb{K}$-labelling f of P for which the w-tuple $(\mathbf{w}_0(f), \mathbf{w}_1(f), ..., \mathbf{w}_n(f))$ of f consists of nonzero elements of $\mathbb{K}$ satisfies $R^\ell f = \operatorname{id} f$. This is what we are going to do now.

So let f be a zero-free $\mathbb{K}$-labelling of P for which the w-tuple $(\mathbf{w}_0(f), \mathbf{w}_1(f), ..., \mathbf{w}_n(f))$ of f consists of nonzero elements of $\mathbb{K}$. We will prove that $R^\ell f = \operatorname{id} f$.

From Proposition 4.4, we know that the map R changes the w-tuple of a $\mathbb{K}$-labelling by shifting it cyclically. Hence, for every $k \in \mathbb{N}$, the map R^k changes the w-tuple of a $\mathbb{K}$-labelling by shifting it cyclically k times. If this k is divisible by $n+1$, then this obviously means that the map R^k preserves the w-tuple of a $\mathbb{K}$-labelling (because the w-tuple has $n+1$ entries, and thus shifting it cyclically for a multiple of $n+1$ times leaves it invariant). Hence, the w-tuple of f equals the w-tuple of $R^\ell f$. Recalling the definition of a w-tuple, we can rewrite this as follows:

$$(\mathbf{w}_0(f), \mathbf{w}_1(f), ..., \mathbf{w}_n(f)) = \left(\mathbf{w}_0\left(R^\ell f\right), \mathbf{w}_1\left(R^\ell f\right), ..., \mathbf{w}_n\left(R^\ell f\right)\right).$$

[23] *Proof.* We will prove a slightly better result: Almost every $f \in \mathbb{K}^{\widehat{P}}$ is a zero-free $\mathbb{K}$-labelling of P with the property that

$$(\mathbf{w}_i(f) \text{ is well-defined and nonzero for every } i \in \{0, 1, ..., n\}). \tag{18}$$

In fact, the condition (18) on an $f \in \mathbb{K}^{\widehat{P}}$ is a requirement saying that certain rational expressions in the values of f do not vanish (namely, the denominators in $\mathbf{w}_i(f)$ and the sums $\mathbf{w}_i(f)$ themselves). If we can prove that none of these expressions is identically zero, then it will follow that for almost every $f \in \mathbb{K}^{\widehat{P}}$, none of these expressions vanishes (because there are only finitely many expressions whose vanishing we are trying to avoid, and the infiniteness of $\mathbb{K}$ allows us to avoid them all if none of them is identically zero); thus (18) will follow and we will be done. Hence, it remains to show that none of these expressions is identically zero.

Assume the contrary. Then, one of our rational expressions – either a denominator in one of the $\mathbf{w}_i(f)$, or one of the sums $\mathbf{w}_i(f)$ – identically vanishes. It must be one of the sums $\mathbf{w}_i(f)$, since the denominators in the $\mathbf{w}_i(f)$ cannot identically vanish (they are simply values $f(y)$). So there exists some $i \in \{0, 1, ..., n\}$ such that every $\mathbb{K}$-labelling f of P (for which $\mathbf{w}_i(f)$ is well-defined) satisfies $\mathbf{w}_i(f) = 0$. Consider this i. Notice that $i \leqslant n$ and thus $1 \notin \widehat{P}_i$.

We have

$$0 = \mathbf{w}_i(f) = \sum_{\substack{x \in \widehat{P}_i;\ y \in \widehat{P}_{i+1}; \\ y \gtrdot x}} \frac{f(x)}{f(y)} = \sum_{x \in \widehat{P}_i} f(x) \sum_{\substack{y \in \widehat{P}_{i+1}; \\ y \gtrdot x}} \frac{1}{f(y)}.$$

This forces the sum $\sum_{\substack{y \in \widehat{P}_{i+1}; \\ y \gtrdot x}} \dfrac{1}{f(y)}$ to be identically 0 for every $x \in \widehat{P}_i$ (because these sums for different values of x are prevented from canceling each other by the completely independent $f(x)$ coefficients in front of them). Fix some $x \in \widehat{P}_i$ (such an x clearly exists since $\deg : \widehat{P} \to \{0, 1, ..., n+1\}$ is surjective), and ponder what it means for the sum $\sum_{\substack{y \in \widehat{P}_{i+1}; \\ y \gtrdot x}} \dfrac{1}{f(y)}$ to be identically 0. It means that this sum is empty, i.e., that there exists no $y \in \widehat{P}_{i+1}$ satisfying $y \gtrdot x$. But this can only happen when $x = 1$, which is not the case in our situation (because $x \in \widehat{P}_i$ and $1 \notin \widehat{P}_i$). So we have obtained a contradiction.

Moreover, by assumption, the w-tuple $(\mathbf{w}_0(f), \mathbf{w}_1(f), ..., \mathbf{w}_n(f))$ of f consists of nonzero elements of $\mathbb{K}$. In other words, no $i \in \{0, 1, ..., n\}$ satisfies $\mathbf{w}_i(f) = 0$.

Furthermore $\pi\left(R^\ell f\right) = \underbrace{\left(\pi \circ R^\ell\right)}_{=\pi} f = \pi(f)$.

Also, Corollary 2.18 (applied to $k = \ell$) yields $\left(R^\ell f\right)(0) = f(0)$.

We now can apply Proposition 6.11 to $g = R^\ell f$. As a result, we obtain $R^\ell f = f$. In other words, $R^\ell f = \operatorname{id} f$.

Now forget that we fixed f. We have thus shown that $R^\ell f = \operatorname{id} f$ for every zero-free $\mathbb{K}$-labelling f of P for which the w-tuple $(\mathbf{w}_0(f), \mathbf{w}_1(f), ..., \mathbf{w}_n(f))$ of f consists of nonzero elements of $\mathbb{K}$. Therefore, we have shown that $R^\ell = \operatorname{id}$ (by what we have said above). Thus, $\operatorname{ord} R \mid \ell = \operatorname{lcm}\left(n+1, \underbrace{m}_{=\operatorname{ord}\overline{R}}\right) = \operatorname{lcm}\left(n+1, \operatorname{ord}\overline{R}\right)$.

3rd step: We now will show that $\operatorname{lcm}\left(n+1, \operatorname{ord}\overline{R}\right) \mid \operatorname{ord} R$.

In order to do that, we assume WLOG that $\operatorname{ord} R \neq \infty$ (because otherwise, $\operatorname{lcm}\left(n+1, \operatorname{ord}\overline{R}\right) \mid \operatorname{ord} R$ is obvious). Hence, $\operatorname{ord} R$ is a positive integer. Denote this positive integer by q. So, $q = \operatorname{ord} R$.

It is easy to see that for almost every (in the sense of Zariski topology) zero-free $\mathbb{K}$-labelling f of P, the entries of the w-tuple $(\mathbf{w}_0(f), \mathbf{w}_1(f), ..., \mathbf{w}_n(f))$ of f are pairwise distinct. Hence, there exists a zero-free $\mathbb{K}$-labelling f of P such that the entries of the w-tuple $(\mathbf{w}_0(f), \mathbf{w}_1(f), ..., \mathbf{w}_n(f))$ of f are pairwise distinct and such that $R^k f$ is well-defined for all $k \in \{0, 1, ..., q\}$. Consider such an f.

Since $q = \operatorname{ord} R$, we have $R^q = \operatorname{id}$, so that $R^q f = f$.

Recall once again (from the 2nd step) that for every $k \in \mathbb{N}$, the map R^k changes the w-tuple of a $\mathbb{K}$-labelling by shifting it cyclically k times. In particular, the map R^q changes the w-tuple of the $\mathbb{K}$-labelling f by shifting it cyclically q times. In other words, the w-tuple of $R^q f$ is obtained from the w-tuple of f by shifting it cyclically q times. Since $R^q f = f$, this rewrites as follows: The w-tuple of f is obtained from the w-tuple of f by shifting it cyclically q times. In other words, the w-tuple of f is fixed under a q-fold cyclic shift. But since the w-tuple of f is an $(n+1)$-tuple of pairwise distinct entries, this can only happen if $n+1 \mid q$. Hence, we have $n+1 \mid q$.

Combining $n+1 \mid q = \operatorname{ord} R$ with $\operatorname{ord}\overline{R} \mid \operatorname{ord} R$, we obtain $\operatorname{lcm}\left(n+1, \operatorname{ord}\overline{R}\right) \mid \operatorname{ord} R$. Combining this with $\operatorname{ord} R \mid \operatorname{lcm}\left(n+1, \operatorname{ord}\overline{R}\right)$, we obtain $\operatorname{ord} R = \operatorname{lcm}\left(n+1, \operatorname{ord}\overline{R}\right)$. This proves Proposition 7.3. □

8 The opposite poset

Before we move on to the first interesting class of posets for which we can compute the order of birational rowmotion, let us prove an easy "symmetry property" of birational rowmotion.

Definition 8.1. Let P be a poset. Then, P^{op} will denote the poset defined on the same ground set as P but with the order relation defined by

$$((a <_{P^{\mathrm{op}}} b \text{ if and only if } b <_P a) \text{ for all } a \in P \text{ and } b \in P)$$

(where $<_P$ denotes the smaller relation of the poset P, and where $<_{P^{\mathrm{op}}}$ denotes the smaller relation of the poset P^{op} which we are defining). The poset P^{op} is called the *opposite poset* of P.

Note that P^{op} is called the *dual* of the poset P in [Stan11].

Remark 8.2. It is clear that $(P^{\mathrm{op}})^{\mathrm{op}} = P$ for any poset P. Also, if $n \in \mathbb{N}$, and if P is an n-graded poset, then P^{op} is an n-graded poset.

Definition 8.3. Let P be a finite poset. Let $\mathbb{K}$ be a field. We denote the maps R and $\overline{R}$ by R_P and $\overline{R}_P$, respectively, so as to make their dependence on P explicit.

We can now state a symmetry property of $\operatorname{ord} R$ (as defined in Definition 7.1):

Proposition 8.4. Let P be a finite poset. Let $\mathbb{K}$ be a field. Then, $\operatorname{ord}(R_{P^{\mathrm{op}}}) = \operatorname{ord}(R_P)$ and $\operatorname{ord}\left(\overline{R}_{P^{\mathrm{op}}}\right) = \operatorname{ord}\left(\overline{R}_P\right)$.

Proof of Proposition 8.4 (sketched). Define a rational map $\kappa : \mathbb{K}^{\widehat{P}} \dashrightarrow \mathbb{K}^{\widehat{P^{\mathrm{op}}}}$ by

$$(\kappa f)(w) = \begin{cases} \dfrac{1}{f(w)}, & \text{if } w \in P; \\ \dfrac{1}{f(1)}, & \text{if } w = 0; \\ \dfrac{1}{f(0)}, & \text{if } w = 1 \end{cases} \qquad \text{for every } w \in \widehat{P^{\mathrm{op}}} \text{ for every } f \in \mathbb{K}^{\widehat{P}}.$$

This map κ is a birational map. (Its inverse map is defined in the same way.)

We claim that $\kappa \circ R_P = R_{P^{\mathrm{op}}}^{-1} \circ \kappa$.

Indeed, it is easy to see (by computation) that every element $v \in P$ satisfies

$$\kappa \circ T_v = T_v \circ \kappa, \tag{19}$$

where the T_v on the left hand side is defined with respect to the poset P, and the T_v on the right hand side is defined with respect to the poset P^{op}. Now, let $(v_1, v_2, ..., v_m)$ be a linear extension of P. Then, $(v_m, v_{m-1}, ..., v_1)$ is a linear extension of P^{op}, so that Proposition 2.20 (applied to P^{op} and $(v_m, v_{m-1}, ..., v_1)$ instead of P and $(v_1, v_2, ..., v_m)$) yields that $R_{P^{\mathrm{op}}}^{-1} = T_{v_1} \circ T_{v_2} \circ ... \circ T_{v_m} : \mathbb{K}^{\widehat{P^{\mathrm{op}}}} \dashrightarrow \mathbb{K}^{\widehat{P^{\mathrm{op}}}}$. On the other hand, the definition of R_P yields $R_P = T_{v_1} \circ T_{v_2} \circ ... \circ T_{v_m} : \mathbb{K}^{\widehat{P}} \dashrightarrow \mathbb{K}^{\widehat{P}}$. Now, using (19), it is easy to see that

$$\kappa \circ (T_{v_1} \circ T_{v_2} \circ ... \circ T_{v_m}) = (T_{v_1} \circ T_{v_2} \circ ... \circ T_{v_m}) \circ \kappa.$$

Since the $T_{v_1} \circ T_{v_2} \circ \ldots \circ T_{v_m}$ on the left hand side equals R_P, and the $T_{v_1} \circ T_{v_2} \circ \ldots \circ T_{v_m}$ on the right hand side equals $R_{P^{\mathrm{op}}}^{-1}$, this rewrites as $\kappa \circ R_P = R_{P^{\mathrm{op}}}^{-1} \circ \kappa$. Since κ is a birational map, this shows that R_P and $R_{P^{\mathrm{op}}}^{-1}$ are birationally equivalent, so that $\operatorname{ord}(R_P) = \operatorname{ord}\left(R_{P^{\mathrm{op}}}^{-1}\right) = \operatorname{ord}(R_{P^{\mathrm{op}}})$. Since κ commutes with homogenization, we also obtain the birational equivalence of the maps $\overline{R}_P$ and $\overline{R}_{P^{\mathrm{op}}}^{-1}$, whence $\operatorname{ord}\left(\overline{R}_P\right) = \operatorname{ord}\left(\overline{R}_{P^{\mathrm{op}}}^{-1}\right) = \operatorname{ord}\left(\overline{R}_{P^{\mathrm{op}}}\right)$. This proves Proposition 8.4. $\square$

9 Skeletal posets

We will now introduce a class of posets which we call "skeletal posets". Roughly speaking, these are graded posets built up inductively from the empty poset by the operations of disjoint union (but only allowing disjoint union of two n-graded posets for one and the same value of n) and "grafting" on an antichain (generalizing the idea of grafting a tree on a new root). In particular, all graded forests (oriented either away from the roots or towards the roots) will belong to this class of posets, but also various other posets. We begin by defining the notions involved:

Definition 9.1. Let $n \in \mathbb{N}$. Let P and Q be two n-graded posets. We denote by PQ the disjoint union of the posets P and Q. (This disjoint union is denoted by $P + Q$ in [Stan11, §3.2]. Its poset structure is defined in such a way that any element of P and any element of Q are incomparable, while P and Q are subposets of PQ.) Clearly, PQ is again an n-graded poset.

Definition 9.2. Let $n \in \mathbb{N}$. Let P be an n-graded poset. Let k be a positive integer. We denote by $B_k P$ the result of adding k new elements to the poset P, and declaring these k elements to be smaller than each of the elements of P (but incomparable with each other). Clearly, $B_k P$ is an $(n+1)$-graded poset.

Definition 9.3. Let $n \in \mathbb{N}$. Let P be an n-graded poset. Let k be a positive integer. We denote by $B'_k P$ the result of adding k new elements to the poset P, and declaring these k elements to be larger than each of the elements of P (but incomparable with each other). Clearly, $B'_k P$ is an $(n+1)$-graded poset.

If P is an n-graded poset and k is a positive integer, then, in the notations of Stanley ([Stan11, §3.2]), we have $B_k P = A_k \oplus P$ and $B'_k P = P \oplus A_k$, where A_k denotes the k-element antichain.

It is easy to see that $B_k P$ and $B'_k P$ are "symmetric" notions with respect to taking the opposite poset:

Proposition 9.4. Let $n \in \mathbb{N}$. Let P be an n-graded poset. Then, $B'_k P = \left(B_k\left(P^{\mathrm{op}}\right)\right)^{\mathrm{op}}$. (Here, we are using the notation introduced in Definition 8.1.)

We now define the notion of a skeletal poset:

Definition 9.5. We define the class of *skeletal posets* inductively by means of the following axioms:

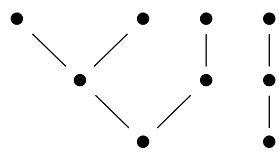
– The empty poset is skeletal.

– If P is an n-graded skeletal poset and k is a positive integer, then the posets $B_k P$ and $B'_k P$ are skeletal.

– If n is a nonnegative integer and P and Q are two n-graded skeletal posets, then the poset PQ is skeletal.

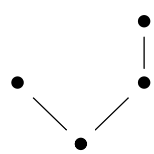
Notice that every skeletal poset is graded. Also, notice that every graded rooted forest (made into a poset by having every node smaller than its children) is a skeletal poset. (Indeed, every graded rooted forest can be constructed from $\varnothing$ using merely the operations $P \mapsto B_1 P$ and $(P, Q) \mapsto PQ$.) Also, every graded rooted arborescence (i.e., the opposite poset of a graded rooted tree) is a skeletal poset (for a similar reason).

Example 9.6. The rooted forest



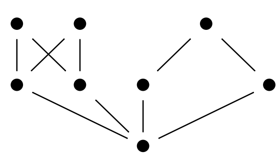
is skeletal, and in fact can be written as $(B_1 ((B_1 (B_2 \varnothing)) (B_1 (B_1 \varnothing)))) (B_1 (B_1 (B_1 \varnothing)))$. (This form of writing is not unique, since $B_2 \varnothing = (B_1 \varnothing) (B_1 \varnothing)$.)

The tree



can be written as $B_1 ((B_1 \varnothing) (B_1 (B_1 \varnothing)))$, but is **not** skeletal because $B_1 \varnothing$ and $B_1 (B_1 \varnothing)$ are not n-graded with one and the same n.

The poset



is neither a tree nor an arborescence, but it has the form $B_1 ((B_2 (B_2 \varnothing)) (B'_1 (B_2 \varnothing)))$ and is skeletal.

Our main result on skeletal posets is the following:

Proposition 9.7. Let P be a skeletal poset. Let $\mathbb{K}$ be a field. Then, $\operatorname{ord}(R_P)$ and $\operatorname{ord}\left(\overline{R}_P\right)$ are finite.

In order to be able to prove this proposition, we first build up some machinery for determining $\operatorname{ord}(R_P)$ and $\operatorname{ord}(\overline{R}_P)$ given such orders in smaller posets. Here is a very basic fact to get started:

Proposition 9.8. Fix $n \in \mathbb{N}$. Let P and Q be two n-graded posets. Let $\mathbb{K}$ be a field. Then, $\operatorname{ord}(R_{PQ}) = \operatorname{lcm}(\operatorname{ord}(R_P), \operatorname{ord}(R_Q))$.

Proof of Proposition 9.8. The proof of this is as easy as it looks: a $\mathbb{K}$-labelling of the disjoint union PQ can be regarded as a pair of a $\mathbb{K}$-labelling of P and a $\mathbb{K}$-labelling of Q (with identical labels at 0 and 1), and the map R (as well as all R_i) acts on these labellings independently. $\square$

The analogue of Proposition 9.8 with all R's replaced by $\overline{R}$'s is false. Instead, $\operatorname{ord}(\overline{R}_{PQ})$ can be computed as follows:[24]

Proposition 9.9. Fix $n \in \mathbb{N}$. Let P and Q be two n-graded posets. Let $\mathbb{K}$ be a field. Then, $\operatorname{ord}(\overline{R}_{PQ}) = \operatorname{lcm}(\operatorname{ord}(R_P), \operatorname{ord}(R_Q))$.

Proof of Proposition 9.9 (sketched). Assume WLOG that $n \neq 0$ (else, everything is obvious). Hence, P and Q are nonempty (being n-graded).

Proposition 7.3 yields $\operatorname{ord}(R_{PQ}) = \operatorname{lcm}(n+1, \operatorname{ord}(\overline{R}_{PQ}))$.

WLOG assume that $\operatorname{ord}(R_P)$ and $\operatorname{ord}(R_Q)$ are finite[25]. Then, Proposition 9.8 shows that $\operatorname{ord}(R_{PQ}) = \operatorname{lcm}(\operatorname{ord}(R_P), \operatorname{ord}(R_Q))$ is finite, so that $\operatorname{ord}(\overline{R}_{PQ})$ is finite (because $\operatorname{ord}(R_{PQ}) = \operatorname{lcm}(n+1, \operatorname{ord}(\overline{R}_{PQ}))$). Let ℓ be $\operatorname{ord}(\overline{R}_{PQ})$. Then, ℓ is finite and satisfies $\overline{R}_{PQ}^{\ell} = \mathrm{id}$. We will show that $n+1 \mid \ell$.

For every $\mathbb{K}$-labelling f of PQ and every $i \in \{0, 1, ..., n\}$, define two elements $\mathbf{w}_i^{(1)}(f)$ and $\mathbf{w}_i^{(2)}(f)$ of $\mathbb{K}$ by

$$\mathbf{w}_i^{(1)}(f) = \sum_{\substack{x \in \widehat{P}_i;\ y \in \widehat{P}_{i+1};\\ y \gtrdot x}} \frac{f(x)}{f(y)} \qquad \text{and} \qquad \mathbf{w}_i^{(2)}(f) = \sum_{\substack{x \in \widehat{Q}_i;\ y \in \widehat{Q}_{i+1};\\ y \gtrdot x}} \frac{f(x)}{f(y)}$$

(where, of course, $\widehat{P}_j$ and $\widehat{Q}_j$ are embedded into $\widehat{PQ}_j$ for every $j \in \{0, 1, ..., n+1\}$ in the obvious way). These elements $\mathbf{w}_i^{(1)}(f)$ and $\mathbf{w}_i^{(2)}(f)$ are defined not for every f, but for "almost every" f in the sense of Zariski topology. We denote the $(n+1)$-tuple

$$\left(\mathbf{w}_0^{(1)}(f) \diagup \mathbf{w}_0^{(2)}(f),\ \mathbf{w}_1^{(1)}(f) \diagup \mathbf{w}_1^{(2)}(f),\ ...,\ \mathbf{w}_n^{(1)}(f) \diagup \mathbf{w}_n^{(2)}(f)\right)$$

as the *comparative w-tuple* of the labelling f. The advantage of comparative w-tuples over usual w-tuples is the following fact: If f and g are two homogeneously equivalent $\mathbb{K}$-labellings of PQ, then

$$(\text{the comparative w-tuple of } f) = (\text{the comparative w-tuple of } g). \qquad (20)$$

[24] The following proposition is, in some sense, uninteresting, as it is a negative result (it merely serves to convince one that $\operatorname{ord}(\overline{R}_{PQ})$ is not lower than what is expected from Propositions 7.3 and 9.8).

[25] Otherwise, $\operatorname{lcm}(\operatorname{ord}(R_P), \operatorname{ord}(R_Q))$ is infinite, whence $\operatorname{ord}(R_{PQ})$ is infinite (by Proposition 9.8), whence $\operatorname{ord}(\overline{R}_{PQ})$ is infinite (because $\operatorname{ord}(R_{PQ}) = \operatorname{lcm}(n+1, \operatorname{ord}(\overline{R}_{PQ}))$), whence Proposition 9.9 is trivial.

(This is easy to check and has no analogue for regular w-tuples.)

It is furthermore easy to see (in analogy to Proposition 4.4) that the map R_{PQ} changes the comparative w-tuple of a $\mathbb{K}$-labelling by shifting it cyclically.

But it is also easy to see (the nonemptiness of P and Q must be used here) that there exists some $f \in \mathbb{K}^{\widehat{PQ}}$ such that the ratios $\mathbf{w}_i^{(1)}(f) \diagup \mathbf{w}_i^{(2)}(f)$ are well-defined and pairwise distinct for all $i \in \{0, 1, ..., n\}$ and such that $R^j f$ is well-defined for every $j \in \{0, 1, ..., \ell\}$. Consider such an f. The ratios $\mathbf{w}_i^{(1)}(f) \diagup \mathbf{w}_i^{(2)}(f)$ are pairwise distinct for all $i \in \{0, 1, ..., n\}$; that is, the comparative w-tuple of f contains no two equal entries.

Since $\overline{R}_{PQ}^{\ell} = \mathrm{id}$, we have $\overline{R}_{PQ}^{\ell}(\pi(f)) = \pi(f)$. The commutativity of the diagram (16) yields $\overline{R}_{PQ}^{\ell} \circ \pi = \pi \circ R_{PQ}^{\ell}$. Now,

$$\pi(f) = \overline{R}_{PQ}^{\ell}(\pi(f)) = \left(\underbrace{\overline{R}_{PQ}^{\ell} \circ \pi}_{= \pi \circ R_{PQ}^{\ell}}\right)(f) = \left(\pi \circ R_{PQ}^{\ell}\right)(f) = \pi\left(R_{PQ}^{\ell} f\right).$$

In other words, the labellings f and $R_{PQ}^{\ell} f$ are homogeneously equivalent. Thus,

$$(\text{the comparative w-tuple of } f) = \left(\text{the comparative w-tuple of } R_{PQ}^{\ell} f\right) \tag{21}$$

(by (20)).

Now, recall that the map R_{PQ} changes the comparative w-tuple of a $\mathbb{K}$-labelling by shifting it cyclically. Hence, for every $k \in \mathbb{N}$, the map R_{PQ}^{k} changes the comparative w-tuple of a $\mathbb{K}$-labelling by shifting it cyclically k times. Applying this to the $\mathbb{K}$-labelling f and to $k = \ell$, we see that the comparative w-tuple of $R_{PQ}^{\ell} f$ is obtained from the comparative w-tuple of f by an ℓ-fold cyclic shift. Due to (21), this rewrites as follows: The comparative w-tuple of f is obtained from the comparative w-tuple of f by an ℓ-fold cyclic shift. In other words, the comparative w-tuple of f is invariant under an ℓ-fold cyclic shift. But since the comparative w-tuple of f consists of $n+1$ pairwise distinct entries, this is impossible unless $n + 1 \mid \ell$. Hence, we must have $n + 1 \mid \ell$.

Now,

$$\mathrm{ord}(R_{PQ}) = \mathrm{lcm}\left(n+1, \mathrm{ord}\left(\overline{R}_{PQ}\right)\right) = \mathrm{ord}\left(\overline{R}_{PQ}\right)$$

(since $n + 1 \mid \ell = \mathrm{ord}\left(\overline{R}_{PQ}\right)$). Hence,

$$\mathrm{ord}\left(\overline{R}_{PQ}\right) = \mathrm{ord}(R_{PQ}) = \mathrm{lcm}(\mathrm{ord}(R_P), \mathrm{ord}(R_Q)).$$

This proves Proposition 9.9. $\square$

Now, let us track the effect of B_k on the order of $\overline{R}$:

Proposition 9.10. Let $n \in \mathbb{N}$. Let P be an n-graded poset. Let $\mathbb{K}$ be a field.

(a) We have $\mathrm{ord}\left(\overline{R}_{B_1 P}\right) = \mathrm{ord}\left(\overline{R}_P\right)$.

(b) For every integer $k > 1$, we have $\mathrm{ord}\left(\overline{R}_{B_k P}\right) = \mathrm{lcm}\left(2, \mathrm{ord}\left(\overline{R}_P\right)\right)$.

Proof of Proposition 9.10 (sketched). We will be proving parts **(a)** and **(b)** together. Let k be a positive integer (this has to be 1 for proving part **(a)**). We need to prove that

$$\operatorname{ord}\left(\overline{R}_{B_kP}\right) = \begin{cases} \operatorname{lcm}\left(2, \operatorname{ord}\left(\overline{R}_P\right)\right), & \text{if } k > 1; \\ \operatorname{ord}\left(\overline{R}_P\right), & \text{if } k = 1 \end{cases}. \tag{22}$$

Proving this clearly will prove both parts **(a)** and **(b)** of Proposition 9.10.

Let us make some conventions:

- For any n-tuple $(\alpha_1, \alpha_2, ..., \alpha_n)$ and any object β, let $\beta \lessdot \alpha$ denote the $(n+1)$-tuple $(\beta, \alpha_1, \alpha_2, ..., \alpha_n)$.
- We are going to identify P with a subposet of B_kP in the obvious way. But of course, the degree map of B_kP restricted to P is not identical with the degree map of P (but rather differs from it by 1), so we will have to distinguish between "degree in P" and "degree in B_kP". We identify the elements 0 and 1 of $\widehat{P}$ with the elements 0 and 1 of $\widehat{B_kP}$, respectively. Thus, $\widehat{P}$ becomes a subposet of $\widehat{B_kP}$. However, it is not generally true that every $u \lessdot v$ in $\widehat{P}$ must satisfy $u \lessdot v$ in $\widehat{B_kP}$.
- We have a rational map $\pi : \mathbb{K}^{\widehat{P}} \dashrightarrow \overline{\mathbb{K}^{\widehat{P}}}$ and a rational map $\pi : \mathbb{K}^{\widehat{B_kP}} \dashrightarrow \overline{\mathbb{K}^{\widehat{B_kP}}}$ denoted by the same letter. This is not problematic, because these two maps can be distinguished by their different domains. We will also use the letter π to denote the rational map $\mathbb{K}^k \dashrightarrow \mathbb{P}\left(\mathbb{K}^k\right)$ obtained from the canonical projection $\mathbb{K}^k \setminus \{0\} \to \mathbb{P}\left(\mathbb{K}^k\right)$ of the nonzero vectors in $\mathbb{K}^k$ onto the projective space.

Now, we recall that the construction of B_kP from P involved adding k new (pairwise incomparable) elements smaller than all existing elements of P to the poset. This operation clearly raises the degree of every element of P by 1 [26], whereas the k newly added elements all obtain degree 1 in B_kP. Formally speaking, this means that $\widehat{B_kP}_i = \widehat{P}_{i-1}$ for every $i \in \{2, 3, ..., n+1\}$, while $\widehat{B_kP}_1$ is a k-element set. Moreover, for any $i \in \{2, 3, ..., n+1\}$, any $u \in \widehat{B_kP}_i = \widehat{P}_{i-1}$ and any $v \in \widehat{B_kP}_{i+1} = \widehat{P}_i$, we have

$$u \lessdot v \text{ in } \widehat{B_kP} \text{ if and only if } u \lessdot v \text{ in } \widehat{P}.$$

(This would not be true if we would allow $i = 1$, $u \in \widehat{P}_0$ and $v \in \widehat{P}_1$.)

We have $\mathbb{K}^{\widehat{B_kP}_i} = \mathbb{K}^{\widehat{P}_{i-1}}$ for every $i \in \{2, 3, ..., n+1\}$ (since $\widehat{B_kP}_i = \widehat{P}_{i-1}$ for every $i \in \{2, 3, ..., n+1\}$), whereas $\mathbb{K}^{\widehat{B_kP}_1} \cong \mathbb{K}^k$ (since $\widehat{B_kP}_1$ is a k-element set). We will

[26] In terms of the Hasse diagram, this can be regarded as the k new elements "bumping up" all existing elements of P by 1 degree.

actually identify $\mathbb{K}^{\widehat{B_kP}_1}$ with $\mathbb{K}^k$. Now,

$$\begin{aligned}
\overline{\mathbb{K}^{\widehat{B_kP}}} &= \prod_{i=1}^{n+1} \mathbb{P}\left(\mathbb{K}^{\widehat{B_kP}_i}\right) = \mathbb{P}\left(\underbrace{\mathbb{K}^{\widehat{B_kP}_1}}_{=\mathbb{K}^k}\right) \times \prod_{i=2}^{n+1} \mathbb{P}\left(\underbrace{\mathbb{K}^{\widehat{B_kP}_i}}_{=\mathbb{K}^{\widehat{P}_{i-1}}}\right) \\
&= \mathbb{P}\left(\mathbb{K}^k\right) \times \prod_{i=2}^{n+1} \mathbb{P}\left(\mathbb{K}^{\widehat{P}_{i-1}}\right) = \mathbb{P}\left(\mathbb{K}^k\right) \times \underbrace{\prod_{i=1}^{n} \mathbb{P}\left(\mathbb{K}^{\widehat{P}_i}\right)}_{=\overline{\mathbb{K}^{\widehat{P}}}} = \mathbb{P}\left(\mathbb{K}^k\right) \times \overline{\mathbb{K}^{\widehat{P}}}. \qquad (23)
\end{aligned}$$

Thus, the elements of $\overline{\mathbb{K}^{\widehat{B_kP}}}$ have the form $\widetilde{p} \curlywedge \widetilde{g}$, where $\widetilde{p} \in \mathbb{P}\left(\mathbb{K}^k\right)$ and $\widetilde{g} \in \overline{\mathbb{K}^{\widehat{P}}}$.

On the other hand, recall that $\widehat{P}$ is a subposet of $\widehat{B_kP}$. More precisely, $\widehat{P}$ is the subposet $\widehat{B_kP} \setminus \widehat{B_kP}_1$ of $\widehat{B_kP}$. Thus, we can define a map $\Phi : \mathbb{K}^k \times \mathbb{K}^{\widehat{P}} \to \mathbb{K}^{\widehat{B_kP}}$ by setting

$$(\Phi(p,g))(v) = \begin{cases} p(v), & \text{if } v \in \widehat{B_kP}_1; \\ g(v), & \text{if } v \notin \widehat{B_kP}_1 \end{cases} \qquad \text{for every } v \in \widehat{B_kP}$$

for every $(p,g) \in \mathbb{K}^k \times \mathbb{K}^{\widehat{P}}$. Here, the term $p(v)$ is to be understood by means of regarding p as an element of $\mathbb{K}^{\widehat{B_kP}_1}$ (since $p \in \mathbb{K}^k = \mathbb{K}^{\widehat{B_kP}_1}$). Clearly, Φ is a bijection. Moreover, it is easy to see that

$$\pi(\Phi(p,g)) = \pi(p) \curlywedge \pi(g) \qquad \text{for all } p \in \mathbb{K}^k \text{ and } g \in \mathbb{K}^{\widehat{P}} \qquad (24)$$

(where the π on the left hand side is the map $\pi : \mathbb{K}^{\widehat{B_kP}} \dashrightarrow \overline{\mathbb{K}^{\widehat{B_kP}}}$, whereas the π in "$\pi(p)$" is the map $\pi : \mathbb{K}^k \dashrightarrow \mathbb{P}\left(\mathbb{K}^k\right)$, and the π in "$\pi(g)$" is the map $\pi : \mathbb{K}^{\widehat{P}} \dashrightarrow \overline{\mathbb{K}^{\widehat{P}}}$).

Now, we claim that every $\widetilde{p} \in \mathbb{P}\left(\mathbb{K}^k\right)$ and $\widetilde{g} \in \overline{\mathbb{K}^{\widehat{P}}}$ satisfy

$$\left(\overline{R_i}\right)_{B_kP}(\widetilde{p} \curlywedge \widetilde{g}) = \widetilde{p} \curlywedge \left(\overline{R_{i-1}}\right)_P(\widetilde{g}) \qquad \text{for all } i \in \{2,3,...,n+1\} \qquad (25)$$

and

$$\left(\overline{R_1}\right)_{B_kP}(\widetilde{p} \curlywedge \widetilde{g}) = \widetilde{p}^{-1} \curlywedge \widetilde{g}. \qquad (26)$$

Proof of (25) and (26): In order to prove (25), it is clearly enough to show that every $p \in \mathbb{K}^k$ and $g \in \mathbb{K}^{\widehat{P}}$ satisfy

$$(R_i)_{B_kP}(p \curlywedge g) \sim p \curlywedge (R_{i-1})_P(g) \qquad \text{for all } i \in \{2,3,...,n+1\}, \qquad (27)$$

where the sign $\sim$ stands for homogeneous equivalence.

It is easy to prove the relation (27) for $i > 2$ (because if $i > 2$, then the elements of $\widehat{B_kP}$ having degrees $i-1$, i and $i+1$ are precisely the elements of $\widehat{P}$ having degrees $i-2$, $i-1$ and i, and therefore toggling the elements of $\widehat{B_kP}_i$ in $p \curlywedge g$ has precisely the same effect as toggling the elements of $\widehat{P}_{i-1}$ in g while leaving p fixed, so that we even get the stronger assertion that $(R_i)_{B_kP}(p \curlywedge g) = p \curlywedge (R_{i-1})_P(g)$). It is not much harder to check that it also holds for $i = 2$ (indeed, for $i = 2$, the only difference between toggling the elements of $\widehat{B_k P}_i$ in $p \lessdot g$ and toggling the elements of $\widehat{P}_{i-1}$ in g while leaving p fixed is a scalar factor which is identical across all elements being toggled in either poset[27]; therefore the results are the same up to homogeneous equivalence).

Finally, (26) is trivial to check (e.g., using Corollary 6.14).

But recall that $\overline{R} = \overline{R_1} \circ \overline{R_2} \circ ... \circ \overline{R_n}$ for any n-graded poset. Hence, $\overline{R}_{B_k P} = \left(\overline{R_1}\right)_{B_k P} \circ \left(\overline{R_2}\right)_{B_k P} \circ \left(\overline{R_3}\right)_{B_k P} \circ ... \circ \left(\overline{R_{n+1}}\right)_{B_k P}$ (because $B_k P$ is an $(n+1)$-graded poset) and $\overline{R}_P = \left(\overline{R_1}\right)_P \circ \left(\overline{R_2}\right)_P \circ ... \circ \left(\overline{R_n}\right)_P$ (because P is an n-graded poset). Because of these equalities, and because of (25) and (26), it is now easy to see that every $\widetilde{p} \in \mathbb{P}\left(\mathbb{K}^k\right)$ and $\widetilde{g} \in \overline{\mathbb{K}^{\widehat{P}}}$ satisfy

$$\overline{R}_{B_k P}\left(\widetilde{p} \lessdot \widetilde{g}\right) = \widetilde{p}^{-1} \lessdot \overline{R}_P\left(\widetilde{g}\right). \tag{28}$$

Furthermore, every $\widetilde{p} \in \mathbb{P}\left(\mathbb{K}^k\right)$ and $\widetilde{g} \in \overline{\mathbb{K}^{\widehat{P}}}$ satisfy

$$\overline{R}^{\ell}_{B_k P}\left(\widetilde{p} \lessdot \widetilde{g}\right) = \widetilde{p}^{(-1)^{\ell}} \lessdot \overline{R}^{\ell}_P\left(\widetilde{g}\right) \qquad \text{for all } \ell \in \mathbb{N}. \tag{29}$$

(This is proven by induction over ℓ, using (28).)

We know that the elements of $\overline{\mathbb{K}^{\widehat{B_k P}}}$ have the form $\widetilde{p} \lessdot \widetilde{g}$, where $\widetilde{p} \in \mathbb{P}\left(\mathbb{K}^k\right)$ and $\widetilde{g} \in \overline{\mathbb{K}^{\widehat{P}}}$. Conversely, every element $\widetilde{p} \lessdot \widetilde{g}$ with $\widetilde{p} \in \mathbb{P}\left(\mathbb{K}^k\right)$ and $\widetilde{g} \in \overline{\mathbb{K}^{\widehat{P}}}$ lies in $\overline{\mathbb{K}^{\widehat{B_k P}}}$. Hence, for

[27]because every $u \in \widehat{B_k P}_1$ and every $v \in \widehat{B_k P}_2$ satisfy $u \lessdot v$

every $\ell \in \mathbb{N}$, we have the following equivalence of assertions:

$$\begin{aligned}
&\left(\text{we have } \overline{R}_{B_k P}^{\ell} = \operatorname{id}\right) \\
\Longleftrightarrow\ &\left(\text{every } \widetilde{p} \in \mathbb{P}\left(\mathbb{K}^k\right) \text{ and } \widetilde{g} \in \overline{\mathbb{K}^{\widehat{P}}} \text{ satisfy } \overline{R}_{B_k P}^{\ell}\left(\widetilde{p} \curlywedge \widetilde{g}\right) = \widetilde{p} \curlywedge \widetilde{g}\right) \\
\Longleftrightarrow\ &\left(\text{every } \widetilde{p} \in \mathbb{P}\left(\mathbb{K}^k\right) \text{ and } \widetilde{g} \in \overline{\mathbb{K}^{\widehat{P}}} \text{ satisfy } \widetilde{p}^{(-1)^{\ell}} \curlywedge \overline{R}_{P}^{\ell}\left(\widetilde{g}\right) = \widetilde{p} \curlywedge \widetilde{g}\right) \\
&\qquad (\text{because of (29)}) \\
\Longleftrightarrow\ &\left(\text{every } \widetilde{p} \in \mathbb{P}\left(\mathbb{K}^k\right) \text{ and } \widetilde{g} \in \overline{\mathbb{K}^{\widehat{P}}} \text{ satisfy } \widetilde{p}^{(-1)^{\ell}} = \widetilde{p} \text{ and } \overline{R}_{P}^{\ell}\left(\widetilde{g}\right) = \widetilde{g}\right) \\
\Longleftrightarrow\ &\left(\underbrace{\text{every } \widetilde{p} \in \mathbb{P}\left(\mathbb{K}^k\right) \text{ satisfies } \widetilde{p}^{(-1)^{\ell}} = \widetilde{p}}_{\text{this is equivalent to } (2\mid\ell \text{ if } k>1)}, \text{ and } \underbrace{\text{every } \widetilde{g} \in \overline{\mathbb{K}^{\widehat{P}}} \text{ satisfies } \overline{R}_{P}^{\ell}\left(\widetilde{g}\right) = \widetilde{g}}_{\text{this is equivalent to } \overline{R}_P^{\ell}=\operatorname{id}}\right) \\
&\qquad \left(\text{since the sets } \mathbb{P}\left(\mathbb{K}^k\right) \text{ and } \overline{\mathbb{K}^{\widehat{P}}} \text{ are nonempty}\right) \\
\Longleftrightarrow\ &\left(\text{we have } (2 \mid \ell \text{ if } k > 1) \text{ and } \underbrace{\overline{R}_{P}^{\ell} = \operatorname{id}}_{\text{this is equivalent to } \operatorname{ord}\left(\overline{R}_P\right)\mid\ell}\right) \\
\Longleftrightarrow\ &\left(\text{we have } (2 \mid \ell \text{ if } k > 1) \text{ and } \operatorname{ord}\left(\overline{R}_P\right) \mid \ell\right) \\
\Longleftrightarrow\ &\begin{cases} \left(\text{we have } 2 \mid \ell \text{ and } \operatorname{ord}\left(\overline{R}_P\right) \mid \ell\right), & \text{if } k > 1; \\ \left(\text{we have } \operatorname{ord}\left(\overline{R}_P\right) \mid \ell\right), & \text{if } k = 1 \end{cases} \\
\Longleftrightarrow\ &\begin{cases} \left(\text{we have } \operatorname{lcm}\left(2, \operatorname{ord}\left(\overline{R}_P\right)\right) \mid \ell\right), & \text{if } k > 1; \\ \left(\text{we have } \operatorname{ord}\left(\overline{R}_P\right) \mid \ell\right), & \text{if } k = 1 \end{cases} \\
\Longleftrightarrow\ &\left(\text{we have } \begin{cases} \operatorname{lcm}\left(2, \operatorname{ord}\left(\overline{R}_P\right)\right), & \text{if } k > 1; \\ \operatorname{ord}\left(\overline{R}_P\right), & \text{if } k = 1 \end{cases} \mid \ell\right).
\end{aligned}$$

Hence, for every $\ell \in \mathbb{N}$, we have the following equivalence of assertions:

$$\begin{aligned}
\left(\text{we have } \operatorname{ord}\left(\overline{R}_{B_k P}\right) \mid \ell\right) \Longleftrightarrow\ &\left(\text{we have } \overline{R}_{B_k P}^{\ell} = \operatorname{id}\right) \\
\Longleftrightarrow\ &\left(\text{we have } \begin{cases} \operatorname{lcm}\left(2, \operatorname{ord}\left(\overline{R}_P\right)\right), & \text{if } k > 1; \\ \operatorname{ord}\left(\overline{R}_P\right), & \text{if } k = 1 \end{cases} \mid \ell\right).
\end{aligned}$$

Consequently, $\operatorname{ord}\left(\overline{R}_{B_k P}\right) = \begin{cases} \operatorname{lcm}\left(2, \operatorname{ord}\left(\overline{R}_P\right)\right), & \text{if } k > 1; \\ \operatorname{ord}\left(\overline{R}_P\right), & \text{if } k = 1 \end{cases}$. This is exactly what (22) claims. Thus, (22) is proven, and with it Proposition 9.10. $\square$

Here is an analogue of Proposition 9.10:

Proposition 9.11. Let $n \in \mathbb{N}$. Let P be an n-graded poset. Let $\mathbb{K}$ be a field.

(a) We have $\operatorname{ord}\left(\overline{R}_{B'_1 P}\right) = \operatorname{ord}\left(\overline{R}_P\right)$.

(b) For every integer $k > 1$, we have $\operatorname{ord}\left(\overline{R}_{B'_k P}\right) = \operatorname{lcm}\left(2, \operatorname{ord}\left(\overline{R}_P\right)\right)$.

The proof of this is very similar (though not exactly identical) to that of Proposition 9.10. Alternatively, it is easy to deduce Proposition 9.11 from Proposition 9.10 using Proposition 8.4 and Proposition 9.4. $\square$

Proposition 9.7 is easily shown by induction using Propositions 9.8, 9.10, 9.11 and 7.3. Moreover, using Propositions 9.8, 9.9, 9.10, 9.11 and 7.3, we can recursively compute (rather than just bound from the above) the orders of R_P and $\overline{R}_P$ for any skeletal poset P without doing any computations in $\mathbb{K}$. (This also shows that the orders of R_P and $\overline{R}_P$ don't depend on the base field $\mathbb{K}$ as long as $\mathbb{K}$ is infinite and P is skeletal.)

In the case of forests and trees we can also use this induction to establish a concrete bound:

Corollary 9.12. Let $n \in \mathbb{N}$. Let P be an n-graded poset. Let $\mathbb{K}$ be a field. Assume that P is a rooted forest (made into a poset by having every node smaller than its children).

(a) Then, $\operatorname{ord}(R_P) \mid \operatorname{lcm}(1, 2, ..., n+1)$.

(b) Moreover, if P is a tree, then $\operatorname{ord}\left(\overline{R}_P\right) \mid \operatorname{lcm}(1, 2, ..., n)$.

Corollary 9.12 is also valid if we replace "every node smaller than its children" by "every node larger than its children", and the proof is exactly analogous.

Proof of Corollary 9.12 (sketched). **(a)** Corollary 9.12 **(a)** can be proven by strong induction over $|P|$. Indeed, if P is an n-graded poset and a rooted forest, then we must be in one of the following three cases:

Case 1: We have $P = \varnothing$.

Case 2: The rooted forest P is a tree.

Case 3: The rooted forest P is a disjoint union of more than one tree.

The validity of Corollary 9.12 is trivial in Case 1, and in Case 3 it follows from the induction hypothesis using Proposition 9.8. In Case 2, we have $P = B_1 Q$ for some rooted forest Q, which is necessarily $(n-1)$-graded; thus, the induction hypothesis (applied to Q instead of P) yields $\operatorname{ord}(R_Q) \mid \operatorname{lcm}(1, 2, ..., (n-1)+1) = \operatorname{lcm}(1, 2, ..., n)$, and we obtain

$$\begin{aligned} \operatorname{ord}\left(\overline{R}_P\right) &= \operatorname{ord}\left(\overline{R}_{B_1 Q}\right) = \operatorname{ord}\left(\overline{R}_Q\right) \qquad \text{(by Proposition 9.10 (a))} \\ &\mid \operatorname{lcm}(1, 2, ..., n) \end{aligned}$$

and

$$\begin{aligned} \operatorname{ord}(R_P) &= \operatorname{lcm}\left(n+1, \underbrace{\operatorname{ord}\left(\overline{R}_P\right)}_{\mid \operatorname{lcm}(1,2,...,n)}\right) \qquad \text{(by Proposition 7.3)} \\ &\mid \operatorname{lcm}(n+1, \operatorname{lcm}(1, 2, ..., n)) = \operatorname{lcm}(1, 2, ..., n+1). \end{aligned}$$

Thus, the induction step is complete in each of the three Cases.

(b) If P is a tree, then we must be in Case 2 of the above case distinction, and thus we have $\operatorname{ord}\left(\overline{R}_P\right) \mid \operatorname{lcm}(1, 2, ..., n)$ as shown above. Corollary 9.12 is therefore proven. $\square$

10 Interlude: Classical rowmotion on skeletal posets

The above results concerning birational rowmotion on skeletal posets suggest the question of what can be said about **classical** rowmotion (on the set of order ideals) on this class of posets. Indeed, while the classical rowmotion map (as opposed to the birational one) has been the object of several studies (e.g., [StWi11] and [CaFl95]), it seems that this rather simple case has never been explicitly covered. Let us therefore go on a tangent to bridge this gap and derive the counterparts of Propositions 9.10 and 9.7 and Corollary 9.12 for classical rowmotion. Nothing of what we do in this Section 10 will be relevant to later sections, so this section can be skipped.

First, we define the maps involved.

Definition 10.1. Let P be a poset.

(a) An *order ideal* of P means a subset S of P such that every $s \in S$ and $p \in P$ with $p \leqslant s$ satisfy $p \in S$.

(b) The set of all order ideals of P will be denoted by $J(P)$.

Here is the definition of (classical) toggles on order ideals (an analogue of Definition 2.6):

Definition 10.2. Let P be a finite poset. Let $v \in P$. Define a map $\mathbf{t}_v : J(P) \to J(P)$ by

$$\mathbf{t}_v(S) = \begin{cases} S \cup \{v\}, \text{ if } v \notin S \text{ and } S \cup \{v\} \in J(P); \\ S \setminus \{v\}, \text{ if } v \in S \text{ and } S \setminus \{v\} \in J(P); \\ S, \text{ otherwise} \end{cases} \qquad \text{for every } S \in J(P).$$

(This is clearly well-defined.) This map $\mathbf{t}_v$ will be called the *classical v-toggle*.

We can rewrite this definition in more "local" terms, by replacing the conditions "$S \cup \{v\} \in J(P)$" and "$S \setminus \{v\} \in J(P)$" by the respectively equivalent conditions "every element $u \in P$ satisfying $u \lessdot v$ lies in S" and "no element $u \in P$ satisfying $u \gtrdot v$ lies in S" (in fact, the equivalence of these conditions is easily seen). Hence, we obtain the following analogue to Proposition 2.9:

Proposition 10.3. Let P be a finite poset. Let $v \in P$. For every $S \in J(P)$, we have:

(a) If w is an element of P such that $w \neq v$, then we have $w \in \mathbf{t}_v(S)$ if and only if $w \in S$.

(b) We have $v \in \mathbf{t}_v(S)$ if and only if

$$\begin{aligned} &(v \in S \text{ and not } (\text{no element } u \in P \text{ satisfying } u \gtrdot v \text{ lies in } S)) \\ \text{or } &(v \notin S \text{ and } (\text{every element } u \in P \text{ satisfying } u \lessdot v \text{ lies in } S)). \end{aligned}$$

While the complicated logical statement in Proposition 10.3 **(b)** can be simplified, the form we have stated it in exhibits its similarity to Proposition 2.9 particularly well. This, in fact, is more than a similarity: If we allow $\mathbb{K}$ to be a semifield rather than a field, we can regard the classical v-toggle $\mathbf{t}_v$ as a restriction of the birational toggle T_v (when $\mathbb{K}$ is chosen appropriately)[28]. Hence, some theorems about birational toggles can be used to derive analogous theorems about classical toggles[29]. We will not use this tactic in the following, because often it will be easier to study the classical v-toggles on their own. However, many of the properties of classical toggles (and classical rowmotion) that we are going to discuss will have proofs that are parallel to the proofs of the analogous results about birational toggles. We will omit these proofs when the analogy is glaring enough.

We have the following easily-verified analogues of Proposition 2.7, Proposition 2.10 and Corollary 2.12:

Proposition 10.4. Let P be a finite poset. Let $v \in P$. Then, the map $\mathbf{t}_v$ is an involution on $J(P)$ (that is, we have $\mathbf{t}_v^2 = \mathrm{id}$).

Proposition 10.5. Let P be a finite poset. Let $v \in P$ and $w \in P$. Then, $\mathbf{t}_v \circ \mathbf{t}_w = \mathbf{t}_w \circ \mathbf{t}_v$, unless we have either $v \lessdot w$ or $w \lessdot v$.

Corollary 10.6. Let P be a finite poset. Let $(v_1, v_2, ..., v_m)$ be a linear extension of P. Then, the map $\mathbf{t}_{v_1} \circ \mathbf{t}_{v_2} \circ ... \circ \mathbf{t}_{v_m} : J(P) \to J(P)$ is well-defined and independent of the choice of the linear extension $(v_1, v_2, ..., v_m)$.

The three results above are observations made on [CaFl95, page 546] (in somewhat different notation).

Two convenient advantages of the classical setup are that we don't have to worry about denominators becoming zero, so our maps are actual maps rather than partial maps, and that we don't have to pass to the poset $\widehat{P}$.

We can now define rowmotion in analogy to Definition 2.13:

[28]Here are the details: Let $\operatorname{Trop}\mathbb{Z}$ be the tropical semiring over $\mathbb{Z}$, that is, the semiring obtained by endowing the set $\mathbb{Z} \cup \{-\infty\}$ with the binary operation $(a, b) \mapsto \max\{a, b\}$ as "addition" and the binary operation $(a, b) \mapsto a + b$ as "multiplication" (where the usual rules for sums involving $-\infty$ apply). Then, $\operatorname{Trop}\mathbb{Z}$ is a semifield, with $(a, b) \mapsto a - b$ serving as "subtraction", with $-\infty$ serving as "zero" and with the integer 0 serving as "one". Now, to every order ideal $S \in J(P)$, we can assign a $(\operatorname{Trop}\mathbb{Z})$-labelling $\operatorname{tlab} S \in (\operatorname{Trop}\mathbb{Z})^{\widehat{P}}$, defined by

$$(\operatorname{tlab} S)(v) = \begin{cases} 1, & \text{if } v \notin S \cup \{0\}; \\ 0, & \text{if } v \in S \cup \{0\} \end{cases}.$$

This yields a map $\operatorname{tlab} : J(P) \to (\operatorname{Trop}\mathbb{Z})^{\widehat{P}}$, obviously injective. This map tlab satisfies $T_v \circ \operatorname{tlab} = \operatorname{tlab} \circ \mathbf{t}_v$ for every $v \in P$. This allows us to regard the classical toggles $\mathbf{t}_v$ as restrictions of the birational toggles T_v, if we consider this map tlab as an inclusion. This reasoning goes back to Einstein and Propp [EiPr13].

[29]For example, we could derive Proposition 10.5 from Proposition 2.10 using this tactic. However, we could not derive (say) Proposition 10.27 from Proposition 9.7 this way, because the order of a restriction of a permutation could be a proper divisor of the order of the permutation.

Definition 10.7. Let P be a finite poset. *Classical rowmotion* (simply called "rowmotion" in existing literature) is defined as the map $\mathbf{t}_{v_1} \circ \mathbf{t}_{v_2} \circ ... \circ \mathbf{t}_{v_m} : J(P) \to J(P)$, where $(v_1, v_2, ..., v_m)$ is a linear extension of P. This map is well-defined (in particular, it does not depend on the linear extension $(v_1, v_2, ..., v_m)$ chosen) because of Corollary 10.6 (and also because of the fact that a linear extension of P exists; this is Theorem 1.4). This map will be denoted by $\mathbf{r}$.

To highlight the similarities between the classical and birational cases, let us state the analogue of Proposition 2.16:

Proposition 10.8. Let P be a finite poset. Let $v \in P$. Let $S \in J(P)$. Then, $v \in \mathbf{r}(S)$ holds if and only if the following two conditions hold:

Condition 1: Every $u \in P$ satisfying $u \lessdot v$ belongs to S.

Condition 2: Either $v \notin S$, or there exists an $u \in \mathbf{r}(S)$ satisfying $u \gtrdot v$. (Recall that the expression "either/or" is meant non-exclusively.)

This proposition is easily seen to be equivalent to the following well-known equivalent description of rowmotion ([CaFl95, Lemma 1], translated into our notation):

Proposition 10.9. Let P be a finite poset. Let $S \in J(P)$. Then, the maximal elements of $\mathbf{r}(S)$ are precisely the minimal elements of $P \setminus S$.

We record the analogue of Proposition 2.19:

Proposition 10.10. Let P be a finite poset. Let S and T be two order ideals of P. Assume that for every $v \in P$, the relation $v \in T$ holds if and only if Conditions 1 and 2 of Proposition 10.8 hold with $\mathbf{r}(S)$ replaced by T. Then, $T = \mathbf{r}(S)$.

In analogy to Proposition 2.20, we have:

Proposition 10.11. Let P be a finite poset. Then, classical rowmotion $\mathbf{r}$ is invertible. Its inverse $\mathbf{r}^{-1}$ is $\mathbf{t}_{v_m} \circ \mathbf{t}_{v_{m-1}} \circ ... \circ \mathbf{t}_{v_1} : J(P) \to J(P)$, where $(v_1, v_2, ..., v_m)$ is a linear extension of P.

We can study graded posets again. In analogy to Corollary 3.6, Definition 3.7, Proposition 3.8 and Proposition 3.9, we have:

Corollary 10.12. Let $n \in \mathbb{N}$. Let P be an n-graded poset. Let $i \in \{1, 2, ..., n\}$. Let $(u_1, u_2, ..., u_k)$ be any list of the elements of $\widehat{P}_i$ with every element of $\widehat{P}_i$ appearing exactly once in the list. (Note that $\widehat{P}_i$ is simply $\{v \in P \mid \deg v = i\}$, because i equals neither 0 nor $n+1$. We are using the notation $\widehat{P}_i$ despite not working with $\widehat{P}$ merely to stress some analogies.) Then, the map $\mathbf{t}_{u_1} \circ \mathbf{t}_{u_2} \circ ... \circ \mathbf{t}_{u_k} : J(P) \to J(P)$ is well-defined and independent of the choice of the list $(u_1, u_2, ..., u_k)$.

Definition 10.13. Let $n \in \mathbb{N}$. Let P be an n-graded poset. Let $i \in \{1, 2, ..., n\}$. Then, let $\mathbf{r}_i$ denote the map $\mathbf{t}_{u_1} \circ \mathbf{t}_{u_2} \circ ... \circ \mathbf{t}_{u_k} : J(P) \to J(P)$, where $(u_1, u_2, ..., u_k)$ is any list of the elements of $\widehat{P}_i$ with every element of $\widehat{P}_i$ appearing exactly once in the list. This map $\mathbf{t}_{u_1} \circ \mathbf{t}_{u_2} \circ ... \circ \mathbf{t}_{u_k}$ is well-defined (in particular, it does not depend on the list $(u_1, u_2, ..., u_k)$) because of Corollary 10.12.

Proposition 10.14. Let $n \in \mathbb{N}$. Let P be an n-graded poset. Then,

$$\mathbf{r} = \mathbf{r}_1 \circ \mathbf{r}_2 \circ ... \circ \mathbf{r}_n.$$

Proposition 10.15. Let $n \in \mathbb{N}$. Let P be an n-graded poset. Let $i \in \{1, 2, ..., n\}$. Then, $\mathbf{r}_i$ is an involution on $J(P)$ (that is, $\mathbf{r}_i^2 = \mathrm{id}$).

A parody of w-tuples can also be defined. The following is analogous to Definition 4.1:

Definition 10.16. Let $n \in \mathbb{N}$. Let P be an n-graded poset. Let $S \in J(P)$. Let $i \in \{0, 1, ..., n\}$. Then, $\mathbf{w}_i(S)$ will denote the integer

$$\begin{cases} 1, & \text{if } P_i \subseteq S \text{ and } P_{i+1} \cap S = \varnothing \\ 0, & \text{otherwise} \end{cases}.$$

Here, we are using the notation P_j for the subset $\deg^{-1}(\{j\})$ of P; this subset is empty if $j = 0$ and also empty if $j = n + 1$.

Analogues of Proposition 4.3 and Proposition 4.4 are easily found:

Proposition 10.17. Let $n \in \mathbb{N}$. Let P be an n-graded poset. Let $i \in \{1, 2, ..., n\}$. Then, every $S \in J(P)$ satisfies

$$\begin{aligned} &(\mathbf{w}_0(\mathbf{r}_i(S)), \mathbf{w}_1(\mathbf{r}_i(S)), ..., \mathbf{w}_n(\mathbf{r}_i(S))) \\ &= (\mathbf{w}_0(S), \mathbf{w}_1(S), ..., \mathbf{w}_{i-2}(S), \mathbf{w}_i(S), \mathbf{w}_{i-1}(S), \mathbf{w}_{i+1}(S), \mathbf{w}_{i+2}(S), ..., \mathbf{w}_n(S)). \end{aligned}$$

Proposition 10.18. Let $n \in \mathbb{N}$. Let P be an n-graded poset. Then, every $S \in J(P)$ satisfies

$$(\mathbf{w}_0(\mathbf{r}(S)), \mathbf{w}_1(\mathbf{r}(S)), ..., \mathbf{w}_n(\mathbf{r}(S))) = (\mathbf{w}_n(S), \mathbf{w}_0(S), \mathbf{w}_1(S), ..., \mathbf{w}_{n-1}(S)).$$

However, the $(n+1)$-tuple $(\mathbf{w}_0(S), \mathbf{w}_1(S), ..., \mathbf{w}_n(S))$ obtained from an order ideal S is not particularly informative. In fact, it is $(0, 0, ..., 0)$ for "most" order ideals; here is what this means precisely:

Definition 10.19. Let $n \in \mathbb{N}$. Let P be an n-graded poset. An order ideal of P is said to be *level* if and only if it has the form $P_1 \cup P_2 \cup ... \cup P_i$ for some $i \in \{0, 1, ..., n\}$.

Easy properties of level order ideals are:

Proposition 10.20. Let $n \in \mathbb{N}$. Let P be an n-graded poset.

(a) There exist precisely $n+1$ level order ideals of P, and those form an orbit under classical rowmotion $\mathbf{r}$. Namely, one has

$$\mathbf{r}\left(P_1 \cup P_2 \cup ... \cup P_i\right) = \begin{cases} P_1 \cup P_2 \cup ... \cup P_{i+1}, & \text{if } i < n; \\ \varnothing, & \text{if } i = n \end{cases}.$$

(b) If $S \in J(P)$, then $(\mathbf{w}_0(S), \mathbf{w}_1(S), ..., \mathbf{w}_n(S)) = (0, 0, ..., 0)$ unless S is level.

Now, we can define an (arguably toylike, but, as we will see, rather useful) analogue of homogeneous equivalence. In somewhat questionable analogy with Definition 6.2, we set:

Definition 10.21. Let $n \in \mathbb{N}$. Let P be an n-graded poset.

Two order ideals S and T of P are said to be *homogeneously equivalent* if and only if either both S and T are level or we have $S = T$. Clearly, being homogeneously equivalent is an equivalence relation. Let $\overline{J(P)}$ denote the set of equivalence classes of elements of $J(P)$ modulo this relation. Let π denote the canonical projection $J(P) \to \overline{J(P)}$. (We distinguish this map π from the map π defined in Definition 6.2 by the fact that they act on different objects.)

The following analogue of Corollary 6.7 is almost trivial:

Corollary 10.22. Let $n \in \mathbb{N}$. Let P be an n-graded poset. If S and T are two homogeneously equivalent order ideals of P, then $\mathbf{r}(S)$ is homogeneously equivalent to $\mathbf{r}(T)$.

(An analogue of Corollary 6.6 exists as well.) We also have the following analogue of Proposition 6.11:

Proposition 10.23. Let $n \in \mathbb{N}$. Let P be an n-graded poset. Let S and T be two order ideals of P such that $(\mathbf{w}_0(S), \mathbf{w}_1(S), ..., \mathbf{w}_n(S)) = (\mathbf{w}_0(T), \mathbf{w}_1(T), ..., \mathbf{w}_n(T))$. Also assume that $\pi(S) = \pi(T)$. Then, $S = T$.

We can furthermore state analogues of Definitions 6.9 and 6.10:

Definition 10.24. Let $n \in \mathbb{N}$. Let P be an n-graded poset. Let $i \in \{1, 2, ..., n\}$. The map $\mathbf{r}_i : J(P) \to J(P)$ descends (through the projection $\pi : J(P) \to \overline{J(P)}$) to a map $\overline{J(P)} \to \overline{J(P)}$. We denote this map $\overline{J(P)} \to \overline{J(P)}$ by $\overline{\mathbf{r}_i}$. Thus, the diagram

$$\begin{tikzcd} J(P) \arrow[r, "\mathbf{r}_i"] \arrow[d, "\pi"'] & J(P) \arrow[d, "\pi"] \\ \overline{J(P)} \arrow[r, "\overline{\mathbf{r}_i}"'] & \overline{J(P)} \end{tikzcd}$$

is commutative.

Definition 10.25. Let $n \in \mathbb{N}$. Let P be an n-graded poset. We define the map $\overline{\mathbf{r}} : \overline{J(P)} \to \overline{J(P)}$ by

$$\overline{\mathbf{r}} = \overline{\mathbf{r}_1} \circ \overline{\mathbf{r}_2} \circ ... \circ \overline{\mathbf{r}_n}.$$

Then, the diagram

$$\begin{tikzcd} J(P) \arrow[r, "\mathbf{r}"] \arrow[d, "\pi"'] & J(P) \arrow[d, "\pi"] \\ \overline{J(P)} \arrow[r, "\overline{\mathbf{r}}"'] & \overline{J(P)} \end{tikzcd} \qquad (30)$$

is commutative. In other words, $\overline{\mathbf{r}}$ is the map $\overline{J(P)} \to \overline{J(P)}$ to which the map $\mathbf{r} : J(P) \to J(P)$ descends (through the projection $\pi : J(P) \to \overline{J(P)}$).

It might seem that the map $\overline{\mathbf{r}}$ is not worth considering, since its cycle structure differs from the cycle structure of $\mathbf{r}$ only in the collapsing of an $(n+1)$-cycle (the one formed by all level order ideals) to a point. However, triviality in combinatorics does not preclude usefulness, and we will employ the "projective" version $\overline{\mathbf{r}}$ of classical rowmotion as a stirrup in determining the order of classical rowmotion $\mathbf{r}$ on skeletal posets.

We have the following simple relation between the orders of $\mathbf{r}$ and $\overline{\mathbf{r}}$:

Proposition 10.26. Let $n \in \mathbb{N}$. Let P be an n-graded poset. Then, $\operatorname{ord} \mathbf{r} = \operatorname{lcm}(n+1, \operatorname{ord} \overline{\mathbf{r}})$.

Notice that our convention to define $\operatorname{lcm}(n+1, \infty)$ as ∞ is irrelevant for Proposition 10.26: In fact, in the situation of Proposition 10.26, both $\operatorname{ord} \mathbf{r}$ and $\operatorname{ord} \overline{\mathbf{r}}$ are clearly (finite) positive integers[30].

Proof of Proposition 10.26 (sketched). We know that $\mathbf{r}$ is an invertible map $J(P) \to J(P)$, thus a permutation of the finite set $J(P)$. Hence, $\operatorname{ord} \mathbf{r}$ is the lcm of the lengths of the cycles of this permutation $\mathbf{r}$. Similarly, $\operatorname{ord} \overline{\mathbf{r}}$ is the lcm of the lengths of the cycles of the permutation $\overline{\mathbf{r}}$ of the finite set $\overline{J(P)}$.

[30] Indeed, the maps $\mathbf{r}$ and $\overline{\mathbf{r}}$ are permutations of finite sets (namely, of the sets $J(P)$ and $\overline{J(P)}$) and thus have finite orders.

Let Z_1, Z_2, ..., Z_k be the cycles of the permutation $\mathbf{r}$ of $J(P)$. We assume WLOG that Z_1 is the cycle consisting of the $n+1$ level order ideals (because we know that they form a cycle). Thus, $|Z_1| = n+1$. Since $\operatorname{ord}\mathbf{r}$ is the lcm of the lengths of the cycles of the permutation $\mathbf{r}$, we have $\operatorname{ord}\mathbf{r} = \operatorname{lcm}(|Z_1|, |Z_2|, ..., |Z_k|)$.

Now, let us recall that $\overline{J(P)}$ is the quotient of $J(P)$ modulo homogeneous equivalence. But homogeneous equivalence merely identifies the $n+1$ level order ideals, while all other elements of $J(P)$ are still pairwise non-equivalent. Hence, the cycles of the permutation $\overline{\mathbf{r}}$ of $\overline{J(P)}$ are $\pi(Z_1)$, $\pi(Z_2)$, ..., $\pi(Z_k)$, and while $\pi(Z_2)$, $\pi(Z_3)$, ..., $\pi(Z_k)$ are isomorphic to Z_2, Z_3, ..., Z_k, respectively, the first cycle $\pi(Z_1)$ (being the projection of the cycle of the level order ideals) now has length 1. Now, $\operatorname{ord}\overline{\mathbf{r}}$ is the lcm of the lengths of the cycles of the permutation $\overline{\mathbf{r}}$ of the finite set $\overline{J(P)}$. Since these cycles are $\pi(Z_1)$, $\pi(Z_2)$, ..., $\pi(Z_k)$, this yields

$$\begin{aligned}
\operatorname{ord}\overline{\mathbf{r}} &= \operatorname{lcm}(|\pi(Z_1)|, |\pi(Z_2)|, ..., |\pi(Z_k)|) = \operatorname{lcm}\left(\underbrace{|\pi(Z_1)|}_{=1}, |\pi(Z_2)|, |\pi(Z_3)|, ..., |\pi(Z_k)|\right) \\
&= \operatorname{lcm}(1, |\pi(Z_2)|, |\pi(Z_3)|, ..., |\pi(Z_k)|) = \operatorname{lcm}(|\pi(Z_2)|, |\pi(Z_3)|, ..., |\pi(Z_k)|) \\
&= \operatorname{lcm}(|Z_2|, |Z_3|, ..., |Z_k|) \qquad \left(\begin{array}{c}\text{since } \pi(Z_2), \ \pi(Z_3), \ ..., \ \pi(Z_k) \text{ are} \\ \text{isomorphic to } Z_2, Z_3, ..., Z_k, \text{ respectively}\end{array}\right).
\end{aligned}$$

Now,

$$\begin{aligned}
\operatorname{ord}\mathbf{r} &= \operatorname{lcm}(|Z_1|, |Z_2|, ..., |Z_k|) = \operatorname{lcm}\left(\underbrace{|Z_1|}_{=n+1}, \underbrace{\operatorname{lcm}(|Z_2|, |Z_3|, ..., |Z_k|)}_{=\operatorname{ord}\overline{\mathbf{r}}}\right) \\
&= \operatorname{lcm}(n+1, \operatorname{ord}\overline{\mathbf{r}}).
\end{aligned}$$

This proves Proposition 10.26. □

Our goal is to make a statement about the order of classical rowmotion on skeletal posets. Of course, the finiteness of these orders is obvious in this case, because $J(P)$ is a finite set. However, we can make stronger claims:

Proposition 10.27. Let P be a skeletal poset. Let $\mathbb{K}$ be a field. Then, $\operatorname{ord}(R_P) = \operatorname{ord}(\mathbf{r}_P)$ and $\operatorname{ord}\left(\overline{R}_P\right) = \operatorname{ord}(\overline{\mathbf{r}}_P)$.

Here, we are using the following convention:

Definition 10.28. Let P be a finite poset. We denote the maps $\mathbf{r}$ and $\overline{\mathbf{r}}$ by $\mathbf{r}_P$ and $\overline{\mathbf{r}}_P$, respectively, so as to make their dependence on P explicit.

Proposition 10.27 yields (in particular) that the order of classical rowmotion coincides with the order of birational rowmotion (whatever the base field) for skeletal posets. This was conjectured by James Propp (private communication) for the case of P a tree. We are going to prove Proposition 10.27 by exhibiting further analogies between classical and birational rowmotion. First of all, the following proposition is just as trivial as its birational counterpart (Proposition 9.8):

Proposition 10.29. Let $n \in \mathbb{N}$. Let P and Q be two n-graded posets. Then, $\operatorname{ord}(\mathbf{r}_{PQ}) = \operatorname{lcm}(\operatorname{ord}(\mathbf{r}_P), \operatorname{ord}(\mathbf{r}_Q))$.

We can show a simple counterpart of this proposition for $\operatorname{ord}(\overline{\mathbf{r}}_{PQ})$ (but still with $\operatorname{ord}(\mathbf{r}_P)$ and $\operatorname{ord}(\mathbf{r}_Q)$ on the right hand side!):

Proposition 10.30. Let $n \in \mathbb{N}$. Let P and Q be two n-graded posets. Then, $\operatorname{ord}(\overline{\mathbf{r}}_{PQ}) = \operatorname{lcm}(\operatorname{ord}(\mathbf{r}_P), \operatorname{ord}(\mathbf{r}_Q))$.

Proof of Proposition 10.30 (sketched). WLOG, assume that $n \neq 0$ (else, the statement is trivial). Hence, P and Q are nonempty.

Consider the order ideal P of PQ. Then, one can easily see (by induction) that every $i \in \{0, 1, ..., n+1\}$ satisfies

$$\mathbf{r}_{PQ}^{i}(P) = \left(\begin{cases} P_1 \cup P_2 \cup ... \cup P_{i-1}, & \text{if } i > 0; \\ P, & \text{if } i = 0 \end{cases} \right) \cup \left(\begin{cases} Q_1 \cup Q_2 \cup ... \cup Q_i, & \text{if } i \leqslant n; \\ \varnothing, & \text{if } i = n+1 \end{cases} \right).$$

From this, it follows that the smallest positive integer k satisfying $\mathbf{r}_{PQ}^{k}(P) = P$ is $n+1$. Since P is not level, this does not change under applying π; that is, the smallest positive integer k satisfying $\overline{\mathbf{r}}_{PQ}^{k}(P) = P$ is still $n+1$. Hence, $n+1 \mid \operatorname{ord}(\overline{\mathbf{r}}_{PQ})$. But Proposition 10.26 (applied to PQ instead of P) yields

$$\operatorname{ord}(\mathbf{r}_{PQ}) = \operatorname{lcm}(n+1, \operatorname{ord}(\overline{\mathbf{r}}_{PQ})) = \operatorname{ord}(\overline{\mathbf{r}}_{PQ})$$

(since $n+1 \mid \operatorname{ord}(\overline{\mathbf{r}}_{PQ})$), so that

$$\operatorname{ord}(\overline{\mathbf{r}}_{PQ}) = \operatorname{ord}(\mathbf{r}_{PQ}) = \operatorname{lcm}(\operatorname{ord}(\mathbf{r}_P), \operatorname{ord}(\mathbf{r}_Q))$$

(by Proposition 10.29). This proves Proposition 10.30. $\square$

More interesting is the analogue of Proposition 9.10:

Proposition 10.31. Let $n \in \mathbb{N}$. Let P be an n-graded poset.

(a) We have $\operatorname{ord}(\overline{\mathbf{r}}_{B_1 P}) = \operatorname{ord}(\overline{\mathbf{r}}_P)$.

(b) For every integer $k > 1$, we have $\operatorname{ord}(\overline{\mathbf{r}}_{B_k P}) = \operatorname{lcm}(2, \operatorname{ord}(\overline{\mathbf{r}}_P))$.

Proof of Proposition 10.31 (sketched). We will be proving parts **(a)** and **(b)** together. Let k be a positive integer (this has to be 1 for proving part **(a)**). We need to prove that

$$\operatorname{ord}(\overline{\mathbf{r}}_{B_k P}) = \begin{cases} \operatorname{lcm}(2, \operatorname{ord}(\overline{\mathbf{r}}_P)), & \text{if } k > 1; \\ \operatorname{ord}(\overline{\mathbf{r}}_P), & \text{if } k = 1 \end{cases}. \tag{31}$$

Proving this clearly will prove both parts **(a)** and **(b)** of Proposition 10.31.

Notice that $B_k P$ is an $(n+1)$-graded poset. For every $\ell \in \{1, 2, ..., n+1\}$, let $(B_k P)_\ell$ be the subset $\deg^{-1}(\{\ell\})$ of $B_k P$. Thus, $(B_k P)_\ell = \{v \in B_k P \mid \deg v = \ell\}$. In particular,

$(B_kP)_1$ is the set of all minimal elements of B_kP, so that $(B_kP)_1$ is an antichain of size k (by the construction of B_kP). We also have $w < v$ for every $w \in (B_kP)_1$ and $v \in P$.

For every finite poset Q, the map $\overline{\mathbf{r}}_Q$ is an invertible map $\overline{J(Q)} \to \overline{J(Q)}$, that is, a permutation of the finite set $\overline{J(Q)}$. Hence, the order $\operatorname{ord}(\overline{\mathbf{r}}_Q)$ is the lcm of the lengths of the orbits of this map $\overline{\mathbf{r}}_Q$. We are going to compare the orbits of the maps $\overline{\mathbf{r}}_{B_kP}$ and $\overline{\mathbf{r}}_P$.

Define a map $\phi : J(P) \to J(B_kP)$ by

$$\phi(S) = (B_kP)_1 \cup S \qquad \text{for every } S \in J(P).$$

It is easy to see that this map ϕ is well-defined (that is, $(B_kP)_1 \cup S$ is an order ideal of B_kP for every $S \in J(P)$), and that it sends level order ideals of P to level order ideals of B_kP. Hence, it preserves homogeneous equivalence, so that it induces a map $\overline{J(P)} \to \overline{J(B_kP)}$. Denote this map $\overline{J(P)} \to \overline{J(B_kP)}$ by $\overline{\phi}$. Thus, $\overline{\phi} \circ \pi = \pi \circ \phi$.

It is moreover easy to see that $\overline{\mathbf{r}}_{B_kP} \circ \overline{\phi} = \overline{\phi} \circ \overline{\mathbf{r}}_P$ [31]. Hence, the subset $\overline{\phi}\left(\overline{J(P)}\right)$ is closed under application of the map $\overline{\mathbf{r}}_{B_kP}$.

The map $\overline{\phi}$ also is injective (this is very easy to see again, since the only order ideals of P which are mapped to level order ideals by ϕ are themselves level). Thus, $\operatorname{ord}\left(\overline{\mathbf{r}}_{B_kP} \mid_{\overline{\phi}\left(\overline{J(P)}\right)}\right) = \operatorname{ord}(\overline{\mathbf{r}}_P)$ (because the injectivity of $\overline{\phi}$ allows us to identify $\overline{J(P)}$ with $\overline{\phi}\left(\overline{J(P)}\right)$ along the map $\overline{\phi}$, and then the equality $\overline{\mathbf{r}}_{B_kP} \circ \overline{\phi} = \overline{\phi} \circ \overline{\mathbf{r}}_P$ rewrites as $\overline{\mathbf{r}}_{B_kP} \mid_{\overline{\phi}\left(\overline{J(P)}\right)} = \overline{\mathbf{r}}_P$, so that $\operatorname{ord}\left(\overline{\mathbf{r}}_{B_kP} \mid_{\overline{\phi}\left(\overline{J(P)}\right)}\right) = \operatorname{ord}(\overline{\mathbf{r}}_P)$).

Let H be the set of all nonempty proper subsets of $(B_kP)_1$. It is clear that $H \subseteq J(B_kP)$. Notice that $H = \varnothing$ if $k = 1$. Every $T \in H$ satisfies

$$\mathbf{r}_{B_kP}(T) = (B_kP)_1 \setminus T$$

(this is easy to see from any definition of classical rowmotion, or from Proposition 10.8). Hence, the set H is closed under application of the map $\mathbf{r}_{B_kP}$, and this map $\mathbf{r}_{B_kP}$ maps every element of H to its complement in $(B_kP)_1$. In particular, this shows that $\operatorname{ord}(\mathbf{r}_{B_kP} \mid_H) = \begin{cases} 2, & \text{if } k > 1; \\ 1, & \text{if } k = 1 \end{cases}$.

We now use the map π to identify the set H with its projection $\pi(H)$ under π (this is allowed because π is injective on H). This identification entails $\overline{\mathbf{r}}_{B_kP} \mid_H = \mathbf{r}_{B_kP} \mid_H$. In particular, the set H is closed under application of the map $\overline{\mathbf{r}}_{B_kP}$.

[31] *Proof.* In order to prove this, it is enough to show that for every $S \in J(P)$, the order ideals $(\mathbf{r}_{B_kP} \circ \phi)(S)$ and $(\phi \circ \mathbf{r}_P)(S)$ are homogeneously equivalent. This is clear in the case when S is level (because both $(\mathbf{r}_{B_kP} \circ \phi)(S)$ and $(\phi \circ \mathbf{r}_P)(S)$ are level in this case), so let us WLOG assume that S is not level. Then, we can actually show that $(\mathbf{r}_{B_kP} \circ \phi)(S)$ and $(\phi \circ \mathbf{r}_P)(S)$ are identical. Indeed, it is easy to see that:

- for every $T \in J(P)$ and every $v \in P$, we have $(\mathbf{t}_v \circ \phi)(T) = (\phi \circ \mathbf{t}_v)(T)$;
- for every nonempty $T \in J(P)$ and every $w \in (B_kP)_1$, we have $(\mathbf{t}_w \circ \phi)(T) = \phi(T)$.

Using these facts, and the definition of classical rowmotion as a composition of classical toggle maps $\mathbf{t}_v$, we can then easily see that $(\mathbf{r}_{B_kP} \circ \phi)(S) = (\phi \circ \mathbf{r}_P)(S)$. This completes the proof of $\overline{\mathbf{r}}_{B_kP} \circ \overline{\phi} = \overline{\phi} \circ \overline{\mathbf{r}}_P$.

But it is easy to see that $J(B_kP)$ is the union of the two subsets H and $\phi(J(P))$ (because every order ideal of B_kP either contains the whole $(B_kP)_1$, or it does not, in which case it cannot contain **any** element of degree > 1). Hence, $\overline{J(B_kP)}$ is the union of the two subsets $\pi(H) = H$ and $\pi(\phi(J(P))) = \overline{\phi}\left(\overline{J(P)}\right)$. Moreover, these two subsets are disjoint and each of them is closed under application of the map $\overline{\mathbf{r}}_{B_kP}$. Hence,

$$\begin{aligned}
\operatorname{ord}(\overline{\mathbf{r}}_{B_kP}) &= \operatorname{lcm}\left(\operatorname{ord}\left(\underbrace{\overline{\mathbf{r}}_{B_kP}\mid_H}_{=\mathbf{r}_{B_kP}\mid_H}\right), \operatorname{ord}\left(\overline{\mathbf{r}}_{B_kP}\mid_{\overline{\phi}\left(\overline{J(P)}\right)}\right)\right) \\
&= \operatorname{lcm}\left(\underbrace{\operatorname{ord}(\mathbf{r}_{B_kP}\mid_H)}_{=\begin{cases} 2, & \text{if } k > 1; \\ 1, & \text{if } k = 1 \end{cases}}, \underbrace{\operatorname{ord}\left(\overline{\mathbf{r}}_{B_kP}\mid_{\overline{\phi}\left(\overline{J(P)}\right)}\right)}_{=\operatorname{ord}(\overline{\mathbf{r}}_P)}\right) \\
&= \operatorname{lcm}\left(\begin{cases} 2, & \text{if } k > 1; \\ 1, & \text{if } k = 1 \end{cases}, \operatorname{ord}(\overline{\mathbf{r}}_P)\right) \\
&= \begin{cases} \operatorname{lcm}(2, \operatorname{ord}(\overline{\mathbf{r}}_P)), & \text{if } k > 1; \\ \operatorname{ord}(\overline{\mathbf{r}}_P), & \text{if } k = 1 \end{cases}.
\end{aligned}$$

This proves (31). Thus, the proof of Proposition 10.31 is complete. □

We can also formulate an analogue of Proposition 9.11:

Proposition 10.32. Let $n \in \mathbb{N}$. Let P be an n-graded poset.

(a) We have $\operatorname{ord}\left(\overline{\mathbf{r}}_{B'_1P}\right) = \operatorname{ord}(\overline{\mathbf{r}}_P)$.

(b) For every integer $k > 1$, we have $\operatorname{ord}\left(\overline{\mathbf{r}}_{B'_kP}\right) = \operatorname{lcm}(2, \operatorname{ord}(\overline{\mathbf{r}}_P))$.

The proof of this is fairly similar to that of Proposition 10.31. □

We can now prove Proposition 10.27:

Proof of Proposition 10.27 (sketched). For any skeletal poset T, we can compute $\operatorname{ord}(R_T)$ and $\operatorname{ord}\left(\overline{R}_T\right)$ inductively using Proposition 9.8, Proposition 9.9, Proposition 9.10 and Proposition 9.11 (and the fact that $\operatorname{ord}(R_\varnothing) = 1$ and $\operatorname{ord}\left(\overline{R}_\varnothing\right) = 1$). More precisely:

- If T is the empty poset $\varnothing$, then $\operatorname{ord}(R_T) = \operatorname{ord}(R_\varnothing) = 1$ and $\operatorname{ord}\left(\overline{R}_T\right) = \operatorname{ord}\left(\overline{R}_\varnothing\right) = 1$.

- If T has the form B_kP for some n-graded skeletal poset P and some positive integer k, then Proposition 9.10 yields

$$\operatorname{ord}\left(\overline{R}_T\right) = \operatorname{ord}\left(\overline{R}_{B_kP}\right) = \begin{cases} \operatorname{lcm}\left(2, \operatorname{ord}\left(\overline{R}_P\right)\right), & \text{if } k > 1; \\ \operatorname{ord}\left(\overline{R}_P\right), & \text{if } k = 1 \end{cases},$$

  and Proposition 7.3 yields $\operatorname{ord}(R_T) = \operatorname{lcm}\left(n+1, \operatorname{ord}\left(\overline{R}_T\right)\right)$.

- Analogously one can compute $\operatorname{ord}(R_T)$ and $\operatorname{ord}\left(\overline{R}_T\right)$ if T has the form $B'_k P$.
- If T has the form PQ for two WLOG nonempty n-graded skeletal posets P and Q, then Proposition 9.8 yields $\operatorname{ord}(R_{PQ}) = \operatorname{lcm}(\operatorname{ord}(R_P), \operatorname{ord}(R_Q))$, and Proposition 9.9 yields $\operatorname{ord}\left(\overline{R}_{PQ}\right) = \operatorname{lcm}(\operatorname{ord}(R_P), \operatorname{ord}(R_Q))$.

This gives an algorithm for inductively computing $\operatorname{ord}(R_T)$ and $\operatorname{ord}\left(\overline{R}_T\right)$ for a skeletal poset T. Using Proposition 10.29, Proposition 10.30, Proposition 10.31 and Proposition 10.32 (and the fact that $\operatorname{ord}(\mathbf{r}_\varnothing) = 1$ and $\operatorname{ord}(\overline{\mathbf{r}}_\varnothing) = 1$) instead, we could similarly obtain an algorithm for inductively computing $\operatorname{ord}(\mathbf{r}_T)$ and $\operatorname{ord}(\overline{\mathbf{r}}_T)$ for a skeletal poset T. And these two algorithms are **the same**, because of the direct analogy between the propositions that are used in the first algorithm and those used in the second one. Therefore, $\operatorname{ord}(R_P) = \operatorname{ord}(\mathbf{r}_P)$ and $\operatorname{ord}\left(\overline{R}_P\right) = \operatorname{ord}(\overline{\mathbf{r}}_P)$. This proves Proposition 10.27. $\square$

Proposition 10.27 does not generalize to arbitrary graded posets. Counterexamples to such a generalization can be found in Section 20.

Finally, in analogy to Corollary 9.12, we can now show:

Corollary 10.33. Let $n \in \mathbb{N}$. Let P be an n-graded poset. Assume that P is a rooted forest (made into a poset by having every node smaller than its children).

(a) Then, $\operatorname{ord}(\mathbf{r}_P) \mid \operatorname{lcm}(1, 2, ..., n+1)$.

(b) Moreover, if P is a tree, then $\operatorname{ord}(\overline{\mathbf{r}}_P) \mid \operatorname{lcm}(1, 2, ..., n)$.

Corollary 10.33 is also valid if we replace "every node smaller than its children" by "every node larger than its children", and the proof is exactly analogous.

Let us notice that the algorithm described in the proof of Proposition 10.27 can be turned into an explicit formula (not just an upper bound as in Corollary 10.33), whose inductive proof we leave to the reader:

Proposition 10.34. Let $n \in \mathbb{N}$. Let P be an n-graded poset. Assume that P is a rooted forest (made into a poset by having every node smaller than its children). Notice that $\left|\widehat{P}_i\right| \leqslant \left|\widehat{P}_{i+1}\right|$ for every $i \in \{0, 1, ..., n-1\}$ (where $\widehat{P}_i$ and $\widehat{P}_{i+1}$ are defined as in Definition 3.4). Then,

$$\operatorname{ord}(\overline{\mathbf{r}}_P) = \operatorname{lcm}\left\{n - i \mid i \in \{0, 1, ..., n-1\}; \left|\widehat{P}_i\right| < \left|\widehat{P}_{i+1}\right|\right\}.$$

(Of course, $\operatorname{ord}(\mathbf{r}_P)$ can now be computed by $\operatorname{ord}(\mathbf{r}_P) = \operatorname{lcm}(n+1, \operatorname{ord}(\overline{\mathbf{r}}_P))$.)

The same property therefore holds for birational rowmotion R_P and its homogeneous version $\overline{R}_P$.

11 The rectangle: statements of the results

Definition 11.1. Let p and q be two positive integers. The $p \times q$*-rectangle* will denote the poset $\{1, 2, ..., p\} \times \{1, 2, ..., q\}$ with order defined as follows: For two elements (i, k) and (i', k') of $\{1, 2, ..., p\} \times \{1, 2, ..., q\}$, we set $(i, k) \leqslant (i', k')$ if and only if $(i \leqslant i'$ and $k \leqslant k')$.

Example 11.2. Here is the Hasse diagram of the 2×3-rectangle:

$$\begin{tikzcd}[column sep=small]
 & & (2,3) \arrow[dl, dash] \arrow[dr, dash] & \\
 & (2,2) \arrow[dl, dash] \arrow[dr, dash] & & (1,3) \arrow[dl, dash] \\
(2,1) \arrow[dr, dash] & & (1,2) \arrow[dl, dash] & \\
 & (1,1) & &
\end{tikzcd} .$$

Remark 11.3. Let p and q be positive integers. The $p \times q$-rectangle is denoted by $[p] \times [q]$ in the papers [StWi11], [EiPr13], [PrRo13] and [PrRo14].

Remark 11.4. Let p and q be two positive integers. Let $\operatorname{Rect}(p, q)$ denote the $p \times q$-rectangle.

(a) The $p \times q$-rectangle is a $(p + q - 1)$-graded poset, with $\deg((i, k)) = i + k - 1$ for all $(i, k) \in \operatorname{Rect}(p, q)$.

(b) Let (i, k) and (i', k') be two elements of $\operatorname{Rect}(p, q)$. Then, $(i, k) \lessdot (i', k')$ if and only if either $(i' = i$ and $k' = k + 1)$ or $(k' = k$ and $i' = i + 1)$.

We are going to use Remark 11.4 without explicit mention.

The following theorem was conjectured by James Propp and the second author:

Theorem 11.5. Let $\operatorname{Rect}(p, q)$ denote the $p \times q$-rectangle. Let $\mathbb{K}$ be a field. Then, $\operatorname{ord}\left(R_{\operatorname{Rect}(p,q)}\right) = p + q$.

This is a birational analogue (and, using the reasoning of [EiPr13], generalization) of the classical fact (appearing in [StWi11, Theorem 3.1] and [Flaa93, Theorem 2]) that $\operatorname{ord}\left(\mathbf{r}_{\operatorname{Rect}(p,q)}\right) = p + q$ (using the notations of Definition 10.7 and Definition 10.28).

Notice that Proposition 7.3 yields that $p + q \mid \operatorname{ord}\left(R_{\operatorname{Rect}(p,q)}\right)$, so all that needs to be proven in order to verify Theorem 11.5 is showing that $R_{\operatorname{Rect}(p,q)}^{p+q} = \operatorname{id}$.

Notice also that in the case when $p \leqslant 2$ and $q \leqslant 2$, Theorem 11.5 follows rather easily from Propositions 9.10 **(a)**, 9.11 **(a)** and 7.3 (because $\operatorname{Rect}(p, q)$ is a skeletal poset in this case), but this approach does not generalize to any interesting cases.

Remark 11.6. Theorem 11.5 generalizes a well-known property of promotion on semistandard Young tableaux of rectangular shape, albeit not in an obvious way. Here are some details (which a reader unacquainted with Young tableaux can freely skip):

Let N be a nonnegative integer, and let λ be a partition. Let $\mathrm{SSYT}_N \lambda$ denote the set of all semistandard Young tableaux of shape λ whose entries are all $\leqslant N$. One can define a map $\mathrm{Pro} : \mathrm{SSYT}_N \lambda \to \mathrm{SSYT}_N \lambda$ called *jeu-de-taquin promotion* (or *Schützenberger promotion*, or simply *promotion* when no ambiguities can arise); see [Russ13, §5.1] for a precise definition. (The definition in [Rhoa10, §2] is different – it defines the *inverse* of this map. Conventions differ.) This map has some interesting properties already for arbitrary λ, but the most interesting situation is that of λ being a rectangular partition (i.e., a partition all of whose nonzero parts are equal). In this situation, a folklore theorem states that $\mathrm{Pro}^N = \mathrm{id}$. (The particular case of this theorem when Pro is applied only to **standard** Young tableaux is well-known – see, e.g., [Haiman92, Theorem 4.4] –, but the only proof of the general theorem that we were able to find in literature is Rhoades's – [Rhoa10, Corollary 5.6] –, which makes use of Kazhdan-Lusztig theory.)

Theorem 11.5 can be used to give an alternative proof of this $\mathrm{Pro}^N = \mathrm{id}$ theorem. See a future version of [EiPr13] (or, for the time being, [EiPr14, §2, pp. 4–5]) for how this works.

Note that [Russ13, §5.1], [Rhoa10, §2] and [EiPr13] give three different definitions of promotion. The definitions in [Russ13, §5.1] and in [EiPr13] are equivalent, while the definition in [Rhoa10, §2] defines the inverse of the map defined in the other two sources. Unfortunately, we were unable to find the proofs of these facts in existing literature; they are claimed in [KiBe95, Propositions 2.5 and 2.6], and can be proven using the concept of tableau switching [Leeu01, Definition 2.2.1].

Besides Theorem 11.5, we can also state some kind of symmetry property of birational rowmotion on the $p \times q$-rectangle (referred to as the "pairing property" in [EiPr13]), which was also conjectured by Propp and the second author:

Theorem 11.7. Let $\mathrm{Rect}\,(p,q)$ denote the $p \times q$-rectangle. Let $\mathbb{K}$ be a field. Let $f \in \mathbb{K}^{\widehat{\mathrm{Rect}(p,q)}}$. Assume that $R_{\mathrm{Rect}(p,q)}^{\ell} f$ is well-defined for every $\ell \in \{0, 1, ..., i+k-1\}$. Let $(i,k) \in \mathrm{Rect}\,(p,q)$. Then,

$$f\left(\left(p+1-i, q+1-k\right)\right) = \frac{f\left(0\right) f\left(1\right)}{\left(R_{\mathrm{Rect}(p,q)}^{i+k-1} f\right)\left(\left(i,k\right)\right)}.$$

This Theorem generalizes the "reciprocity phenomenon" observed on the 2×2-rectangle in Example 2.15.

Remark 11.8. While Theorem 11.5 only makes a statement about $R_{\operatorname{Rect}(p,q)}$, it can be used (in combination with Proposition 9.10 and others) to derive upper bounds on the orders of R_P and $\overline{R}_P$ for some other posets P. Here is an example: Let $\mathbb{K}$ be a field. For the duration of this remark, let us denote the poset $\operatorname{Rect}(2,3) \setminus \{(1,1), (2,3)\}$ by N. (The Hasse diagram of this poset has the rather simple form
$$\begin{tikzcd}[column sep=small] & (2,2) \arrow[dl, no head] \arrow[dr, no head] & & (1,3) \arrow[dl, no head] \\ (2,1) & & (1,2) & \end{tikzcd},$$
which explains why we have chosen to call it N here.) Then, $\operatorname{ord}(R_N) \mid 15$ and $\operatorname{ord}\left(\overline{R}_N\right) \mid 5$. This can be proven as follows: We have $\operatorname{Rect}(2,3) \cong B_1(B_1'N)$ and therefore

$$\begin{aligned}\operatorname{ord}\left(\overline{R}_{\operatorname{Rect}(2,3)}\right) &= \operatorname{ord}\left(\overline{R}_{B_1(B_1'N)}\right) = \operatorname{ord}\left(\overline{R}_{B_1'N}\right) \qquad \text{(by Proposition 9.10 } \textbf{(a)}\text{)} \\ &= \operatorname{ord}\left(\overline{R}_N\right) \qquad \text{(by Proposition 9.11 } \textbf{(a)}\text{)},\end{aligned}$$

so that

$$\begin{aligned}\operatorname{ord}\left(\overline{R}_N\right) &= \operatorname{ord}\left(\overline{R}_{\operatorname{Rect}(2,3)}\right) \\ &\mid \operatorname{lcm}\left(4+1, \operatorname{ord}\left(\overline{R}_{\operatorname{Rect}(2,3)}\right)\right) = \operatorname{ord}\left(R_{\operatorname{Rect}(2,3)}\right) \\ &\qquad \text{(by Proposition 7.3, since } \operatorname{Rect}(2,3) \text{ is 4-graded)} \\ &= 2+3 \qquad \text{(by Theorem 11.5)} \\ &= 5\end{aligned}$$

and thus

$$\begin{aligned}\operatorname{ord}(R_N) &= \operatorname{lcm}\left(\underbrace{2+1}_{=3}, \underbrace{\operatorname{ord}\left(\overline{R}_N\right)}_{\mid 5}\right) \qquad \text{(by Proposition 7.3, since } N \text{ is 2-graded)} \\ &\mid \operatorname{lcm}(3,5) = 15.\end{aligned}$$

It can actually be shown that $\operatorname{ord}(R_N) = 15$ and $\operatorname{ord}\left(\overline{R}_N\right) = 5$ by direct computation.

In the same vein it can be shown that $\operatorname{ord}\left(\overline{R}_{\operatorname{Rect}(p,q)\setminus\{(1,1),(p,q)\}}\right) \mid p+q$ and $\operatorname{ord}\left(R_{\operatorname{Rect}(p,q)\setminus\{(1,1),(p,q)\}}\right) \mid \operatorname{lcm}(p+q-2, p+q)$ for any integers $p > 1$ and $q > 1$. This doesn't, however, generalize to arbitrary posets obtained by removing some ranks from $\operatorname{Rect}(p,q)$ (indeed, sometimes birational rowmotion doesn't even have finite order on such posets, cf. Section 20).

12 Reduced labellings

The proof that we give for Theorem 11.5 and Theorem 11.7 is largely inspired by the proof of Zamolodchikov's conjecture in case AA given by Volkov in [Volk06][32]. This is not very surprising because the orbit of a $\mathbb{K}$-labelling under birational rowmotion appears superficially similar to a solution of a Y-system of type AA. Yet we do not see a way to derive Theorem 11.5 from Zamolodchikov's conjecture or vice versa. (It should be noticed that Zamolodchikov's Y-system has an obvious "reducibility property", namely consisting of two decoupled subsystems – a property at least not obviously satisfied in the case of birational rowmotion.)

The first step towards our proof of Theorem 11.5 is to restrict attention to so-called *reduced labellings*. Let us define these first:

Definition 12.1. Let $\operatorname{Rect}(p,q)$ denote the $p \times q$-rectangle. Let $\mathbb{K}$ be a field. A labelling $f \in \mathbb{K}^{\widehat{\operatorname{Rect}(p,q)}}$ is said to be *reduced* if $f(0) = f(1) = 1$. The set of all reduced labellings $f \in \mathbb{K}^{\widehat{\operatorname{Rect}(p,q)}}$ will be identified with $\mathbb{K}^{\operatorname{Rect}(p,q)}$ in the obvious way.

Note that fixing the values of $f(0)$ and $f(1)$ like this makes f "less generic", but still the operator $R_{\operatorname{Rect}(p,q)}$ restricts to a rational map from the variety of all reduced labellings $f \in \mathbb{K}^{\widehat{\operatorname{Rect}(p,q)}}$ to itself. (This is because the operator $R_{\operatorname{Rect}(p,q)}$ does not change the values at 0 and 1, and does not degenerate from setting $f(0) = f(1) = 1$.)

Reduced labellings are not much less general than arbitrary labellings: In fact, every zero-free $\mathbb{K}$-labelling f of a graded poset P is homogeneously equivalent to a reduced labelling. Thus, many results can be proven for all labellings by means of proving them for reduced labellings first, and then extending them to general labellings by means of homogeneous equivalence.[33] We will use this tactic in our proof of Theorem 11.5. Here is how this works:

Proposition 12.2. Let $\operatorname{Rect}(p,q)$ denote the $p \times q$-rectangle. Let $\mathbb{K}$ be a field. Assume that almost every (in the Zariski sense) reduced labelling $f \in \mathbb{K}^{\widehat{\operatorname{Rect}(p,q)}}$ satisfies $R^{p+q}_{\operatorname{Rect}(p,q)} f = f$. Then, $\operatorname{ord}\left(R_{\operatorname{Rect}(p,q)}\right) = p + q$.

Proof of Proposition 12.2 (sketched). Let $g \in \mathbb{K}^{\widehat{\operatorname{Rect}(p,q)}}$ be any $\mathbb{K}$-labelling of $\operatorname{Rect}(p,q)$ which is sufficiently generic for $R^{p+q}_{\operatorname{Rect}(p,q)} g$ to be well-defined.

We use the notation of Definition 5.2. Recall that $\operatorname{Rect}(p,q)$ is a $(p+q-1)$-graded poset. We can easily find a $(p+q+1)$-tuple $(a_0, a_1, ..., a_{p+q}) \in (\mathbb{K}^\times)^{p+q+1}$ such that $(a_0, a_1, ..., a_{p+q}) \flat g$ is a reduced $\mathbb{K}$-labelling (in fact, set $a_0 = \dfrac{1}{g(0)}$ and $a_{p+q} = \dfrac{1}{g(1)}$, and

[32] "Case AA" refers to the Cartesian product of the Dynkin diagrams of two type-A root systems. This, of course, is a rectangle, just as in our Theorem 11.5.

[33] A slightly different way to reduce the case of a general labelling to that of a reduced one is taken in [EiPr13, §4].

choose all other a_i arbitrarily). Corollary 5.7 (applied to $p+q-1$, $\operatorname{Rect}(p,q)$ and g instead of n, P and f) then yields

$$R^{p+q}_{\operatorname{Rect}(p,q)} \left((a_0, a_1, ..., a_{p+q}) \flat g \right) = (a_0, a_1, ..., a_{p+q}) \flat \left(R^{p+q}_{\operatorname{Rect}(p,q)} g \right). \tag{32}$$

We have assumed that almost every (in the Zariski sense) reduced labelling $f \in \mathbb{K}^{\widehat{\operatorname{Rect}(p,q)}}$ satisfies $R^{p+q}_{\operatorname{Rect}(p,q)} f = f$. Thus, every reduced labelling $f \in \mathbb{K}^{\widehat{\operatorname{Rect}(p,q)}}$ for which $R^{p+q}_{\operatorname{Rect}(p,q)} f$ is well-defined satisfies $R^{p+q}_{\operatorname{Rect}(p,q)} f = f$ (because $R^{p+q}_{\operatorname{Rect}(p,q)} f = f$ can be written as an equality between rational functions in the labels of f, and thus it must hold everywhere if it holds on a Zariski-dense open subset). Applying this to $f = (a_0, a_1, ..., a_{p+q}) \flat g$, we obtain that $R^{p+q}_{\operatorname{Rect}(p,q)} \left((a_0, a_1, ..., a_{p+q}) \flat g \right) = (a_0, a_1, ..., a_{p+q}) \flat g$. Thus,

$$\begin{aligned} (a_0, a_1, ..., a_{p+q}) \flat g &= R^{p+q}_{\operatorname{Rect}(p,q)} \left((a_0, a_1, ..., a_{p+q}) \flat g \right) \\ &= (a_0, a_1, ..., a_{p+q}) \flat \left(R^{p+q}_{\operatorname{Rect}(p,q)} g \right) \qquad \text{(by (32))}. \end{aligned} \tag{33}$$

We can cancel the "$(a_0, a_1, ..., a_{p+q}) \flat$" from both sides of this equality (because all the a_i are nonzero), and thus obtain $g = R^{p+q}_{\operatorname{Rect}(p,q)} g$.

Now, forget that we fixed g. We thus have proven that $g = R^{p+q}_{\operatorname{Rect}(p,q)} g$ holds for every $\mathbb{K}$-labelling $g \in \mathbb{K}^{\widehat{\operatorname{Rect}(p,q)}}$ of $\operatorname{Rect}(p,q)$ which is sufficiently generic for $R^{p+q}_{\operatorname{Rect}(p,q)} g$ to be well-defined. In other words, $R^{p+q}_{\operatorname{Rect}(p,q)} = \operatorname{id}$ as partial maps. Hence, $\operatorname{ord}\left(R_{\operatorname{Rect}(p,q)}\right) \mid p+q$.

On the other hand, Proposition 7.3 (applied to $P = \operatorname{Rect}(p,q)$ and $n = p+q-1$) yields $\operatorname{ord}\left(R_{\operatorname{Rect}(p,q)}\right) = \operatorname{lcm}\left((p+q-1)+1, \operatorname{ord}\left(\overline{R}_{\operatorname{Rect}(p,q)}\right)\right)$. Hence, $\operatorname{ord}\left(R_{\operatorname{Rect}(p,q)}\right)$ is divisible by $(p+q-1)+1 = p+q$. Combined with $\operatorname{ord}\left(R_{\operatorname{Rect}(p,q)}\right) \mid p+q$, this yields $\operatorname{ord}\left(R_{\operatorname{Rect}(p,q)}\right) = p+q$. This proves Proposition 12.2. $\square$

Let us also formulate the particular case of Theorem 11.7 for reduced labellings:

Theorem 12.3. Let $\operatorname{Rect}(p,q)$ denote the $p \times q$-rectangle. Let $\mathbb{K}$ be a field. Let $f \in \mathbb{K}^{\widehat{\operatorname{Rect}(p,q)}}$ be reduced. Assume that $R^{\ell}_{\operatorname{Rect}(p,q)} f$ is well-defined for every $\ell \in \{0, 1, ..., i+k-1\}$. Let $(i,k) \in \operatorname{Rect}(p,q)$. Then,

$$f\left((p+1-i, q+1-k)\right) = \frac{1}{\left(R^{i+k-1}_{\operatorname{Rect}(p,q)} f\right)\left((i,k)\right)}.$$

We will prove this before we prove the general form (Theorem 11.7), and in fact we are going to derive Theorem 11.7 from its particular case, Theorem 12.3. We are not going to encumber this section with the derivation; its details can be found in Section 16.

13 The Grassmannian parametrization: statements

In this section, we are going to introduce the main actor in our proof of Theorem 11.5: an assignment of a reduced $\mathbb{K}$-labelling of $\operatorname{Rect}(p,q)$, denoted $\operatorname{Grasp}_j A$, to any integer

j and almost any matrix $A \in \mathbb{K}^{p\times(p+q)}$ (Definition 13.9). This assignment will give us a family of $\mathbb{K}$-labellings of $\operatorname{Rect}(p,q)$ which is large enough to cover almost all reduced $\mathbb{K}$-labellings of $\operatorname{Rect}(p,q)$ (this is formalized in Proposition 13.14), while at the same time the construction of this assignment makes it easy to track the behavior of the $\mathbb{K}$-labellings in this family through multiple iterations of birational rowmotion. Indeed, we will see that birational rowmotion has a very simple effect on the reduced $\mathbb{K}$-labelling $\operatorname{Grasp}_j A$ (Proposition 13.13).

Definition 13.1. Let $\mathbb{K}$ be a commutative ring. Let $A \in \mathbb{K}^{u\times v}$ be a $u\times v$-matrix for some nonnegative integers u and v. (This means, at least in this paper, a matrix with u rows and v columns.)

(a) For every $i \in \{1,2,...,v\}$, let A_i denote the i-th column of A.

(b) Moreover, we extend this definition to all $i \in \mathbb{Z}$ as follows: For every $i \in \mathbb{Z}$, let

$$A_i = (-1)^{(u-1)(i-i')/v} \cdot A_{i'},$$

where i' is the element of $\{1,2,...,v\}$ which is congruent to i modulo v. (Thus, $A_{v+i} = (-1)^{u-1} A_i$ for every $i \in \mathbb{Z}$. Consequently, the sequence $(A_i)_{i\in\mathbb{Z}}$ is periodic with period dividing $2v$, and if u is odd, the period also divides v.)

(c) For any four integers a, b, c and d satisfying $a \leqslant b$ and $c \leqslant d$, we let $A[a:b \mid c:d]$ be the matrix whose columns (from left to right) are A_a, A_{a+1}, ..., A_{b-1}, A_c, A_{c+1}, ..., A_{d-1}. (This matrix has $b-a+d-c$ columns.)[34] When $b-a+d-c=u$, this matrix $A[a:b \mid c:d]$ is a square matrix (with u rows and u columns), and thus has a determinant $\det(A[a:b \mid c:d])$.

(d) We extend the definition of $\det(A[a:b \mid c:d])$ to encompass the case when $b=a-1$ or $d=c-1$, by setting $\det(A[a:b \mid c:d])=0$ in this case (although the matrix $A[a:b \mid c:d]$ itself is not defined in this case).

The reader should be warned that, for $\det(A[a:b \mid c:d])$ to be defined, we need $b-a+d-c=u$ (not just $b-a+d-c \equiv u \bmod v$, despite the apparent periodicity in the construction of the matrix A.)

Example 13.2. If A is the 2×3-matrix $\begin{pmatrix} 3 & 5 & 7 \\ 4 & 1 & 9 \end{pmatrix}$, then Definition 13.1 **(b)** yields (for instance) $A_5 = (-1)^{(2-1)(5-2)/3} \cdot A_2 = -A_2 = -\begin{pmatrix} 5 \\ 1 \end{pmatrix} = \begin{pmatrix} -5 \\ -1 \end{pmatrix}$ and $A_{-4} = (-1)^{(2-1)((-4)-2)/3} \cdot A_2 = A_2 = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$.

[34] Some remarks on this matrix $A[a:b \mid c:d]$ are appropriate at this point.

1. We notice that we allow the case $a=b$. In this case, obviously, the columns of the matrix $A[a:b \mid c:d]$ are A_c, A_{c+1}, ..., A_{d-1}, so the precise value of $a=b$ does not matter. Similarly, the case $c=d$ is allowed.

2. The matrix $A[a:b \mid c:d]$ is not always a submatrix of A. Its columns are columns of A multiplied with 1 or -1; they can appear several times and need not appear in the same order as they appear in A.

If A is the 3×2-matrix $\begin{pmatrix} 1 & 2 \\ 3 & 2 \\ -5 & 4 \end{pmatrix}$, then Definition 13.1 **(b)** yields (for instance) $A_0 = (-1)^{(3-1)(0-2)/2} \cdot A_2 = A_2 = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}$.

Remark 13.3. Some parts of Definition 13.1 might look accidental and haphazard; here are some motivations and aide-memoires:

The choice of sign in Definition 13.1 **(b)** is not only the "right" one for what we are going to do below, but also naturally appears in [Post06, Remark 3.3]. It guarantees, among other things, that if $A \in \mathbb{R}^{u \times v}$ is totally nonnegative, then the matrix having columns A_{1+i}, A_{2+i}, ..., A_{v+i} is totally nonnegative for every $i \in \mathbb{Z}$.

The notation $A[a : b \mid c : d]$ in Definition 13.1 **(c)** borrows from Python's notation $[x : y]$ for taking indices from the interval $\{x, x+1, ..., y-1\}$.

The convention to define $\det(A[a : b \mid c : d])$ as 0 in Definition 13.1 **(d)** can be motivated using exterior algebra as follows: If we identify $\wedge^u(\mathbb{K}^u)$ with $\mathbb{K}$ by equating with $1 \in \mathbb{K}$ the wedge product $e_1 \wedge e_2 \wedge ... \wedge e_u$ of the standard basis vectors, then $\det(A[a : b \mid c : d]) = A_a \wedge A_{a+1} \wedge ... \wedge A_{b-1} \wedge A_c \wedge A_{c+1} \wedge ... \wedge A_{d-1}$; this belongs to the product of $\wedge^{b-a}(\mathbb{K}^u)$ with $\wedge^{d-c}(\mathbb{K}^u)$ in $\wedge^u(\mathbb{K}^u)$. If $b = a-1$, then this product is 0 (since $\wedge^{b-a}(\mathbb{K}^u) = \wedge^{-1}(\mathbb{K}^u) = 0$), so $\det(A[a : b \mid c : d])$ has to be 0 in this case.

Before we go any further, we make several straightforward observations about the notations we have just introduced.

Proposition 13.4. Let $\mathbb{K}$ be a field. Let $A \in \mathbb{K}^{u \times v}$ be a $u \times v$-matrix for some nonnegative integers u and v. Let a, b, c and d be four integers satisfying $a \leqslant b$ and $c \leqslant d$ and $b - a + d - c = u$. Assume that some element of the interval $\{a, a+1, ..., b-1\}$ is congruent to some element of the interval $\{c, c+1, ..., d-1\}$ modulo v. Then, $\det(A[a : b \mid c : d]) = 0$.

Proof of Proposition 13.4. The assumption yields that the matrix $A[a : b \mid c : d]$ has two columns which are proportional to each other by a factor of ± 1. Hence, this matrix has determinant 0. $\square$

Proposition 13.5. Let $\mathbb{K}$ be a field. Let $A \in \mathbb{K}^{u \times v}$ be a $u \times v$-matrix for some nonnegative integers u and v. Let a, b, c and d be four integers satisfying $a \leqslant b$ and $c \leqslant d$ and $b - a + d - c = u$. Then,

$$\det(A[a : b \mid c : d]) = (-1)^{(b-a)(d-c)} \det(A[c : d \mid a : b]).$$

Proof of Proposition 13.5. This follows from the fact that permuting the columns of a matrix multiplies its determinant by the sign of the corresponding permutation. $\square$

Proposition 13.6. Let $\mathbb{K}$ be a field. Let $A \in \mathbb{K}^{u \times v}$ be a $u \times v$-matrix for some nonnegative integers u and v. Let a, b_1, b_2 and c be four integers satisfying $a \leqslant b_1 \leqslant c$ and $a \leqslant b_2 \leqslant c$. Then,

$$A[a : b_1 \mid b_1 : c] = A[a : b_2 \mid b_2 : c].$$

Proof of Proposition 13.6. Both matrices $A[a : b_1 \mid b_1 : c]$ and $A[a : b_2 \mid b_2 : c]$ are simply the matrix with columns A_a, A_{a+1}, ..., A_{c-1}. □

Proposition 13.7. Let $\mathbb{K}$ be a field. Let $A \in \mathbb{K}^{u \times v}$ be a $u \times v$-matrix for some nonnegative integers u and v. Let c and d be two integers satisfying $c \leqslant d$. Then:

(a) Any integers a_1 and a_2 satisfy

$$A[a_1 : a_1 \mid c : d] = A[a_2 : a_2 \mid c : d].$$

(b) Any integers a_1 and a_2 satisfy

$$A[c : d \mid a_1 : a_1] = A[c : d \mid a_2 : a_2].$$

(c) If a and b are integers satisfying $c \leqslant b \leqslant d$, then

$$A[c : b \mid b : d] = A[c : d \mid a : a].$$

Proof of Proposition 13.7. All six matrices in the above equalities are simply the matrix with columns A_c, A_{c+1}, ..., A_{d-1}. □

Proposition 13.8. Let $\mathbb{K}$ be a field. Let $A \in \mathbb{K}^{u \times v}$ be a $u \times v$-matrix for some nonnegative integers u and v. Let a, b, c and d be four integers satisfying $a \leqslant b$ and $c \leqslant d$ and $b - a + d - c = u$.

(a) We have

$$\det(A[v + a : v + b \mid v + c : v + d]) = \det(A[a : b \mid c : d]).$$

(b) We have

$$\det(A[a : b \mid v + c : v + d]) = (-1)^{(u-1)(d-c)} \det(A[a : b \mid c : d]).$$

(c) We have

$$\det(A[a : b \mid v + c : v + d]) = \det(A[c : d \mid a : b]).$$

Proof of Proposition 13.8 (sketched). Nothing about this is anything more than trivial. Part **(a)** and **(b)** follow from the fact that $A_{v+i} = (-1)^{u-1} A_i$ for every $i \in \mathbb{Z}$ (which is owed to Definition 13.1 **(b)**) and the multilinearity of the determinant. The proof of part **(c)** additionally uses Proposition 13.5 and a careful sign computation (notice that $(-1)^{(d-c-1)(d-c)} = 1$ because $(d-c-1)(d-c)$ is even, no matter what the parities of c and d are). All details can be easily filled in by the reader. $\square$

Definition 13.9. Let $\mathbb{K}$ be a field. Let p and q be two positive integers. Let $A \in \mathbb{K}^{p\times(p+q)}$. Let $j \in \mathbb{Z}$.

(a) We define a map $\operatorname{Grasp}_j A \in \mathbb{K}^{\operatorname{Rect}(p,q)}$ by

$$\left(\operatorname{Grasp}_j A\right)((i,k)) = \frac{\det\left(A\left[j+1 : j+i \mid j+i+k-1 : j+p+k\right]\right)}{\det\left(A\left[j : j+i \mid j+i+k : j+p+k\right]\right)} \qquad \text{for every } (i,k) \in \operatorname{Rect}(p,q) = \{1,2,...,p\} \times \{1,2,...,q\} \tag{34}$$

(this is well-defined when the matrix A is sufficiently generic (in the sense of Zariski topology), since the matrix $A\left[j : j+i \mid j+i+k : j+p+k\right]$ is obtained by picking p distinct columns out of A, some possibly multiplied with $(-1)^{u-1}$). This map $\operatorname{Grasp}_j A$ will be considered as a reduced $\mathbb{K}$-labelling of $\operatorname{Rect}(p,q)$ (since we are identifying the set of all reduced labellings $f \in \mathbb{K}^{\widehat{\operatorname{Rect}(p,q)}}$ with $\mathbb{K}^{\operatorname{Rect}(p,q)}$).

(b) It will be handy to extend the map $\operatorname{Grasp}_j A$ to a slightly larger domain by blindly following (34) (and using Definition 13.1 **(d)**), accepting the fact that outside $\{1,2,...,p\} \times \{1,2,...,q\}$ its values can be "infinity":

$$\begin{aligned}
\left(\operatorname{Grasp}_j A\right)((0,k)) &= 0 && \text{for all } k \in \{1,2,...,q\}; \\
\left(\operatorname{Grasp}_j A\right)((p+1,k)) &= \infty && \text{for all } k \in \{1,2,...,q\}; \\
\left(\operatorname{Grasp}_j A\right)((i,0)) &= 0 && \text{for all } i \in \{1,2,...,p\}; \\
\left(\operatorname{Grasp}_j A\right)((i,q+1)) &= \infty && \text{for all } i \in \{1,2,...,p\}.
\end{aligned}$$

(We treat ∞ as a symbol with the properties $\frac{1}{0} = \infty$ and $\frac{1}{\infty} = 0$.)

The notation "Grasp" harkens back to "Grassmannian parametrization", as we will later parametrize (generic) reduced labellings on $\operatorname{Rect}(p,q)$ by matrices via this map Grasp_0. The reason for the word "Grassmannian" is that, while we have defined Grasp_j as a rational map from the matrix space $\mathbb{K}^{p\times(p+q)}$, it actually is not defined outside of the Zariski-dense open subset $\mathbb{K}^{p\times(p+q)}_{\mathrm{rk}=p}$ of $\mathbb{K}^{p\times(p+q)}$ formed by all matrices whose rank is p, and on that subset $\mathbb{K}^{p\times(p+q)}_{\mathrm{rk}=p}$ it factors through the quotient of $\mathbb{K}^{p\times(p+q)}_{\mathrm{rk}=p}$ by the left multiplication action of $\operatorname{GL}_p \mathbb{K}$ (because it is easy to see that $\operatorname{Grasp}_j A$ is invariant under row transformations of A); this quotient is a well-known avatar of the Grassmannian.

The formula (34) is inspired by the Y_{ijk} of Volkov's [Volk06]; similar expressions (in a different context) also appear in [Kiri00, Theorem 4.21].

Example 13.10. If $p = 2$, $q = 2$ and $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix}$, then

$$(\operatorname{Grasp}_0 A)((1,1)) = \frac{\det(A[1:1 \mid 1:3])}{\det(A[0:1 \mid 2:3])} = \frac{\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}}{\det\begin{pmatrix} -a_{14} & a_{12} \\ -a_{24} & a_{22} \end{pmatrix}} = \frac{a_{11}a_{22} - a_{12}a_{21}}{a_{12}a_{24} - a_{14}a_{22}}$$

and

$$(\operatorname{Grasp}_1 A)((1,2)) = \frac{\det(A[2:2 \mid 3:5])}{\det(A[1:2 \mid 4:5])} = \frac{\det\begin{pmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{pmatrix}}{\det\begin{pmatrix} a_{11} & a_{14} \\ a_{21} & a_{24} \end{pmatrix}} = \frac{a_{13}a_{24} - a_{14}a_{23}}{a_{11}a_{24} - a_{14}a_{21}}.$$

We will see more examples of values of $\operatorname{Grasp}_0 A$ in Example 15.1.

Proposition 13.11. Let $\mathbb{K}$ be a field. Let p and q be two positive integers. Let $A \in \mathbb{K}^{p \times (p+q)}$ be a matrix. Then, $\operatorname{Grasp}_j A = \operatorname{Grasp}_{p+q+j} A$ for every $j \in \mathbb{Z}$ (provided that A is sufficiently generic in the sense of Zariski topology for $\operatorname{Grasp}_j A$ to be well-defined).

Proof of Proposition 13.11 (sketched). We need to show that

$$\left(\operatorname{Grasp}_j A\right)((i,k)) = \left(\operatorname{Grasp}_{p+q+j} A\right)((i,k))$$

for every $(i,k) \in \{1,2,...,p\} \times \{1,2,...,q\}$. But we have

$$\begin{aligned} &A[p+q+j : p+q+j+i \mid p+q+j+i+k : p+q+j+p+k] \\ &= A[j : j+i \mid j+i+k : j+p+k] \end{aligned}$$

(by Proposition 13.8 **(a)**, applied to $u = p$, $v = p+q$, $a = j$, $b = j+i$, $c = j+i+k$ and $d = j+p+k$) and

$$\begin{aligned} &A[p+q+j+1 : p+q+j+i \mid p+q+j+i+k-1 : p+q+j+p+k] \\ &= A[j+1 : j+i \mid j+i+k-1 : j+p+k] \end{aligned}$$

(by Proposition 13.8 **(a)**, applied to $u = p$, $v = p+q$, $a = j+1$, $b = j+i$, $c = j+i+k-1$ and $d = j+p+k$). Using these equalities, we immediately obtain $\left(\operatorname{Grasp}_j A\right)((i,k)) = \left(\operatorname{Grasp}_{p+q+j} A\right)((i,k))$ from the definition of $\operatorname{Grasp}_j A$. Proposition 13.11 is proven. $\square$

Proposition 13.12. Let $\mathbb{K}$ be a field. Let p and q be two positive integers. Let $A \in \mathbb{K}^{p \times (p+q)}$ be a matrix. Let $(i,k) \in \operatorname{Rect}(p,q)$ and $j \in \mathbb{Z}$. Then,

$$\left(\operatorname{Grasp}_j A\right)((i,k)) = \frac{1}{\left(\operatorname{Grasp}_{j+i+k-1} A\right)((p+1-i, q+1-k))}$$

(provided that A is sufficiently generic in the sense of Zariski topology for $\left(\operatorname{Grasp}_j A\right)((i,k))$ and $\left(\operatorname{Grasp}_{j+i+k-1} A\right)((p+1-i, q+1-k))$ to be well-defined).

Proof. The proof of Proposition 13.12 is completely straightforward: one merely needs to expand the definitions of $\left(\operatorname{Grasp}_j A\right)((i,k))$ and $\left(\operatorname{Grasp}_{j+i+k-1} A\right)((p+1-i, q+1-k))$ and to apply Proposition 13.8 **(c)** twice. $\square$

Now, let us state the two facts which will combine to a proof of Theorem 11.5:

Proposition 13.13. Let $\mathbb{K}$ be a field. Let p and q be two positive integers. Let $A \in \mathbb{K}^{p\times(p+q)}$ be a matrix. Let $j \in \mathbb{Z}$. Then,

$$\operatorname{Grasp}_j A = R_{\operatorname{Rect}(p,q)}\left(\operatorname{Grasp}_{j+1} A\right)$$

(provided that A is sufficiently generic in the sense of Zariski topology for the two sides of this equality to be well-defined).

Proposition 13.14. Let $\mathbb{K}$ be a field. Let p and q be two positive integers. For almost every (in the Zariski sense) $f \in \mathbb{K}^{\operatorname{Rect}(p,q)}$, there exists a matrix $A \in \mathbb{K}^{p\times(p+q)}$ satisfying $f = \operatorname{Grasp}_0 A$.

Once these propositions are proven, Theorems 11.5, 12.3 and 11.7 will be rather easy to obtain. We delay the details of this until Section 16.

14 The Plücker-Ptolemy relation

This section is devoted to proving Proposition 13.13. Before we proceed to the proof, we will need some fundamental identities concerning determinants of matrices. Our main tool is the following fact, which we call the *Plücker-Ptolemy relation*:

Theorem 14.1. Let $\mathbb{K}$ be a field. Let $A \in \mathbb{K}^{u\times v}$ be a $u\times v$-matrix for some nonnegative integers u and v. Let a, b, c and d be four integers satisfying $a \leqslant b+1$ and $c \leqslant d+1$ and $b-a+d-c = u-2$. Then,

$$\begin{aligned}
&\det\left(A\left[a-1:b \mid c:d+1\right]\right)\cdot\det\left(A\left[a:b+1 \mid c-1:d\right]\right)\\
&+\det\left(A\left[a:b \mid c-1:d+1\right]\right)\cdot\det\left(A\left[a-1:b+1 \mid c:d\right]\right)\\
&=\det\left(A\left[a-1:b \mid c-1:d\right]\right)\cdot\det\left(A\left[a:b+1 \mid c:d+1\right]\right).
\end{aligned}$$

Notice that the special case of this theorem for $v = u+2$, $a = 2$, $b = p$, $c = p+2$ and $d = p+q$ is the following lemma:

Lemma 14.2. Let $\mathbb{K}$ be a field. Let $u \in \mathbb{N}$. Let $B \in \mathbb{K}^{u \times (u+2)}$ be a $u \times (u+2)$-matrix. Let p and q be two integers $\geqslant 2$ satisfying $p + q = u + 2$. Then,

$$\begin{aligned} &\det(B[1:p \mid p+2:p+q+1]) \cdot \det(B[2:p+1 \mid p+1:p+q]) \\ &+ \det(B[2:p \mid p+1:p+q+1]) \cdot \det(B[1:p+1 \mid p+2:p+q]) \\ &= \det(B[1:p \mid p+1:p+q]) \cdot \det(B[2:p+1 \mid p+2:p+q+1]). \end{aligned} \tag{35}$$

Proof of Theorem 14.1 (sketched). If $a = b-1$ or $c = d-1$, then Theorem 14.1 degenerates to a triviality (namely, $0 + 0 = 0$). Hence, for the rest of this proof, we assume WLOG that neither $a = b - 1$ nor $c = d - 1$. Hence, $a \leqslant b$ and $c \leqslant d$.

Now, Theorem 14.1 follows from the Plücker relations (see, e.g., [KlLa72, (QR)]) applied to the $u \times (u+2)$-matrix $A[a-1 : b+1 \mid c-1 : d+1]$. But let us show an alternative proof of Theorem 14.1 which avoids the use of the Plücker relations:

Let $p = b - a + 2$ and $q = d - c + 2$. Then, $p \geqslant 2$, $q \geqslant 2$ and $p + q = u + 2$.

Let B be the matrix whose columns (from left to right) are A_{a-1}, A_a, ..., A_b, A_{c-1}, A_c, ..., A_d. Then, B is a $u \times (u+2)$-matrix and satisfies

$$\begin{aligned} A[a-1:b \mid c:d+1] &= B[1:p-1 \mid p+2:p+q+1]; \\ A[a:b+1 \mid c-1:d] &= B[2:p \mid p+1:p+q]; \\ A[a:b \mid c-1:d+1] &= B[2:p-1 \mid p+1:p+q+1]; \\ A[a-1:b+1 \mid c:d] &= B[1:p \mid p+2:p+q]; \\ A[a-1:b \mid c-1:d] &= B[1:p-1 \mid p+1:p+q]; \\ A[a:b+1 \mid c:d+1] &= B[2:p \mid p+2:p+q+1]. \end{aligned}$$

Hence, the equality that we have to prove, namely

$$\begin{aligned} &\det(A[a-1:b \mid c:d+1]) \cdot \det(A[a:b+1 \mid c-1:d]) \\ &+ \det(A[a:b \mid c-1:d+1]) \cdot \det(A[a-1:b+1 \mid c:d]) \\ &= \det(A[a-1:b \mid c-1:d]) \cdot \det(A[a:b+1 \mid c:d+1]), \end{aligned}$$

rewrites as

$$\begin{aligned} &\det(B[1:p \mid p+2:p+q+1]) \cdot \det(B[2:p+1 \mid p+1:p+q]) \\ &+ \det(B[2:p \mid p+1:p+q+1]) \cdot \det(B[1:p+1 \mid p+2:p+q]) \\ &= \det(B[1:p \mid p+1:p+q]) \cdot \det(B[2:p+1 \mid p+2:p+q+1]). \end{aligned}$$

But this equality follows from Lemma 14.2. Hence, in order to complete the proof of Theorem 14.1, we only need to verify Lemma 14.2. □

Proof of Lemma 14.2 (sketched). Let $(e_1, e_2, ..., e_u)$ be the standard basis of the $\mathbb{K}$-vector space $\mathbb{K}^u$.

Let α and β be the $(p-1)$-st entries of the columns B_1 and B_{p+q} of B. Let γ and δ be the p-th entries of the columns B_1 and B_{p+q} of B.

We need to prove (35). Since (35) is a polynomial identity in the entries of B, let us WLOG assume that the columns B_2, B_3, ..., B_{p+q-1} of B (these are the middle u among the altogether $u+2=p+q$ columns of B) are linearly independent (since u vectors in $\mathbb{K}^u$ in general position are linearly independent). Then, by applying row transformations to the matrix B, we can transform these columns into the basis vectors e_1, e_2, ..., e_u of $\mathbb{K}^u$. Since the equality (35) is preserved under row transformations of B (indeed, row transformations of B amount to row transformations of all six matrices appearing in (35), and thus their only effect on the equality (35) is to multiply the six determinants appearing in (35) by certain scalar factors, but these scalar factors are all equal and thus don't affect the validity of the equality), we can therefore WLOG assume that the columns B_2, B_3, ..., B_{p+q-1} of B **are** the basis vectors e_1, e_2, ..., e_u of $\mathbb{K}^u$. The matrix B then looks as follows:

$$\begin{pmatrix}
* & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & * \\
* & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & * \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & * \\
* & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & * \\
* & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & * \\
\alpha & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 0 & \cdots & 0 & \beta \\
\gamma & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 0 & \cdots & 0 & \delta \\
* & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 & \cdots & 0 & * \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
* & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & *
\end{pmatrix},$$

where asterisks ($*$) signify entries which we are not concerned with.

Now, there is a method to simplify the determinant of a matrix if some columns of this matrix are known to belong to the standard basis $(e_1, e_2, ..., e_u)$. Indeed, such a matrix can first be brought to a block-triangular form by permuting columns (which affects the determinant by $(-1)^\sigma$, with σ being the sign of the permutation used), and then its determinant can be evaluated using the fact that the determinant of a block-triangular matrix is the product of the determinants of its diagonal blocks. Applying this method to each of the six matrices appearing in (35), we obtain

$$\begin{aligned}
\det\left(B\left[1:p\mid p+2:p+q+1\right]\right) &= (-1)^{p+q}\left(\alpha\delta-\beta\gamma\right); \\
\det\left(B\left[2:p+1\mid p+1:p+q\right]\right) &= 1; \\
\det\left(B\left[2:p\mid p+1:p+q+1\right]\right) &= (-1)^{q-1}\,\beta; \\
\det\left(B\left[1:p+1\mid p+2:p+q\right]\right) &= (-1)^{p-1}\,\gamma; \\
\det\left(B\left[1:p\mid p+1:p+q\right]\right) &= (-1)^{p-2}\,\alpha; \\
\det\left(B\left[2:p+1\mid p+2:p+q+1\right]\right) &= (-1)^{q-2}\,\delta.
\end{aligned}$$

Hence, (35) rewrites as

$$(-1)^{p+q}\left(\alpha\delta-\beta\gamma\right)\cdot 1+(-1)^{q-1}\,\beta\cdot(-1)^{p-1}\,\gamma=(-1)^{p-2}\,\alpha\cdot(-1)^{q-2}\,\delta.$$

Upon cancelling the signs, this simplifies to $(\alpha\delta - \beta\gamma) + \beta\gamma = \alpha\delta$, which is trivially true. Thus we have proven (35). Hence, Lemma 14.2 is proven. $\square$

> **Remark 14.3.** Instead of transforming the middle $p+q$ columns of the matrix B to the standard basis vectors e_1, e_2, ..., e_u of $\mathbb{K}^u$ as we did in the proof of Lemma 14.2, we could have transformed the first and last columns of B into the two last standard basis vectors e_{u-1} and e_u. The resulting identity would have been Dodgson's condensation identity (which appears, e.g., in [Zeil98, *(Alice)*]), applied to the matrix formed by the remaining u columns of B and after some interchange of rows and columns.

Proof of Proposition 13.13. Let $f = \operatorname{Grasp}_{j+1} A$ and $g = \operatorname{Grasp}_j A$.

Clearly, $f(0) = 1 = g(0)$ and $f(1) = 1 = g(1)$.

We want to show that $\operatorname{Grasp}_j A = R_{\operatorname{Rect}(p,q)} \left(\operatorname{Grasp}_{j+1} A\right)$. In other words, we want to show that $g = R_{\operatorname{Rect}(p,q)}(f)$ (because $g = \operatorname{Grasp}_j A$ and $f = \operatorname{Grasp}_{j+1} A$). According to Proposition 2.19 (applied to $P = \operatorname{Rect}(p,q)$), this will follow once we can show that

$$g(v) = \frac{1}{f(v)} \cdot \frac{\sum\limits_{\substack{u \in \widehat{\operatorname{Rect}(p,q)};\\ u \lessdot v}} f(u)}{\sum\limits_{\substack{u \in \widehat{\operatorname{Rect}(p,q)};\\ u \gtrdot v}} \frac{1}{g(u)}} \qquad \text{for every } v \in \operatorname{Rect}(p,q). \tag{36}$$

So let $v \in \operatorname{Rect}(p,q)$. Thus, $v = (i,k)$ for some $i \in \{1,2,...,p\}$ and $k \in \{1,2,...,q\}$. Consider these i and k. We must prove (36).

We are clearly in one of the following four cases:

Case 1: We have $v \neq (1,1)$ and $v \neq (p,q)$.

Case 2: We have $v = (1,1)$ and $v \neq (p,q)$.

Case 3: We have $v \neq (1,1)$ and $v = (p,q)$.

Case 4: We have $v = (1,1)$ and $v = (p,q)$.

Let us consider Case 1 first. In this case, we have $v \neq (1,1)$ and $v \neq (p,q)$. As a consequence, all elements $u \in \widehat{\operatorname{Rect}(p,q)}$ satisfying $u \lessdot v$ belong to $\operatorname{Rect}(p,q)$, and the same holds for all $u \in \widehat{\operatorname{Rect}(p,q)}$ satisfying $u \gtrdot v$.

Due to the specific form of the poset $\operatorname{Rect}(p,q)$, there are at most two elements u of $\widehat{\operatorname{Rect}(p,q)}$ satisfying $u \lessdot v$, namely $(i,k-1)$ (which exists only if $k \neq 1$) and $(i-1,k)$ (which exists only if $i \neq 1$). Hence, the sum $\sum\limits_{\substack{u \in \widehat{\operatorname{Rect}(p,q)};\\ u \lessdot v}} f(u)$ takes one of the three forms $f((i,k-1)) + f((i-1,k))$, $f((i,k-1))$ and $f((i-1,k))$. Due to Definition 13.9 **(b)**, all of these three forms can be rewritten uniformly as $f((i,k-1)) + f((i-1,k))$ (because if $(i,k-1) \notin \operatorname{Rect}(p,q)$ then Definition 13.9 **(b)** guarantees that $f((i,k-1)) = 0$, and similarly $f((i-1,k)) = 0$ if $(i-1,k) \notin \operatorname{Rect}(p,q)$). So we have

$$\sum_{\substack{u \in \widehat{\operatorname{Rect}(p,q)};\\ u \lessdot v}} f(u) = f((i,k-1)) + f((i-1,k)). \tag{37}$$

Similarly,

$$\sum_{\substack{u\in\widehat{\operatorname{Rect}(p,q)};\\ u\gtrdot v}}\frac{1}{g\left(u\right)}=\frac{1}{g\left(\left(i,k+1\right)\right)}+\frac{1}{g\left(\left(i+1,k\right)\right)},\tag{38}$$

where we set $\dfrac{1}{\infty}=0$ as usual.

But $f=\operatorname{Grasp}_{j+1}A$. Hence,

$$\begin{aligned}
&f\left(\left(i,k-1\right)\right)\\
&=\left(\operatorname{Grasp}_{j+1}A\right)\left(\left(i,k-1\right)\right)\\
&=\frac{\det\left(A\left[\left(j+1\right)+1:\left(j+1\right)+i\mid\left(j+1\right)+i+\left(k-1\right)-1:\left(j+1\right)+p+\left(k-1\right)\right]\right)}{\det\left(A\left[j+1:\left(j+1\right)+i\mid\left(j+1\right)+i+\left(k-1\right):\left(j+1\right)+p+\left(k-1\right)\right]\right)}\\
&\qquad\qquad\left(\text{by the definition of }\operatorname{Grasp}_{j+1}A\right)\\
&=\frac{\det\left(A\left[j+2:j+i+1\mid j+i+k-1:j+p+k\right]\right)}{\det\left(A\left[j+1:j+i+1\mid j+i+k:j+p+k\right]\right)}
\end{aligned}$$

and

$$\begin{aligned}
&f\left(\left(i-1,k\right)\right)\\
&=\left(\operatorname{Grasp}_{j+1}A\right)\left(\left(i-1,k\right)\right)\\
&=\frac{\det\left(A\left[\left(j+1\right)+1:\left(j+1\right)+\left(i-1\right)\mid\left(j+1\right)+\left(i-1\right)+k-1:\left(j+1\right)+p+k\right]\right)}{\det\left(A\left[j+1:\left(j+1\right)+\left(i-1\right)\mid\left(j+1\right)+\left(i-1\right)+k:\left(j+1\right)+p+k\right]\right)}\\
&\qquad\qquad\left(\text{by the definition of }\operatorname{Grasp}_{j+1}A\right)\\
&=\frac{\det\left(A\left[j+2:j+i\mid j+i+k-1:j+p+k+1\right]\right)}{\det\left(A\left[j+1:j+i\mid j+i+k:j+p+k+1\right]\right)}.
\end{aligned}$$

Due to these two equalities, (37) becomes

$$\begin{aligned}
&\sum_{\substack{u\in\widehat{\operatorname{Rect}(p,q)};\\ u\lessdot v}} f(u)\\
&=\frac{\det\left(A\left[j+2:j+i+1\mid j+i+k-1:j+p+k\right]\right)}{\det\left(A\left[j+1:j+i+1\mid j+i+k:j+p+k\right]\right)}\\
&\qquad+\frac{\det\left(A\left[j+2:j+i\mid j+i+k-1:j+p+k+1\right]\right)}{\det\left(A\left[j+1:j+i\mid j+i+k:j+p+k+1\right]\right)}\\
&=\left(\det\left(A\left[j+1:j+i+1\mid j+i+k:j+p+k\right]\right)\right)^{-1}\\
&\qquad\cdot\left(\det\left(A\left[j+1:j+i\mid j+i+k:j+p+k+1\right]\right)\right)^{-1}\\
&\qquad\cdot\left(\det\left(A\left[j+1:j+i\mid j+i+k:j+p+k+1\right]\right)\right.\\
&\qquad\qquad\cdot\det\left(A\left[j+2:j+i+1\mid j+i+k-1:j+p+k\right]\right)\\
&\qquad+\det\left(A\left[j+2:j+i\mid j+i+k-1:j+p+k+1\right]\right)\\
&\qquad\qquad\left.\cdot\det\left(A\left[j+1:j+i+1\mid j+i+k:j+p+k\right]\right)\right)\\
&=\left(\det\left(A\left[j+1:j+i+1\mid j+i+k:j+p+k\right]\right)\right)^{-1}\\
&\qquad\cdot\left(\det\left(A\left[j+1:j+i\mid j+i+k:j+p+k+1\right]\right)\right)^{-1}\\
&\qquad\cdot\det\left(A\left[j+1:j+i\mid j+i+k-1:j+p+k\right]\right)\\
&\qquad\cdot\det\left(A\left[j+2:j+i+1\mid j+i+k:j+p+k+1\right]\right)
\end{aligned}\tag{39}$$

(because applying Theorem 14.1 to $a=j+2$, $b=j+i$, $c=j+i+k$ and $d=j+p+k$ yields

$$\begin{aligned}
&\det\left(A\left[j+1:j+i\mid j+i+k:j+p+k+1\right]\right)\\
&\qquad\cdot\det\left(A\left[j+2:j+i+1\mid j+i+k-1:j+p+k\right]\right)\\
&+\det\left(A\left[j+2:j+i\mid j+i+k-1:j+p+k+1\right]\right)\\
&\qquad\cdot\det\left(A\left[j+1:j+i+1\mid j+i+k:j+p+k\right]\right)\\
&=\det\left(A\left[j+1:j+i\mid j+i+k-1:j+p+k\right]\right)\\
&\qquad\cdot\det\left(A\left[j+2:j+i+1\mid j+i+k:j+p+k+1\right]\right)
\end{aligned}$$

).

On the other hand, $g=\operatorname{Grasp}_j A$, so that

$$\begin{aligned}
&g\left((i,k+1)\right)\\
&=\left(\operatorname{Grasp}_j A\right)\left((i,k+1)\right)=\frac{\det\left(A\left[j+1:j+i\mid j+i+(k+1)-1:j+p+(k+1)\right]\right)}{\det\left(A\left[j:j+i\mid j+i+(k+1):j+p+(k+1)\right]\right)}\\
&\qquad\left(\text{by the definition of } \operatorname{Grasp}_j A\right)\\
&=\frac{\det\left(A\left[j+1:j+i\mid j+i+k:j+p+k+1\right]\right)}{\det\left(A\left[j:j+i\mid j+i+k+1:j+p+k+1\right]\right)}
\end{aligned}$$

and therefore

$$\frac{1}{g\left(\left(i,k+1\right)\right)}=\frac{\det\left(A\left[j:j+i\mid j+i+k+1:j+p+k+1\right]\right)}{\det\left(A\left[j+1:j+i\mid j+i+k:j+p+k+1\right]\right)}. \tag{40}$$

Also, from $g=\operatorname{Grasp}_j A$, we obtain

$$\begin{aligned}
&g\left(\left(i+1,k\right)\right)\\
&=\left(\operatorname{Grasp}_j A\right)\left(\left(i-1,k\right)\right)=\frac{\det\left(A\left[j+1:j+\left(i+1\right)\mid j+\left(i+1\right)+k-1:j+p+k\right]\right)}{\det\left(A\left[j:j+\left(i+1\right)\mid j+\left(i+1\right)+k:j+p+k\right]\right)}\\
&\qquad\qquad\left(\text{by the definition of } \operatorname{Grasp}_j A\right)\\
&=\frac{\det\left(A\left[j+1:j+i+1\mid j+i+k:j+p+k\right]\right)}{\det\left(A\left[j:j+i+1\mid j+i+k+1:j+p+k\right]\right)},
\end{aligned}$$

so that

$$\frac{1}{g\left(\left(i+1,k\right)\right)}=\frac{\det\left(A\left[j:j+i+1\mid j+i+k+1:j+p+k\right]\right)}{\det\left(A\left[j+1:j+i+1\mid j+i+k:j+p+k\right]\right)}. \tag{41}$$

Due to (40) and (41), the equality (38) becomes

$$\begin{aligned}
&\sum_{\substack{u\in\widehat{\operatorname{Rect}(p,q)};\\ u\gtrdot v}}\frac{1}{g\left(u\right)}\\
&=\frac{\det\left(A\left[j:j+i\mid j+i+k+1:j+p+k+1\right]\right)}{\det\left(A\left[j+1:j+i\mid j+i+k:j+p+k+1\right]\right)}\\
&\qquad+\frac{\det\left(A\left[j:j+i+1\mid j+i+k+1:j+p+k\right]\right)}{\det\left(A\left[j+1:j+i+1\mid j+i+k:j+p+k\right]\right)}\\
&=\left(\det\left(A\left[j+1:j+i\mid j+i+k:j+p+k+1\right]\right)\right)^{-1}\\
&\qquad\cdot\left(\det\left(A\left[j+1:j+i+1\mid j+i+k:j+p+k\right]\right)\right)^{-1}\\
&\qquad\cdot\left(\det\left(A\left[j:j+i\mid j+i+k+1:j+p+k+1\right]\right)\right.\\
&\qquad\qquad\cdot\det\left(A\left[j+1:j+i+1\mid j+i+k:j+p+k\right]\right)\\
&\qquad+\det\left(A\left[j+1:j+i\mid j+i+k:j+p+k+1\right]\right)\\
&\qquad\qquad\left.\cdot\det\left(A\left[j:j+i+1\mid j+i+k+1:j+p+k\right]\right)\right)\\
&=\left(\det\left(A\left[j+1:j+i\mid j+i+k:j+p+k+1\right]\right)\right)^{-1}\\
&\qquad\cdot\left(\det\left(A\left[j+1:j+i+1\mid j+i+k:j+p+k\right]\right)\right)^{-1}\\
&\qquad\cdot\det\left(A\left[j:j+i\mid j+i+k:j+p+k\right]\right)\\
&\qquad\cdot\det\left(A\left[j+1:j+i+1\mid j+i+k+1:j+p+k+1\right]\right)
\end{aligned} \tag{42}$$

(because applying Theorem 14.1 to $a=j+1$, $b=j+i$, $c=j+i+k+1$ and $d=j+p+k$

yields

$$\begin{aligned}
&\det\left(A\left[j:j+i\mid j+i+k+1:j+p+k+1\right]\right)\\
&\qquad\cdot\det\left(A\left[j+1:j+i+1\mid j+i+k:j+p+k\right]\right)\\
&+\det\left(A\left[j+1:j+i\mid j+i+k:j+p+k+1\right]\right)\\
&\qquad\cdot\det\left(A\left[j:j+i+1\mid j+i+k+1:j+p+k\right]\right)\\
&=\det\left(A\left[j:j+i\mid j+i+k:j+p+k\right]\right)\\
&\qquad\cdot\det\left(A\left[j+1:j+i+1\mid j+i+k+1:j+p+k+1\right]\right)
\end{aligned}$$

).

Since $v=(i,k)$ and $g=\operatorname{Grasp}_j A$, we have

$$\begin{aligned}
&g(v)\\
&=\left(\operatorname{Grasp}_j A\right)((i,k))=\frac{\det\left(A\left[j+1:j+i\mid j+i+k-1:j+p+k\right]\right)}{\det\left(A\left[j:j+i\mid j+i+k:j+p+k\right]\right)}\\
&\qquad\left(\text{by the definition of }\operatorname{Grasp}_j A\right).
\end{aligned}\tag{43}$$

Since $v=(i,k)$ and $f=\operatorname{Grasp}_{j+1} A$, we have

$$\begin{aligned}
&f(v)\\
&=\left(\operatorname{Grasp}_{j+1} A\right)((i,k))\\
&=\frac{\det\left(A\left[(j+1)+1:(j+1)+i\mid (j+1)+i+k-1:(j+1)+p+k\right]\right)}{\det\left(A\left[j+1:(j+1)+i\mid (j+1)+i+k:(j+1)+p+k\right]\right)}\\
&\qquad\left(\text{by the definition of }\operatorname{Grasp}_{j+1} A\right)\\
&=\frac{\det\left(A\left[j+2:j+i+1\mid j+i+k:j+p+k+1\right]\right)}{\det\left(A\left[j+1:j+i+1\mid j+i+k+1:j+p+k+1\right]\right)}.
\end{aligned}\tag{44}$$

Now, we can rewrite the terms $\sum\limits_{\substack{u\in\widehat{\operatorname{Rect}(p,q)};\\ u\lessdot v}} f(u)$, $\sum\limits_{\substack{u\in\widehat{\operatorname{Rect}(p,q)};\\ u\gtrdot v}} \dfrac{1}{g(u)}$, $g(v)$ and $f(v)$ in (36) using the equalities (39), (42), (43) and (44), respectively. The resulting equation is a tautology because all determinants cancel out (this can be checked by the reader). Hence, (36) is proven in Case 1.

Let us now consider Case 3. In this case, we have $v\neq(1,1)$ and $v=(p,q)$. Hence, (39), (43) and (44) are still valid, whereas (42) gets superseded by the simpler equality

$$\sum_{\substack{u\in\widehat{\operatorname{Rect}(p,q)};\\ u\gtrdot v}}\frac{1}{g(u)}=\frac{1}{g(1)}=\frac{1}{1}=1.\tag{45}$$

Now, we can rewrite the terms $\sum\limits_{\substack{u\in\widehat{\operatorname{Rect}(p,q)};\\ u\lessdot v}} f(u)$, $\sum\limits_{\substack{u\in\widehat{\operatorname{Rect}(p,q)};\\ u\gtrdot v}} \dfrac{1}{g(u)}$, $g(v)$ and $f(v)$ in (36) using the equalities (39), (45), (43) and (44), respectively. The resulting equation (after multiplying through with all denominators and cancelling terms appearing on both sides) simplifies to

$$\begin{aligned}
&\det\left(A\left[j+1:j+i+1\mid j+i+k:j+p+k\right]\right) \\
&\qquad\qquad\cdot\det\left(A\left[j+1:j+i\mid j+i+k:j+p+k+1\right]\right) \\
&=\det\left(A\left[j+1:j+i+1\mid j+i+k+1:j+p+k+1\right]\right) \\
&\qquad\qquad\cdot\det\left(A\left[j:j+i\mid j+i+k:j+p+k\right]\right).
\end{aligned}$$

Since $i=p$ and $k=q$ (because $(i,k)=v=(p,q)$), this rewrites as

$$\begin{aligned}
&\det\left(A\left[j+1:j+p+1\mid j+p+q:j+p+q\right]\right) \\
&\qquad\qquad\cdot\det\left(A\left[j+1:j+p\mid j+p+q:j+p+q+1\right]\right) \\
&=\det\left(A\left[j+1:j+p+1\mid j+p+q+1:j+p+q+1\right]\right) \\
&\qquad\qquad\cdot\det\left(A\left[j:j+p\mid j+p+q:j+p+q\right]\right).
\end{aligned}$$

But this follows from

$$\begin{aligned}
&\det\left(A\left[j+1:j+p+1\mid j+p+q:j+p+q\right]\right) \\
&=\det\left(A\left[j+1:j+p+1\mid j+p+q+1:j+p+q+1\right]\right)
\end{aligned}$$

(which is clear from Proposition 13.7 **(b)**) and

$$\begin{aligned}
&\det\left(A\left[j+1:j+p\mid j+p+q:j+p+q+1\right]\right) \\
&=\det\left(A\left[j:j+p\mid j+p+q:j+p+q\right]\right)
\end{aligned}$$

(which can be easily proven[35]). Thus, (36) is proven in Case 3.

Let us next consider Case 2. In this case, we have $v=(1,1)$ and $v\neq(p,q)$. Hence, (42), (43) and (44) are still valid, whereas (39) gets superseded by the simpler equality

$$\sum_{\substack{u\in\widehat{\operatorname{Rect}(p,q)};\\ u<v}} f(u)=f(0)=1. \tag{46}$$

[35] *Proof.* We have

$$\begin{aligned}
&\det\left(A\left[j+1:j+p\mid j+p+q:j+p+q+1\right]\right) \\
&=\det\left(A\left[j+1:j+p\mid p+q+j:p+q+j+1\right]\right)=\det\left(\underbrace{A\left[j:j+1\mid j+1:j+p\right]}_{\substack{=A[j:j+p\mid j+p+q:j+p+q]\\ \text{(by Proposition 13.7 } \textbf{(c)}\text{)}}}\right) \\
&\qquad\text{(by Proposition 13.8 } \textbf{(c)}\text{, applied to } u=p,\ v=p+q,\ a=j+1,\ b=j+p,\ c=j \text{ and } d=j+1\text{)} \\
&=\det\left(A\left[j:j+p\mid j+p+q:j+p+q\right]\right),
\end{aligned}$$

qed.

Now, we can rewrite the terms $\sum\limits_{\substack{u\in\widehat{\operatorname{Rect}(p,q)};\\ u\lessdot v}} f(u)$, $\sum\limits_{\substack{u\in\widehat{\operatorname{Rect}(p,q)};\\ u\gtrdot v}} \dfrac{1}{g(u)}$, $g(v)$ and $f(v)$ in (36) using the equalities (46), (42), (43) and (44), respectively. The resulting equation (after multiplying through with all denominators and cancelling terms appearing on both sides) simplifies to

$$\begin{aligned}
&\det\left(A\left[j+1:j+i\mid j+i+k-1:j+p+k\right]\right)\\
&\qquad\cdot\det\left(A\left[j+2:j+i+1\mid j+i+k:j+p+k+1\right]\right)\\
&=\det\left(A\left[j+1:j+i+1\mid j+i+k:j+p+k\right]\right)\\
&\qquad\cdot\det\left(A\left[j+1:j+i\mid j+i+k:j+p+k+1\right]\right).
\end{aligned}$$

Since $i=1$ and $k=1$ (because $(i,k)=v=(1,1)$), this rewrites as

$$\begin{aligned}
&\det\left(A\left[j+1:j+1\mid j+1+1-1:j+p+1\right]\right)\\
&\qquad\cdot\det\left(A\left[j+2:j+1+1\mid j+1+1:j+p+1+1\right]\right)\\
&=\det\left(A\left[j+1:j+1+1\mid j+1+1:j+p+1\right]\right)\\
&\qquad\cdot\det\left(A\left[j+1:j+1\mid j+1+1:j+p+1+1\right]\right).
\end{aligned}$$

In other words, this rewrites as

$$\begin{aligned}
&\det\left(A\left[j+1:j+1\mid j+1:j+p+1\right]\right)\\
&\qquad\cdot\det\left(A\left[j+2:j+2\mid j+2:j+p+2\right]\right)\\
&=\det\left(A\left[j+1:j+2\mid j+2:j+p+1\right]\right)\\
&\qquad\cdot\det\left(A\left[j+1:j+1\mid j+2:j+p+2\right]\right).
\end{aligned}$$

But this trivially follows from

$$\det\left(A\left[j+1:j+1\mid j+1:j+p+1\right]\right)=\det\left(A\left[j+1:j+2\mid j+2:j+p+1\right]\right)$$

(this is because of Proposition 13.6) and

$$\det\left(A\left[j+2:j+2\mid j+2:j+p+2\right]\right)=\det\left(A\left[j+1:j+1\mid j+2:j+p+2\right]\right)$$

(this is because of Proposition 13.7 **(a)**). This proves (36) in Case 2.

We have now proven (36) in each of the Cases 1, 2 and 3. We leave the proof in Case 4 to the reader (this case is completely straightforward, since it has $(p,q)=v=(1,1)$). Thus, we now know that (36) holds in each of the four Cases 1, 2, 3 and 4. Since these four Cases cover all possibilities, this yields that (36) always holds. As we have seen, this completes the proof of Proposition 13.13. □

A remark seems in order, about why we paid so much attention to the "degenerate" Cases 2, 3 and 4. Indeed, only in Cases 3 and 4 have we used the fact that the sequence $(A_n)_{n\in\mathbb{Z}}$ is "$(p+q)$-periodic up to sign" rather than just an arbitrary sequence of length-p column vectors. Had we left out these seemingly straightforward cases, it would have seemed that the proof showed a result too good to be true (because it is rather clear that the periodicity in the definition of A_n for general $n\in\mathbb{Z}$ is needed).

15 Dominance of the Grassmannian parametrization

Let us show an example before we start proving Proposition 13.14.

Example 15.1. For this example, let $p = 2$ and $q = 2$. Let $f \in \mathbb{K}^{\widehat{\operatorname{Rect}(2,2)}}$ be a generic reduced labelling. We want to construct a matrix $A \in \mathbb{K}^{2\times(2+2)}$ satisfying $f = \operatorname{Grasp}_0 A$.

Clearly, the condition $f = \operatorname{Grasp}_0 A$ imposes 4 equations on the entries of A (one for every element of $\operatorname{Rect}(2,2)$). Since the matrix A we want to find has a total of 8 entries, we are therefore trying to solve an underdetermined system. However, we can get rid of the superfluous freedom if we additionally try to ensure that our matrix A has the form $(I_p \mid B)$ for some $B \in \mathbb{K}^{2\times 2}$ (where $(I_p \mid B)$ means the matrix obtained from the $p \times p$ identity matrix I_p by attaching the matrix B to it on the right). Let us do this now. So we are looking for a matrix $B \in \mathbb{K}^{2\times 2}$ satisfying $f = \operatorname{Grasp}_0 (I_p \mid B)$. This puts 4 conditions on 4 unknowns. Write $B = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$. Then, $(I_p \mid B) = \begin{pmatrix} 1 & 0 & x & y \\ 0 & 1 & z & w \end{pmatrix}$. Now,

$$(\operatorname{Grasp}_0 (I_p \mid B))((1,1)) = \frac{\det((I_p \mid B)[1:1 \mid 1:3])}{\det((I_p \mid B)[0:1 \mid 2:3])} = \frac{\det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}{\det\begin{pmatrix} -y & 0 \\ -w & 1 \end{pmatrix}} = \frac{-1}{y};$$

$$(\operatorname{Grasp}_0 (I_p \mid B))((1,2)) = \frac{\det((I_p \mid B)[1:1 \mid 2:4])}{\det((I_p \mid B)[0:1 \mid 3:4])} = \frac{\det\begin{pmatrix} 0 & x \\ 1 & z \end{pmatrix}}{\det\begin{pmatrix} -y & x \\ -w & z \end{pmatrix}} = \frac{-x}{wx - yz};$$

$$(\operatorname{Grasp}_0 (I_p \mid B))((2,1)) = \frac{\det((I_p \mid B)[1:2 \mid 2:3])}{\det((I_p \mid B)[0:2 \mid 3:3])} = \frac{\det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}{\det\begin{pmatrix} -y & 1 \\ -w & 0 \end{pmatrix}} = \frac{1}{w};$$

$$(\operatorname{Grasp}_0 (I_p \mid B))((2,2)) = \frac{\det((I_p \mid B)[1:2 \mid 3:4])}{\det((I_p \mid B)[0:2 \mid 4:4])} = \frac{\det\begin{pmatrix} 1 & x \\ 0 & z \end{pmatrix}}{\det\begin{pmatrix} -y & 1 \\ -w & 0 \end{pmatrix}} = \frac{z}{w}.$$

The requirement $f = \operatorname{Grasp}_0 (I_p \mid B)$ therefore translates into the system

$$\begin{cases} f((1,1)) & = & \dfrac{-1}{y}; \\ f((1,2)) & = & \dfrac{-x}{wx - yz}; \\ f((2,1)) & = & \dfrac{1}{w}; \\ f((2,2)) & = & \dfrac{z}{w} \end{cases}.$$

This system can be solved by elimination: First, compute w using $f((2,1)) = \frac{1}{w}$, obtaining $w = \frac{1}{f((2,1))}$; then, compute y using $f((1,1)) = \frac{-1}{y}$, obtaining $y = \frac{-1}{f((1,1))}$; then, compute z using $f((2,2)) = \frac{z}{w}$ and the already eliminated w, obtaining $z = \frac{f((2,2))}{f((2,1))}$; finally, compute x using $f((1,2)) = \frac{-x}{wx - yz}$ and the already eliminated w, y, z, obtaining $x = \frac{-f((1,2))\, f((2,2))}{(f((1,2)) + f((2,1)))\, f((1,1))}$. While the denominators in these fractions can vanish, leading to underdetermination or unsolvability, this will not happen for **generic** f.

This approach to solving $f = \operatorname{Grasp}_0 A$ generalizes to arbitrary p and q, and motivates the following proof.

We are now going to outline the proof of Proposition 13.14. As shouldn't be surprising after Example 15.1, the underlying idea of the proof is the following: For any fixed $f \in \mathbb{K}^{\operatorname{Rect}(p,q)}$, the equation $f = \operatorname{Grasp}_0 A$ (with A an unknown matrix in $\mathbb{K}^{p\times(p+q)}$) can be considered as a system of pq equations on $p(p+q)$ unknowns (the entries of A). While this system is usually underdetermined, we can restrict the entries of A by requiring that the leftmost p columns of A form the $p \times p$ identity matrix. Upon this restriction, we are left with pq unknowns only, and for f sufficiently generic, the resulting system will be uniquely solvable by "triangular elimination" (i.e., there is an equation containing only one unknown; then, when this unknown is eliminated, the resulting system again contains an equation with only one unknown, and once this one is eliminated, one gets a further system containing an equation with only one unknown, and so forth) – like a triangular system of linear equations with nonzero entries on the diagonal, but without the linearity. Of course, this is not a complete proof because the applicability of "triangular elimination" has to be proven, not merely claimed. We are only going to sketch the ideas of this proof, leaving all straightforward details to the reader to fill in. For the sake of clarity, we are going to word the argument using algebraic properties of families of rational functions instead of using the algorithmic nature of "triangular elimination" (similarly to how most applications of linear algebra use the language of bases of vector spaces rather than talk about the process of solving systems by Gaussian elimination). While this clarity comes at the cost of a slight disconnect from the motivation of the proof, we hope that the reader will still see how the wind blows.

We first introduce some notation to capture the essence of "triangular elimination" without having to talk about actually moving around variables in equations:

Definition 15.2. Let $\mathbb{F}$ be a field. Let $\mathbf{P}$ be a finite set.

(a) Let $x_{\mathbf{p}}$ be a new symbol for every $\mathbf{p} \in \mathbf{P}$. We will denote by $\mathbb{F}(x_{\mathbf{P}})$ the field of rational functions over $\mathbb{F}$ in the indeterminates $x_{\mathbf{p}}$ with $\mathbf{p}$ ranging over all elements of $\mathbf{P}$ (hence altogether $|\mathbf{P}|$ indeterminates). We also will denote by $\mathbb{F}[x_{\mathbf{P}}]$ the ring of polynomials over $\mathbb{F}$ in the indeterminates $x_{\mathbf{p}}$ with $\mathbf{p}$ ranging over all elements of $\mathbf{P}$.

(Thus, $\mathbb{F}(x_{\mathbf{P}}) = \mathbb{F}(x_{\mathbf{p}_1}, x_{\mathbf{p}_2}, ..., x_{\mathbf{p}_n})$ and $\mathbb{F}[x_{\mathbf{P}}] = \mathbb{F}[x_{\mathbf{p}_1}, x_{\mathbf{p}_2}, ..., x_{\mathbf{p}_n}]$ if $\mathbf{P}$ is written in the form $\mathbf{P} = \{\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_n\}$.) The symbols $x_{\mathbf{p}}$ are understood to be distinct, and are used as commuting indeterminates. We regard $\mathbb{F}[x_{\mathbf{P}}]$ as a subring of $\mathbb{F}(x_{\mathbf{P}})$, and $\mathbb{F}(x_{\mathbf{P}})$ as the field of quotients of $\mathbb{F}[x_{\mathbf{P}}]$.

(b) If $\mathbf{Q}$ is a subset of $\mathbf{P}$, then $\mathbb{F}(x_{\mathbf{Q}})$ can be canonically embedded into $\mathbb{F}(x_{\mathbf{P}})$, and $\mathbb{F}[x_{\mathbf{Q}}]$ can be canonically embedded into $\mathbb{F}[x_{\mathbf{P}}]$. We regard these embeddings as inclusions.

(c) Let $\mathbb{K}$ be a field extension of $\mathbb{F}$. Let f be an element of $\mathbb{F}(x_{\mathbf{P}})$. If $(a_{\mathbf{p}})_{\mathbf{p}\in\mathbf{P}} \in \mathbb{K}^{\mathbf{P}}$ is a family of elements of $\mathbb{K}$ indexed by elements of $\mathbf{P}$, then we let $f\left((a_{\mathbf{p}})_{\mathbf{p}\in\mathbf{P}}\right)$ denote the element of $\mathbb{K}$ obtained by substituting $a_{\mathbf{p}}$ for $x_{\mathbf{p}}$ for each $\mathbf{p} \in \mathbf{P}$ in the rational function f. This $f\left((a_{\mathbf{p}})_{\mathbf{p}\in\mathbf{P}}\right)$ is defined only if the substitution does not render the denominator equal to 0. If $\mathbb{K}$ is infinite, this shows that $f\left((a_{\mathbf{p}})_{\mathbf{p}\in\mathbf{P}}\right)$ is defined for almost all $(a_{\mathbf{p}})_{\mathbf{p}\in\mathbf{P}} \in \mathbb{K}^{\mathbf{P}}$ (with respect to the Zariski topology).

(d) Let $\mathbf{P}$ now be a finite totally ordered set, and let $\lessdot$ be the smaller relation of $\mathbf{P}$. For every $\mathbf{p} \in \mathbf{P}$, let $\mathbf{p} \Downarrow$ denote the subset $\{\mathbf{v} \in \mathbf{P} \mid \mathbf{v} \lessdot \mathbf{p}\}$ of $\mathbf{P}$. For every $\mathbf{p} \in \mathbf{P}$, let $Q_{\mathbf{p}}$ be an element of $\mathbb{F}(x_{\mathbf{P}})$.

We say that the family $(Q_{\mathbf{p}})_{\mathbf{p}\in\mathbf{P}}$ is $\mathbf{P}$-*triangular* if and only if the following condition holds:

Algebraic triangularity condition: For every $\mathbf{p} \in \mathbf{P}$, there exist elements $\alpha_{\mathbf{p}}$, $\beta_{\mathbf{p}}$, $\gamma_{\mathbf{p}}$, $\delta_{\mathbf{p}}$ of $\mathbb{F}(x_{\mathbf{p}\Downarrow})$ such that $\alpha_{\mathbf{p}}\delta_{\mathbf{p}} - \beta_{\mathbf{p}}\gamma_{\mathbf{p}} \neq 0$ and $Q_{\mathbf{p}} = \dfrac{\alpha_{\mathbf{p}}x_{\mathbf{p}} + \beta_{\mathbf{p}}}{\gamma_{\mathbf{p}}x_{\mathbf{p}} + \delta_{\mathbf{p}}}$. [36]

We will use $\mathbf{P}$-triangularity via the following fact:

Lemma 15.3. Let $\mathbb{F}$ be a field. Let $\mathbf{P}$ be a finite totally ordered set. For every $\mathbf{p} \in \mathbf{P}$, let $Q_{\mathbf{p}}$ be an element of $\mathbb{F}(x_{\mathbf{P}})$. Assume that $(Q_{\mathbf{p}})_{\mathbf{p}\in\mathbf{P}}$ is a $\mathbf{P}$-triangular family. Then:

(a) The family $(Q_{\mathbf{p}})_{\mathbf{p}\in\mathbf{P}} \in (\mathbb{F}(x_{\mathbf{P}}))^{\mathbf{P}}$ is algebraically independent (over $\mathbb{F}$).

(b) There exists a $\mathbf{P}$-triangular family $(R_{\mathbf{p}})_{\mathbf{p}\in\mathbf{P}} \in (\mathbb{F}(x_{\mathbf{P}}))^{\mathbf{P}}$ such that every $\mathbf{q} \in \mathbf{P}$ satisfies $Q_{\mathbf{q}}\left((R_{\mathbf{p}})_{\mathbf{p}\in\mathbf{P}}\right) = x_{\mathbf{q}}$.

Proof of Lemma 15.3 (sketched). As in the definition of $\mathbf{P}$-triangularity, we let $\mathbf{p} \Downarrow$ denote the subset $\{\mathbf{v} \in \mathbf{P} \mid \mathbf{v} \lessdot \mathbf{p}\}$ of $\mathbf{P}$ for every $\mathbf{p} \in \mathbf{P}$.

(a) Assume that the family $(Q_{\mathbf{p}})_{\mathbf{p}\in\mathbf{P}} \in (\mathbb{F}(x_{\mathbf{P}}))^{\mathbf{P}}$ is not algebraically independent (over $\mathbb{F}$). Then, some nonzero polynomial $P \in \mathbb{F}[x_{\mathbf{P}}]$ satisfies $P\left((Q_{\mathbf{p}})_{\mathbf{p}\in\mathbf{P}}\right) = 0$. Fix such a P, and let $\mathbf{u}$ be the maximal (with respect to the order on $\mathbf{P}$) element of $\mathbf{P}$ such that $x_{\mathbf{u}}$ appears in P (meaning that the degree of P with respect to the variable $x_{\mathbf{u}}$ is > 0). Then, P can be construed as a non-constant polynomial in the variable $x_{\mathbf{u}}$ over the ring

[36] Notice that the fraction $\dfrac{\alpha_{\mathbf{p}}x_{\mathbf{p}} + \beta_{\mathbf{p}}}{\gamma_{\mathbf{p}}x_{\mathbf{p}} + \delta_{\mathbf{p}}}$ is well-defined for any four elements $\alpha_{\mathbf{p}}$, $\beta_{\mathbf{p}}$, $\gamma_{\mathbf{p}}$, $\delta_{\mathbf{p}}$ of $\mathbb{F}(x_{\mathbf{p}\Downarrow})$ such that $\alpha_{\mathbf{p}}\delta_{\mathbf{p}} - \beta_{\mathbf{p}}\gamma_{\mathbf{p}} \neq 0$. (Indeed, $\gamma_{\mathbf{p}}x_{\mathbf{p}} + \delta_{\mathbf{p}} \neq 0$ in this case, as can easily be checked.)

$\mathbb{F}[x_{\mathbf{u}\Downarrow}]$. Hence, $P\left((Q_{\mathbf{p}})_{\mathbf{p}\in\mathbf{P}}\right) = 0$ shows that $Q_{\mathbf{u}}$ is algebraic over the subfield of $\mathbb{F}(x_{\mathbf{P}})$ generated by the elements $Q_{\mathbf{v}}$ for $\mathbf{v} \in \mathbf{u}\Downarrow$.

Now, notice that every $\mathbf{v} \in \mathbf{u}\Downarrow$ satisfies $Q_{\mathbf{v}} \in \mathbb{F}(x_{\mathbf{u}\Downarrow})$ [37]. Hence, the subfield of $\mathbb{F}(x_{\mathbf{P}})$ generated by the elements $Q_{\mathbf{v}}$ for $\mathbf{v} \in \mathbf{u}\Downarrow$ is a subfield of $\mathbb{F}(x_{\mathbf{u}\Downarrow})$. Since $Q_{\mathbf{u}}$ is algebraic over the former field, we thus conclude that $Q_{\mathbf{u}}$ is "all the more" algebraic over the latter field. But by the algebraic triangularity condition, there exist elements $\alpha_{\mathbf{u}}$, $\beta_{\mathbf{u}}$, $\gamma_{\mathbf{u}}$, $\delta_{\mathbf{u}}$ of $\mathbb{F}(x_{\mathbf{u}\Downarrow})$ such that $\alpha_{\mathbf{u}}\delta_{\mathbf{u}} - \beta_{\mathbf{u}}\gamma_{\mathbf{u}} \neq 0$ and $Q_{\mathbf{u}} = \dfrac{\alpha_{\mathbf{u}}x_{\mathbf{u}} + \beta_{\mathbf{u}}}{\gamma_{\mathbf{u}}x_{\mathbf{u}} + \delta_{\mathbf{u}}}$. We can easily solve the equation $Q_{\mathbf{u}} = \dfrac{\alpha_{\mathbf{u}}x_{\mathbf{u}} + \beta_{\mathbf{u}}}{\gamma_{\mathbf{u}}x_{\mathbf{u}} + \delta_{\mathbf{u}}}$ for $x_{\mathbf{u}}$ and obtain $x_{\mathbf{u}} = \dfrac{Q_{\mathbf{u}}\delta_{\mathbf{u}} - \beta_{\mathbf{u}}}{\alpha_{\mathbf{u}} - Q_{\mathbf{u}}\gamma_{\mathbf{u}}}$ (and the denominator here does not vanish because of $\alpha_{\mathbf{u}}\delta_{\mathbf{u}} - \beta_{\mathbf{u}}\gamma_{\mathbf{u}} \neq 0$). Therefore, $x_{\mathbf{u}}$ is algebraic over the field $\mathbb{F}(x_{\mathbf{u}\Downarrow})$ (because we know $Q_{\mathbf{u}}$ to be algebraic over this field, whereas $\alpha_{\mathbf{u}}$, $\beta_{\mathbf{u}}$, $\gamma_{\mathbf{u}}$, $\delta_{\mathbf{u}}$ lie in that field). But this is absurd since $\mathbf{u} \notin \mathbf{u}\Downarrow$. This contradiction shows that our assumption was wrong, and Lemma 15.3 **(a)** is proven.

(b) We will construct the required family $(R_{\mathbf{p}})_{\mathbf{p}\in\mathbf{P}} \in (\mathbb{F}(x_{\mathbf{P}}))^{\mathbf{P}}$ by induction. Of course, this is trivial if $\mathbf{P} = \varnothing$, so let us assume that $\mathbf{P}$ is nonempty. Let $\mathbf{m}$ be the maximum element of $\mathbf{P}$, and let us assume that we have already constructed a $\mathbf{P}\setminus\{\mathbf{m}\}$-triangular family $(R_{\mathbf{p}})_{\mathbf{p}\in\mathbf{P}\setminus\{\mathbf{m}\}} \in \left(\mathbb{F}\left(x_{\mathbf{P}\setminus\{\mathbf{m}\}}\right)\right)^{\mathbf{P}\setminus\{\mathbf{m}\}}$ such that every $\mathbf{q} \in \mathbf{P}\setminus\{\mathbf{m}\}$ satisfies $Q_{\mathbf{q}}\left((R_{\mathbf{p}})_{\mathbf{p}\in\mathbf{P}\setminus\{\mathbf{m}\}}\right) = x_{\mathbf{q}}$. We now only need to find an element $R_{\mathbf{m}} \in \mathbb{F}(x_{\mathbf{P}})$ such that the resulting family $(R_{\mathbf{p}})_{\mathbf{p}\in\mathbf{P}} \in (\mathbb{F}(x_{\mathbf{P}}))^{\mathbf{P}}$ will be $\mathbf{P}$-triangular and satisfy $Q_{\mathbf{m}}\left((R_{\mathbf{p}})_{\mathbf{p}\in\mathbf{P}}\right) = x_{\mathbf{m}}$.

Since $\mathbf{m}$ is maximum, we have $\mathbf{m}\Downarrow = \mathbf{P}\setminus\{\mathbf{m}\}$.

We know that the family $(R_{\mathbf{p}})_{\mathbf{p}\in\mathbf{P}\setminus\{\mathbf{m}\}}$ is $\mathbf{P}\setminus\{\mathbf{m}\}$-triangular. Hence, Lemma 15.3 **(a)** (applied to this family) yields that the family $(R_{\mathbf{p}})_{\mathbf{p}\in\mathbf{P}\setminus\{\mathbf{m}\}}$ is algebraically independent. This yields that it can be substituted into any rational function in $\mathbb{F}\left(x_{\mathbf{P}\setminus\{\mathbf{m}\}}\right)$ (without running the risk of denominators becoming 0).

The family $(Q_{\mathbf{p}})_{\mathbf{p}\in\mathbf{P}}$ is $\mathbf{P}$-triangular, so that (by the algebraic triangularity condition) there exist elements $\alpha_{\mathbf{m}}$, $\beta_{\mathbf{m}}$, $\gamma_{\mathbf{m}}$, $\delta_{\mathbf{m}}$ of $\mathbb{F}(x_{\mathbf{m}\Downarrow})$ such that $\alpha_{\mathbf{m}}\delta_{\mathbf{m}} - \beta_{\mathbf{m}}\gamma_{\mathbf{m}} \neq 0$ and $Q_{\mathbf{m}} = \dfrac{\alpha_{\mathbf{m}}x_{\mathbf{m}} + \beta_{\mathbf{m}}}{\gamma_{\mathbf{m}}x_{\mathbf{m}} + \delta_{\mathbf{m}}}$. Consider these $\alpha_{\mathbf{m}}$, $\beta_{\mathbf{m}}$, $\gamma_{\mathbf{m}}$, $\delta_{\mathbf{m}}$. Now, define four elements $\alpha'_{\mathbf{m}}$, $\beta'_{\mathbf{m}}$, $\gamma'_{\mathbf{m}}$, $\delta'_{\mathbf{m}}$ of $\mathbb{F}\left(x_{\mathbf{P}\setminus\{\mathbf{m}\}}\right)$ by

$$\alpha'_{\mathbf{m}} = \delta_{\mathbf{m}}\left((R_{\mathbf{p}})_{\mathbf{p}\in\mathbf{P}\setminus\{\mathbf{m}\}}\right), \qquad \beta'_{\mathbf{m}} = -\beta_{\mathbf{m}}\left((R_{\mathbf{p}})_{\mathbf{p}\in\mathbf{P}\setminus\{\mathbf{m}\}}\right),$$
$$\gamma'_{\mathbf{m}} = -\gamma_{\mathbf{m}}\left((R_{\mathbf{p}})_{\mathbf{p}\in\mathbf{P}\setminus\{\mathbf{m}\}}\right), \qquad \delta'_{\mathbf{m}} = \alpha_{\mathbf{m}}\left((R_{\mathbf{p}})_{\mathbf{p}\in\mathbf{P}\setminus\{\mathbf{m}\}}\right).$$

[37] *Proof.* Let $\mathbf{v} \in \mathbf{u}\Downarrow$. Then, $\mathbf{v} \lhd \mathbf{u}$, so that $\mathbf{v}\Downarrow \subseteq \mathbf{u}\Downarrow$, hence $\mathbb{F}(x_{\mathbf{v}\Downarrow}) \subseteq \mathbb{F}(x_{\mathbf{u}\Downarrow})$.

By the algebraic triangularity condition, we know that there exist elements $\alpha_{\mathbf{v}}$, $\beta_{\mathbf{v}}$, $\gamma_{\mathbf{v}}$, $\delta_{\mathbf{v}}$ of $\mathbb{F}(x_{\mathbf{v}\Downarrow})$ such that $\alpha_{\mathbf{v}}\delta_{\mathbf{v}} - \beta_{\mathbf{v}}\gamma_{\mathbf{v}} \neq 0$ and $Q_{\mathbf{v}} = \dfrac{\alpha_{\mathbf{v}}x_{\mathbf{v}} + \beta_{\mathbf{v}}}{\gamma_{\mathbf{v}}x_{\mathbf{v}} + \delta_{\mathbf{v}}}$. These elements $\alpha_{\mathbf{v}}$, $\beta_{\mathbf{v}}$, $\gamma_{\mathbf{v}}$, $\delta_{\mathbf{v}}$ belong to $\mathbb{F}(x_{\mathbf{u}\Downarrow})$ (by virtue of lying in $\mathbb{F}(x_{\mathbf{v}\Downarrow}) \subseteq \mathbb{F}(x_{\mathbf{u}\Downarrow})$), and so does $x_{\mathbf{v}}$ (since $\mathbf{v} \in \mathbf{u}\Downarrow$). Hence, the fraction $\dfrac{\alpha_{\mathbf{v}}x_{\mathbf{v}} + \beta_{\mathbf{v}}}{\gamma_{\mathbf{v}}x_{\mathbf{v}} + \delta_{\mathbf{v}}}$ also lies in $\mathbb{F}(x_{\mathbf{u}\Downarrow})$. Since this fraction is $Q_{\mathbf{v}}$, we thus have shown $Q_{\mathbf{v}} \in \mathbb{F}(x_{\mathbf{u}\Downarrow})$, qed.

Note that these are well-defined (because $\alpha_{\mathbf{m}}$, $\beta_{\mathbf{m}}$, $\gamma_{\mathbf{m}}$, $\delta_{\mathbf{m}}$ belong to $\mathbb{F}(x_{\mathbf{m}\Downarrow}) = \mathbb{F}\left(x_{\mathbf{P}\setminus\{\mathbf{m}\}}\right)$ and because the family $(R_{\mathbf{p}})_{\mathbf{p}\in\mathbf{P}\setminus\{\mathbf{m}\}}$ is algebraically independent) and belong to $\mathbb{F}(x_{\mathbf{m}\Downarrow})$ (since $\mathbf{P}\setminus\{\mathbf{m}\} = \mathbf{m}\Downarrow$). They furthermore satisfy

$$\begin{aligned}
&\alpha'_{\mathbf{m}}\delta'_{\mathbf{m}} - \beta'_{\mathbf{m}}\gamma'_{\mathbf{m}} \\
&= \delta_{\mathbf{m}}\left((R_{\mathbf{p}})_{\mathbf{p}\in\mathbf{P}\setminus\{\mathbf{m}\}}\right)\cdot\alpha_{\mathbf{m}}\left((R_{\mathbf{p}})_{\mathbf{p}\in\mathbf{P}\setminus\{\mathbf{m}\}}\right) - \left(-\beta_{\mathbf{m}}\left((R_{\mathbf{p}})_{\mathbf{p}\in\mathbf{P}\setminus\{\mathbf{m}\}}\right)\right)\left(-\gamma_{\mathbf{m}}\left((R_{\mathbf{p}})_{\mathbf{p}\in\mathbf{P}\setminus\{\mathbf{m}\}}\right)\right) \\
&= \underbrace{\left(\delta_{\mathbf{m}}\alpha_{\mathbf{m}} - (-\beta_{\mathbf{m}})(-\gamma_{\mathbf{m}})\right)}_{=\alpha_{\mathbf{m}}\delta_{\mathbf{m}}-\beta_{\mathbf{m}}\gamma_{\mathbf{m}}\neq 0}\left((R_{\mathbf{p}})_{\mathbf{p}\in\mathbf{P}\setminus\{\mathbf{m}\}}\right) \neq 0
\end{aligned}$$

(since $(R_{\mathbf{p}})_{\mathbf{p}\in\mathbf{P}\setminus\{\mathbf{m}\}}$ is algebraically independent). Let us now define $R_{\mathbf{m}} = \dfrac{\alpha'_{\mathbf{m}}x_{\mathbf{m}} + \beta'_{\mathbf{m}}}{\gamma'_{\mathbf{m}}x_{\mathbf{m}} + \delta'_{\mathbf{m}}}$. (This is easily seen to be well-defined because $\alpha'_{\mathbf{m}}\delta'_{\mathbf{m}} - \beta'_{\mathbf{m}}\gamma'_{\mathbf{m}} \neq 0$ entails $(\gamma'_{\mathbf{m}}, \delta'_{\mathbf{m}}) \neq (0,0)$.) Since the family $(R_{\mathbf{p}})_{\mathbf{p}\in\mathbf{P}\setminus\{\mathbf{m}\}}$ is already $\mathbf{P}\setminus\{\mathbf{m}\}$-triangular, and because of the fact that $\alpha'_{\mathbf{m}}$, $\beta'_{\mathbf{m}}$, $\gamma'_{\mathbf{m}}$, $\delta'_{\mathbf{m}}$ are elements of $\mathbb{F}(x_{\mathbf{m}\Downarrow})$ satisfying $\alpha'_{\mathbf{m}}\delta'_{\mathbf{m}} - \beta'_{\mathbf{m}}\gamma'_{\mathbf{m}} \neq 0$ and $R_{\mathbf{m}} = \dfrac{\alpha'_{\mathbf{m}}x_{\mathbf{m}} + \beta'_{\mathbf{m}}}{\gamma'_{\mathbf{m}}x_{\mathbf{m}} + \delta'_{\mathbf{m}}}$, we see that the family $(R_{\mathbf{p}})_{\mathbf{p}\in\mathbf{P}} \in (\mathbb{F}(x_{\mathbf{P}}))^{\mathbf{P}}$ is $\mathbf{P}$-triangular. We are now going to prove that $Q_{\mathbf{m}}\left((R_{\mathbf{p}})_{\mathbf{p}\in\mathbf{P}}\right) = x_{\mathbf{m}}$, and then we will be done.

Since $Q_{\mathbf{m}} = \dfrac{\alpha_{\mathbf{m}}x_{\mathbf{m}} + \beta_{\mathbf{m}}}{\gamma_{\mathbf{m}}x_{\mathbf{m}} + \delta_{\mathbf{m}}}$, we have

$$Q_{\mathbf{m}}\left((R_{\mathbf{p}})_{\mathbf{p}\in\mathbf{P}}\right) = \frac{\alpha_{\mathbf{m}}\left((R_{\mathbf{p}})_{\mathbf{p}\in\mathbf{P}}\right)R_{\mathbf{m}} + \beta_{\mathbf{m}}\left((R_{\mathbf{p}})_{\mathbf{p}\in\mathbf{P}}\right)}{\gamma_{\mathbf{m}}\left((R_{\mathbf{p}})_{\mathbf{p}\in\mathbf{P}}\right)R_{\mathbf{m}} + \delta_{\mathbf{m}}\left((R_{\mathbf{p}})_{\mathbf{p}\in\mathbf{P}}\right)}. \tag{47}$$

But $\alpha_{\mathbf{m}} \in \mathbb{F}\left(x_{\mathbf{P}\setminus\{\mathbf{m}\}}\right)$, so that the variable $x_{\mathbf{m}}$ does not appear in $\alpha_{\mathbf{m}}$ at all. Hence, $\alpha_{\mathbf{m}}\left((R_{\mathbf{p}})_{\mathbf{p}\in\mathbf{P}}\right) = \alpha_{\mathbf{m}}\left((R_{\mathbf{p}})_{\mathbf{p}\in\mathbf{P}\setminus\{\mathbf{m}\}}\right) = \delta'_{\mathbf{m}}$. Using this and the similarly proven equalities $\beta_{\mathbf{m}}\left((R_{\mathbf{p}})_{\mathbf{p}\in\mathbf{P}}\right) = -\beta'_{\mathbf{m}}$, $\gamma_{\mathbf{m}}\left((R_{\mathbf{p}})_{\mathbf{p}\in\mathbf{P}}\right) = -\gamma'_{\mathbf{m}}$ and $\delta_{\mathbf{m}}\left((R_{\mathbf{p}})_{\mathbf{p}\in\mathbf{P}}\right) = \alpha'_{\mathbf{m}}$, we can rewrite the equality (47) as

$$Q_{\mathbf{m}}\left((R_{\mathbf{p}})_{\mathbf{p}\in\mathbf{P}}\right) = \frac{\delta'_{\mathbf{m}}R_{\mathbf{m}} - \beta'_{\mathbf{m}}}{-\gamma'_{\mathbf{m}}R_{\mathbf{m}} + \alpha'_{\mathbf{m}}}.$$

But the right hand side of this equality simplifies to $x_{\mathbf{m}}$ if we recall that $R_{\mathbf{m}} = \dfrac{\alpha'_{\mathbf{m}}x_{\mathbf{m}} + \beta'_{\mathbf{m}}}{\gamma'_{\mathbf{m}}x_{\mathbf{m}} + \delta'_{\mathbf{m}}}$ (the proof of this is mechanical, using no properties of $\alpha'_{\mathbf{m}}$, $\beta'_{\mathbf{m}}$, $\gamma'_{\mathbf{m}}$, $\delta'_{\mathbf{m}}$ and $x_{\mathbf{m}}$ other than lying in a field). Hence, we have shown that $Q_{\mathbf{m}}\left((R_{\mathbf{p}})_{\mathbf{p}\in\mathbf{P}}\right) = x_{\mathbf{m}}$. As explained above, this completes the (inductive) proof of Lemma 15.3 **(b)**. □

We now can proceed to the proof of Proposition 13.14:

Proof of Proposition 13.14 (sketched). Let $\mathbb{F}$ be the prime field of $\mathbb{K}$. (This means either $\mathbb{Q}$ or $\mathbb{F}_p$ depending on the characteristic of $\mathbb{K}$.) In the following, the word "algebraically independent" will always mean "algebraically independent over $\mathbb{F}$" (rather than over $\mathbb{K}$ or over $\mathbb{Z}$).

Let $\mathbf{P}$ be a totally ordered set such that

$$\mathbf{P} = \{1, 2, ..., p\} \times \{1, 2, ..., q\} \text{ as sets,}$$

and such that

$$(i, k) \trianglelefteq (i', k') \text{ for all } (i, k) \in \mathbf{P} \text{ and } (i', k') \in \mathbf{P} \text{ satisfying } (i \geqslant i' \text{ and } k \leqslant k'),$$

where $\trianglelefteq$ denotes the smaller-or-equal relation of $\mathbf{P}$. Such a $\mathbf{P}$ clearly exists (in fact, there usually exist several such $\mathbf{P}$, and it doesn't matter which of them we choose). We denote the smaller relation of $\mathbf{P}$ by $\triangleleft$. We will later see what this total order is good for (intuitively, it is an order in which the variables can be eliminated; in other words, it makes our system behave like a triangular matrix rather than like a triangular matrix with permuted columns), but for now let us notice that it is generally not compatible with $\operatorname{Rect}(p, q)$.

Let $Z : \{1, 2, ..., q\} \to \{1, 2, ..., q\}$ denote the map which sends every $k \in \{1, 2, ..., q-1\}$ to $k+1$ and sends q to 1. Thus, Z is a permutation in the symmetric group S_q, and can be written in cycle notation as $(1, 2, ..., q)$.

Consider the field $\mathbb{F}(x_{\mathbf{P}})$ and the ring $\mathbb{F}[x_{\mathbf{P}}]$ defined as in Definition 15.2.

Recall that we need to prove Proposition 13.14. In other words, we need to show that for almost every $f \in \mathbb{K}^{\operatorname{Rect}(p,q)}$, there exists a matrix $A \in \mathbb{K}^{p \times (p+q)}$ satisfying $f = \operatorname{Grasp}_0 A$.

In order to prove this, it is enough to show that there exists a matrix $\widetilde{D} \in (\mathbb{F}(x_{\mathbf{P}}))^{p \times (p+q)}$ satisfying

$$x_{\mathbf{p}} = \left(\operatorname{Grasp}_0 \widetilde{D}\right)(\mathbf{p}) \qquad \text{for every } \mathbf{p} \in \mathbf{P}. \tag{48}$$

Indeed, once the existence of such a matrix $\widetilde{D}$ is proven, we will be able to obtain a matrix $A \in \mathbb{K}^{p \times (p+q)}$ satisfying $f = \operatorname{Grasp}_0 A$ for almost every $f \in \mathbb{K}^{\operatorname{Rect}(p,q)}$ simply by substituting $f(\mathbf{p})$ for every $x_{\mathbf{p}}$ in all entries of the matrix $\widetilde{D}$ [38]. Hence, all we need to show is the existence of a matrix $\widetilde{D} \in (\mathbb{F}(x_{\mathbf{P}}))^{p \times (p+q)}$ satisfying (48).

Define a matrix $C \in (\mathbb{F}[x_{\mathbf{P}}])^{p \times q}$ by

$$C = \left(x_{(i, Z(k))}\right)_{1 \leqslant i \leqslant p,\ 1 \leqslant k \leqslant q}.$$

This is simply a matrix whose entries are all the indeterminates $x_{\mathbf{p}}$ of the polynomial ring $\mathbb{F}[x_{\mathbf{P}}]$, albeit in a strange order. (The order, again, is tailored to make the "triangularity" argument work nicely. This matrix C is not going to be directly related to the $\widetilde{D}$ we will construct, but will be used in its construction.)

For every $(i, k) \in \mathbf{P}$, define an element $\mathfrak{N}_{(i,k)} \in \mathbb{F}[x_{\mathbf{P}}]$ by

$$\mathfrak{N}_{(i,k)} = \det\left((I_p \mid C)[1 : i \mid i + k - 1 : p + k]\right). \tag{49}$$

[38] Indeed, this matrix A (obtained by substitution of $f(\mathbf{p})$ for $x_{\mathbf{p}}$) will be well-defined for almost every $f \in \mathbb{K}^{\operatorname{Rect}(p,q)}$ (the "almost" is due to the possibility of some denominators becoming 0), and will satisfy $f(\mathbf{p}) = (\operatorname{Grasp}_0 A)(\mathbf{p})$ for every $\mathbf{p} \in \mathbf{P}$ (because $\widetilde{D}$ satisfies (48)), that is, $f = \operatorname{Grasp}_0 A$.

For every $(i,k) \in \mathbf{P}$, define an element $\mathfrak{D}_{(i,k)} \in \mathbb{F}[x_{\mathbf{P}}]$ by

$$\mathfrak{D}_{(i,k)} = \det\left(\left(I_p \mid C\right)[0 : i \mid i+k : p+k]\right). \tag{50}$$

Our plan from here is the following:

Step 1: We will find alternate expressions for the polynomials $\mathfrak{N}_{(i,k)}$ and $\mathfrak{D}_{(i,k)}$ which will give us a better idea of what variables occur in these polynomials.

Step 2: We will show that $\mathfrak{N}_{(i,k)}$ and $\mathfrak{D}_{(i,k)}$ are nonzero for all $(i,k) \in \mathbf{P}$.

Step 3: We will define a $Q_{\mathbf{p}} \in \mathbb{F}(x_{\mathbf{P}})$ for every $\mathbf{p} \in \mathbf{P}$ by $Q_{\mathbf{p}} = \dfrac{\mathfrak{N}_{\mathbf{p}}}{\mathfrak{D}_{\mathbf{p}}}$, and we will show that $Q_{\mathbf{p}} = \left(\operatorname{Grasp}_0\left(I_p \mid C\right)\right)(\mathbf{p})$.

Step 4: We will prove that the family $\left(Q_{\mathbf{p}}\right)_{\mathbf{p}\in\mathbf{P}} \in \left(\mathbb{F}(x_{\mathbf{P}})\right)^{\mathbf{P}}$ is $\mathbf{P}$-triangular.

Step 5: We will use Lemma 15.3 **(b)** and the result of Step 4 to find a matrix $\widetilde{D} \in \left(\mathbb{F}(x_{\mathbf{P}})\right)^{p\times(p+q)}$ satisfying (48).

Let us now go into detail on each specific step (although we won't take that detail very far).

Details of Step 1: Let us introduce three more pieces of notation pertaining to matrices:

- If $\ell \in \mathbb{N}$, and if A_1, A_2, ..., A_k are several matrices with ℓ rows each, then $(A_1 \mid A_2 \mid ... \mid A_k)$ will denote the matrix obtained by starting with an (empty) $\ell\times 0$-matrix, then attaching the matrix A_1 to it on the right, then attaching the matrix A_2 to the result on the right, etc., and finally attaching the matrix A_k to the result on the right. For example, if p is a nonnegative integer, and B is a matrix with p rows, then $(I_p \mid B)$ means the matrix obtained from the $p\times p$ identity matrix I_p by attaching the matrix B to it on the right. (As a concrete example, $\left(I_2 \mid \begin{pmatrix} 1 & -2 \\ 3 & 0 \end{pmatrix}\right) = \begin{pmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & 3 & 0 \end{pmatrix}$.)

- If $\ell \in \mathbb{N}$, if B is a matrix with ℓ rows, and if i_1, i_2, ..., i_k are some elements of $\{1,2,...,\ell\}$, then $\operatorname{rows}_{i_1,i_2,...,i_k} B$ will denote the matrix whose rows (from top to bottom) are the rows labelled i_1, i_2, ..., i_k of the matrix B.

- If u and v are two nonnegative integers, and A is a $u\times v$-matrix, then, for any two integers a and b satisfying $a \leqslant b$, we let $A[a:b]$ be the matrix whose columns (from left to right) are A_a, A_{a+1}, ..., A_{b-1}. This is a natural extension of the notation introduced in Definition 13.1 **(c)** (or, rather, the latter notation is a natural extension of the definition we just made) and has the obvious property that if a, b and c are integers satisfying $a \leqslant b \leqslant c$, then $A[a:c] = A[a:b \mid b:c]$.

We will use without proof a standard fact about determinants:

- Given a commutative ring $\mathbb{L}$, two nonnegative integers a and b satisfying $a \geqslant b$, and a matrix $U \in \mathbb{L}^{a\times b}$, we have

$$\det\left(\begin{pmatrix} I_{a-b} \\ 0_{b\times(a-b)} \end{pmatrix} \mid U\right) = \det\left(\operatorname{rows}_{a-b+1,a-b+2,...,a} U\right) \tag{51}$$

and

$$\det\left(\left(\begin{array}{c} 0_{b\times(a-b)} \\ I_{a-b} \end{array}\right) \mid U\right) = (-1)^{b(a-b)} \det\left(\operatorname{rows}_{1,2,\ldots,b} U\right). \tag{52}$$

(Here, $0_{u\times v}$ denotes the $u \times v$ zero matrix for all $u \in \mathbb{N}$ and $v \in \mathbb{N}$, and $\left(\begin{array}{c} I_{a-b} \\ 0_{b\times(a-b)} \end{array}\right)$ and $\left(\begin{array}{c} 0_{b\times(a-b)} \\ I_{a-b} \end{array}\right)$ are to be read as block matrices.)

Now,

$$(I_p \mid C)[1 : i \mid i+k-1 : p+k] = \left(\left(\begin{array}{c} I_{i-1} \\ 0_{(p-(i-1))\times(i-1)} \end{array}\right) \mid (I_p \mid C)[i+k-1 : p+k]\right),$$

so that

$$\begin{aligned}
&\det\left((I_p \mid C)[1 : i \mid i+k-1 : p+k]\right) \\
&= \det\left(\left(\begin{array}{c} I_{i-1} \\ 0_{(p-(i-1))\times(i-1)} \end{array}\right) \mid (I_p \mid C)[i+k-1 : p+k]\right) \\
&= \det\left(\operatorname{rows}_{i,i+1,\ldots,p}\left((I_p \mid C)[i+k-1 : p+k]\right)\right)
\end{aligned}$$

(by (51)). Thus,

$$\begin{aligned}
\mathfrak{N}_{(i,k)} &= \det\left((I_p \mid C)[1 : i \mid i+k-1 : p+k]\right) \\
&= \det\left(\operatorname{rows}_{i,i+1,\ldots,p}\left((I_p \mid C)[i+k-1 : p+k]\right)\right).
\end{aligned} \tag{53}$$

Also,

$$\begin{aligned}
&(I_p \mid C)[0 : i \mid i+k : p+k] \\
&= \left(\underbrace{(I_p \mid C)_0}_{\substack{=(-1)^{p-1}C_q \\ \text{(by Definition 13.1 } \textbf{(b)}\text{)}}} \mid \left(\begin{array}{c} I_{i-1} \\ 0_{(p-(i-1))\times(i-1)} \end{array}\right) \mid (I_p \mid C)[i+k : p+k]\right) \\
&= \left((-1)^{p-1} C_q \mid \left(\begin{array}{c} I_{i-1} \\ 0_{(p-(i-1))\times(i-1)} \end{array}\right) \mid (I_p \mid C)[i+k : p+k]\right),
\end{aligned}$$

whence

$$\begin{aligned}
&\det\left(\left(I_p \mid C\right)[0:i \mid i+k:p+k]\right)\\
&= \det\left((-1)^{p-1} C_q \;\mid\; \begin{pmatrix} I_{i-1} \\ 0_{(p-(i-1))\times(i-1)} \end{pmatrix} \;\mid\; \left(I_p \mid C\right)[i+k:p+k]\right)\\
&= (-1)^{p-1} \det\left(C_q \;\mid\; \begin{pmatrix} I_{i-1} \\ 0_{(p-(i-1))\times(i-1)} \end{pmatrix} \;\mid\; \left(I_p \mid C\right)[i+k:p+k]\right)\\
&= \underbrace{(-1)^{p-1}(-1)^{i-1}}_{=(-1)^{p-i}} \det\left(\begin{pmatrix} I_{i-1} \\ 0_{(p-(i-1))\times(i-1)} \end{pmatrix} \;\mid\; C_q \;\mid\; \left(I_p \mid C\right)[i+k:p+k]\right)\\
&\qquad \left(\begin{array}{c}\text{since permuting the columns of a matrix multiplies the}\\ \text{determinant by the sign of the permutation}\end{array}\right)\\
&= (-1)^{p-i} \det\left(\begin{pmatrix} I_{i-1} \\ 0_{(p-(i-1))\times(i-1)} \end{pmatrix} \;\mid\; C_q \;\mid\; \left(I_p \mid C\right)[i+k:p+k]\right)\\
&= (-1)^{p-i} \det\left(\operatorname{rows}_{i,i+1,\ldots,p}\left(C_q \;\mid\; \left(I_p \mid C\right)[i+k:p+k]\right)\right)
\end{aligned}$$

(by (51)). Thus,

$$\begin{aligned}
\mathfrak{D}_{(i,k)} &= \det\left(\left(I_p \mid C\right)[0:i \mid i+k:p+k]\right)\\
&= (-1)^{p-i} \det\left(\operatorname{rows}_{i,i+1,\ldots,p}\left(C_q \;\mid\; \left(I_p \mid C\right)[i+k:p+k]\right)\right). \qquad (54)
\end{aligned}$$

We have thus found alternative formulas (53) and (54) for $\mathfrak{N}_{(i,k)}$ and $\mathfrak{D}_{(i,k)}$. While not shorter than the definitions, these formulas involve smaller matrices (unless $i = 1$) and are more useful in understanding the monomials appearing in $\mathfrak{N}_{(i,k)}$ and $\mathfrak{D}_{(i,k)}$.

Details of Step 2: We claim that $\mathfrak{N}_{(i,k)}$ and $\mathfrak{D}_{(i,k)}$ are nonzero for all $(i,k) \in \mathbf{P}$.

Proof. Let $(i,k) \in \mathbf{P}$. Let us first check that $\mathfrak{N}_{(i,k)}$ is nonzero.

There are, in fact, many ways to do this. Here is probably the shortest one: Assume the contrary, i.e., assume that $\mathfrak{N}_{(i,k)} = 0$. Then, every matrix $G \in \mathbb{F}^{p\times(p+q)}$ satisfies $\det\left(G[1:i \mid i+k-1:p+k]\right) = 0$ [39]. But this is absurd, because we can pick G

[39] *Proof.* Let $\widetilde{\mathbb{F}}$ be a field extension of $\mathbb{F}$ such that $\left|\widetilde{\mathbb{F}}\right| = \infty$. (We need this to make sense of Zariski density arguments.) We are going to prove that every matrix $G \in \widetilde{\mathbb{F}}^{p\times(p+q)}$ satisfies $\det\left(G[1:i \mid i+k-1:p+k]\right) = 0$; this will clearly imply the same claim for $G \in \mathbb{F}^{p\times(p+q)}$.

Let $G \in \widetilde{\mathbb{F}}^{p\times(p+q)}$. We want to prove that $\det\left(G[1:i \mid i+k-1:p+k]\right) = 0$. Since this is a polynomial identity in the entries of G, we can WLOG assume that G is generic enough that the first p columns of G are linearly independent (since this just restricts G to a Zariski-dense open subset of $\widetilde{\mathbb{F}}^{p\times(p+q)}$). Assume this. Then, we can write G in the form $(U \mid V)$, with U being the matrix formed by the first p columns of G, and V being the matrix formed by the remaining q columns. Since the first p columns of G are linearly independent, the matrix U is invertible.

Left multiplication by U^{-1} acts on matrices column by column. This yields

$$U^{-1}\cdot\left(G[1:i \mid i+k-1:p+k]\right) = \left(U^{-1}G\right)[1:i \mid i+k-1:p+k].$$

Also, $U^{-1}\underbrace{G}_{=(U\mid V)} = U^{-1}(U \mid V) = \left(U^{-1}U \mid U^{-1}V\right) = \left(I_p \mid U^{-1}V\right)$.

to have the p columns labelled 1, 2, ..., $i-1$, $i+k-1$, $i+k$, ..., $p+k-1$ linearly independent. This contradiction shows that our assumption was wrong. Hence, $\mathfrak{N}_{(i,k)}$ is nonzero. Similarly, $\mathfrak{D}_{(i,k)}$ is nonzero.

Details of Step 3: Define a $Q_{\mathbf{p}} \in \mathbb{F}(x_{\mathbf{P}})$ for every $\mathbf{p} \in \mathbf{P}$ by $Q_{\mathbf{p}} = \dfrac{\mathfrak{N}_{\mathbf{p}}}{\mathfrak{D}_{\mathbf{p}}}$. This is well-defined because Step 2 has shown that $\mathfrak{D}_{\mathbf{p}}$ is nonzero. Moreover, it is easy to see that every $(i,k) \in \mathbf{P}$ satisfies

$$Q_{(i,k)} = (\mathrm{Grasp}_0 (I_p \mid C))((i,k)).$$

[40] In other words, every $\mathbf{p} \in \mathbf{P}$ satisfies

$$Q_{\mathbf{p}} = (\mathrm{Grasp}_0 (I_p \mid C))(\mathbf{p}). \tag{55}$$

Details of Step 4: We are now going to prove that the family $(Q_{\mathbf{p}})_{\mathbf{p}\in\mathbf{P}} \in (\mathbb{F}(x_{\mathbf{P}}))^{\mathbf{P}}$ is $\mathbf{P}$-triangular.

By the definition of $\mathbf{P}$-triangularity, this requires showing that for every $\mathbf{p} \in \mathbf{P}$, there exist elements $\alpha_{\mathbf{p}}$, $\beta_{\mathbf{p}}$, $\gamma_{\mathbf{p}}$, $\delta_{\mathbf{p}}$ of $\mathbb{F}(x_{\mathbf{p}\Downarrow})$ such that $\alpha_{\mathbf{p}}\delta_{\mathbf{p}} - \beta_{\mathbf{p}}\gamma_{\mathbf{p}} \neq 0$ and $Q_{\mathbf{p}} = \dfrac{\alpha_{\mathbf{p}} x_{\mathbf{p}} + \beta_{\mathbf{p}}}{\gamma_{\mathbf{p}} x_{\mathbf{p}} + \delta_{\mathbf{p}}}$ (where $\mathbf{p}\Downarrow$ is defined as in Definition 15.2 **(d)**). So fix $\mathbf{p} \in \mathbf{P}$. Write $\mathbf{p}$ in the form $\mathbf{p} = (i,k)$.

We will actually do something slightly better than we need. We will find elements $\alpha_{\mathbf{p}}$, $\beta_{\mathbf{p}}$, $\gamma_{\mathbf{p}}$, $\delta_{\mathbf{p}}$ of $\mathbb{F}[x_{\mathbf{p}\Downarrow}]$ (not just of $\mathbb{F}(x_{\mathbf{p}\Downarrow})$) such that $\alpha_{\mathbf{p}}\delta_{\mathbf{p}} - \beta_{\mathbf{p}}\gamma_{\mathbf{p}} \neq 0$ and $\mathfrak{N}_{\mathbf{p}} = \alpha_{\mathbf{p}} x_{\mathbf{p}} + \beta_{\mathbf{p}}$ and $\mathfrak{D}_{\mathbf{p}} = \gamma_{\mathbf{p}} x_{\mathbf{p}} + \delta_{\mathbf{p}}$. (Of course, the conditions $\mathfrak{N}_{\mathbf{p}} = \alpha_{\mathbf{p}} x_{\mathbf{p}} + \beta_{\mathbf{p}}$ and $\mathfrak{D}_{\mathbf{p}} = \gamma_{\mathbf{p}} x_{\mathbf{p}} + \delta_{\mathbf{p}}$ combined imply $Q_{\mathbf{p}} = \dfrac{\alpha_{\mathbf{p}} x_{\mathbf{p}} + \beta_{\mathbf{p}}}{\gamma_{\mathbf{p}} x_{\mathbf{p}} + \delta_{\mathbf{p}}}$, hence the yearned-for $\mathbf{P}$-triangularity.)

Now, we have $\mathfrak{N}_{(i,k)} = 0$. Since $\mathfrak{N}_{(i,k)} = \det((I_p \mid C)[1:i \mid i+k-1:p+k])$, this yields that $\det((I_p \mid C)[1:i \mid i+k-1:p+k]) = 0$. But the matrix C is, in some sense, the "most generic matrix": namely, the entries of the matrix C are pairwise distinct commuting indeterminates, and therefore we can obtain any other matrix (over a commutative $\mathbb{F}$-algebra) from C by substituting the corresponding values for the indeterminates. In particular, we can make a substitution that turns C into $U^{-1}V$. Thus, from $\det((I_p \mid C)[1:i \mid i+k-1:p+k]) = 0$, we obtain $\det\left(\left(I_p \mid U^{-1}V\right)[1:i \mid i+k-1:p+k]\right) = 0$.

Now,

$$\begin{aligned}
&(\det U)^{-1} \cdot \det(G[1:i \mid i+k-1:p+k]) \\
&= \det\left(\underbrace{U^{-1} \cdot (G[1:i \mid i+k-1:p+k])}_{=(U^{-1}G)[1:i\mid i+k-1:p+k]}\right) = \det\left(\left(\underbrace{U^{-1}G}_{=(I_p\mid U^{-1}V)}\right)[1:i \mid i+k-1:p+k]\right) \\
&= \det\left(\left(I_p \mid U^{-1}V\right)[1:i \mid i+k-1:p+k]\right) = 0.
\end{aligned}$$

Multiplying this with $\det U$ (which is nonzero since U is invertible), we obtain $\det(G[1:i \mid i+k-1:p+k]) = 0$, qed.

[40] Indeed, the definition of $\mathrm{Grasp}_0 (I_p \mid C)$ yields

$$(\mathrm{Grasp}_0 (I_p \mid C))((i,k)) = \frac{\det((I_p \mid C)[1:i \mid i+k-1:p+k])}{\det((I_p \mid C)[0:i \mid i+k:p+k])} = \frac{\mathfrak{N}_{(i,k)}}{\mathfrak{D}_{(i,k)}}$$

(by (49) and (50)).

Let us first deal with two "boundary" cases: the case when $k=1$, and the case when $k \neq 1$ but $i = p$.

The case when $k = 1$ is very easy. In fact, in this case, it is easy to prove that $\mathfrak{N}_{\mathbf{p}} = 1$ (using (53)) and that $\mathfrak{D}_{\mathbf{p}} = (-1)^{i+p} x_{\mathbf{p}}$ (using (54)). Consequently, we can take $\alpha_{\mathbf{p}} = 0$, $\beta_{\mathbf{p}} = 1$, $\gamma_{\mathbf{p}} = (-1)^{i+p}$ and $\delta_{\mathbf{p}} = 0$, and it is clear that all three requirements $\alpha_{\mathbf{p}}\delta_{\mathbf{p}} - \beta_{\mathbf{p}}\gamma_{\mathbf{p}} \neq 0$ and $\mathfrak{N}_{\mathbf{p}} = \alpha_{\mathbf{p}} x_{\mathbf{p}} + \beta_{\mathbf{p}}$ and $\mathfrak{D}_{\mathbf{p}} = \gamma_{\mathbf{p}} x_{\mathbf{p}} + \delta_{\mathbf{p}}$ are satisfied.

The case when $k \neq 1$ but $i = p$ is not much harder. In this case, (53) simplifies to $\mathfrak{N}_{\mathbf{p}} = x_{\mathbf{p}}$, and (54) simplifies to $\mathfrak{D}_{\mathbf{p}} = x_{(p,1)}$. Hence, we can take $\alpha_{\mathbf{p}} = 1$, $\beta_{\mathbf{p}} = 0$, $\gamma_{\mathbf{p}} = 0$ and $\delta_{\mathbf{p}} = x_{(p,1)}$ to achieve $\alpha_{\mathbf{p}}\delta_{\mathbf{p}} - \beta_{\mathbf{p}}\gamma_{\mathbf{p}} \neq 0$ and $\mathfrak{N}_{\mathbf{p}} = \alpha_{\mathbf{p}} x_{\mathbf{p}} + \beta_{\mathbf{p}}$ and $\mathfrak{D}_{\mathbf{p}} = \gamma_{\mathbf{p}} x_{\mathbf{p}} + \delta_{\mathbf{p}}$. Note that this choice of $\delta_{\mathbf{p}}$ is legitimate because $x_{(p,1)}$ does lie in $\mathbb{F}[x_{\mathbf{p}\Downarrow}]$ (since $(p,1) \in \mathbf{p}\Downarrow$).

Now that these two cases are settled, let us deal with the remaining case. So we have neither $k = 1$ nor $i = p$.

Consider the matrix $\operatorname{rows}_{i,i+1,\ldots,p}((I_p \mid C)[i+k-1 : p+k])$ (this matrix appears on the right hand side of (53)). Each entry of this matrix comes either from the matrix I_p or from the matrix C. If it comes from I_p, it clearly lies in $\mathbb{F}[x_{\mathbf{p}\Downarrow}]$. If it comes from C, it has the form $x_{\mathbf{q}}$ for some $\mathbf{q} \in \mathbf{P}$, and this $\mathbf{q}$ belongs to $\mathbf{p}\Downarrow$ unless the entry is the $(1, p-i+1)$-th entry. Therefore, each entry of the matrix $(I_p \mid C)[i+k-1 : p+k]$ apart from the $(1, p-i+1)$-th entry lies in $\mathbb{F}[x_{\mathbf{p}\Downarrow}]$, whereas the $(1, p-i+1)$-th entry is $x_{\mathbf{p}}$. Hence, if we use Laplace expansion with respect to the first row to compute the determinant of this matrix, we obtain a formula of the form

$$\begin{aligned}
&\det\left(\operatorname{rows}_{i,i+1,\ldots,p}\left((I_p \mid C)[i+k-1 : p+k]\right)\right) \\
&= x_{\mathbf{p}} \cdot (\text{some polynomial in entries lying in } \mathbb{F}[x_{\mathbf{p}\Downarrow}]) \\
&\qquad + (\text{more polynomials in entries lying in } \mathbb{F}[x_{\mathbf{p}\Downarrow}]) \\
&\in \mathbb{F}[x_{\mathbf{p}\Downarrow}] \cdot x_{\mathbf{p}} + \mathbb{F}[x_{\mathbf{p}\Downarrow}].
\end{aligned}$$

In other words, there exist elements $\alpha_{\mathbf{p}}$ and $\beta_{\mathbf{p}}$ of $\mathbb{F}[x_{\mathbf{p}\Downarrow}]$ such that
$\det\left(\operatorname{rows}_{i,i+1,\ldots,p}\left((I_p \mid C)[i+k-1 : p+k]\right)\right) = \alpha_{\mathbf{p}} x_{\mathbf{p}} + \beta_{\mathbf{p}}$. Consider these $\alpha_{\mathbf{p}}$ and $\beta_{\mathbf{p}}$. We have

$$\begin{aligned}
\mathfrak{N}_{\mathbf{p}} = \mathfrak{N}_{(i,k)} &= \det\left(\operatorname{rows}_{i,i+1,\ldots,p}\left((I_p \mid C)[i+k-1 : p+k]\right)\right) \qquad (\text{by (53)}) \qquad (56) \\
&= \alpha_{\mathbf{p}} x_{\mathbf{p}} + \beta_{\mathbf{p}}. \qquad (57)
\end{aligned}$$

We can similarly deal with the matrix $\operatorname{rows}_{i,i+1,\ldots,p}(C_q \mid (I_p \mid C)[i+k : p+k])$ which appears on the right hand side of (54). Again, each entry of this matrix apart from the $(1, p-i+1)$-th entry lies in $\mathbb{F}[x_{\mathbf{p}\Downarrow}]$, whereas the $(1, p-i+1)$-th entry is $x_{\mathbf{p}}$. Using Laplace expansion again, we thus see that

$$\det\left(\operatorname{rows}_{i,i+1,\ldots,p}\left(C_q \mid (I_p \mid C)[i+k : p+k]\right)\right) \in \mathbb{F}[x_{\mathbf{p}\Downarrow}] \cdot x_{\mathbf{p}} + \mathbb{F}[x_{\mathbf{p}\Downarrow}],$$

so that

$$(-1)^{p-i} \det\left(\operatorname{rows}_{i,i+1,\ldots,p}\left(C_q \mid (I_p \mid C)[i+k : p+k]\right)\right) \in \mathbb{F}[x_{\mathbf{p}\Downarrow}] \cdot x_{\mathbf{p}} + \mathbb{F}[x_{\mathbf{p}\Downarrow}].$$

Hence, there exist elements $\gamma_{\mathbf{p}}$ and $\delta_{\mathbf{p}}$ of $\mathbb{F}[x_{\mathbf{p}\Downarrow}]$ such that $(-1)^{p-i} \det(\operatorname{rows}_{i,i+1,\ldots,p}(C_q \mid (I_p \mid C)[i+k:p+k])) = \gamma_{\mathbf{p}} x_{\mathbf{p}} + \delta_{\mathbf{p}}$. Consider these $\gamma_{\mathbf{p}}$ and $\delta_{\mathbf{p}}$. We have

$$\begin{aligned} \mathfrak{D}_{\mathbf{p}} = \mathfrak{D}_{(i,k)} &= (-1)^{p-i} \det(\operatorname{rows}_{i,i+1,\ldots,p}(C_q \mid (I_p \mid C)[i+k:p+k])) \qquad \text{(by (54))} \\ &= \gamma_{\mathbf{p}} x_{\mathbf{p}} + \delta_{\mathbf{p}}. \end{aligned} \tag{58}$$

We thus have found elements $\alpha_{\mathbf{p}}$, $\beta_{\mathbf{p}}$, $\gamma_{\mathbf{p}}$, $\delta_{\mathbf{p}}$ of $\mathbb{F}[x_{\mathbf{p}\Downarrow}]$ satisfying $\mathfrak{N}_{\mathbf{p}} = \alpha_{\mathbf{p}} x_{\mathbf{p}} + \beta_{\mathbf{p}}$ and $\mathfrak{D}_{\mathbf{p}} = \gamma_{\mathbf{p}} x_{\mathbf{p}} + \delta_{\mathbf{p}}$. In order to finish the proof of $\mathbf{P}$-triangularity, we only need to show that $\alpha_{\mathbf{p}}\delta_{\mathbf{p}} - \beta_{\mathbf{p}}\gamma_{\mathbf{p}} \neq 0$.

In order to achieve this goal, we notice that

$$\alpha_{\mathbf{p}} \underbrace{\mathfrak{D}_{\mathbf{p}}}_{=\gamma_{\mathbf{p}} x_{\mathbf{p}} + \delta_{\mathbf{p}}} - \underbrace{\mathfrak{N}_{\mathbf{p}}}_{=\alpha_{\mathbf{p}} x_{\mathbf{p}} + \beta_{\mathbf{p}}} \gamma_{\mathbf{p}} = \alpha_{\mathbf{p}}(\gamma_{\mathbf{p}} x_{\mathbf{p}} + \delta_{\mathbf{p}}) - (\alpha_{\mathbf{p}} x_{\mathbf{p}} + \beta_{\mathbf{p}})\gamma_{\mathbf{p}} = \alpha_{\mathbf{p}}\delta_{\mathbf{p}} - \beta_{\mathbf{p}}\gamma_{\mathbf{p}}.$$

Hence, proving $\alpha_{\mathbf{p}}\delta_{\mathbf{p}} - \beta_{\mathbf{p}}\gamma_{\mathbf{p}} \neq 0$ is equivalent to proving $\alpha_{\mathbf{p}}\mathfrak{D}_{\mathbf{p}} - \mathfrak{N}_{\mathbf{p}}\gamma_{\mathbf{p}} \neq 0$. It is the latter that we are going to do, because $\alpha_{\mathbf{p}}$, $\mathfrak{D}_{\mathbf{p}}$, $\mathfrak{N}_{\mathbf{p}}$ and $\gamma_{\mathbf{p}}$ are easier to get our hands on than $\beta_{\mathbf{p}}$ and $\delta_{\mathbf{p}}$.

So we need to prove that $\alpha_{\mathbf{p}}\mathfrak{D}_{\mathbf{p}} - \mathfrak{N}_{\mathbf{p}}\gamma_{\mathbf{p}} \neq 0$. To do so, we look back at our proof of

$$\det(\operatorname{rows}_{i,i+1,\ldots,p}((I_p \mid C)[i+k-1:p+k])) \in \mathbb{F}[x_{\mathbf{p}\Downarrow}] \cdot x_{\mathbf{p}} + \mathbb{F}[x_{\mathbf{p}\Downarrow}].$$

This proof proceeded by applying Laplace expansion with respect to the first row to the matrix $\operatorname{rows}_{i,i+1,\ldots,p}((I_p \mid C)[i+k-1:p+k])$. The only term involving $x_{\mathbf{p}}$ was

$$x_{\mathbf{p}} \cdot (\text{some polynomial in entries lying in } \mathbb{F}[x_{\mathbf{p}\Downarrow}]).$$

Recalling the statement of Laplace expansion, we notice that "some polynomial in entries lying in $\mathbb{F}[x_{\mathbf{p}\Downarrow}]$" in this term is actually the $(1, p-i+1)$-th cofactor of the matrix $\operatorname{rows}_{i,i+1,\ldots,p}((I_p \mid C)[i+k-1:p+k])$. Hence,

$$\begin{aligned} \alpha_{\mathbf{p}} &= (\text{the } (1, p-i+1)\text{-th cofactor of } \operatorname{rows}_{i,i+1,\ldots,p}((I_p \mid C)[i+k-1:p+k])) \\ &= (-1)^{p-i} \cdot \det(\operatorname{rows}_{i+1,i+2,\ldots,p}((I_p \mid C)[i+k-1:p+k-1])). \end{aligned} \tag{59}$$

Similarly,

$$\gamma_{\mathbf{p}} = \det(\operatorname{rows}_{i+1,i+2,\ldots,p}(C_q \mid (I_p \mid C)[i+k:p+k-1])) \tag{60}$$

(note that we lost the sign $(-1)^{p-i}$ from (58) since it got cancelled against the $(-1)^{p-(i+1)}$ arising from the definition of a cofactor).

Now, recall that we have neither $k = 1$ nor $i = p$. Hence, $(i+1, k-1)$ also belongs to $\mathbf{P}$, so we can apply (53) to $(i+1, k-1)$ in lieu of (i, k), and obtain

$$\mathfrak{N}_{(i+1,k-1)} = \det(\operatorname{rows}_{i+1,i+2,\ldots,p}((I_p \mid C)[i+k-1:p+k-1])).$$

In light of this, (59) becomes

$$\alpha_{\mathbf{p}} = (-1)^{p-i} \cdot \mathfrak{N}_{(i+1,k-1)}.$$

Similarly, we can apply (54) to $(i+1, k-1)$ in lieu of (i, k), and use this to rewrite (60) as

$$\gamma_{\mathbf{p}} = (-1)^{p-(i+1)} \cdot \mathfrak{D}_{(i+1,k-1)}.$$

Hence,

$$\begin{aligned}
&\underbrace{\alpha_{\mathbf{p}}}_{=(-1)^{p-i}\cdot\mathfrak{N}_{(i+1,k-1)}} \mathfrak{D}_{\mathbf{p}} - \mathfrak{N}_{\mathbf{p}} \underbrace{\gamma_{\mathbf{p}}}_{=(-1)^{p-(i+1)}\cdot\mathfrak{D}_{(i+1,k-1)}} \\
&= (-1)^{p-i} \cdot \mathfrak{N}_{(i+1,k-1)} \cdot \mathfrak{D}_{\mathbf{p}} - \mathfrak{N}_{\mathbf{p}} \cdot \underbrace{(-1)^{p-(i+1)}}_{=-(-1)^{p-i}} \cdot \mathfrak{D}_{(i+1,k-1)} \\
&= (-1)^{p-i} \cdot \left(\mathfrak{N}_{(i+1,k-1)} \mathfrak{D}_{\mathbf{p}} + \mathfrak{N}_{\mathbf{p}} \mathfrak{D}_{(i+1,k-1)} \right).
\end{aligned}$$

Thus, we can shift our goal from proving $\alpha_{\mathbf{p}} \mathfrak{D}_{\mathbf{p}} - \mathfrak{N}_{\mathbf{p}} \gamma_{\mathbf{p}} \neq 0$ to proving $\mathfrak{N}_{(i+1,k-1)} \mathfrak{D}_{\mathbf{p}} + \mathfrak{N}_{\mathbf{p}} \mathfrak{D}_{(i+1,k-1)} \neq 0$.

But this turns out to be surprisingly simple: Since $\mathbf{p} = (i, k)$, we have

$$\begin{aligned}
&\mathfrak{N}_{(i+1,k-1)} \mathfrak{D}_{\mathbf{p}} + \mathfrak{N}_{\mathbf{p}} \mathfrak{D}_{(i+1,k-1)} \\
&= \mathfrak{N}_{(i+1,k-1)} \mathfrak{D}_{(i,k)} + \mathfrak{N}_{(i,k)} \mathfrak{D}_{(i+1,k-1)} = \mathfrak{D}_{(i,k)} \cdot \mathfrak{N}_{(i+1,k-1)} + \mathfrak{N}_{(i,k)} \cdot \mathfrak{D}_{(i+1,k-1)} \\
&= \det\left((I_p \mid C) [0 : i \mid i+k : p+k] \right) \cdot \det\left((I_p \mid C) [1 : i+1 \mid i+k-1 : p+k-1] \right) \\
&\qquad + \det\left((I_p \mid C) [1 : i \mid i+k-1 : p+k] \right) \cdot \det\left((I_p \mid C) [0 : i+1 \mid i+k : p+k-1] \right) \\
&\qquad \left(\begin{array}{c} \text{here, we just substituted } \mathfrak{D}_{(i,k)},\ \mathfrak{N}_{(i+1,k-1)},\ \mathfrak{N}_{(i,k)} \text{ and } \mathfrak{D}_{(i+1,k-1)} \\ \text{by their definitions} \end{array} \right) \\
&= \det\left((I_p \mid C) [0 : i \mid i+k-1 : p+k-1] \right) \cdot \det\left((I_p \mid C) [1 : i+1 \mid i+k : p+k] \right) \qquad (61)
\end{aligned}$$

(by Theorem 14.1, applied to p, $p+q$, $(I_p \mid C)$, 1, i, $i+k$ and $p+k-1$ instead of u, v, A, a, b, c and d). On the other hand, $(i, k-1)$ and $(i+1, k)$ also belong to $\mathbf{P}$ and satisfy

$$\mathfrak{D}_{(i,k-1)} = \det\left((I_p \mid C) [0 : i \mid i+k-1 : p+k-1] \right)$$

and

$$\mathfrak{N}_{(i+1,k)} = \det\left((I_p \mid C) [1 : i+1 \mid i+k : p+k] \right)$$

(by the respective definitions of $\mathfrak{D}_{(i,k-1)}$ and $\mathfrak{N}_{(i+1,k)}$). Hence, (61) becomes

$$\begin{aligned}
&\mathfrak{N}_{(i+1,k-1)} \mathfrak{D}_{\mathbf{p}} + \mathfrak{N}_{\mathbf{p}} \mathfrak{D}_{(i+1,k-1)} \\
&= \underbrace{\det\left((I_p \mid C) [0 : i \mid i+k-1 : p+k-1] \right)}_{=\mathfrak{D}_{(i,k-1)}} \cdot \underbrace{\det\left((I_p \mid C) [1 : i+1 \mid i+k : p+k] \right)}_{=\mathfrak{N}_{(i+1,k)}} \\
&= \mathfrak{D}_{(i,k-1)} \cdot \mathfrak{N}_{(i+1,k)} \neq 0
\end{aligned}$$

(since the result of Step 2 shows that $\mathfrak{D}_{(i,k-1)}$ and $\mathfrak{N}_{(i+1,k)}$ are nonzero). This finishes our proof of $\mathfrak{N}_{(i+1,k-1)}\mathfrak{D}_{\mathbf{p}} + \mathfrak{N}_{\mathbf{p}}\mathfrak{D}_{(i+1,k-1)} \neq 0$, thus also of $\alpha_{\mathbf{p}}\mathfrak{D}_{\mathbf{p}} - \mathfrak{N}_{\mathbf{p}}\gamma_{\mathbf{p}} \neq 0$, hence also of $\alpha_{\mathbf{p}}\delta_{\mathbf{p}} - \beta_{\mathbf{p}}\gamma_{\mathbf{p}} \neq 0$, and ultimately of the $\mathbf{P}$-triangularity of the family $(Q_{\mathbf{p}})_{\mathbf{p}\in\mathbf{P}}$.

Details of Step 5: Recall that our goal is to prove the existence of a matrix $\widetilde{D} \in (\mathbb{F}(x_{\mathbf{P}}))^{p\times(p+q)}$ satisfying (48).

Since Step 4, we know that the family $(Q_{\mathbf{p}})_{\mathbf{p}\in\mathbf{P}} \in (\mathbb{F}(x_{\mathbf{P}}))^{\mathbf{P}}$ is $\mathbf{P}$-triangular. Hence, Lemma 15.3 **(b)** shows that there exists a $\mathbf{P}$-triangular family $(R_{\mathbf{p}})_{\mathbf{p}\in\mathbf{P}} \in (\mathbb{F}(x_{\mathbf{P}}))^{\mathbf{P}}$ such that every $\mathbf{q} \in \mathbf{P}$ satisfies $Q_{\mathbf{q}}\left((R_{\mathbf{p}})_{\mathbf{p}\in\mathbf{P}}\right) = x_{\mathbf{q}}$. Consider this $(R_{\mathbf{p}})_{\mathbf{p}\in\mathbf{P}}$. Applying Lemma 15.3 **(a)** to this family $(R_{\mathbf{p}})_{\mathbf{p}\in\mathbf{P}}$, we conclude that $(R_{\mathbf{p}})_{\mathbf{p}\in\mathbf{P}}$ is algebraically independent.

In Step 3, we have shown that $Q_{\mathbf{p}} = (\operatorname{Grasp}_0(I_p \mid C))(\mathbf{p})$ for every $\mathbf{p} \in \mathbf{P}$. Renaming $\mathbf{p}$ as $\mathbf{q}$, we rewrite this as follows:

$$Q_{\mathbf{q}} = (\operatorname{Grasp}_0(I_p \mid C))(\mathbf{q}) \qquad \text{for every } \mathbf{q} \in \mathbf{P}. \tag{62}$$

Now, let $\widetilde{C} \in (\mathbb{F}(x_{\mathbf{P}}))^{p\times(p+q)}$ denote the matrix obtained from the matrix $C \in (\mathbb{F}[x_{\mathbf{P}}])^{p\times(p+q)}$ by substituting $(R_{\mathbf{p}})_{\mathbf{p}\in\mathbf{P}}$ for the variables $(x_{\mathbf{p}})_{\mathbf{p}\in\mathbf{P}}$. Since (62) is an identity between rational functions in the variables $(x_{\mathbf{p}})_{\mathbf{p}\in\mathbf{P}}$, we thus can substitute $(R_{\mathbf{p}})_{\mathbf{p}\in\mathbf{P}}$ for the variables $(x_{\mathbf{p}})_{\mathbf{p}\in\mathbf{P}}$ in (62)[41], and obtain

$$Q_{\mathbf{q}}\left((R_{\mathbf{p}})_{\mathbf{p}\in\mathbf{P}}\right) = \left(\operatorname{Grasp}_0\left(I_p \mid \widetilde{C}\right)\right)(\mathbf{q}) \qquad \text{for every } \mathbf{q} \in \mathbf{P}$$

(since this substitution takes the matrix C to $\widetilde{C}$). But since $Q_{\mathbf{q}}\left((R_{\mathbf{p}})_{\mathbf{p}\in\mathbf{P}}\right) = x_{\mathbf{q}}$ for every $\mathbf{q} \in \mathbf{P}$, this rewrites as

$$x_{\mathbf{q}} = \left(\operatorname{Grasp}_0\left(I_p \mid \widetilde{C}\right)\right)(\mathbf{q}) \qquad \text{for every } \mathbf{q} \in \mathbf{P}.$$

Upon renaming $\mathbf{q}$ as $\mathbf{p}$ again, this becomes

$$x_{\mathbf{p}} = \left(\operatorname{Grasp}_0\left(I_p \mid \widetilde{C}\right)\right)(\mathbf{p}) \qquad \text{for every } \mathbf{p} \in \mathbf{P}.$$

Hence, there exists a matrix $\widetilde{D} \in (\mathbb{F}(x_{\mathbf{P}}))^{p\times(p+q)}$ satisfying (48) (namely, $\widetilde{D} = \left(I_p \mid \widetilde{C}\right)$). Thus, as we know, Proposition 13.14 is proven. $\square$

16 The rectangle: finishing the proofs

As promised, we now use Propositions 13.13 and 13.14 to derive our initially stated results on rectangles. First, we formulate an easy consequence of Proposition 13.13:

[41] The substitution does not suffer from vanishing denominators because $(R_{\mathbf{p}})_{\mathbf{p}\in\mathbf{P}}$ is algebraically independent.

Corollary 16.1. Let $\mathbb{K}$ be a field. Let p and q be two positive integers. Let $A \in \mathbb{K}^{p\times(p+q)}$ be a matrix. Then, every $i \in \mathbb{N}$ satisfies

$$\operatorname{Grasp}_{-i} A = R^{i}_{\operatorname{Rect}(p,q)} \left(\operatorname{Grasp}_0 A\right)$$

(provided that A is sufficiently generic in the sense of Zariski topology that both sides of this equality are well-defined).

Proof of Corollary 16.1 (sketched). We will prove Corollary 16.1 by induction over i:

Induction base: For $i = 0$, the claim of Corollary 16.1 boils down to $\operatorname{Grasp}_0 A = R^{0}_{\operatorname{Rect}(p,q)} \left(\operatorname{Grasp}_0 A\right)$. This is obvious, and so the induction base is complete.

Induction step: Let $j \in \mathbb{N}$. Assume that Corollary 16.1 holds for $i = j$. We need to prove that Corollary 16.1 holds for $i = j + 1$ as well.

Proposition 13.13 (applied to $-(j+1)$ instead of j) yields

$$\begin{aligned}\operatorname{Grasp}_{-(j+1)} A &= R_{\operatorname{Rect}(p,q)} \left(\operatorname{Grasp}_{-(j+1)+1} A\right) = R_{\operatorname{Rect}(p,q)} \left(\underbrace{\operatorname{Grasp}_{-j} A}_{\substack{=R^{j}_{\operatorname{Rect}(p,q)}(\operatorname{Grasp}_0 A) \\ \text{(since Corollary 16.1} \\ \text{holds for } i=j)}}\right) \\ &= R_{\operatorname{Rect}(p,q)} \left(R^{j}_{\operatorname{Rect}(p,q)} \left(\operatorname{Grasp}_0 A\right)\right) = R^{j+1}_{\operatorname{Rect}(p,q)} \left(\operatorname{Grasp}_0 A\right).\end{aligned}$$

In other words, Corollary 16.1 holds for $i = j + 1$. This completes the induction step. The induction proof of Corollary 16.1 is thus finished. □

Proof of Theorem 11.5 (sketched). We need to show that $\operatorname{ord}\left(R_{\operatorname{Rect}(p,q)}\right) = p + q$. According to Proposition 12.2, it is enough to prove that almost every (in the Zariski sense) reduced labelling $f \in \mathbb{K}^{\widehat{\operatorname{Rect}(p,q)}}$ satisfies $R^{p+q}_{\operatorname{Rect}(p,q)} f = f$. So let $f \in \mathbb{K}^{\widehat{\operatorname{Rect}(p,q)}}$ be a sufficiently generic reduced labelling. In other words, f is a sufficiently generic element of $\mathbb{K}^{\operatorname{Rect}(p,q)}$ (because the reduced labellings $\mathbb{K}^{\widehat{\operatorname{Rect}(p,q)}}$ are being identified with the elements of $\mathbb{K}^{\operatorname{Rect}(p,q)}$). Due to Proposition 13.14, there exists a matrix $A \in \mathbb{K}^{p\times(p+q)}$ satisfying $f = \operatorname{Grasp}_0 A$. Consider this A. Due to Corollary 16.1 (applied to $i = p + q$), we have

$$\operatorname{Grasp}_{-(p+q)} A = R^{p+q}_{\operatorname{Rect}(p,q)} \left(\underbrace{\operatorname{Grasp}_0 A}_{=f}\right) = R^{p+q}_{\operatorname{Rect}(p,q)} f.$$

But Proposition 13.11 (applied to $j = -(p+q)$) yields

$$\begin{aligned}\operatorname{Grasp}_{-(p+q)} A &= \operatorname{Grasp}_{p+q+(-(p+q))} A = \operatorname{Grasp}_0 A \qquad (\text{since } p + q + (-(p+q)) = 0) \\ &= f.\end{aligned}$$

Hence, $f = \operatorname{Grasp}_{-(p+q)} A = R^{p+q}_{\operatorname{Rect}(p,q)} f$. In other words, $R^{p+q}_{\operatorname{Rect}(p,q)} f = f$. This (as we know) proves Theorem 11.5. □

Proof of Theorem 12.3 (sketched). Let us regard the reduced labelling $f \in \mathbb{K}^{\widehat{\operatorname{Rect}(p,q)}}$ as an element of $\mathbb{K}^{\operatorname{Rect}(p,q)}$ (because we identify reduced labellings in $\mathbb{K}^{\widehat{\operatorname{Rect}(p,q)}}$ with elements of $\mathbb{K}^{\operatorname{Rect}(p,q)}$). We assume WLOG that this element $f \in \mathbb{K}^{\operatorname{Rect}(p,q)}$ is generic enough (among the reduced labellings) for Proposition 13.14 to apply. By Proposition 13.14, there exists a matrix $A \in \mathbb{K}^{p\times(p+q)}$ satisfying $f = \operatorname{Grasp}_0 A$. Consider this A. Due to Corollary 16.1 (applied to $i + k - 1$ instead of i), we have

$$\operatorname{Grasp}_{-(i+k-1)} A = R^{i+k-1}_{\operatorname{Rect}(p,q)} \left(\underbrace{\operatorname{Grasp}_0 A}_{=f} \right) = R^{i+k-1}_{\operatorname{Rect}(p,q)} f.$$

But Proposition 13.12 (applied to $j = -(i + k - 1)$) yields

$$\begin{aligned}
\left(\operatorname{Grasp}_{-(i+k-1)} A\right)((i,k)) &= \frac{1}{\left(\operatorname{Grasp}_{-(i+k-1)+i+k-1} A\right)((p+1-i, q+1-k))} \\
&= \frac{1}{f((p+1-i, q+1-k))} \\
&\qquad \left(\text{since } \operatorname{Grasp}_{-(i+k-1)+i+k-1} A = \operatorname{Grasp}_0 A = f\right),
\end{aligned}$$

so that

$$f((p+1-i, q+1-k)) = \frac{1}{\left(\operatorname{Grasp}_{-(i+k-1)} A\right)((i,k))} = \frac{1}{\left(R^{i+k-1}_{\operatorname{Rect}(p,q)} f\right)((i,k))}$$

(since $\operatorname{Grasp}_{-(i+k-1)} A = R^{i+k-1}_{\operatorname{Rect}(p,q)} f$). This proves Theorem 12.3. $\square$

Proof of Theorem 11.7 (sketched). We will be using the notation $(a_0, a_1, ..., a_{n+1}) \flat f$ defined in Definition 5.2.

Let $f \in \mathbb{K}^{\widehat{\operatorname{Rect}(p,q)}}$ be arbitrary. By genericity, we assume WLOG that $f(0)$ and $f(1)$ are nonzero.

Let $n = p+q-1$. Then, $\operatorname{Rect}(p,q)$ is an n-graded poset. Also, $i+k-1 \in \{0, 1, ..., n\}$. Moreover, $1 \leqslant n - i - k + 2 \leqslant n$.

Define an $(n + 2)$-tuple $(a_0, a_1, ..., a_{n+1}) \in \mathbb{K}^{n+2}$ by

$$a_r = \begin{cases} \dfrac{1}{f(0)}, & \text{if } r = 0; \\ 1, & \text{if } 1 \leqslant r \leqslant n; \\ \dfrac{1}{f(1)}, & \text{if } r = n+1 \end{cases} \qquad \text{for every } r \in \{0, 1, ..., n+1\}.$$

Thus, $a_{n-i-k+2} = 1$ (since $1 \leqslant n - i - k + 2 \leqslant n$) and $a_0 = \dfrac{1}{f(0)}$ and $a_{n+1} = \dfrac{1}{f(1)}$.

Let $f' = (a_0, a_1, ..., a_{n+1}) \flat f$. Then, it is easy to see from the definition of $(a_0, a_1, ..., a_{n+1}) \flat f$ that $f'(0) = 1$ and $f'(1) = 1$. In other words, f' is a reduced $\mathbb{K}$-labelling. Hence, Theorem

12.3 (applied to f' instead of f) yields

$$f'\left(\left(p+1-i,q+1-k\right)\right) = \frac{1}{\left(R_{\operatorname{Rect}(p,q)}^{i+k-1}\left(f'\right)\right)\left(\left(i,k\right)\right)}. \tag{63}$$

On the other hand, again from the definition of $f' = (a_0, a_1, ..., a_{n+1}) \flat f$, it is easy to see that $f'(v) = f(v)$ for every $v \in \operatorname{Rect}(p,q)$. This yields, in particular, that $f'\left(\left(p+1-i,q+1-k\right)\right) = f\left(\left(p+1-i,q+1-k\right)\right)$.

But let us define an element $\widehat{a}_\kappa^{(\ell)} \in \mathbb{K}^\times$ for every $\ell \in \{0,1,...,n+1\}$ and $\kappa \in \{0,1,...,n+1\}$ as in Proposition 5.5. Then, Proposition 5.5 (applied to $\ell = i+k-1$) yields

$$R_{\operatorname{Rect}(p,q)}^{i+k-1}\left(\left(a_0,a_1,...,a_{n+1}\right)\flat f\right) = \left(\widehat{a}_0^{(i+k-1)}, \widehat{a}_1^{(i+k-1)}, ..., \widehat{a}_{n+1}^{(i+k-1)}\right) \flat \left(R_{\operatorname{Rect}(p,q)}^{i+k-1} f\right).$$

Since $(a_0, a_1, ..., a_{n+1}) \flat f = f'$, this rewrites as

$$R_{\operatorname{Rect}(p,q)}^{i+k-1}\left(f'\right) = \left(\widehat{a}_0^{(i+k-1)}, \widehat{a}_1^{(i+k-1)}, ..., \widehat{a}_{n+1}^{(i+k-1)}\right) \flat \left(R_{\operatorname{Rect}(p,q)}^{i+k-1} f\right).$$

Hence,

$$\begin{aligned}
&\left(R_{\operatorname{Rect}(p,q)}^{i+k-1}\left(f'\right)\right)\left(\left(i,k\right)\right) \\
&= \left(\left(\widehat{a}_0^{(i+k-1)}, \widehat{a}_1^{(i+k-1)}, ..., \widehat{a}_{n+1}^{(i+k-1)}\right) \flat \left(R_{\operatorname{Rect}(p,q)}^{i+k-1} f\right)\right)\left(\left(i,k\right)\right) \\
&= \widehat{a}_{\deg((i,k))}^{(i+k-1)} \cdot \left(R_{\operatorname{Rect}(p,q)}^{i+k-1} f\right)\left(\left(i,k\right)\right) \\
&\qquad \left(\text{by the definition of } \left(\widehat{a}_0^{(i+k-1)}, \widehat{a}_1^{(i+k-1)}, ..., \widehat{a}_{n+1}^{(i+k-1)}\right) \flat \left(R_{\operatorname{Rect}(p,q)}^{i+k-1} f\right)\right) \\
&= \widehat{a}_{i+k-1}^{(i+k-1)} \cdot \left(R_{\operatorname{Rect}(p,q)}^{i+k-1} f\right)\left(\left(i,k\right)\right) \qquad \left(\text{since } \deg\left(\left(i,k\right)\right) = i+k-1\right) \\
&= \frac{1}{f(0)\,f(1)} \cdot \left(R_{\operatorname{Rect}(p,q)}^{i+k-1} f\right)\left(\left(i,k\right)\right)
\end{aligned}$$

(since the definition of $\widehat{a}_{i+k-1}^{(i+k-1)}$ yields

$$\begin{aligned}
\widehat{a}_{i+k-1}^{(i+k-1)} &= \begin{cases} \dfrac{a_{n+1} a_{(i+k-1)-(i+k-1)}}{a_{n+1-(i+k-1)}}, & \text{if } i+k-1 \geqslant i+k-1; \\ \dfrac{a_{n+1+(i+k-1)-(i+k-1)} a_0}{a_{n+1-(i+k-1)}}, & \text{if } i+k-1 < i+k-1 \end{cases} \\
&= \frac{a_{n+1} a_{(i+k-1)-(i+k-1)}}{a_{n+1-(i+k-1)}} \qquad \left(\text{since } i+k-1 \geqslant i+k-1\right) \\
&= \frac{a_{n+1} a_0}{a_{n-i-k+2}} = \underbrace{a_{n+1}}_{=\frac{1}{f(1)}} \underbrace{a_0}_{=\frac{1}{f(0)}} \qquad \left(\text{since } a_{n-i-k+2} = 1\right) \\
&= \frac{1}{f(0)\,f(1)}
\end{aligned}$$

). Thus, (63) rewrites as

$$f'((p+1-i, q+1-k)) = \frac{1}{\frac{1}{f(0)f(1)} \cdot \left(R_{\mathrm{Rect}(p,q)}^{i+k-1} f\right)((i,k))} = \frac{f(0)f(1)}{\left(R_{\mathrm{Rect}(p,q)}^{i+k-1} f\right)((i,k))}.$$

This rewrites as

$$f((p+1-i, q+1-k)) = \frac{f(0)f(1)}{\left(R_{\mathrm{Rect}(p,q)}^{i+k-1} f\right)((i,k))}$$

(since we know that $f'((p+1-i, q+1-k)) = f((p+1-i, q+1-k))$). This proves Theorem 11.7. $\square$

17 The $\triangleright$ triangle

Having proven the main properties of birational rowmotion R on the rectangle $\mathrm{Rect}(p,q)$ and on skeletal posets, we now turn to other posets. We will spend the next three sections discussing the order of birational rowmotion on certain triangle-shaped posets obtained as subsets of the square $\mathrm{Rect}(p,p)$. We start with the easiest case:

Definition 17.1. Let p be a positive integer. Define a subset $\mathrm{Tria}(p)$ of $\mathrm{Rect}(p,p)$ by

$$\mathrm{Tria}(p) = \left\{(i,k) \in \{1,2,...,p\}^2 \mid i \leqslant k\right\}.$$

This subset $\mathrm{Tria}(p)$ inherits a poset structure from $\mathrm{Rect}(p,p)$. In the following, we will consider $\mathrm{Tria}(p)$ as a poset using this structure. This poset $\mathrm{Tria}(p)$ is a $(2p-1)$-graded poset. It has the form of a triangle (either of $\triangleleft$ shape or of $\triangleright$ shape, depending on how you draw the Hasse diagram).

Example 17.2. Here is the Hasse diagram of the poset $\operatorname{Rect}(4,4)$, with the elements that belong to $\operatorname{Tria}(4)$ marked by underlines:

$$\begin{tikzcd}[column sep=tiny]
 & & & \underline{(4,4)} \arrow[dl, dash] \arrow[dr, dash] & & & \\
 & & (4,3) \arrow[dl, dash] \arrow[dr, dash] & & \underline{(3,4)} \arrow[dl, dash] \arrow[dr, dash] & & \\
 & (4,2) \arrow[dl, dash] \arrow[dr, dash] & & \underline{(3,3)} \arrow[dl, dash] \arrow[dr, dash] & & \underline{(2,4)} \arrow[dl, dash] \arrow[dr, dash] & \\
(4,1) \arrow[dr, dash] & & (3,2) \arrow[dl, dash] \arrow[dr, dash] & & \underline{(2,3)} \arrow[dl, dash] \arrow[dr, dash] & & \underline{(1,4)} \arrow[dl, dash] \\
 & (3,1) \arrow[dr, dash] & & \underline{(2,2)} \arrow[dl, dash] \arrow[dr, dash] & & \underline{(1,3)} \arrow[dl, dash] & \\
 & & (2,1) \arrow[dr, dash] & & \underline{(1,2)} \arrow[dl, dash] & & \\
 & & & \underline{(1,1)} & & &
\end{tikzcd}.$$

And here is the Hasse diagram of the poset $\operatorname{Tria}(4)$ itself:

$$\begin{tikzcd}[column sep=tiny]
(4,4) \arrow[dr, dash] & & & \\
 & (3,4) \arrow[dl, dash] \arrow[dr, dash] & & \\
(3,3) \arrow[dr, dash] & & (2,4) \arrow[dl, dash] \arrow[dr, dash] & \\
 & (2,3) \arrow[dl, dash] \arrow[dr, dash] & & (1,4) \arrow[dl, dash] \\
(2,2) \arrow[dr, dash] & & (1,3) \arrow[dl, dash] & \\
 & (1,2) \arrow[dl, dash] & & \\
(1,1) & & &
\end{tikzcd}.$$

Remark 17.3. Let p be a positive integer. The poset $\operatorname{Tria}(p)$ appears in [StWi11, §6.2] under the guise of the poset of order ideals (under inclusion) of the rectangle $\operatorname{Rect}(2, p-1)$. In fact, it is easily checked that the poset of order ideals just mentioned (denoted by $J([2] \times [p-1])$ in [StWi11]) is isomorphic to $\operatorname{Tria}(p)$.

We could also consider the subset $\left\{(i,k) \in \{1,2,...,p\}^2 \ \mid \ i \geqslant k\right\}$, but that would yield a poset isomorphic to $\operatorname{Tria}(p)$ and thus would not be of any further interest.

Theorem 17.4. Let p be a positive integer. Let $\mathbb{K}$ be a field. Then, $\operatorname{ord}\left(R_{\operatorname{Tria}(p)}\right) = 2p$.

This theorem yields $\operatorname{ord}\left(\overline{R}_{\operatorname{Tria}(p)}\right) \mid 2p$. It can be shown that actually $\operatorname{ord}\left(\overline{R}_{\operatorname{Tria}(p)}\right) = 2p$ for $p > 3$, while $\operatorname{ord}\left(\overline{R}_{\operatorname{Tria}(1)}\right) = 1$, $\operatorname{ord}\left(\overline{R}_{\operatorname{Tria}(2)}\right) = 1$ and $\operatorname{ord}\left(\overline{R}_{\operatorname{Tria}(3)}\right) = 2$.

Again, Theorem 17.4 is the birational version of a known result on classical rowmotion: From [StWi11, Theorem 6.2] (and our Remark 17.3), it follows that $\operatorname{ord}\left(\mathbf{r}_{\operatorname{Tria}(p)}\right) = 2p$ (using the notations of Definition 10.7 and Definition 10.28). Theorem 17.4 thus shows that birational rowmotion and classical rowmotion have the same order for $\operatorname{Tria}(p)$.

In order to prove Theorem 17.4, we need a way to turn labellings of $\operatorname{Tria}(p)$ into labellings of $\operatorname{Rect}(p,p)$ in a rowmotion-equivariant way. It turns out that the obvious "unfolding" construction (with some fudge coefficients) works:

Lemma 17.5. Let p be a positive integer. Let $\mathbb{K}$ be a field of characteristic $\neq 2$.

(a) Let $\operatorname{vrefl} : \operatorname{Rect}(p,p) \to \operatorname{Rect}(p,p)$ be the map sending every $(i,k) \in \operatorname{Rect}(p,p)$ to (k,i). This map vrefl is an involutive poset automorphism of $\operatorname{Rect}(p,p)$. (In intuitive terms, vrefl is simply reflection across the vertical axis.) We have $\operatorname{vrefl}(v) \in \operatorname{Tria}(p)$ for every $v \in \operatorname{Rect}(p,p) \setminus \operatorname{Tria}(p)$.

We extend vrefl to an involutive poset automorphism of $\widehat{\operatorname{Rect}(p,p)}$ by setting $\operatorname{vrefl}(0) = 0$ and $\operatorname{vrefl}(1) = 1$.

(b) Define a map $\operatorname{dble} : \mathbb{K}^{\widehat{\operatorname{Tria}(p)}} \to \mathbb{K}^{\widehat{\operatorname{Rect}(p,p)}}$ by setting

$$(\operatorname{dble} f)(v) = \begin{cases} \dfrac{1}{2} f(1), & \text{if } v = 1; \\ 2 f(0), & \text{if } v = 0; \\ f(v), & \text{if } v \in \operatorname{Tria}(p); \\ f(\operatorname{vrefl}(v)), & \text{otherwise} \end{cases}$$

for all $v \in \widehat{\operatorname{Rect}(p,p)}$ for all $f \in \mathbb{K}^{\widehat{\operatorname{Tria}(p)}}$. This is well-defined. We have

$$(\operatorname{dble} f)(v) = f(v) \qquad \text{for every } v \in \operatorname{Tria}(p). \tag{64}$$

Also,

$$(\operatorname{dble} f)(\operatorname{vrefl}(v)) = f(v) \qquad \text{for every } v \in \operatorname{Tria}(p). \tag{65}$$

(c) We have

$$R_{\operatorname{Rect}(p,p)} \circ \operatorname{dble} = \operatorname{dble} \circ R_{\operatorname{Tria}(p)}.$$

The coefficients $\dfrac{1}{2}$ and 2 in the definition of dble ensure that the equality $R_{\operatorname{Rect}(p,p)} \circ \operatorname{dble} = \operatorname{dble} \circ R_{\operatorname{Tria}(p)}$ in part **(c)** of the Lemma holds on the level of labellings and not just up to homogeneous equivalence.

Proof of Lemma 17.5 (sketched). **(a)** Obvious.

(b) The well-definedness of dble is pretty obvious. The relation (64) follows from the definition of dble. The relation (65) follows from the fact that every $v \in \operatorname{Tria}(p)$

satisfies either $\operatorname{vrefl}(v) \notin \operatorname{Tria}(p) \cup \{0,1\}$ (in which case the definition of $\operatorname{dble} f$ yields $(\operatorname{dble} f)(\operatorname{vrefl}(v)) = f\left(\underbrace{\operatorname{vrefl}(\operatorname{vrefl}(v))}_{=v}\right) = f(v)$) or $\operatorname{vrefl}(v) = v$ (in which case $(\operatorname{dble} f)\left(\underbrace{\operatorname{vrefl}(v)}_{=v}\right) = (\operatorname{dble} f)(v) = f(v)$ again by the definition of $\operatorname{dble} f$). This proves Lemma 17.5 **(b)**.

(c) We need to check that $\operatorname{dble} \circ R_{\operatorname{Tria}(p)} = R_{\operatorname{Rect}(p,p)} \circ \operatorname{dble}$. In other words, we have to prove that $\left(\operatorname{dble} \circ R_{\operatorname{Tria}(p)}\right) f = \left(R_{\operatorname{Rect}(p,p)} \circ \operatorname{dble}\right) f$ for every $f \in \mathbb{K}^{\widehat{\operatorname{Tria}(p)}}$ for which $R_{\operatorname{Tria}(p)}(f)$ is well-defined. So let $f \in \mathbb{K}^{\widehat{\operatorname{Tria}(p)}}$ be such that $R_{\operatorname{Tria}(p)}(f)$ is well-defined.

Set $f' = \operatorname{dble} f$ and $g = R_{\operatorname{Tria}(p)} f$. Set $g' = \operatorname{dble} g$. Then,

$$\left(\operatorname{dble} \circ R_{\operatorname{Tria}(p)}\right) f = \operatorname{dble}\left(\underbrace{R_{\operatorname{Tria}(p)} f}_{=g}\right) = \operatorname{dble} g = g'$$

and

$$\left(R_{\operatorname{Rect}(p,p)} \circ \operatorname{dble}\right) f = R_{\operatorname{Rect}(p,p)}\left(\underbrace{\operatorname{dble} f}_{=f'}\right) = R_{\operatorname{Rect}(p,p)} f'.$$

Thus, our goal (namely, to prove that $\left(\operatorname{dble} \circ R_{\operatorname{Tria}(p)}\right) f = \left(R_{\operatorname{Rect}(p,p)} \circ \operatorname{dble}\right) f$) is equivalent to showing that $g' = R_{\operatorname{Rect}(p,p)} f'$.

So we need to prove that $g' = R_{\operatorname{Rect}(p,p)} f'$. Since $f'(0) = g'(0)$ (because the operation dble multiplies the label at 0 with 2, while the operation $R_{\operatorname{Tria}(p)}$ leaves it unchanged) and $f'(1) = g'(1)$ (for a similar reason), we know from Proposition 2.19 (applied to $\operatorname{Rect}(p,p)$, f' and g' instead of P, f and g) that this will be done if we can show that

$$g'(v) = \frac{1}{f'(v)} \cdot \frac{\sum\limits_{\substack{u \in \widehat{\operatorname{Rect}(p,p)};\\ u \lessdot v}} f'(u)}{\sum\limits_{\substack{u \in \widehat{\operatorname{Rect}(p,p)};\\ u \gtrdot v}} \frac{1}{g'(u)}} \qquad \text{for every } v \in \operatorname{Rect}(p,p). \tag{66}$$

Our goal is therefore to prove (66).

But every $v \in \operatorname{Tria}(p)$ satisfies

$$g(v) = \left(R_{\operatorname{Tria}(p)} f\right)(v) = \frac{1}{f(v)} \cdot \frac{\sum\limits_{\substack{u \in \widehat{\operatorname{Tria}(p)};\\ u \lessdot v}} f(u)}{\sum\limits_{\substack{u \in \widehat{\operatorname{Tria}(p)};\\ u \gtrdot v}} \frac{1}{\left(R_{\operatorname{Tria}(p)} f\right)(u)}}$$

(by Proposition 2.16, applied to $\operatorname{Tria}(p)$ instead of P). Since $R_{\operatorname{Tria}(p)} f = g$, this rewrites as

$$g(v) = \frac{1}{f(v)} \cdot \frac{\sum\limits_{\substack{u \in \widehat{\operatorname{Tria}(p)};\\ u \lessdot v}} f(u)}{\sum\limits_{\substack{u \in \widehat{\operatorname{Tria}(p)};\\ u \gtrdot v}} \frac{1}{g(u)}}. \tag{67}$$

Now, let us prove (66). So fix $v \in \operatorname{Rect}(p,p)$. Write v in the form $v = (i,k) \in \{1,2,...,p\}^2$. We distinguish between three cases:

Case 1: We have $i < k$.

Case 2: We have $i = k$.

Case 3: We have $i > k$.

Let us first consider Case 1. In this case, $i < k$. As a consequence, every $u \in \widehat{\operatorname{Rect}(p,p)}$ satisfying $u \lessdot v$ lies in $\operatorname{Tria}(p)$. Hence, every $u \in \widehat{\operatorname{Rect}(p,p)}$ satisfying $u \lessdot v$ satisfies

$$\underbrace{f'}_{=\operatorname{dble} f}(u) = (\operatorname{dble} f)(u) = f(u) \tag{68}$$

(by (64)). Thus,

$$\sum_{\substack{u \in \widehat{\operatorname{Rect}(p,p)};\\ u \lessdot v}} \underbrace{f'(u)}_{=f(u)} = \sum_{\substack{u \in \widehat{\operatorname{Rect}(p,p)};\\ u \lessdot v}} f(u) = \sum_{\substack{u \in \widehat{\operatorname{Tria}(p)};\\ u \lessdot v}} f(u) \tag{69}$$

(since the elements $u \in \widehat{\operatorname{Tria}(p)}$ such that $u \lessdot v$ are precisely the elements $u \in \widehat{\operatorname{Rect}(p,p)}$ such that $u \lessdot v$).

Also, every $u \in \widehat{\operatorname{Rect}(p,p)}$ satisfying $u \gtrdot v$ lies in $\operatorname{Tria}(p)$. Hence, every $u \in \widehat{\operatorname{Rect}(p,p)}$ satisfying $u \gtrdot v$ satisfies

$$\underbrace{g'}_{=\operatorname{dble} g}(u) = (\operatorname{dble} g)(u) = g(u) \tag{70}$$

(by (64)). Hence,

$$\sum_{\substack{u \in \widehat{\operatorname{Rect}(p,p)};\\ u \gtrdot v}} \frac{1}{g'(u)} = \sum_{\substack{u \in \widehat{\operatorname{Rect}(p,p)};\\ u \gtrdot v}} \frac{1}{g(u)} = \sum_{\substack{u \in \widehat{\operatorname{Tria}(p)};\\ u \gtrdot v}} \frac{1}{g(u)} \tag{71}$$

(because the elements $u \in \widehat{\operatorname{Tria}(p)}$ such that $u \gtrdot v$ are precisely the elements $u \in \widehat{\operatorname{Rect}(p,p)}$ such that $u \gtrdot v$).

Finally, from $i < k$, we have $v \in \operatorname{Tria}(p)$, so that $\underbrace{f'}_{=\operatorname{dble} f}(v) = (\operatorname{dble} f)(v) = f(v)$ (by (64)) and similarly $g'(v) = g(v)$.

Using the equalities (69) and (71) as well as $f'(v) = f(v)$ and $g'(v) = g(v)$, we can rewrite (66) as

$$g(v) = \frac{1}{f(v)} \cdot \frac{\sum\limits_{\substack{u \in \widehat{\operatorname{Tria}(p)};\\ u \lessdot v}} f(u)}{\sum\limits_{\substack{u \in \widehat{\operatorname{Tria}(p)};\\ u \gtrdot v}} \frac{1}{g(u)}}.$$

But this follows from (67). Since (67) is known to hold, we thus have proven (66) in Case 1.

Let us next consider Case 3. It is very easy to check that every $h \in \operatorname{dble}\left(\mathbb{K}^{\widehat{\operatorname{Tria}(p)}}\right)$ satisfies $h(\operatorname{vrefl}(w)) = h(w)$ for every $w \in \widehat{\operatorname{Rect}(p,p)}$. Applied to $h = f'$ (which belongs to $\operatorname{dble}\left(\mathbb{K}^{\widehat{\operatorname{Tria}(p)}}\right)$ because $f' = \operatorname{dble} f$), this yields $f'(\operatorname{vrefl}(w)) = f'(w)$ for every $w \in \widehat{\operatorname{Rect}(p,p)}$. But applied to $h = g'$ (which belongs to $\operatorname{dble}\left(\mathbb{K}^{\widehat{\operatorname{Tria}(p)}}\right)$ because $g' = \operatorname{dble} g$), the same property yields $g'(\operatorname{vrefl}(w)) = g'(w)$ for every $w \in \widehat{\operatorname{Rect}(p,p)}$. We thus can rewrite the equality (66) (which we desire to prove) by replacing each $g'(w)$ by $g'(\operatorname{vrefl}(w))$ and by replacing each $f'(w)$ by $f'(\operatorname{vrefl}(w))$. Additionally, we can replace "$u \lessdot v$" by "$\operatorname{vrefl}(u) \lessdot \operatorname{vrefl}(v)$", and replace "$u \gtrdot v$" by "$\operatorname{vrefl}(u) \gtrdot \operatorname{vrefl}(v)$". Consequently, (66) rewrites as

$$g'(\operatorname{vrefl}(v)) = \frac{1}{f'(\operatorname{vrefl}(v))} \cdot \frac{\sum\limits_{\substack{u \in \widehat{\operatorname{Rect}(p,p)};\\ \operatorname{vrefl}(u) \lessdot \operatorname{vrefl}(v)}} f'(\operatorname{vrefl}(u))}{\sum\limits_{\substack{u \in \widehat{\operatorname{Rect}(p,p)};\\ \operatorname{vrefl}(u) \gtrdot \operatorname{vrefl}(v)}} \frac{1}{g'(\operatorname{vrefl}(u))}}. \tag{72}$$

This equality can be simplified further by substituting u for $\operatorname{vrefl}(u)$ on its right hand side:

$$g'(\operatorname{vrefl}(v)) = \frac{1}{f'(\operatorname{vrefl}(v))} \cdot \frac{\sum\limits_{\substack{u \in \widehat{\operatorname{Rect}(p,p)};\\ u \lessdot \operatorname{vrefl}(v)}} f'(u)}{\sum\limits_{\substack{u \in \widehat{\operatorname{Rect}(p,p)};\\ u \gtrdot \operatorname{vrefl}(v)}} \frac{1}{g'(u)}}. \tag{73}$$

This is precisely the statement of (66) with $\operatorname{vrefl}(v)$ instead of v. But since we are in Case 3 with our element v, we have $i > k$, so that $k < i$, and thus the element $\operatorname{vrefl}(v) = (k,i)$ of $\operatorname{Rect}(p,p)$ is in Case 1. Having already verified (66) in Case 1, we can thus apply (66) to $\operatorname{vrefl}(v)$ instead of v, and conclude that (73) holds. This, as we know, is equivalent to (66), and so (66) is proven in Case 3.

Let us finally consider Case 2. In this case, $i = k$. Thus, $v = \left(i, \underbrace{k}_{=i}\right) = (i,i)$. Hence, $v \in \widehat{\operatorname{Tria}(p)}$. Thus, $\underbrace{f'}_{=\operatorname{dble} f}(v) = (\operatorname{dble} f)(v) = f(v)$ (by (64), since $v \in \operatorname{Tria}(p)$).

Similarly, $g'(v) = g(v)$.

We should now consider four subcases, depending on whether $i \notin \{1, p\}$ or $i = 1 \neq p$ or $i = p \neq 1$ or $i = 1 = p$. But we are only going to deal with the first of these subcases here, leaving the other three to the reader. So let us consider the subcase when $i \notin \{1, p\}$.

We have $v = (i, i)$. Thus, the only element $u \in \widehat{\operatorname{Tria}(p)}$ such that $u \gtrdot v$ is $(i, i+1)$, and the only element $u \in \widehat{\operatorname{Tria}(p)}$ such that $u \lessdot v$ is $(i-1, i)$. Thus, (67) simplifies to

$$g(v) = \frac{1}{f(v)} \cdot \frac{f((i-1, i))}{\left(\dfrac{1}{g((i, i+1))}\right)}. \tag{74}$$

Now, recall that $g' = \operatorname{dble} g$. From the definition of $\operatorname{dble} g$, it therefore follows easily that $g'((i, i+1)) = g((i, i+1))$ and $g'((i+1, i)) = g((i, i+1))$.

Also, $f' = \operatorname{dble} f$. From the definition of $\operatorname{dble} f$, we thus obtain $f'((i-1, i)) = f((i-1, i))$ and $f'((i, i-1)) = f((i-1, i))$.

But the elements $u \in \widehat{\operatorname{Rect}(p, p)}$ such that $u \gtrdot v$ are precisely $(i+1, i)$ and $(i, i+1)$, and the elements $u \in \widehat{\operatorname{Rect}(p, p)}$ such that $u \lessdot v$ are precisely $(i-1, i)$ and $(i, i-1)$. Thus, the right hand side of (66) simplifies as follows:

$$\begin{aligned}
&\frac{1}{f'(v)} \cdot \frac{\sum\limits_{\substack{u \in \widehat{\operatorname{Rect}(p,p)};\\ u \lessdot v}} f'(u)}{\sum\limits_{\substack{u \in \widehat{\operatorname{Rect}(p,p)};\\ u \gtrdot v}} \dfrac{1}{g'(u)}} \\
&= \frac{1}{f'(v)} \cdot \frac{f'((i-1, i)) + f'((i, i-1))}{\dfrac{1}{g'((i+1, i))} + \dfrac{1}{g'((i, i+1))}} = \frac{1}{f(v)} \cdot \frac{f((i-1, i)) + f((i-1, i))}{\dfrac{1}{g((i, i+1))} + \dfrac{1}{g((i, i+1))}} \\
&\qquad \left(\begin{array}{c} \text{since } f'((i-1, i)) = f((i-1, i)),\ f'((i, i-1)) = f((i-1, i)), \\ g'((i+1, i)) = g((i, i+1)),\ g'((i, i+1)) = g((i, i+1)) \\ \text{and } f'(v) = f(v) \end{array}\right) \\
&= \frac{1}{f(v)} \cdot \frac{2 \cdot f((i-1, i))}{2 \cdot \dfrac{1}{g((i, i+1))}} = \frac{1}{f(v)} \cdot \frac{f((i-1, i))}{\left(\dfrac{1}{g((i, i+1))}\right)} = g(v) \qquad \text{(by (74))} \\
&= g'(v).
\end{aligned}$$

In other words, (66) is proven in Case 2.

We have now proven (66) in all three cases (not counting the subcases which we left to the reader to "enjoy"). Thus, (66) holds, and as we know this yields that $g' = R_{\operatorname{Rect}(p,p)} f'$. Lemma 17.5 **(c)** is thus proven. □

Proof of Theorem 17.4 (sketched). Applying Proposition 7.3 to $2p-1$ and $\operatorname{Tria}(p)$ instead of n and P, we obtain $\operatorname{ord}\left(R_{\operatorname{Tria}(p)}\right) = \operatorname{lcm}\left(2p-1+1, \operatorname{ord}\left(\overline{R}_{\operatorname{Tria}(p)}\right)\right)$. Hence,

$\operatorname{ord}\left(R_{\operatorname{Tria}(p)}\right)$ is divisible by $2p-1+1=2p$. Now, if we can prove that $\operatorname{ord}\left(R_{\operatorname{Tria}(p)}\right) \mid 2p$, then we will immediately obtain $\operatorname{ord}\left(R_{\operatorname{Tria}(p)}\right)=2p$, and Theorem 17.4 will be proven.

So let us show that $\operatorname{ord}\left(R_{\operatorname{Tria}(p)}\right) \mid 2p$. This means showing that $R_{\operatorname{Tria}(p)}^{2p}=\operatorname{id}$. Since this statement boils down to a collection of polynomial identities in the labels of an arbitrary $\mathbb{K}$-labelling of $\operatorname{Tria}(p)$, it is clear that it is enough to prove it in the case when $\mathbb{K}$ is a field of rational functions in finitely many variables over $\mathbb{Q}$. So let us WLOG assume that $\mathbb{K}$ is a field of rational functions in finitely many variables over $\mathbb{Q}$. Then, the characteristic of $\mathbb{K}$ is $\neq 2$ (it is 0 indeed), so that we can apply Lemma 17.5.

Let us use the notations of Lemma 17.5. Lemma 17.5 **(c)** yields

$$R_{\operatorname{Rect}(p,p)} \circ \operatorname{dble} = \operatorname{dble} \circ R_{\operatorname{Tria}(p)}.$$

From this, it follows (by induction over k) that

$$R_{\operatorname{Rect}(p,p)}^{k} \circ \operatorname{dble} = \operatorname{dble} \circ R_{\operatorname{Tria}(p)}^{k}$$

for every $k \in \mathbb{N}$. Applied to $k=2p$, this yields

$$R_{\operatorname{Rect}(p,p)}^{2p} \circ \operatorname{dble} = \operatorname{dble} \circ R_{\operatorname{Tria}(p)}^{2p}. \tag{75}$$

But Theorem 11.5 (applied to $q=p$) yields $\operatorname{ord}\left(R_{\operatorname{Rect}(p,p)}\right)=p+p=2p$, so that $R_{\operatorname{Rect}(p,p)}^{2p}=\operatorname{id}$. Hence, (75) simplifies to

$$\operatorname{dble} = \operatorname{dble} \circ R_{\operatorname{Tria}(p)}^{2p}.$$

We can cancel dble from this equation, because dble is an injective and therefore left-cancellable map. As a consequence, we obtain $\operatorname{id}=R_{\operatorname{Tria}(p)}^{2p}$. In other words, $R_{\operatorname{Tria}(p)}^{2p}=\operatorname{id}$. This proves Theorem 17.4. $\square$

18 The Δ and ∇ triangles

The next kind of triangle-shaped posets is more interesting.

Definition 18.1. Let p be a positive integer. Define three subsets $\Delta(p)$, $\operatorname{Eq}(p)$ and $\nabla(p)$ of $\operatorname{Rect}(p,p)=\{1,2,...,p\}\times\{1,2,...,p\}=\{1,2,...,p\}^2$ by

$$\begin{aligned}
\Delta(p) &= \left\{(i,k)\in\{1,2,...,p\}^2 \ \mid\ i+k>p+1\right\};\\
\operatorname{Eq}(p) &= \left\{(i,k)\in\{1,2,...,p\}^2 \ \mid\ i+k=p+1\right\};\\
\nabla(p) &= \left\{(i,k)\in\{1,2,...,p\}^2 \ \mid\ i+k<p+1\right\}.
\end{aligned}$$

These subsets $\Delta(p)$, $\operatorname{Eq}(p)$ and $\nabla(p)$ inherit a poset structure from $\operatorname{Rect}(p,p)$. In the following, we will consider $\Delta(p)$, $\operatorname{Eq}(p)$ and $\nabla(p)$ as posets using this structure.

Clearly, $\operatorname{Eq}(p)$ is an antichain with p elements. (The name Eq comes from "equator".) The posets $\Delta(p)$ and $\nabla(p)$ are $(p-1)$-graded posets. They have the form of a "Delta-shaped triangle" and a "Nabla-shaped triangle", respectively (whence the names).

Example 18.2. Here is the Hasse diagram of the poset $\operatorname{Rect}(4,4)$, where the elements belonging to $\Delta(4)$ have been underlined and the elements belonging to $\operatorname{Eq}(4)$ have been boxed:

$$\begin{tikzcd}[column sep=tiny]
& & & \underline{(4,4)} \arrow[dl, dash] \arrow[dr, dash] & & & \\
& & \underline{(4,3)} \arrow[dl, dash] \arrow[dr, dash] & & \underline{(3,4)} \arrow[dl, dash] \arrow[dr, dash] & & \\
& \underline{(4,2)} \arrow[dl, dash] \arrow[dr, dash] & & \underline{(3,3)} \arrow[dl, dash] \arrow[dr, dash] & & \underline{(2,4)} \arrow[dl, dash] \arrow[dr, dash] & \\
\boxed{(4,1)} \arrow[dr, dash] & & \boxed{(3,2)} \arrow[dl, dash] \arrow[dr, dash] & & \boxed{(2,3)} \arrow[dl, dash] \arrow[dr, dash] & & \boxed{(1,4)} \arrow[dl, dash] \\
& (3,1) \arrow[dr, dash] & & (2,2) \arrow[dl, dash] \arrow[dr, dash] & & (1,3) \arrow[dl, dash] & \\
& & (2,1) \arrow[dr, dash] & & (1,2) \arrow[dl, dash] & & \\
& & & (1,1) & & &
\end{tikzcd} .$$

And here is the Hasse diagram of the poset $\Delta(4)$ itself:

$$\begin{tikzcd}[column sep=tiny]
& & (4,4) \arrow[dl, dash] \arrow[dr, dash] & & \\
& (4,3) \arrow[dl, dash] \arrow[dr, dash] & & (3,4) \arrow[dl, dash] \arrow[dr, dash] & \\
(4,2) & & (3,3) & & (2,4)
\end{tikzcd} .$$

Here, on the other hand, is the Hasse diagram of the poset $\nabla(4)$:

$$\begin{tikzcd}[column sep=tiny]
(3,1) \arrow[dr, dash] & & (2,2) \arrow[dl, dash] \arrow[dr, dash] & & (1,3) \arrow[dl, dash] \\
& (2,1) \arrow[dr, dash] & & (1,2) \arrow[dl, dash] & \\
& & (1,1) & &
\end{tikzcd} .$$

Remark 18.3. Let p be a positive integer. The poset $\Delta(p)$ is isomorphic to the poset $\Phi^+(A_{p-1})$ of [StWi11, §3.2].

Remark 18.4. For every positive integer p, we have $\nabla(p) \cong (\Delta(p))^{\mathrm{op}}$ as posets. This follows immediately from the poset antiautomorphism

$$\begin{aligned} \operatorname{hrefl} : \operatorname{Rect}(p,p) &\to \operatorname{Rect}(p,p), \\ (i,k) &\mapsto (p+1-k, p+1-i) \end{aligned}$$

sending $\nabla(p)$ to $\Delta(p)$.

Here we are using the following notions:

Definition 18.5. **(a)** If P and Q are two posets, then a map $f : P \to Q$ is called a *poset antihomomorphism* if and only if every $p_1 \in P$ and $p_2 \in P$ satisfying $p_1 \leqslant p_2$ in P satisfy $f(p_1) \geqslant f(p_2)$ in Q. It is easy to see that the poset antihomomorphisms $P \to Q$ are precisely the poset homomorphisms $P \to Q^{\mathrm{op}}$.

(b) If P and Q are two posets, then an invertible map $f : P \to Q$ is called a *poset antiisomorphism* if and only if both f and f^{-1} are poset antihomomorphisms.

(c) If P is a poset and $f : P \to P$ is an invertible map, then f is said to be a *poset antiautomorphism* if f is a poset antiisomorphism.

We now state the main property of birational rowmotion R on the posets $\nabla(p)$ and $\Delta(p)$:

Theorem 18.6. Let p be an integer $\geqslant 1$. Let $\mathbb{K}$ be a field. For every $(i,k) \in \nabla(p)$ and every $f \in \mathbb{K}^{\widehat{\nabla(p)}}$, we have

$$\left(R^p_{\nabla(p)} f\right)((i,k)) = f((k,i)).$$

Theorem 18.7. Let p be an integer $\geqslant 1$. Let $\mathbb{K}$ be a field. For every $(i,k) \in \Delta(p)$ and every $f \in \mathbb{K}^{\widehat{\Delta(p)}}$, we have

$$\left(R^p_{\Delta(p)} f\right)((i,k)) = f((k,i)).$$

The following two corollaries follow easily from these two theorems:

Corollary 18.8. Let p be an integer > 1. Let $\mathbb{K}$ be a field. Then:

(a) We have $\operatorname{ord}\left(R_{\nabla(p)}\right) \mid 2p$.

(b) If $p > 2$, then $\operatorname{ord}\left(R_{\nabla(p)}\right) = 2p$.

Corollary 18.9. Let p be an integer > 1. Let $\mathbb{K}$ be a field. Then:

(a) We have $\operatorname{ord}\left(R_{\Delta(p)}\right) \mid 2p$.

(b) If $p > 2$, then $\operatorname{ord}\left(R_{\Delta(p)}\right) = 2p$.

Corollary 18.9 is analogous to a known result for classical rowmotion. In fact, from [StWi11, Conjecture 3.6] (originally a conjecture of Panyushev, then proven by Armstrong, Stump and Thomas) and our Remark 18.3, it can be seen that (using the notations of Definition 10.7 and Definition 10.28) every integer $p > 2$ satisfies $\operatorname{ord}\left(\mathbf{r}_{\Delta(p)}\right) = 2p$.

We now prepare for the proofs of Theorems 18.6 and 18.7.

First of all, Corollary 18.8 is clearly equivalent to Corollary 18.9 (because of Remark 18.4 and Proposition 8.4). It is a bit more complicated to see that Theorem 18.6 is equivalent to Theorem 18.7; we will show this later. But let us first prove Theorem 18.7. The proof will use a mapping that transforms labellings of $\Delta(p)$ into labellings of $\operatorname{Rect}(p,p)$ in a way that is rowmotion-equivariant up to homogeneous equivalence. This mapping is similar in its function to the mapping dble of Lemma 17.5, but its definition is more intricate:[42]

Lemma 18.10. Let p be a positive integer. Clearly, $\operatorname{Rect}(p,p)$ is the disjoint union of the sets $\Delta(p)$, $\nabla(p)$ and $\operatorname{Eq}(p)$.

Let $\mathbb{K}$ be a field of characteristic $\neq 2$.

(a) Let $\operatorname{hrefl} : \operatorname{Rect}(p,p) \to \operatorname{Rect}(p,p)$ be the map sending every $(i,k) \in \operatorname{Rect}(p,p)$ to $(p+1-k, p+1-i)$. This map hrefl is an involution and a poset antiautomorphism of $\operatorname{Rect}(p,p)$. (In intuitive terms, hrefl is simply reflection across the horizontal axis (i.e., the line $\operatorname{Eq}(p)$).) We have $\operatorname{hrefl}|_{\operatorname{Eq}(p)} = \operatorname{id}$ and $\operatorname{hrefl}(\Delta(p)) = \nabla(p)$.

We extend hrefl to an involutive poset antiautomorphism of $\widehat{\operatorname{Rect}(p,p)}$ by setting $\operatorname{hrefl}(0) = 1$ and $\operatorname{hrefl}(1) = 0$.

(b) Define a rational map $\operatorname{wing} : \mathbb{K}^{\widehat{\Delta(p)}} \dashrightarrow \mathbb{K}^{\widehat{\operatorname{Rect}(p,p)}}$ by setting

$$(\operatorname{wing} f)(v) = \begin{cases} f(v), & \text{if } v \in \Delta(p) \cup \{1\}; \\ 1, & \text{if } v \in \operatorname{Eq}(p); \\ \dfrac{1}{\left(R_{\Delta(p)}^{p-\deg v} f\right)(\operatorname{hrefl} v)}, & \text{if } v \in \nabla(p) \cup \{0\} \end{cases}$$

for all $v \in \widehat{\operatorname{Rect}(p,p)}$ for all $f \in \mathbb{K}^{\widehat{\Delta(p)}}$. This is well-defined.

(c) There exists a rational map $\overline{\operatorname{wing}} : \overline{\mathbb{K}^{\widehat{\Delta(p)}}} \dashrightarrow \overline{\mathbb{K}^{\widehat{\operatorname{Rect}(p,p)}}}$ such that the diagram

$$\begin{tikzcd} \mathbb{K}^{\widehat{\Delta(p)}} \arrow[r, dashed, "\operatorname{wing}"] \arrow[d, dashed, "\pi"'] & \mathbb{K}^{\widehat{\operatorname{Rect}(p,p)}} \arrow[d, dashed, "\pi"] \\ \overline{\mathbb{K}^{\widehat{\Delta(p)}}} \arrow[r, dashed, "\overline{\operatorname{wing}}"'] & \overline{\mathbb{K}^{\widehat{\operatorname{Rect}(p,p)}}} \end{tikzcd} \qquad (76)$$

commutes.

(d) The rational map $\overline{\operatorname{wing}}$ defined in Lemma 18.10 **(c)** satisfies

$$\overline{R}_{\operatorname{Rect}(p,p)} \circ \overline{\operatorname{wing}} = \overline{\operatorname{wing}} \circ \overline{R}_{\Delta(p)}.$$

(e) Consider the map $\operatorname{vrefl} : \operatorname{Rect}(p,p) \to \operatorname{Rect}(p,p)$ defined in Lemma 17.5. Define a map $\operatorname{vrefl}^* : \mathbb{K}^{\widehat{\operatorname{Rect}(p,p)}} \to \mathbb{K}^{\widehat{\operatorname{Rect}(p,p)}}$ by setting

$$(\operatorname{vrefl}^* f)(v) = f(\operatorname{vrefl}(v)) \qquad \text{for all } v \in \widehat{\operatorname{Rect}(p,p)}$$

for all $f \in \mathbb{K}^{\widehat{\operatorname{Rect}(p,p)}}$. Also, define a map $\operatorname{vrefl}^* : \mathbb{K}^{\widehat{\Delta(p)}} \to \mathbb{K}^{\widehat{\Delta(p)}}$ by setting

$$(\operatorname{vrefl}^* f)(v) = f(\operatorname{vrefl}(v)) \qquad \text{for all } v \in \widehat{\Delta(p)}$$

[42] See also Lemma 18.12 further below for a generalization of parts of this construction.

for all $f \in \mathbb{K}^{\widehat{\Delta(p)}}$. Then,

$$\operatorname{vrefl}^* \circ R_{\Delta(p)} = R_{\Delta(p)} \circ \operatorname{vrefl}^* \qquad (77)$$

(as rational maps $\mathbb{K}^{\widehat{\Delta(p)}} \dashrightarrow \mathbb{K}^{\widehat{\Delta(p)}}$). Furthermore,

$$\operatorname{vrefl}^* \circ R_{\operatorname{Rect}(p,p)} = R_{\operatorname{Rect}(p,p)} \circ \operatorname{vrefl}^* \qquad (78)$$

(as rational maps $\mathbb{K}^{\widehat{\operatorname{Rect}(p,p)}} \dashrightarrow \mathbb{K}^{\widehat{\operatorname{Rect}(p,p)}}$). Finally,

$$\operatorname{vrefl}^* \circ \operatorname{wing} = \operatorname{wing} \circ \operatorname{vrefl}^* \qquad (79)$$

(as rational maps $\mathbb{K}^{\widehat{\Delta(p)}} \dashrightarrow \mathbb{K}^{\widehat{\operatorname{Rect}(p,p)}}$).

(f) Almost every (in the sense of Zariski topology) labelling $f \in \mathbb{K}^{\widehat{\Delta(p)}}$ satisfying $f(0) = 2$ satisfies

$$R_{\operatorname{Rect}(p,p)} (\operatorname{wing} f) = \operatorname{wing} \left(R_{\Delta(p)} f \right).$$

(g) If f and g are two homogeneously equivalent zero-free $\mathbb{K}$-labellings of $\Delta(p)$, then $\operatorname{vrefl}^* f$ is homogeneously equivalent to $\operatorname{vrefl}^* g$.

Proof of Lemma 18.10 (sketched). We will not delve into the details of this tedious and yet straightforward proof. Let us merely make some comments on the few interesting parts of it. Parts **(a)**, **(b)**, **(c)** and **(g)** are obvious. Part **(f)** can be verified label-by-label using Propositions 2.16 and 2.19 and some nasty casework. Part **(d)** won't be used in the following, but can easily be derived from part **(f)**. Part **(e)** more or less follows from the fact that the definitions of $R_{\Delta(p)}$, $R_{\operatorname{Rect}(p,p)}$ and wing are all "invariant" under the vertical reflection vrefl; but proving part **(e)** in a pedestrian way might be even more straightforward than formalizing this invariance argument[43]. □

For easier reference, let us record a corollary of Lemma 18.10 **(f)**:

Corollary 18.11. Let p be a positive integer. Let $\mathbb{K}$ be a field of characteristic $\neq 2$. Consider the map wing defined in Lemma 18.10. Let $\ell \in \mathbb{N}$.

Then, almost every (in the sense of Zariski topology) labelling $f \in \mathbb{K}^{\widehat{\Delta(p)}}$ satisfying $f(0) = 2$ satisfies

$$R^{\ell}_{\operatorname{Rect}(p,p)} (\operatorname{wing} f) = \operatorname{wing} \left(R^{\ell}_{\Delta(p)} f \right).$$

Proof of Corollary 18.11 (sketched). The proof of Corollary 18.11 is an easy induction over ℓ (details left to the reader), using Lemma 18.10 **(f)** and the fact that $R_{\Delta(p)}$ does not change the label at 1. □

[43] Again, Propositions 2.16 and 2.19 come in handy for proving (77) and (78). Then, one can prove (by induction over ℓ) that $\operatorname{vrefl}^* \circ R^{\ell}_{\Delta(p)} = R^{\ell}_{\Delta(p)} \circ \operatorname{vrefl}^*$ for all $\ell \in \mathbb{N}$. Using this, (79) is straightforward to check.

We can now proceed to the proof of the theorems stated at the beginning of this section:

Proof of Theorem 18.7 (sketched). The result that we are striving to prove is a collection of identities between rational functions, hence boils down to a collection of polynomial identities in the labels of an arbitrary $\mathbb{K}$-labelling of $\Delta(p)$. Therefore, it is enough to prove it in the case when $\mathbb{K}$ is a field of rational functions in finitely many variables over $\mathbb{Q}$. So let us WLOG assume that we are in this case. Then, the characteristic of $\mathbb{K}$ is $\neq 2$ (it is 0 indeed), so that we can apply Lemma 18.10 and Corollary 18.11.

Consider the maps hrefl, wing, vrefl and vrefl* defined in Lemma 18.10. Clearly, it will be enough to prove that

$$R^p_{\Delta(p)} = \mathrm{vrefl}^*$$

as rational maps $\mathbb{K}^{\widehat{\Delta(p)}} \dashrightarrow \mathbb{K}^{\widehat{\Delta(p)}}$. In other words, it will be enough to prove that $R^p_{\Delta(p)} g = \mathrm{vrefl}^* g$ for almost every $g \in \mathbb{K}^{\widehat{\Delta(p)}}$.

So let $g \in \mathbb{K}^{\widehat{\Delta(p)}}$ be any sufficiently generic zero-free labelling of $\Delta(p)$. We need to show that $R^p_{\Delta(p)} g = \mathrm{vrefl}^* g$.

Let us use Definition 5.2. The poset $\Delta(p)$ is $(p-1)$-graded. We can find a $(p+1)$-tuple $(a_0, a_1, ..., a_p) \in (\mathbb{K}^\times)^{p+1}$ such that $((a_0, a_1, ..., a_p) \flat g)(0) = 2$ (by setting $a_0 = \dfrac{2}{g(0)}$, and choosing all other a_i arbitrarily). Fix such a $(p+1)$-tuple, and set $f = (a_0, a_1, ..., a_p) \flat g$. Then, $f(0) = 2$. We are going to prove that $R^p_{\Delta(p)} f = \mathrm{vrefl}^* f$. Until we have done this, we can forget about g; all we need to know is that f is a sufficiently generic $\mathbb{K}$-labelling of $\Delta(p)$ satisfying $f(0) = 2$.

Let $(i,k) \in \Delta(p)$ be arbitrary. Then, $i+k > p+1$ (since $(i,k) \in \Delta(p)$). Consequently, $2p - (i+k-1)$ is a well-defined element of $\{1, 2, ..., p-1\}$. Denote this element by h. Thus, $h \in \{1, 2, ..., p-1\}$ and $i+k-1+h = 2p$. Moreover, $(k,i) = \mathrm{vrefl}\, v \in \Delta(p)$.

Let $v = (p+1-k, p+1-i)$. Then, $v = \mathrm{hrefl}((i,k)) \in \nabla(p)$ (since $(i,k) \in \Delta(p)$) and $\deg v = h$ (this follows by simple computation). Moreover, $\mathrm{hrefl}\, v = (i,k)$.

Applying Corollary 18.11 to $\ell = h$, we obtain $R^h_{\mathrm{Rect}(p,p)}(\mathrm{wing}\, f) = \mathrm{wing}\left(R^h_{\Delta(p)} f\right)$, hence

$$\begin{aligned}
&\left(R^h_{\operatorname{Rect}(p,p)}(\operatorname{wing} f)\right)(v)\\
&= \left(\operatorname{wing}\left(R^h_{\Delta(p)} f\right)\right)(v) = \frac{1}{\left(R^{p-\deg v}_{\Delta(p)}\left(R^h_{\Delta(p)} f\right)\right)(\operatorname{hrefl} v)}\\
&\qquad\qquad (\text{by the definition of } \operatorname{wing}, \text{ since } v \in \nabla(p) \subseteq \nabla(p) \cup \{0\})\\
&= \frac{1}{\left(R^{p-h}_{\Delta(p)}\left(R^h_{\Delta(p)} f\right)\right)((i,k))} \qquad\qquad (\text{since } \deg v = h \text{ and } \operatorname{hrefl} v = (i,k))\\
&= \frac{1}{\left(R^p_{\Delta(p)} f\right)((i,k))} \qquad\qquad \left(\text{since } R^{p-h}_{\Delta(p)}\left(R^h_{\Delta(p)} f\right) = \underbrace{\left(R^{p-h}_{\Delta(p)} \circ R^h_{\Delta(p)}\right)}_{=R^p_{\Delta(p)}} f = R^p_{\Delta(p)} f\right).
\end{aligned} \tag{80}$$

But Theorem 11.7 (applied to p, $R^h_{\operatorname{Rect}(p,p)}(\operatorname{wing} f)$ and (k,i) instead of q, f and (i,k)) yields

$$\begin{aligned}
&\left(R^h_{\operatorname{Rect}(p,p)}(\operatorname{wing} f)\right)((p+1-k, p+1-i))\\
&= \frac{\left(R^h_{\operatorname{Rect}(p,p)}(\operatorname{wing} f)\right)(0) \cdot \left(R^h_{\operatorname{Rect}(p,p)}(\operatorname{wing} f)\right)(1)}{\left(R^{i+k-1}_{\operatorname{Rect}(p,p)}\left(R^h_{\operatorname{Rect}(p,p)}(\operatorname{wing} f)\right)\right)((k,i))}.
\end{aligned}$$

Since $(p+1-k, p+1-i) = v$ and

$$\begin{aligned}
R^{i+k-1}_{\operatorname{Rect}(p,p)}\left(R^h_{\operatorname{Rect}(p,p)}(\operatorname{wing} f)\right) &= \left(\underbrace{R^{i+k-1}_{\operatorname{Rect}(p,p)} \circ R^h_{\operatorname{Rect}(p,p)}}_{\substack{=R^{i+k-1+h}_{\operatorname{Rect}(p,p)} = R^{2p}_{\operatorname{Rect}(p,p)} \\ (\text{since } i+k-1+h=2p)}}\right)(\operatorname{wing} f)\\
&= \underbrace{R^{2p}_{\operatorname{Rect}(p,p)}}_{\substack{=\operatorname{id} \\ (\text{since Theorem 11.5 (applied to } q=p) \\ \text{yields } \operatorname{ord}\left(R_{\operatorname{Rect}(p,p)}\right)=p+p=2p)}}(\operatorname{wing} f) = \operatorname{wing} f,
\end{aligned}$$

this equality rewrites as

$$\left(R^h_{\operatorname{Rect}(p,p)}(\operatorname{wing} f)\right)(v) = \frac{\left(R^h_{\operatorname{Rect}(p,p)}(\operatorname{wing} f)\right)(0) \cdot \left(R^h_{\operatorname{Rect}(p,p)}(\operatorname{wing} f)\right)(1)}{(\operatorname{wing} f)((k,i))}.$$

Since

$$\begin{aligned}
&\underbrace{\left(R^h_{\operatorname{Rect}(p,p)}(\operatorname{wing} f)\right)(0)}_{\substack{=(\operatorname{wing} f)(0) \\ \text{(by Corollary 2.18)}}} \cdot \underbrace{\left(R^h_{\operatorname{Rect}(p,p)}(\operatorname{wing} f)\right)(1)}_{\substack{=(\operatorname{wing} f)(1) \\ \text{(by Corollary 2.18)}}} \\
&= \underbrace{(\operatorname{wing} f)(0)}_{\substack{=\dfrac{1}{\left(R^{p-\deg 0}_{\Delta(p)} f\right)(\operatorname{hrefl} 0)} \\ \text{(by the definition of wing)}}} \cdot \underbrace{(\operatorname{wing} f)(1)}_{\substack{=f(1) \\ \text{(by the definition of wing)}}} \\
&= \frac{1}{\left(R^{p-\deg 0}_{\Delta(p)} f\right)(\operatorname{hrefl} 0)} \cdot f(1) = 1
\end{aligned}$$

(since Corollary 2.18 yields $\left(R^{p-\deg 0}_{\Delta(p)} f\right)(\operatorname{hrefl} 0) = f(\operatorname{hrefl} 0) = f(1)$), this simplifies to

$$\left(R^h_{\operatorname{Rect}(p,p)}(\operatorname{wing} f)\right)(v) = \frac{1}{(\operatorname{wing} f)((k,i))}.$$

Compared with (80), this yields $\dfrac{1}{\left(R^p_{\Delta(p)} f\right)((i,k))} = \dfrac{1}{(\operatorname{wing} f)((k,i))}$. Taking inverses in this equality, we get

$$\begin{aligned}
\left(R^p_{\Delta(p)} f\right)((i,k)) &= (\operatorname{wing} f)((k,i)) = f\left(\underbrace{(k,i)}_{=\operatorname{vrefl}(i,k)}\right) \\
&\qquad \left(\text{by the definition of wing, since } (k,i) \in \Delta(p) \subseteq \Delta(p) \cup \{1\}\right) \\
&= f(\operatorname{vrefl}(i,k)) = (\operatorname{vrefl}^* f)((i,k)) \\
&\qquad \left(\text{since } (\operatorname{vrefl}^* f)((i,k)) = f(\operatorname{vrefl}(i,k)) \text{ by the definition of } \operatorname{vrefl}^*\right).
\end{aligned}$$

Now, we have shown this for **every** $(i,k) \in \Delta(p)$. In other words, we have shown that $R^p_{\Delta(p)} f = \operatorname{vrefl}^* f$.

Now, recall that $f = (a_0, a_1, ..., a_p) \flat g$. Hence,

$$R^p_{\Delta(p)} f = R^p_{\Delta(p)}((a_0, a_1, ..., a_p) \flat g) = (a_0, a_1, ..., a_p) \flat \left(R^p_{\Delta(p)} g\right) \tag{81}$$

(by Corollary 5.7, applied to $\Delta(p)$, $p-1$ and g instead of P, n and f). On the other hand, $f = (a_0, a_1, ..., a_p) \flat g$ yields

$$\operatorname{vrefl}^* f = \operatorname{vrefl}^*((a_0, a_1, ..., a_p) \flat g) = (a_0, a_1, ..., a_p) \flat (\operatorname{vrefl}^* g) \tag{82}$$

(this is easy to check directly using the definitions of $\flat$ and vrefl^*, since vrefl preserves degrees). In light of (81) and (82), the equality $R^p_{\Delta(p)} f = \operatorname{vrefl}^* f$ becomes

$(a_0, a_1, ..., a_p) \flat \left(R^p_{\Delta(p)} g \right) = (a_0, a_1, ..., a_p) \flat (\operatorname{vrefl}^* g)$. We can cancel the "$(a_0, a_1, ..., a_p) \flat$" from both sides of this equation (since all a_i are nonzero), and thus obtain $R^p_{\Delta(p)} g = \operatorname{vrefl}^* g$. As we have seen, this is all we need to prove Theorem 18.7. $\square$

We can now obtain Theorem 18.6 from Theorem 18.7 using a construction from the proof of Proposition 8.4:

Proof of Theorem 18.6 (sketched). The poset antiautomorphism hrefl of $\operatorname{Rect}(p, p)$ defined in Remark 18.4 restricts to a poset antiisomorphism $\operatorname{hrefl} : \nabla(p) \to \Delta(p)$, that is, to a poset homomorphism $\operatorname{hrefl} : \nabla(p) \to (\Delta(p))^{\mathrm{op}}$. We will use this isomorphism to identify the poset $\nabla(p)$ with the opposite poset $(\Delta(p))^{\mathrm{op}}$ of $\Delta(p)$.

Set $P = \Delta(p)$. Define a rational map $\kappa : \mathbb{K}^{\widehat{P}} \dashrightarrow \mathbb{K}^{\widehat{P^{\mathrm{op}}}}$ as in the proof of Proposition 8.4. Then, as in said proof, it can be shown that the map κ is a birational map and satisfies $\kappa \circ R_P = R^{-1}_{P^{\mathrm{op}}} \circ \kappa$. Since $P = \Delta(p)$ and $P^{\mathrm{op}} = (\Delta(p))^{\mathrm{op}} = \nabla(p)$, this rewrites as $\kappa \circ R_{\Delta(p)} = R^{-1}_{\nabla(p)} \circ \kappa$. For the same reason, we know that κ is a rational map $\mathbb{K}^{\widehat{\Delta(p)}} \dashrightarrow \mathbb{K}^{\widehat{\nabla(p)}}$.

From $\kappa \circ R_{\Delta(p)} = R^{-1}_{\nabla(p)} \circ \kappa$, we can easily obtain $\kappa \circ R^m_{\Delta(p)} = R^{-m}_{\nabla(p)} \circ \kappa$ for every $m \in \mathbb{N}$. In particular, $\kappa \circ R^p_{\Delta(p)} = R^{-p}_{\nabla(p)} \circ \kappa$.

Now, consider the map $\operatorname{vrefl}^* : \mathbb{K}^{\widehat{\Delta(p)}} \to \mathbb{K}^{\widehat{\Delta(p)}}$ defined in Lemma 18.10 **(e)**, and also consider the similarly defined map $\operatorname{vrefl}^* : \mathbb{K}^{\widehat{\nabla(p)}} \to \mathbb{K}^{\widehat{\nabla(p)}}$. Both squares of the diagram

$$\begin{tikzcd}
\mathbb{K}^{\widehat{\Delta(p)}} \arrow[r, dashed, "R^p_{\Delta(p)}"] \arrow[d, dashed, "\kappa"] & \mathbb{K}^{\widehat{\Delta(p)}} \arrow[r, "\operatorname{vrefl}^*"] \arrow[d, dashed, "\kappa"] & \mathbb{K}^{\widehat{\Delta(p)}} \arrow[d, dashed, "\kappa"] \\
\mathbb{K}^{\widehat{\nabla(p)}} \arrow[r, dashed, "R^{-p}_{\nabla(p)}"'] & \mathbb{K}^{\widehat{\nabla(p)}} \arrow[r, "\operatorname{vrefl}^*"'] & \mathbb{K}^{\widehat{\nabla(p)}}
\end{tikzcd}$$

commute (the left square does so because of $\kappa \circ R^p_{\Delta(p)} = R^{-p}_{\nabla(p)} \circ \kappa$, and the commutativity of the right square follows from a simple calculation), and so the whole diagram commutes. In other words,

$$\kappa \circ \left(\operatorname{vrefl}^* \circ R^p_{\Delta(p)} \right) = \left(\operatorname{vrefl}^* \circ R^{-p}_{\nabla(p)} \right) \circ \kappa. \tag{83}$$

But the statement of Theorem 18.7 can be rewritten as $R^p_{\Delta(p)} = \operatorname{vrefl}^*$. Since vrefl^* is an involution (this is clear by inspection), we have $\operatorname{vrefl}^* = (\operatorname{vrefl}^*)^{-1}$, so that $\underbrace{\operatorname{vrefl}^*}_{=(\operatorname{vrefl}^*)^{-1}} \circ \underbrace{R^p_{\Delta(p)}}_{=\operatorname{vrefl}^*} = (\operatorname{vrefl}^*)^{-1} \circ \operatorname{vrefl}^* = \mathrm{id}$. Thus, (83) simplifies to $\kappa \circ \mathrm{id} = \left(\operatorname{vrefl}^* \circ R^{-p}_{\nabla(p)} \right) \circ \kappa$. In other words, $\kappa = \left(\operatorname{vrefl}^* \circ R^{-p}_{\nabla(p)} \right) \circ \kappa$. Since κ is a birational map, we can cancel κ from this identity, obtaining $\mathrm{id} = \operatorname{vrefl}^* \circ R^{-p}_{\nabla(p)}$. In other words, $R^p_{\nabla(p)} = \operatorname{vrefl}^*$. But this is precisely the statement of Theorem 18.6. $\square$

Proof of Corollary 18.9 (sketched). **(a)** Let $f \in \mathbb{K}^{\widehat{\Delta(p)}}$ be sufficiently generic. Then, every $(i,k) \in \Delta(p)$ satisfies

$$\begin{aligned}
&\left(\underbrace{R_{\Delta(p)}^{2p}}_{=R_{\Delta(p)}^{p} \circ R_{\Delta(p)}^{p}} f \right) ((i,k)) \\
&= \left(\left(R_{\Delta(p)}^{p} \circ R_{\Delta(p)}^{p} \right) f \right) ((i,k)) = \left(R_{\Delta(p)}^{p} \left(R_{\Delta(p)}^{p} f \right) \right) ((i,k)) \\
&= \left(R_{\Delta(p)}^{p} f \right) ((k,i)) \qquad \left(\text{by Theorem 18.7, applied to } R_{\Delta(p)}^{p} f \text{ instead of } f \right) \\
&= f((i,k)) \qquad (\text{by Theorem 18.7, applied to } (k,i) \text{ instead of } (i,k)).
\end{aligned}$$

Hence, the two labellings $R_{\Delta(p)}^{2p} f$ and f are equal on every element of $\Delta(p)$. Since these two labellings are also equal on 0 and 1 (because Corollary 2.18 yields $\left(R_{\Delta(p)}^{2p} f \right)(0) = f(0)$ and $\left(R_{\Delta(p)}^{2p} f \right)(1) = f(1)$), this yields that the two labellings $R_{\Delta(p)}^{2p} f$ and f are equal on every element of $\Delta(p) \cup \{0,1\} = \widehat{\Delta(p)}$. Hence, $R_{\Delta(p)}^{2p} f = f = \operatorname{id} f$.

Now, forget that we fixed f. We thus have shown that $R_{\Delta(p)}^{2p} f = \operatorname{id} f$ for every sufficiently generic $f \in \mathbb{K}^{\widehat{\Delta(p)}}$. Hence, $R_{\Delta(p)}^{2p} = \operatorname{id}$. In other words, $\operatorname{ord}\left(R_{\Delta(p)} \right) \mid 2p$. This proves Corollary 18.9 **(a)**.

(b) Proving Corollary 18.9 **(b)** is left to the reader. □

Proof of Corollary 18.8 (sketched). Corollary 18.8 can be deduced from Theorem 18.6 in the same way as Corollary 18.9 is deduced from Theorem 18.7. We won't dwell on the details. □

Let us conclude this section by stating a generalization of parts **(b)**, **(c)**, **(d)** and **(f)** of Lemma 18.10 that was pointed out by a referee. Rather than restricting itself to $\operatorname{Rect}(p,p)$, it is concerned with an arbitrary $(2p-1)$-graded poset satisfying certain axioms (which can be informally subsumed under the slogan "symmetric with respect to degree p and regular near the middle"):[44]

[44]We choose to label the parts of Lemma 18.12 by **(b)**, **(c)**, **(d)** and **(f)**, since they generalize the parts **(b)**, **(c)**, **(d)** and **(f)** of Lemma 18.10, respectively.

Lemma 18.12. Let p be a positive integer. Let P be a $(2p-1)$-graded finite poset. Let $\operatorname{hrefl} : P \to P$ be an involution such that hrefl is a poset antiautomorphism of P. (This hrefl has nothing to do with the hrefl defined in Lemma 18.10, although of course it is analogous to the latter.) We extend hrefl to an involutive poset antiautomorphism of $\widehat{P}$ by setting $\operatorname{hrefl}(0) = 1$ and $\operatorname{hrefl}(1) = 0$.

Assume that every $v \in \widehat{P}$ satisfies

$$\deg(\operatorname{hrefl} v) = 2p - \deg v. \tag{84}$$

Let N be a positive integer. Assume that, for every $v \in P$ satisfying $\deg v = p - 1$, there exist precisely N elements u of P satisfying $u \gtrdot v$.

Define three subsets Δ, Eq and ∇ of P by

$$\Delta = \{v \in P \mid \deg v > p\};$$
$$\mathrm{Eq} = \{v \in P \mid \deg v = p\};$$
$$\nabla = \{v \in P \mid \deg v < p\}.$$

Clearly, Δ, Eq and ∇ become subposets of P. The poset Eq is an antichain, while the posets Δ and ∇ are $(p-1)$-graded.

Assume that $\operatorname{hrefl}|_{\mathrm{Eq}} = \mathrm{id}$. It is easy to see that $\operatorname{hrefl}(\Delta) = \nabla$.

Let $\mathbb{K}$ be a field such that N is invertible in $\mathbb{K}$.

(b) Define a rational map $\operatorname{wing} : \mathbb{K}^{\widehat{\Delta}} \dashrightarrow \mathbb{K}^{\widehat{P}}$ by setting

$$(\operatorname{wing} f)(v) = \begin{cases} f(v), & \text{if } v \in \Delta \cup \{1\}; \\ 1, & \text{if } v \in \mathrm{Eq}; \\ \dfrac{1}{\left(R_{\Delta}^{p-\deg v} f\right)(\operatorname{hrefl} v)}, & \text{if } v \in \nabla \cup \{0\} \end{cases}$$

for all $v \in \widehat{P}$ for all $f \in \mathbb{K}^{\widehat{\Delta}}$. This is well-defined.

(c) There exists a rational map $\overline{\operatorname{wing}} : \overline{\mathbb{K}^{\widehat{\Delta}}} \dashrightarrow \overline{\mathbb{K}^{\widehat{P}}}$ such that the diagram

$$\begin{tikzcd} \mathbb{K}^{\widehat{\Delta}} \arrow[r, dashed, "\operatorname{wing}"] \arrow[d, dashed, "\pi"'] & \mathbb{K}^{\widehat{P}} \arrow[d, dashed, "\pi"] \\ \overline{\mathbb{K}^{\widehat{\Delta}}} \arrow[r, dashed, "\overline{\operatorname{wing}}"'] & \overline{\mathbb{K}^{\widehat{P}}} \end{tikzcd}$$

commutes.

(d) The rational map $\overline{\operatorname{wing}}$ defined in Lemma 18.12 **(b)** satisfies

$$\overline{R_P} \circ \overline{\operatorname{wing}} = \overline{\operatorname{wing}} \circ \overline{R_\Delta}.$$

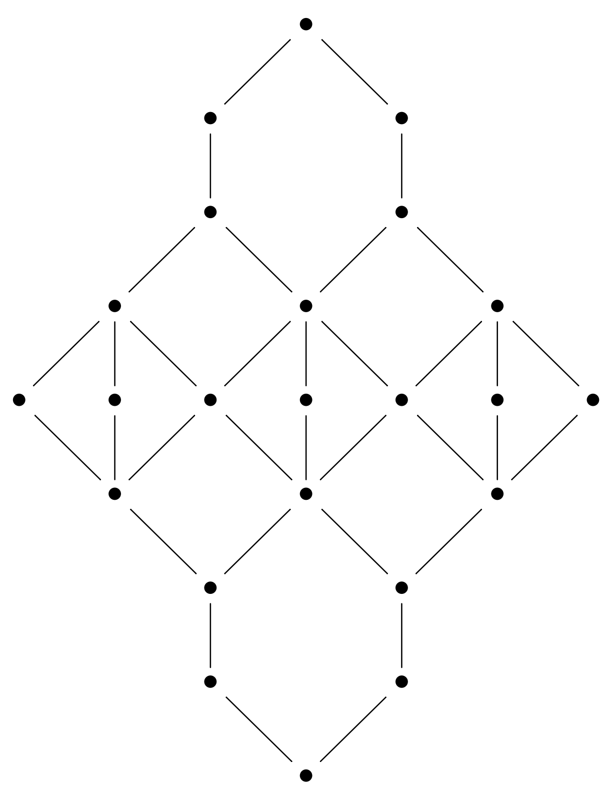
(f) Almost every (in the sense of Zariski topology) labelling $f \in \mathbb{K}^{\widehat{\Delta}}$ satisfying $f(0) = N$ satisfies

$$R_P(\operatorname{wing} f) = \operatorname{wing}(R_\Delta f).$$

Notice that the hypothesis (84) is actually redundant (it follows from the other requirements), but we have chosen to state it because it is easily checked in practice and used in the proof.

Example 18.13. Let P be a positive integer, and let $\mathbb{K}$ be a field of characteristic $\neq 2$. The hypotheses of Lemma 18.12 are satisfied if we set $P = \text{Rect}\,(p, p)$, $\text{hrefl} = \text{hrefl}$ (by this, we mean that we define hrefl to be the map hrefl defined in Lemma 18.10) and $N = 2$. In this case, the posets Δ, Eq and ∇ defined in Lemma 18.12 are precisely the posets $\Delta\,(p)$, $\text{Eq}\,(p)$ and $\nabla\,(p)$ introduced in Definition 18.1. Hence, Lemma 18.12 (when applied to this setting) yields the parts **(b)**, **(c)**, **(d)** and **(f)** of Lemma 18.10.

Example 18.14. Here is another example of a situation in which Lemma 18.12 applies. Namely, the hypotheses of Lemma 18.12 are satisfied when $p = 5$, $N = 3$ and P is the poset with Hasse diagram



(with $\text{hrefl} : P \to P$ being the reflection with respect to the horizontal axis of symmetry of this diagram).

Proof of Lemma 18.12 (sketched). The proof of Lemma 18.12 is almost completely analogous to the proof of parts **(b)**, **(c)**, **(d)** and **(f)** of Lemma 18.10. Of course, several changes need to be made to the latter proof to make it apply to Lemma 18.12: for instance,

- every appearance of $\operatorname{Rect}(p,p)$, $\Delta(p)$, $\nabla(p)$ or $\operatorname{Eq}(p)$ must be replaced by P, Δ, ∇ or Eq, respectively;
- many (but not all) appearances of the number 2 (such as its appearance in the definition of a_i) have to be replaced by N;
- various properties of P now no longer follow from the definition of $\operatorname{Rect}(p,p)$ (because P is no longer $\operatorname{Rect}(p,p)$), but instead have to be derived from the hypotheses of Lemma 18.12[45];
- checking the case when $p \leqslant 2$ is no longer trivial, but needs a bit more work[46].

[45]Most of the time, this is obvious. For instance, the fact that $\operatorname{hrefl}(\Delta) = \nabla$ follows from (84). The only fact that is not completely trivial is that, for every $v \in P$ satisfying $\deg v = p+1$, there exist precisely N elements u of P satisfying $u \lessdot v$. Let us prove this fact.

We know that hrefl is a poset antiautomorphism of $\widehat{P}$. Hence, if u and v are two elements of $\widehat{P}$, then we have the following equivalence of statements:

$$(u \lessdot v) \Longleftrightarrow (\operatorname{hrefl} u \gtrdot \operatorname{hrefl} v). \tag{85}$$

We also have assumed that, for every $v \in P$ satisfying $\deg v = p-1$, there exist precisely N elements u of P satisfying $u \gtrdot v$. In other words, for every $v \in P$ satisfying $\deg v = p-1$, we have

$$(\text{the number of elements } u \text{ of } P \text{ satisfying } u \gtrdot v) = N. \tag{86}$$

Now, let $v \in P$ be such that $\deg v = p+1$. We need to show that there exist precisely N elements u of P satisfying $u \lessdot v$.

From (84), we obtain $\deg(\operatorname{hrefl} v) = 2p - \underbrace{\deg v}_{=p+1} = 2p - (p+1) = p-1$. Hence, (86) (applied to $\operatorname{hrefl} v$ instead of v) yields

$$(\text{the number of elements } u \text{ of } P \text{ satisfying } u \gtrdot \operatorname{hrefl} v) = N. \tag{87}$$

But $\operatorname{hrefl} : P \to P$ is a bijection (since hrefl is an involution). Thus, we can substitute $\operatorname{hrefl} u$ for u in "(the number of elements u of P satisfying $u \gtrdot \operatorname{hrefl} v$)". We thus obtain

$$\begin{aligned}
&(\text{the number of elements } u \text{ of } P \text{ satisfying } u \gtrdot \operatorname{hrefl} v) \\
&= \left(\text{the number of elements } u \text{ of } P \text{ satisfying } \underbrace{\operatorname{hrefl} u \gtrdot \operatorname{hrefl} v}_{\substack{\text{this is equivalent to } (u \lessdot v) \\ \text{(due to (85))}}}\right) \\
&= (\text{the number of elements } u \text{ of } P \text{ satisfying } u \lessdot v).
\end{aligned}$$

Thus,

$$\begin{aligned}
&(\text{the number of elements } u \text{ of } P \text{ satisfying } u \lessdot v) \\
&= (\text{the number of elements } u \text{ of } P \text{ satisfying } u \gtrdot \operatorname{hrefl} v) = N
\end{aligned}$$

(by (87)). In other words, there exist precisely N elements u of P satisfying $u \lessdot v$.

This completes our proof of the fact that, for every $v \in P$ satisfying $\deg v = p+1$, there exist precisely N elements u of P satisfying $u \lessdot v$.

[46]The case when $p = 1$ is still obvious (since Δ and ∇ are empty sets in this case). The case when

□

19 The quarter-triangles

We have now studied the order of birational rowmotion on all four triangles (two of which are isomorphic as posets) which are obtained by cutting the rectangle $\operatorname{Rect}(p,p)$ along one of its diagonals. But we can also cut $\operatorname{Rect}(p,p)$ along **both** diagonals into four smaller triangles. These are isomorphic in pairs, and we will analyze them now. The following definition is an analogue of Definition 18.1 but using $\operatorname{Tria}(p)$ instead of $\operatorname{Rect}(p,p)$:

Definition 19.1. Let p be a positive integer. Define three subsets $\operatorname{NEtri}(p)$, $\operatorname{Eqtri}(p)$ and $\operatorname{SEtri}(p)$ of $\operatorname{Tria}(p)$ by

$$\begin{aligned} \operatorname{NEtri}(p) &= \{(i,k) \in \operatorname{Tria}(p) \mid i+k > p+1\}; \\ \operatorname{Eqtri}(p) &= \{(i,k) \in \operatorname{Tria}(p) \mid i+k = p+1\}; \\ \operatorname{SEtri}(p) &= \{(i,k) \in \operatorname{Tria}(p) \mid i+k < p+1\}. \end{aligned}$$

These subsets $\operatorname{NEtri}(p)$, $\operatorname{Eqtri}(p)$ and $\operatorname{SEtri}(p)$ inherit a poset structure from $\operatorname{Tria}(p)$. In the following, we will consider $\operatorname{NEtri}(p)$, $\operatorname{Eqtri}(p)$ and $\operatorname{SEtri}(p)$ as posets using this structure.

Clearly, $\operatorname{Eqtri}(p)$ is an antichain. The posets $\operatorname{NEtri}(p)$ and $\operatorname{SEtri}(p)$ are $(p-1)$-graded posets having the form of right-angled triangles.

$p = 2$ can be handled by the same arguments that were used to deal with the case when $p > 2$ (in particular, the same definition of the $(2p+1)$-tuple $(a_0, a_1, ..., a_{2p})$ applies), but the details are slightly different (instead of the seven cases, there are now only three cases: $\deg v = 3$, $\deg v = 2$ and $\deg v = 1$).

Example 19.2. Here is the Hasse diagram of the poset Tria (4), where the elements belonging to NEtri (4) have been underlined and the elements belonging to Eqtri (4) have been boxed:

$$\begin{tikzcd}
\underline{(4,4)} \arrow[dr, dash] & & & \\
& \underline{(3,4)} \arrow[dl, dash] \arrow[dr, dash] & & \\
\underline{(3,3)} \arrow[dr, dash] & & \underline{(2,4)} \arrow[dl, dash] \arrow[dr, dash] & \\
& \boxed{(2,3)} \arrow[dl, dash] \arrow[dr, dash] & & \boxed{(1,4)} \arrow[dl, dash] \\
(2,2) \arrow[dr, dash] & & (1,3) \arrow[dl, dash] & \\
& (1,2) \arrow[dl, dash] & & \\
(1,1) & & &
\end{tikzcd}.$$

And here is the Hasse diagram of the poset NEtri (4) itself:

$$\begin{tikzcd}
(4,4) \arrow[dr, dash] & & \\
& (3,4) \arrow[dl, dash] \arrow[dr, dash] & \\
(3,3) & & (2,4)
\end{tikzcd}.$$

Here, on the other hand, is the Hasse diagram of the poset SEtri (4):

$$\begin{tikzcd}
(2,2) \arrow[dr, dash] & & (1,3) \arrow[dl, dash] \\
& (1,2) \arrow[dl, dash] & \\
(1,1) & &
\end{tikzcd}.$$

Remark 19.3. Let p be an **even** positive integer. The poset NEtri (p) is isomorphic to the poset $\Phi^+ \left(B_{p/2}\right)$ of [StWi11, §3.2]. (For odd p, the poset NEtri (p) does not seem to appear in [StWi11, §3.2].)

The following conjectures have been verified using Sage for small values of p:

Conjecture 19.4. Let p be an integer > 1. Then, $\operatorname{ord}\left(R_{\mathrm{SEtri}(p)}\right) = p$.

Conjecture 19.5. Let p be an integer > 1. Then, $\operatorname{ord}\left(R_{\mathrm{NEtri}(p)}\right) = p$.

The approach used to prove Theorem 17.4 allows proving these two conjectures in the case of odd p, but in the even-p case it fails (although the order of **classical** rowmotion is again known to be p in the even-p case – see [StWi11, Conjecture 3.6]). Here is how the proof proceeds in the case of odd p:

Proposition 19.6. Let p be an odd integer > 1. Let $\mathbb{K}$ be a field. Then, $\operatorname{ord}\left(R_{\mathrm{SEtri}(p)}\right) = p$.

Proposition 19.7. Let p be an odd integer > 1. Let $\mathbb{K}$ be a field. Then, $\operatorname{ord}\left(R_{\mathrm{NEtri}(p)}\right) = p$.

Our proof of Proposition 19.7 rests upon the following fact:

Lemma 19.8. Let $\mathbb{K}$ be a field of characteristic $\neq 2$.

Let p be a positive integer.

(a) Let $\operatorname{vrefl} : \Delta(p) \to \Delta(p)$ be the map sending every $(i, k) \in \Delta(p)$ to (k, i). This map vrefl is an involutive poset automorphism of $\Delta(p)$. (In intuitive terms, vrefl is simply reflection across the vertical axis.) We have $\operatorname{vrefl}(v) \in \mathrm{NEtri}(p)$ for every $v \in \Delta(p) \setminus \mathrm{NEtri}(p)$.

We extend vrefl to an involutive poset automorphism of $\widehat{\Delta(p)}$ by setting $\operatorname{vrefl}(0) = 0$ and $\operatorname{vrefl}(1) = 1$.

(b) Define a map $\operatorname{dble} : \mathbb{K}^{\widehat{\mathrm{NEtri}(p)}} \to \mathbb{K}^{\widehat{\Delta(p)}}$ by setting

$$(\operatorname{dble} f)(v) = \begin{cases} \dfrac{1}{2} f(1), & \text{if } v = 1; \\ f(0), & \text{if } v = 0; \\ f(v), & \text{if } v \in \mathrm{NEtri}(p); \\ f(\operatorname{vrefl}(v)), & \text{otherwise} \end{cases}$$

for all $v \in \widehat{\Delta(p)}$ for all $f \in \mathbb{K}^{\widehat{\mathrm{NEtri}(p)}}$. This is well-defined. We have

$$(\operatorname{dble} f)(v) = f(v) \qquad \text{for every } v \in \mathrm{NEtri}(p). \tag{88}$$

Also,

$$(\operatorname{dble} f)(\operatorname{vrefl}(v)) = f(v) \qquad \text{for every } v \in \mathrm{NEtri}(p). \tag{89}$$

(c) Assume that p is odd. Then,

$$R_{\Delta(p)} \circ \operatorname{dble} = \operatorname{dble} \circ R_{\mathrm{NEtri}(p)}.$$

We omit the proofs of Lemma 19.8, Proposition 19.7 and Proposition 19.6 since neither of them involves any new ideas. The first is analogous to that of Lemma 17.5 (with $\Delta(p)$

and $\operatorname{NEtri}(p)$ taking the roles of $\operatorname{Rect}(p,p)$ and $\operatorname{Tria}(p)$, respectively)[47]. The proof of Proposition 19.7 combines Lemma 19.8 with Theorem 18.7. Proposition 19.6 is derived from Proposition 19.7 using Proposition 8.4. $\square$

Nathan Williams suggested that the following generalization of Conjecture 19.5 might hold:

Conjecture 19.9. Let p be an integer > 1. Let $s \in \mathbb{N}$. Let $\operatorname{NEtri}'(p)$ be the subposet $\{(i,k) \in \operatorname{NEtri}(p) \mid k \geqslant s\}$ of $\operatorname{NEtri}(p)$. Then, $\operatorname{ord}\left(R_{\operatorname{NEtri}'(p)}\right) \mid p$.

This conjecture has been verified using Sage for all $p \leqslant 7$. Williams (based on a philosophy from his thesis [Will13]) suspects there could be a birational map between $\mathbb{K}^{\widehat{\operatorname{NEtri}'(p)}}$ and $\mathbb{K}^{\operatorname{Rect}(\widehat{s-1,p}-s+1)}$ which commutes with the respective birational rowmotion operators for all $s > \dfrac{p}{2}$; this, if shown, would obviously yield a proof of Conjecture 19.9. This already is an interesting question for classical rowmotion; a bijection between the antichains (and thus between the order ideals) of $\operatorname{NEtri}'(p)$ and those of $\operatorname{Rect}(s-1, p-s+1)$ was found by Stembridge [Stem86, Theorem 5.4], but does not commute with classical rowmotion.

20 Negative results

Generally, it is not true that if P is an n-graded poset, then $\operatorname{ord}(R_P)$ is necessarily finite. When $\operatorname{char}\mathbb{K} = 0$, the authors have proven the following[48]:

- If P is the poset $\{x_1, x_2, x_3, x_4, x_5\}$ with relations $x_1 < x_3$, $x_1 < x_4$, $x_1 < x_5$, $x_2 < x_4$ and $x_2 < x_5$ (this is a 5-element 2-graded poset), then $\operatorname{ord}(R_P) = \infty$.
- If P is the "chain-link fence" poset /\/\/\ (that is, the subposet $\{(i,k) \in \operatorname{Rect}(4,4) \mid 5 \leqslant i+k \leqslant 6\}$ of $\operatorname{Rect}(4,4)$), then $\operatorname{ord}(R_P) = \infty$.
- If P is the Boolean lattice $[2] \times [2] \times [2]$, then $\operatorname{ord}(R_P) = \infty$.

The situation seems even more hopeless for non-graded posets.

[47] The only non-straightforward change that must be made to the proof is the following: In Case 2 of the proof of Lemma 17.5, we used the (obvious) observation that $(i-1,i)$ and $(i,i-1)$ are elements of $\operatorname{Rect}(p,p)$ for every $(i,i) \in \operatorname{Rect}(p,p)$ satisfying $i \neq 1$. The analogous observation that we need for proving Lemma 19.8 is still true in the case of odd p, but a bit less obvious. In fact, it is the observation that $(i-1,i)$ and $(i,i-1)$ are elements of $\Delta(p)$ for every $(i,i) \in \Delta(p)$. This uses the oddness of p.

[48] See the ancillary files of the present arXiv preprint (arXiv:1402.6178) for an outline of the (rather technical) proofs.

21 The root system connection

A question naturally suggesting itself is: What is it that makes certain posets P have finite $\operatorname{ord}(R_P)$, while others have not? Can we characterize the former posets? It might be too optimistic to expect a full classification, given that our examples are already rather diverse (skeletal posets, rectangles, triangles, posets like that in Remark 11.8). As a first step (and inspired by the general forms of the Zamolodchikov conjecture), we were tempted to study posets arising from Dynkin diagrams. It appears that, unlike in the Zamolodchikov conjecture, the interesting cases are not those having P be a product of Dynkin diagrams, but those having P be a positive root poset of a root system, or a parabolic quotient thereof. The idea is not new, as it was already conjectured by Panyushev [Pan08, Conjecture 2.1] and proven by Armstrong, Stump and Thomas [AST11, Theorem 1.2] that if W is a finite Weyl group with Coxeter number h, then classical rowmotion on the set $J(\Phi^+(W))$ (where $\Phi^+(W)$ is the poset of positive roots of W) has order h or $2h$ (along with a few more properties, akin to our "reciprocity" statements)[49].

In the case of birational rowmotion, the situation is less simple. Specifically, the following can be said about positive root posets of crystallographic root systems (as considered in [StWi11, §3.2])[50]:

- If $P = \Phi^+(A_n)$ for $n \geqslant 2$, then $\operatorname{ord}(R_P) = 2(n+1)$. This is just the assertion of Corollary 18.9. Note that for $n = 1$, the order $\operatorname{ord}(R_P)$ is 2 instead of $2(1+1) = 4$.
- If $P = \Phi^+(B_n)$ for $n \geqslant 1$, then Conjecture 19.4 claims that $\operatorname{ord}(R_P) = 2n$. Note that $\Phi^+(B_n) \cong \Phi^+(C_n)$.
- We have $\operatorname{ord}(R_P) = 2$ for $P = \Phi^+(D_2)$, and we have $\operatorname{ord}(R_P) = 8$ for $P = \Phi^+(D_3)$. However, $\operatorname{ord}(R_P) = \infty$ in the case when $P = \Phi^+(D_4)$. This should not come as a surprise, since $\Phi^+(D_4)$ has a property that none of the $\Phi^+(A_n)$ or $\Phi^+(B_n) \cong \Phi^+(C_n)$ have, namely an element covered by three other elements. On the other hand, the finite orders in the $\Phi^+(D_2)$ and $\Phi^+(D_3)$ cases can be explained by $\Phi^+(D_2) \cong \Phi^+(A_1 \times A_1) \cong$ (two-element antichain) and $\Phi^+(D_3) \cong \Phi^+(A_3)$.

Nathan Williams has suggested that the behavior of $\Phi^+(A_n)$ and $\Phi^+(B_n) \cong \Phi^+(C_n)$ to have finite orders of R_P could generalize to the "positive root posets" of the other "coincidental types" H_3 and $I_2(m)$ (see, for example, Table 2.2 in [Will13]). And indeed, computations in Sage have established that $\operatorname{ord}(R_P) = 10$ for $P = \Phi^+(H_3)$, and we also have $\operatorname{ord}(R_P) = \operatorname{lcm}(2, m)$ for $P = \Phi^+(I_2(m))$ (this is a very easy consequence of Proposition 7.3).

It seems that minuscule heaps, as considered e.g. in [RuSh12, §6], also lead to small $\operatorname{ord}(R_P)$ values. Namely:

[49] Neither [Pan08] nor [AST11] work directly with order ideals and rowmotion, but instead they study antichains of the poset $\Phi^+(W)$ (which are called "nonnesting partitions" in [AST11]) and an operation on these antichains called Panyushev complementation. There is, however, a simple bijection between the set of antichains of a poset P and the set $J(P)$, and the conjugate of Panyushev complementation with respect to this bijection is precisely classical rowmotion.

[50] We refer to [StWi11, Definition 3.4] for notations.

- The heap $P_{w_0^J}$ in [RuSh12, Figure 8 (b)] satisfies $\operatorname{ord}(R_P) = 12$.
- The heap $P_{w_0^J}$ in [RuSh12, Figure 9 (b)] seems to satisfy $\operatorname{ord}(R_P) = 18$ (this was verified on numerical examples, as the poset is too large for efficient general computations).

(These two posets also appear as posets corresponding to the "Cayley plane" and the "Freudenthal variety" in [ThoYo07, p. 2].)

Various other families of posets related to root systems (minuscule posets, d-complete posets, rc-posets, alternating sign matrix posets) remain to be studied.

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