MA377 Rings and Modules
Samir Siksek
Notes taught in 2019 and 2021; 96-page PDF
`https://samirsiksek.github.io/siksek.github.io/teaching/rm/rmversion9.pdf`
Errata and comments by Darij Grinberg

The page numbers below refer to the printed page numbers in the supplied PDF (thus the first page of Chapter 1 is page 1). This list is not claimed to be exhaustive. Items marked "substantive" correct a mathematical assertion or repair a proof gap. Items marked "pedagogical" are not necessarily errors, but seem likely to cause avoidable difficulty for a student meeting rings and modules for the first time. The remaining items are local corrections or clarifications.

Most of the errata below were found with assistance from GPT-5.6 Sol.

A. Corrections and comments

1. **Page 1, list of references and last sentence:** "Dummitt" should be "Dummit". Also hyphenate the compound adjective in "our approach will be more hands-on".

2. **Page 3, paragraph after Example 2:** Replace "subtlties" by "subtleties".

3. **Page 4, end of Example 6:** Replace "and that it contained in $\mathbb{R}^2$" by "and that it is contained in $\mathbb{R}^2$".

4. **Page 5, proof of Lemma 12:** *Pedagogical point:* From
$$a^n(a^{m-n} - 1) = 0$$
the proof concludes that $a^{m-n} = 1$. It would help to justify this in more detail: From $a \neq 0$, we get $a^n \neq 0$, since an integral domain has no zero divisors. Merely saying "as $a \neq 0$" skips the small induction (or repeated cancellation) that justifies this.

5. **Page 5, Section 4:** Replace "such at" by "such as" and "powerseries" by "power series". Later, replace "an multiplicative inverse" by "a multiplicative inverse".

6. **Page 7, Example 22:** Replace "f' denote the derivate of f" by "f' denotes the derivative of f". Later in the same sentence, insert "a" in "it is homomorphism".

7. **Page 7, Example 24:** Replace "as $1 \notin \mathbb{Z}$" by "as $1 \notin 2\mathbb{Z}$".

8. **Page 8, definition of AB:** Replace "the empty sum with $n = 0$ as the 0" by "the empty sum with $n = 0$ as 0".

9. **Page 8, Example 28:** Replace "as following exercise shows" by "as the following exercise shows".

10. **Page 9, Exercise 30(ii):** In the description of the summands, replace "$y_j \in J$" by "$y_i \in J$".

11. **Page 9, Exercise 31:** Replace "$\mathfrak{b} = (b_1, \ldots, b_m)$" by "$\mathfrak{b} = (b_1, \ldots, b_n)$".

12. **Page 9, Example 33:** Insert a full stop in "So $\mathfrak{a} = \mathbb{Z}[X]$ Hence $1 \in \mathfrak{a}$".

13. **Page 11, first line:** Replace "Workout" by "Work out".

14. **Page 11, paragraph after Theorem 38:** The customary spelling is "Bézout identities".

15. **Page 12, proof of Theorem 40(b):** In the forward implication, replace "Hence by (b), $\bar{a} \in (\mathbb{Z}/m\mathbb{Z})^*$" by "Hence by (a), ..."; part (b) is the statement currently being proved. In the next sentence, delete one "and" from "and and so".

16. **Page 13, proof of Theorem 44:** After writing $a = qb + r$, it should perhaps be said that
$$r = a - qb \in \mathfrak{a}.$$
Only then does the minimality of $\partial(b)$ among the nonzero elements of $\mathfrak{a}$ imply that $r = 0$.

17. **Page 13, Example 47 and definition of a coset:** Replace "It turns that" by "It turns out that". A set $r + \mathfrak{a}$ is a "coset of $\mathfrak{a}$ in R", rather than a "coset of R".

18. **Page 14, Theorem 50:** Replace "the addition and multiplicative identity elements" by "the additive and multiplicative identity elements".

19. **Page 15, first remark after Theorem 53:** Replace "corollaries for the first one" by "corollaries of the first one".

20. **Page 16, Example 54 and proof of Theorem 56:** Replace "In otherwords" by "In other words", and "a ideal" by "an ideal".

21. **Pages 17–18, Lemma 60 and its proof:** Clarify that "$\deg(r) < \deg(f)$" uses the conventions that $\deg(0) = -\infty$ and that $-\infty$ is smaller than any integer.

22. **Page 19, proof of Theorem 65:** Replace "lets show" by "let's show", and replace "if f is composite" by "if f is reducible".

23. **Page 19, same proof:** The equality "$\gcd(f, f_1) = f_1$" need not be literally true under the stated convention that polynomial gcds are monic, since f_1 was not chosen monic. What is needed is simply that $\gcd(f, f_1)$ has positive degree and hence is not 1.

24. **Page 20, §14:** Again: "composite" should be "reducible".

25. **Page 22, reduction of θ^{n+1}:** On the third line of the long-ish computation, replace "$-a_1\theta - a_1\theta^2 - \cdots - -a_{n-2}\theta^{n-1}$" by "$-a_0\theta - a_1\theta^2 - \cdots - a_{n-2}\theta^{n-1}$".

26. **Page 22, summary after that computation:** Insert "of" in "as linear combinations $1, \theta, \ldots, \theta^{n-1}$".

27. **Page 25, Theorem 73(i):** Insert "a" in "is 2-sided ideal".

28. **Pages 25–26, proof of Theorem 73:** In each of the two subgroup arguments, the proof establishes that the set contains 0 and is closed under addition, then calls it a subgroup without checking additive inverses. Fortunately, the latter is indeed unnecessary if the ideal property ($ra \in \mathfrak{a}$ for all $r \in R$ and $a \in \mathfrak{a}$) has been proved (since $-a = (-1)\,a$ for all a), but this should be said somewhere.

29. **Page 26, Example 76:** Replace "$\mathrm{Ann}_{\mathbb{Z}/15\mathbb{Z}} =$" by "$\mathrm{Ann}_{\mathbb{Z}/15\mathbb{Z}}(10) =$".

30. **Page 27, definition of group-ring multiplication:** Replace "where gh denotes multiplication in g" by "where gh denotes multiplication in G".

31. **Page 29, second paragraph:** Replace "maybe expressed" by "may be expressed".

32. **Page 33, Example 95:** Add a full stop after "$\mathbb{R}$-algebra".

33. **Page 33, Example 96:** Replace "indentifying" by "identifying".

34. **Page 33, proof of Lemma 98:** Replace "Let take" by "Let's take".

35. **Page 34, Theorem 99:** In the evaluation map, replace "$f(X) \mapsto f(a)$" by "$f(X) \mapsto f(\alpha)$".

36. **Page 35, first bullet about χ_α:** The expression "$\det(XI_n - \phi_\alpha)$" subtracts a linear map from a matrix. Either choose a basis and write "$\chi_\alpha(X) = \det(XI_n - [\phi_\alpha])$", or use the basis-free operator notation "$\det(XI_A - \phi_\alpha)$".

37. **Page 35, Lemma 101:** Replace "possitive" by "positive".

38. **Page 35, proof of Lemma 101:** Replace "applying the above with $\beta = 1$ show" by "applying the above with $\beta = 1$ shows".

39. **Page 36, Example 102:** The characteristic-polynomial display concerns **i**, so replace "χ_α" and "M_α" by "$\chi_{\mathbf{i}}$" and "$M_{\mathbf{i}}$".

40. **Page 37, proof of Theorem 107:** Insert "you" in "it is easier to do this if think of quaternions as 2×2 matrices".

41. **Page 38, Theorem 111:** Require $R \neq 0$. In fact, the zero ring has no nonzero proper left ideals, but it is not a division ring under the definition on page 37.

42. **Page 39, sentence before Lemma 115:** Replace "The following lemma say no" by "The following lemma says no".

43. **Page 41, Exercise 119:** Replace "satisfes" by "satisfies".

44. **Page 43, proof of Lemma 128:** "for some positive integer n" should be "for some positive integer r".

45. **Page 43, paragraph before Lemma 130:** Close the parenthesis in "(recall $\dim(V) = n - 1$, where $n = \dim(A)$".

46. **Page 45, penultimate paragraph:** Replace "$\mathbb{R}$ belongs to the centre" by "$\mathbb{R}$ is contained in the centre".

47. **Page 47, proof of Lemma 137:** Replace the malformed "As $D \supseteq C_s$ is a division ring" by "As D is a division ring". (The centralizer in this lemma is C_x, not C_s.)

48. **Page 48, proof of Lemma 137:** Replace "are required" by "as required".

49. **Page 48, §4:** Replace "orbit-stablizer" by "orbit-stabilizer".

50. **Page 49, proof of Lemma 139:** Replace "Orb_x" by "$\mathrm{Orb}\,(x)$".

51. **Page 50, proof of Theorem 140:** Delete one "the" from "all the the orbits".

52. **Page 50, proof of Corollary 141:** In the stabilizer calculation, replace "$gy_ig^{-1} = y$" by "$gy_ig^{-1} = y_i$".

53. **Page 51, definition of Φ_n:** To make the displayed quotient unambiguous, specify that the least common multiple is chosen monic. For $n = 1$, also state the convention that the lcm of the empty family is 1; otherwise the first answer in Exercise 143 is not determined by the definition.

54. **Pages 51–52, Theorem 148:** The product

$$\prod_{\substack{1 \leq r < n, \\ (r,n)=1}} (X - \zeta_n^r)$$

is empty when $n = 1$, although $\Phi_1(X) = X - 1$. Either state the theorem for $n \geq 2$, or replace the "$1 \leq r < n$" under the product sign by "$1 \leq r \leq n$" (or "$0 \leq r < n$").

55. **Page 52, proof of Theorem 148:** Replace "enough ot prove" by "enough to prove" and "Coversely" by "Conversely".

56. **Page 52, proof of Wedderburn's Little Theorem:** Replace "But the definition (13) of $\Phi_n(X)$, we know" by "By definition (13), we know". Later replace "contradiciton" by "contradiction".

57. **Page 55, definition of a right module and Example 153:** Replace "multipliction" by "multiplication", and insert "an" in "This is $n \times 1$ matrix".

58. **Page 56, continuation of Example 153:** The sentence says that for column vectors, multiplication by matrices on the left is not defined, immediately after using precisely that multiplication. Replace it by either "for column vectors, multiplication by matrices on the right is not defined" or "for row vectors, multiplication by matrices on the left is not defined".

59. **Page 56, end of Example 154:** Replace "one-one correspondence" by "one-to-one correspondence".

60. **Page 57, Example 158:** Insert "a" in "the same as subspace of V".

61. **Page 58, Lemma 163 and the following paragraph:** Replace "addition and scalar multiplication is defined" by "addition and scalar multiplication are defined", and hyphenate "R-module".

62. **Page 59, Example 165:** Insert the missing "if" twice: "if and only they are K-vector spaces" and "if and only it is a linear transformation".

63. **Page 59, Exercise 169:** Replace "submodules of the R-submodule M/N" by "submodules of the R-module M/N".

64. **Page 60, Lemma 170:** Replace "decomposted" by "decomposed".

65. **Page 61, definition of span:** Insert "is" in "This the set of all finite linear combinations", and (optionally) insert "an" in "spans (or generates) M as R-module".

66. **Page 61, Example 175:** Replace "for any M-module R" by "for any R-module M".

67. **Page 61, definition of linear independence:** Require $x_1, \ldots, x_m$ to be *distinct* elements of X. If repetitions are allowed, every nonempty set is declared dependent by writing $1x + (-1)x = 0$.

    That said, it is probably better to define linear independence for families or tuples rather than for sets.

68. **Page 62, Example 180:** Delete "a" from "an R-basis for a M".

69. **Page 63, Example 183:** Replace "indepdendent" by "independent".

70. **Page 64, Exercise 188 and Section 5:** Hyphenate "$\mathbb{R}[T]$-module". In the definition of $\mathrm{Hom}_R(M, N)$, delete the duplicated "h :" from

$$\{h : h : M \to N \text{ is a homomorphism}\}.$$

Also insert "the" in "the additive identity is trivial homomorphism".

71. **Page 65, Exercise 191:** "Check that $rf \in \mathrm{Hom}_R(M, M)$" should be "Check that $rf \in \mathrm{Hom}_R(M, N)$".

72. **Page 65, Example 192:** Replace "contruct" by "construct". More importantly, in "This will be a homomorphism as ϕ, f, ϕ^{-1} are isomorphisms", replace "are isomorphisms" by "are homomorphisms"; $f \in \mathrm{End}(M)$ need not be invertible. A comma after the subordinate clause "as $\phi : M \to N$ is an isomorphism" would also smooth the preceding paragraph.

73. **Page 65, Exercise 195:** In both module actions, replace

$$a_0 + a_1 + a_2X^2 + \cdots + a_rX^r \quad \text{by} \quad a_0 + a_1X + a_2X^2 + \cdots + a_rX^r.$$

74. **Page 66, first paragraph of Section 6:** Replace "in terms of the M" by "in terms of M", and "beyond know that it is a homomorphism" by "beyond knowing that it is a homomorphism".

75. **Page 67, paragraph after (18):** Replace "the product of the matrices to two homomorphism" by "the product of the matrices of two homomorphisms".

76. **Page 68, associativity calculation:** Two closing parentheses are missing in the left-hand calculation. It should be

$$(f \circ (g \circ h))(\mathbf{v}) = f((g \circ h)(\mathbf{v})) = f(g(h(\mathbf{v}))).$$

Also, the right calculation should perhaps go on its separate line, since it currently bleeds out into the margins.

77. **Page 68, discussion of Theorem 197:** Insert "you" in "to convince of the truth".

78. **Page 68, proof of Theorem 197:** In the last bullet, replace "given a matrix how to you write down" by "given a matrix, how do you write down".

79. **Page 69, Example 200:** After "If $\#A \geq 2$", the "the" should be a "then".

80. **Pages 70–71, proof of Theorem 205:** To prove that $\mathfrak{b} = \bigcup_i \mathfrak{b}_i$ is an ideal, the proof checks 0, addition, and multiplication by ring elements, but not additive inverses. Again, this is legitimate, but the reason should be mentioned.

81. **Page 71, Exercise 206:** This holds only if R is nonzero.

82. **Page 71, Exercise 209:** Replace "containining" by "containing".

83. **Page 72, Exercise 210:** In Exercise 210(v), delete one copy of "that" before "$\mathbb{Q}$ has no maximal $\mathbb{Z}$-submodules".

84. **Page 72, Theorem 211 and Corollary 212:** Use "a" rather than "an" before consonant sounds: "a D-module", "a D-basis", "a D-linearly independent set", and similarly "a K-basis".

85. **Page 73, proof of Theorem 211(ii):** Delete one copy of "combination" in "a finite linear combination combination of elements of T".

    In the next sentence, after "Since T is maximal, $T \cup \{\mathbf{v}\}$ does not belong to $\mathcal{P}$", add "(unless $\mathbf{v} \in T$, in which case our claim is obvious)".

86. **Page 75, Example 218:** Delete "the" from "the submodules of the $\mathbb{Z}/m\mathbb{Z}$", and insert "if" in "if and only m is prime".

87. **Page 76, Example 221:** Replace "extended to a K-basis ... for K" by "extended to a K-basis ... for K^n".

88. **Page 76, Theorem 222:** Delete "a" from "D^n is a simple".

89. **Page 76, proof of Theorem 222:** Replace "Let $A \in M_n(K)$" by "Let $A \in M_n(D)$".

90. **Page 76, proof of Theorem 224:** Insert "is" in "Thus there some isomorphism $g = f^{-1}$".

91. **Page 77, proof of Lemma 228:** Replace "R does not have any left R-submodules" and "R has no left ideals" by "R has no nonzero proper left R-submodules" and "R has no nonzero proper left ideals". Both 0 and R itself are always left ideals.

92. **Page 78, proof of Theorem 227:** Theorem 73 concerns 2-sided ideals of a quotient ring. Here $\mathfrak{m}$ is only a left ideal and $R/\mathfrak{m}$ is a quotient module, so cite the module Correspondence Theorem, Exercise 169, instead.

93. **Page 79, Example 229:** Replace each of the three occurrences of "V" by "W".

94. **Page 79, Example 230:** Replace "has not complementary ideal" by "has no complementary ideal".

95. **Page 80, Exercise 234:** Replace "the $\mathbb{R}[X]$-module were" by "the $\mathbb{R}[X]$-module where".

96. **Page 81, Exercise 240:** Replace "let $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ are" by "let $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ be".

97. **Page 81, proof of Theorem 241:** The induction begins with $\dim_K(V) = 1$, but the theorem also allows $V = 0$. Add that when $\dim_K(V) = 0$, V is the empty direct sum of simple modules. (On the other hand, the case $\dim_K(V) = 1$ does not require separate treatment.)

98. **Page 81, proof of Theorem 241:** Replace "there is a A-submodule" by "there is an A-submodule".

99. **Page 81, end of the same proof:** Replace "the U_i and V_j are simple" by "the U_i and W_j are simple".

100. **Page 82, Lemma 243:** Replace "$= \psi h$" by "$= \psi$".

101. **Page 83, Lemma 245 and Example 246:** Since the subject is plural, write "$\phi_1, \ldots, \phi_r$ form a K-basis" and "ϕ_1, ϕ_2, ϕ_3 form a basis". In the proof, insert "We" before "can write".

102. **Page 85, proof of Lemma 249:** Replace "We compute $Ae_{u,v}$ and $Ae_{v,u}$ and compare" by "We compute $Ae_{u,v}$ and $e_{u,v}A$ and compare", as in the calculation that follows.

103. **Page 86, Section 6 and Examples 252–253:** Replace "otherwords" by "other words" and hyphenate "$K[G]$-module". In both examples, insert "we" in "so can apply Maschke's theorem".

104. **Page 87, proof of Maschke's Theorem:** Replace the semicolon after claim (iii) by a full stop, use the plural in "prove claim (i), (ii), and (iii)", and replace "is is easy" by "it is easy".

105. **Page 88, proof of Theorem 256:** The m in the isomorphism "$\mathbb{C}[G] \cong \prod_{i=1}^{m} M_{n_i}(D_i)$" (which is obtained from Artin–Wedderburn) is a-priori only some nonnegative integer; that it equals the m from Theorem 256 (that is, the number of conjugacy classes of G) will only become clear at the end of the proof. Thus, a different letter (or a warning) is warranted.

106. **Page 88, same proof:** The direct result saying that a finite-dimensional complex division algebra is $\mathbb{C}$ is Theorem 118, not Frobenius' real classification in Theorem 123. Citing Theorem 118 is both shorter and clearer.

107. **Page 92, proof of Theorem 268 and Exercise 270:** Replace "an 2-sided ideal" by "a 2-sided ideal", replace "the kernel ... contain" by "the kernel ... contains", and replace "endormphism" by "endomorphism".