The Redei–Berge symmetric function of a directed graph

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Abstract. Let $D = (V, A)$ be a digraph with n vertices, where each arc $a \in A$ is a pair (u, v) of two vertices. We study the *Redei–Berge symmetric function U_D*, defined as the quasisymmetric function

$$\sum L_{\operatorname{Des}(w,D),\ n} \in \operatorname{QSym}.$$

Here, the sum ranges over all lists $w = (w_1, w_2, \ldots, w_n)$ that contain each vertex of D exactly once, and the corresponding addend is

$$L_{\operatorname{Des}(w,D),\ n} := \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_n; \\ i_p < i_{p+1} \text{ for each } p \text{ satisfying } (w_p, w_{p+1}) \in A}} x_{i_1} x_{i_2} \cdots x_{i_n}$$

(an instance of Gessel's fundamental quasisymmetric functions).

While U_D is a specialization of Chow's path-cycle symmetric function, which has been studied before, we prove some new formulas that express U_D in terms of the power-sum symmetric functions. We show that U_D is always p-integral, and furthermore is p-positive whenever D has no 2-cycles. When D is a tournament, U_D can be written as a polynomial in $p_1, 2p_3, 2p_5, 2p_7, \ldots$ with nonnegative integer coefficients. By specializing these results, we obtain the famous theorems of Redei and Berge on the number of Hamiltonian paths in digraphs and tournaments, as well as a modulo-4 refinement of Redei's theorem.

Keywords: directed graph, symmetric function, tournament, Hamiltonian path, power sum symmetric function.

Mathematics Subject Classification 2020: 05A05, 05A19, 05E05, 05C30.

Contents

1. Definitions and the main theorems

We begin with introducing the notations, some of which come from [EC2sup22, Problem 120]. We use standard notations as defined, e.g., in [Stanle01, Chapter 7] and [GriRei20, Chapters 2 and 5].

1.1. Digraphs, V-listings and D-descents

We let $\mathbb{N} := \{0, 1, 2, \ldots\}$ and $\mathbb{P} := \{1, 2, 3, \ldots\}$. We set $[n] := \{1, 2, \ldots, n\}$ for each $n \in \mathbb{Z}$. (This set $[n]$ is empty if $n \leq 0$.)

The words "list" and "tuple" will be used interchangeably, and will always mean finite ordered tuples.

We shall next introduce some basic notations regarding digraphs (i.e., directed graphs):

Definition 1.1. A *digraph* means a pair (V, A), where V is a finite set and where A is a subset of $V \times V$. The elements of V are called the *vertices* of this digraph, and the elements of A are called the *arcs* of this digraph. For any further notations, we refer to standard literature (the definitions in [Grinbe17, §1.1-§1.2] should suffice) and common sense. (Our digraphs are allowed to have loops, but this has no effect on what follows.)

Definition 1.2. Let $D = (V, A)$ be a digraph. Then, the digraph $(V, (V \times V) \setminus A)$ will be denoted by $\overline{D}$ and called the *complement* of the digraph D. Its arcs will be called the *non-arcs* of D (since they are precisely the pairs $(u, v) \in V \times V$ that are not arcs of D).

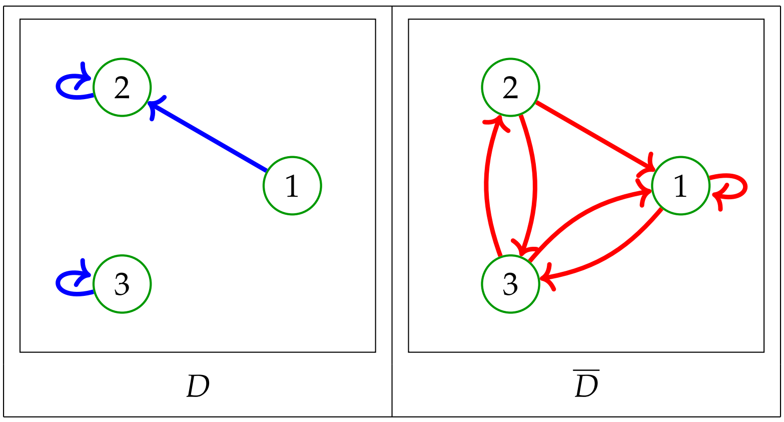
Example 1.3. If D is the digraph

$$(\{1,2,3\}, \{(1,2), (2,2), (3,3)\}),$$

then its complement $\overline{D}$ is the digraph

$$(\{1,2,3\}, \{(1,1), (1,3), (2,1), (2,3), (3,1), (3,2)\}).$$

Here are the two digraphs, drawn side by side:



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Definition 1.4. Let V be a finite set. A *V-listing* will mean a list of elements of V that contains each element of V exactly once.

For example, $(2, 1, 3)$ is a $\{1, 2, 3\}$-listing.

Of course, if V is a finite set, then there are exactly $|V|!$ many V-listings. They are in a canonical bijection with the bijective maps from $[|V|]$ to V, and in a non-canonical bijection with the permutations of V.

Convention 1.5. If w is any list (i.e., tuple), and if i is a positive integer, then w_i shall denote the i-th entry of w. (Thus, $w = (w_1, w_2, \ldots, w_k)$, where k is the length of w.)

Definition 1.6. Let $D = (V, A)$ be a digraph. Let $w = (w_1, w_2, \ldots, w_n)$ be a V-listing. Then:

(a) A *D-descent* of w means an $i \in [n-1]$ satisfying $(w_i, w_{i+1}) \in A$.

(b) We let $\operatorname{Des}(w, D)$ denote the set of all D-descents of w.

Example 1.7. Let D be the digraph D from Example 1.3, and let w be the V-listing $(3, 1, 2)$. Then, 2 is a D-descent of w (since $(w_2, w_3) = (1, 2) \in A$), but 1 is not a D-descent of w (since $(w_1, w_2) = (3, 1) \notin A$). Hence, $\operatorname{Des}(w, D) = \{2\}$.

Example 1.8. Let $n \in \mathbb{N}$, and let $V = [n]$. Let D be the digraph whose vertices are the elements of V and whose arcs are all the pairs $(i, j) \in [n]^2$ satisfying $i > j$. Let w be a V-listing. Then, the D-descents of w are exactly the descents of w in the usual sense (i.e., the numbers $i \in [n-1]$ satisfying $w_i > w_{i+1}$).

We note that D-descents for general digraphs D have already implicitly appeared in the work of Foata and Zeilberger [FoaZei96], which considers the number $\operatorname{maj}'_D w := \sum\limits_{i \in \operatorname{Des}(w,D)} i$ for each V-listing w. We would not be surprised if what follows can shed some new light on the results of [FoaZei96], but so far we have not found any deeper connections.

1.2. Quasisymmetric functions

Next, we introduce some notations from the theory of quasisymmetric functions (see, e.g., [Stanle01, §7.19] or [GriRei20, Chapter 5]):

Definition 1.9.

(a) A *composition* means a finite list of positive integers. If $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ is a composition, then the number k is called the *length* of α, whereas the number $\alpha_1 + \alpha_2 + \cdots + \alpha_k$ is called the *size* of α. If $n \in \mathbb{N}$, then a *composition of n* shall mean a composition having size n.

(b) A *partition* (or *integer partition*) means a composition that is weakly decreasing.

For example, $(2,5,3)$ is a composition of 10 that has length 3 and is not a partition (since $2 < 5$).

Definition 1.10. Let $n \in \mathbb{N}$. For any subset I of $[n-1]$, we let $\operatorname{comp}(I,n)$ be the list

$$(i_1 - i_0,\ i_2 - i_1,\ i_3 - i_2,\ \ldots,\ i_k - i_{k-1}),$$

where $i_0, i_1, \ldots, i_k$ are the elements of $\{0\} \cup I \cup \{n\}$ listed in strictly increasing order. This list $\operatorname{comp}(I,n)$ is a composition of n.

Example 1.11. If $n = 6$ and $I = \{2,3,5\}$, then $\operatorname{comp}(I,n) = (2,1,2,1)$.

Note that $\operatorname{comp}(I,n)$ is denoted by $\operatorname{co}(I)$ in [Stanle01, §7.19], but we prefer to make the dependence on n explicit here. In the notation of [GriRei20, Definition 5.1.10], the composition $\operatorname{comp}(I,n)$ is the preimage of I under the bijection $D : \operatorname{Comp}_n \to 2^{[n-1]}$.

For any $n \in \mathbb{N}$, the map

$$\begin{aligned} \{\text{subsets of } [n-1]\} &\to \{\text{compositions of } n\}, \\ I &\mapsto \operatorname{comp}(I,n) \end{aligned}$$

is a bijection.

Definition 1.12. Consider the ring $\mathbb{Z}[[x_1, x_2, x_3, \ldots]]$ of formal power series in countably many indeterminates $x_1, x_2, x_3, \ldots$. Two subrings of this ring $\mathbb{Z}[[x_1, x_2, x_3, \ldots]]$ are:

- the ring Λ of *symmetric functions* (defined, e.g., in [Stanle01, §7.1] or in [GriRei20, §2.1]);
- the ring QSym of *quasisymmetric functions* (defined, e.g., in [Stanle01, §7.19] or in [GriRei20, §5.1]).

We will not actually use any properties of these rings Λ and QSym anywhere except in Sections 8, 6 and 7 (and even there, only Λ will be used); thus, a reader unfamiliar with symmetric functions can read $\mathbb{Z}[[x_1, x_2, x_3, \ldots]]$ instead of Λ or QSym everywhere else.

Definition 1.13. Let α be a composition. Then, L_α will denote the *fundamental quasisymmetric function* corresponding to α. This is a formal power series in

QSym, and is defined as follows: Let I be the unique subset of $[n-1]$ satisfying $\alpha = \operatorname{comp}(I,n)$. Then, we set

$$L_\alpha = \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_n; \\ i_p < i_{p+1} \text{ for each } p \in I}} x_{i_1} x_{i_2} \cdots x_{i_n} \in \mathrm{QSym}$$

(where the summation indices $i_1, i_2, \ldots, i_n$ range over $\mathbb{P}$).

See [Stanle01, §7.19] or [GriRei20, §5] for more about these fundamental quasisymmetric functions L_α (originally introduced by Ira Gessel)[1]. We will actually find it easier to index them not by the compositions α but rather by the corresponding subsets I of $[n-1]$. Thus, we define:

Definition 1.14. Let $n \in \mathbb{N}$, and let I be a subset of $[n-1]$. Then, we will use the notation $L_{I,n}$ for $L_{\operatorname{comp}(I,n)}$. Explicitly, we have

$$L_{I,n} = \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_n; \\ i_p < i_{p+1} \text{ for each } p \in I}} x_{i_1} x_{i_2} \cdots x_{i_n} \in \mathrm{QSym} \qquad (1)$$

(where the summation indices $i_1, i_2, \ldots, i_n$ range over $\mathbb{P}$).

Example 1.15. If $n = 3$ and $I = \{2\}$, then

$$\begin{aligned} L_{I,n} = L_{\{2\},3} &= \sum_{\substack{i_1 \leq i_2 \leq i_3; \\ i_p < i_{p+1} \text{ for each } p \in \{2\}}} x_{i_1} x_{i_2} x_{i_3} = \sum_{i_1 \leq i_2 < i_3} x_{i_1} x_{i_2} x_{i_3} \\ &= x_1x_1x_2 + x_1x_1x_3 + \cdots + x_1x_2x_3 + x_1x_2x_4 + \cdots + \cdots + x_2x_2x_3 + \cdots. \end{aligned}$$

1.3. The Redei–Berge symmetric function

We are now ready to define the main protagonist of this paper:

Definition 1.16. Let $n \in \mathbb{N}$. Let $D = (V, A)$ be a digraph with n vertices. We define the *Redei–Berge symmetric function* U_D to be the quasisymmetric function

$$\sum_{w \text{ is a } V\text{-listing}} L_{\operatorname{Des}(w,D),\ n} \in \mathrm{QSym}.$$

[1]Note that the definition of L_α given in [GriRei20, Definition 5.2.4] differs from ours. However, it is equivalent to ours, since [GriRei20, Proposition 5.2.9] shows that the L_α defined in [GriRei20, Definition 5.2.4] satisfy the same formula that we used to define our L_α.

Example 1.17. Let D be the digraph D from Example 1.3. Then,

$$\begin{aligned}
U_D &= \sum_{w \text{ is a } V\text{-listing}} L_{\operatorname{Des}(w,D),\ 3} \\
&= L_{\operatorname{Des}((1,2,3),D),\ 3} + L_{\operatorname{Des}((1,3,2),D),\ 3} + L_{\operatorname{Des}((2,1,3),D),\ 3} \\
&\qquad + L_{\operatorname{Des}((2,3,1),D),\ 3} + L_{\operatorname{Des}((3,1,2),D),\ 3} + L_{\operatorname{Des}((3,2,1),D),\ 3} \\
&= L_{\{1\},\ 3} + L_{\varnothing,\ 3} + L_{\varnothing,\ 3} + L_{\varnothing,\ 3} + L_{\{2\},\ 3} + L_{\varnothing,\ 3} \\
&= 4 \cdot \underbrace{L_{\varnothing,\ 3}}_{= \sum_{i_1 \leq i_2 \leq i_3} x_{i_1} x_{i_2} x_{i_3}} + \underbrace{L_{\{1\},\ 3}}_{= \sum_{i_1 < i_2 \leq i_3} x_{i_1} x_{i_2} x_{i_3}} + \underbrace{L_{\{2\},\ 3}}_{= \sum_{i_1 \leq i_2 < i_3} x_{i_1} x_{i_2} x_{i_3}} \\
&= 4 \cdot \sum_{i_1 \leq i_2 \leq i_3} x_{i_1} x_{i_2} x_{i_3} + \sum_{i_1 < i_2 \leq i_3} x_{i_1} x_{i_2} x_{i_3} + \sum_{i_1 \leq i_2 < i_3} x_{i_1} x_{i_2} x_{i_3}.
\end{aligned}$$

From this expression, we can easily see that U_D is actually a symmetric function, and can be written (e.g.) as $p_1^3 + 2p_1p_2 + p_3$, where $p_k := x_1^k + x_2^k + x_3^k + \cdots$ is the k-th power-sum symmetric function for each $k \geq 1$.

The name "Redei–Berge symmetric function" for the power series U_D was chosen because (as we will soon see) it is actually a symmetric function and is related to two classical results of Redei and Berge on the number of Hamiltonian paths in digraphs. In [EC2sup22, Problem 120], it is called U_X, where X is what we call A (that is, the set of arcs of D); but we shall here put the entire digraph D into the subscript.

The Redei–Berge symmetric function U_D equals the quasisymmetric function $\Xi_{\overline{D}}(x,0)$ from Chow's [Chow96].[2] It also is denoted by $\Pi_{\overline{D}}$ in [Wisema07].[3] Several properties of U_D have been shown in [Chow96] and in [Wisema07], and some of them will be reproved here for the sake of self-containedness and variety. However, our main results – Theorems 1.31, 1.39 and 1.41 – appear to be new.

Question 1.18. Can these results be extended to the more general functions $\Xi_D(x,y)$ from [Chow96]?

1.4. Arcs and cyclic arcs

The main results of this paper are explicit (albeit not, in general, subtraction-free) expansions of U_D in terms of the power-sum symmetric functions. To state these, we need some more notations. We shall soon define cycles of digraphs and cycles of permutations, and we will then connect the two notions. First, some auxiliary notations:

[2] Indeed, this equality follows immediately from [Chow96, Proposition 7], since the quasisymmetric function we call $L_{I,n}$ appears under the name of $Q_{I,n}$ in [Chow96], and since our $\operatorname{Des}(w,\overline{D})$ is what is called $S(w)$ in [Chow96].

[3] Indeed, comparing the definition of Π_D in [Wisema07, Definition 2.2] with the definition of Ξ_D in [Chow96, §2] shows that $\Pi_D = \Xi_D(x,0)$. Thus, $\Pi_{\overline{D}} = \Xi_{\overline{D}}(x,0) = U_D$ (as we already know).

Definition 1.19. Let V be a set. Let $v = (v_1, v_2, \ldots, v_k) \in V^k$ be a nonempty tuple of elements of V.

(a) We define a subset $\operatorname{Arcs} v$ of $V \times V$ by

$$\begin{aligned}\operatorname{Arcs} v &:= \{(v_i, v_{i+1}) \mid i \in [k-1]\} \\ &= \{(v_1, v_2), (v_2, v_3), \ldots, (v_{k-1}, v_k)\} \\ &\subseteq V \times V.\end{aligned} \tag{2}$$

This subset $\operatorname{Arcs} v$ is called the *arc set* of the tuple v. Its elements (v_i, v_{i+1}) are called the *arcs* of v.

(b) We define a subset $\operatorname{CArcs} v$ of $V \times V$ by

$$\begin{aligned}\operatorname{CArcs} v &:= \{(v_i, v_{i+1}) \mid i \in [k]\} \\ &= \{(v_1, v_2), (v_2, v_3), \ldots, (v_{k-1}, v_k), (v_k, v_1)\} \\ &\subseteq V \times V,\end{aligned} \tag{3}$$

where we set $v_{k+1} := v_1$. This subset $\operatorname{CArcs} v$ is called the *cyclic arc set* of the tuple v. Its elements (v_i, v_{i+1}) are called the *cyclic arcs* of v.

(c) The *reversal* of v is defined to be the tuple $(v_k, v_{k-1}, \ldots, v_1) \in V^k$.

Example 1.20. Let $V = \mathbb{N}$ and $v = (1, 4, 2, 6) \in V^4$. Then,

$$\begin{aligned}\operatorname{Arcs} v &= \{(1,4), (4,2), (2,6)\} \qquad \text{and} \\ \operatorname{CArcs} v &= \{(1,4), (4,2), (2,6), (6,1)\}.\end{aligned}$$

Note that if we cyclically rotate a nonempty tuple $v \in V^k$, then the set $\operatorname{CArcs} v$ remains unchanged: i.e., for any $(v_1, v_2, \ldots, v_k) \in V^k$, we have

$$\operatorname{CArcs}(v_1, v_2, \ldots, v_k) = \operatorname{CArcs}(v_2, v_3, \ldots, v_k, v_1).$$

1.5. Permutations and their cycles

Now, let us discuss permutations and their cycles. We start with some basic notations:

Definition 1.21. Let V be a finite set. Then, $\mathfrak{S}_V$ shall denote the symmetric group of V (that is, the group of all permutations of V).

Note that the order of this group is $|\mathfrak{S}_V| = |V|!$.

Definition 1.22. Let V be a set.

(a) Two tuples $v \in V^k$ and $w \in V^\ell$ of elements of V are said to be *rotation-equivalent* if w can be obtained from v by cyclic rotation, i.e., if $\ell = k$ and $w = (v_i, v_{i+1}, \ldots, v_k, v_1, v_2, \ldots, v_{i-1})$ for some $i \in [k]$.

(b) The relation "rotation-equivalent" is an equivalence relation on the set of all nonempty tuples of elements of V. Its equivalence classes are called the *rotation-equivalence classes*. In other words, the rotation-equivalence classes are the orbits of the operation

$$(a_1, a_2, \ldots, a_k) \mapsto (a_2, a_3, \ldots, a_k, a_1)$$

on the set of all nonempty tuples of elements of V.

(c) The rotation-equivalence class that contains a given nonempty tuple $v \in V^k$ will be denoted by $v_\sim$.

For instance, the tuple $(1,2,3,4)$ is rotation-equivalent to $(3,4,1,2)$, but not to $(4,3,2,1)$. Thus,

$$(1,2,3,4)_\sim = (3,4,1,2)_\sim \neq (4,3,2,1)_\sim .$$

Also,

$$(1,3,6)_\sim = \{(1,3,6),\ (3,6,1),\ (6,1,3)\} .$$

Definition 1.23. Let V be a set. Let γ be a rotation-equivalence class (of nonempty tuples of elements of V). Then:

(a) All tuples $v \in \gamma$ have the same length (i.e., number of entries). This length is denoted by $\ell(\gamma)$, and is called the *length* of γ. Thus, if $\gamma = v_\sim$ for some tuple $v \in V^k$, then $\ell(\gamma) = k$.

(b) All tuples $v \in \gamma$ have the same cyclic arc set $\operatorname{CArcs} v$ (since $\operatorname{CArcs} v$ remains unchanged if we cyclically rotate v). This cyclic arc set is denoted by $\operatorname{CArcs} \gamma$, and is called the *cyclic arc set* of γ. Thus, the cyclic arc set of a rotation-equivalence class $\gamma = (v_1, v_2, \ldots, v_k)_\sim$ is

$$\operatorname{CArcs} \gamma = \{(v_1, v_2),\ (v_2, v_3),\ \ldots,\ (v_{k-1}, v_k),\ (v_k, v_1)\} .$$

(c) All tuples $v \in \gamma$ have the same entries (up to order). These entries are called the *entries* of γ. Thus, the entries of a rotation-equivalence class $\gamma = (v_1, v_2, \ldots, v_k)_\sim$ are $v_1, v_2, \ldots, v_k$.

(d) The reversals of all tuples $v \in \gamma$ are the elements of a single rotation-equivalence class $\operatorname{rev} \gamma$. This latter class will be called the *reversal* of γ. Thus, the reversal of a rotation-equivalence class $\gamma = (v_1, v_2, \ldots, v_k)_\sim$ is the rotation-equivalence class $(v_k, v_{k-1}, \ldots, v_1)_\sim$.

(e) We say that γ is *nontrivial* if $\ell(\gamma) > 1$.

For instance, the rotation-equivalence class $(3,1,4)_\sim$ has length 3, cyclic arc set $\{(3,1), (1,4), (4,3)\}$, and entries $3,1,4$. Its reversal is $(4,1,3)_\sim$, and it is nontrivial (since $\ell((3,1,4)_\sim) = 3 > 1$).

Definition 1.24. Let V be a finite set. Let $\sigma \in \mathfrak{S}_V$ be any permutation.

(a) The *cycles* of σ are the rotation-equivalence classes of the tuples of the form

$$\left(\sigma^0(i), \sigma^1(i), \ldots, \sigma^{k-1}(i)\right),$$

where i is some element of V, and where k is the smallest positive integer satisfying $\sigma^k(i) = i$.

For example, the permutation $w_0 \in \mathfrak{S}_{[7]}$ that sends each $i \in [7]$ to $8-i$ has cycles $(1,7)_\sim$, $(2,6)_\sim$, $(3,5)_\sim$ and $(4)_\sim$. (Note that we do allow a cycle to have length 1.)

(b) The *cycle type* of σ means the partition whose entries are the lengths of the cycles of σ. We denote this cycle type by $\operatorname{type}\sigma$. It is a partition of the number $|V|$.

(c) We let $\operatorname{Cycs}\sigma$ denote the set of all cycles of σ.

Example 1.25. Let $w_0 \in \mathfrak{S}_{[7]}$ be the permutation that sends each $i \in [7]$ to $8-i$. We have already seen that w_0 has cycles $(1,7)_\sim$, $(2,6)_\sim$, $(3,5)_\sim$ and $(4)_\sim$. Their respective lengths are $2,2,2,1$. Thus, the cycle type of w_0 is $\operatorname{type} w_0 = (2,2,2,1)$. We have $\operatorname{Cycs}\sigma = \{(1,7)_\sim, (2,6)_\sim, (3,5)_\sim, (4)_\sim\}$. The first three of the four cycles $(1,7)_\sim$, $(2,6)_\sim$, $(3,5)_\sim$ and $(4)_\sim$ are nontrivial.

1.6. D-paths and D-cycles

Next, we define paths and cycles in a digraph:

Definition 1.26. Let $D = (V, A)$ be a digraph.

(a) A *D-path* (or *path of D*) shall mean a nonempty tuple v of distinct elements of V such that $\operatorname{Arcs} v \subseteq A$.

(b) A *D-cycle* (or *cycle of D*) shall mean a rotation-equivalence class γ of nonempty tuples of distinct elements of V such that $\operatorname{CArcs}\gamma \subseteq A$.

We note that our notion of "cycle of D" differs slightly from the common one used in graph theory[4].

Example 1.27. Let D be the digraph D from Example 1.3. Then:

(a) The pair $(1,2)$ as well as the three 1-tuples (1), (2) and (3) are D-paths. The triple $(1,2,2)$ is not a D-path (even though it satisfies the "$\operatorname{Arcs} v \subseteq A$" condition), since its entries $1,2,2$ are not distinct. The triple $(1,2,3)$ is not a D-path, since $(2,3)$ is not an arc of D.

The triple $(2,3,1)$ is a $\overline{D}$-path (and there are several others).

(b) The only D-cycles are the rotation-equivalence classes $(2)_\sim$ and $(3)_\sim$. The $\overline{D}$-cycles are $(1)_\sim$, $(1,3)_\sim$, $(2,3)_\sim$ and $(2,1,3)_\sim$.

1.7. The sets $\mathfrak{S}_V(D)$ and $\mathfrak{S}_V\left(D,\overline{D}\right)$

Now, we can connect digraphs with permutations by comparing their cycles:

Definition 1.28. Let $D=(V,A)$ be a digraph. Then, we define

$$\mathfrak{S}_V(D) := \{\sigma \in \mathfrak{S}_V \mid \text{each nontrivial cycle of } \sigma \text{ is a } D\text{-cycle}\}$$

and

$$\mathfrak{S}_V\left(D,\overline{D}\right) := \left\{\sigma \in \mathfrak{S}_V \mid \text{each cycle of } \sigma \text{ is a } D\text{-cycle or a } \overline{D}\text{-cycle}\right\}.$$

Note that we could just as well replace "each cycle" by "each nontrivial cycle" in the definition of $\mathfrak{S}_V\left(D,\overline{D}\right)$, since a cycle of length 1 is always a D-cycle or a $\overline{D}$-cycle (depending on whether its only cyclic arc belongs to A or not). However, we could not replace "nontrivial cycle" by "cycle" in the definition of $\mathfrak{S}_V(D)$.

Example 1.29. Let D be the digraph D from Example 1.3. Let $V=\{1,2,3\}$ be its set of vertices. Then:

(a) We have $\mathfrak{S}_V(D)=\{\mathrm{id}_V\}$, since the only D-cycles have length 1.

(b) We have

$$\mathfrak{S}_V\left(D,\overline{D}\right)=\left\{\mathrm{id}_V,\ \mathrm{cyc}_{1,3},\ \mathrm{cyc}_{2,3},\ \mathrm{cyc}_{1,3,2}\right\},$$

where $\mathrm{cyc}_{i_1,i_2,\ldots,i_k}$ denotes the permutation that cyclically permutes the elements $i_1,i_2,\ldots,i_k$ while leaving all other elements of V unchanged.

[4]Namely, cycles in graph theory have their first vertex repeated at the end, whereas our cycles don't. However, this difference is purely notational: A cycle $(v_1,v_2,\ldots,v_k)_\sim$ in our sense corresponds to the cycle $(v_1,v_2,\ldots,v_k,v_1)$ in the graph-theorists' terminology.

1.8. Formulas for U_D

1.8.1. The power-sum symmetric functions

We now introduce some of the best-known (and easiest to define) symmetric functions:

Definition 1.30.

(a) For each positive integer n, we define the *power-sum symmetric function*

$$p_n := x_1^n + x_2^n + x_3^n + \cdots \in \Lambda.$$

(b) If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is a partition with k positive entries, then we set

$$p_\lambda := p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_k} \in \Lambda.$$

For instance, $p_{(2,2,1)} = p_2 p_2 p_1 = \left(x_1^2 + x_2^2 + x_3^2 + \cdots\right)^2 (x_1 + x_2 + x_3 + \cdots)$.

1.8.2. The first main theorem: general digraphs

We now state our first main theorem (which will be proved in Section 2):

Theorem 1.31. Let $D = (V, A)$ be a digraph. Set

$$\varphi(\sigma) := \sum_{\substack{\gamma \in \operatorname{Cycs}\sigma; \\ \gamma \text{ is a } D\text{-cycle}}} (\ell(\gamma) - 1) \qquad \text{for each } \sigma \in \mathfrak{S}_V.$$

Then,

$$U_D = \sum_{\sigma \in \mathfrak{S}_V\left(D, \overline{D}\right)} (-1)^{\varphi(\sigma)} p_{\operatorname{type}\sigma}.$$

Example 1.32. Let $V = \{1, 2, 3, 4, 5, 6\}$ and $D = (V,\ V \times V)$. Let $\sigma \in \mathfrak{S}_V$ be the permutation whose cycles are $(1,3)_\sim$, $(2,4,5)_\sim$ and $(6)_\sim$. Then, every cycle of σ is a D-cycle, and the number $\varphi(\sigma)$ (as defined in Theorem 1.31) is

$$\begin{aligned}&(\ell((1,3)_\sim) - 1) + (\ell((2,4,5)_\sim) - 1) + (\ell((6)_\sim) - 1) \\ &= (2-1) + (3-1) + (1-1) = 3.\end{aligned}$$

Example 1.33. Let D be the digraph D from Example 1.3. Recall that $\mathfrak{S}_V\left(D,\overline{D}\right) = \left\{\mathrm{id}_V,\ \mathrm{cyc}_{1,3},\ \mathrm{cyc}_{2,3},\ \mathrm{cyc}_{1,3,2}\right\}$. Thus, Theorem 1.31 yields

$$\begin{aligned} U_D &= \underbrace{(-1)^{\varphi(\mathrm{id}_V)}}_{=(-1)^0=1} \underbrace{p_{\mathrm{type}(\mathrm{id}_V)}}_{=p_{(1,1,1)}=p_1^3} + \underbrace{(-1)^{\varphi\left(\mathrm{cyc}_{1,3}\right)}}_{=(-1)^0=1} \underbrace{p_{\mathrm{type}\left(\mathrm{cyc}_{1,3}\right)}}_{=p_{(2,1)}=p_2p_1} \\ &\qquad + \underbrace{(-1)^{\varphi\left(\mathrm{cyc}_{2,3}\right)}}_{=(-1)^0=1} \underbrace{p_{\mathrm{type}\left(\mathrm{cyc}_{2,3}\right)}}_{=p_{(2,1)}=p_2p_1} + \underbrace{(-1)^{\varphi\left(\mathrm{cyc}_{1,3,2}\right)}}_{=(-1)^0=1} \underbrace{p_{\mathrm{type}\left(\mathrm{cyc}_{1,3,2}\right)}}_{=p_{(3)}=p_3} \\ &= p_1^3 + p_2p_1 + p_2p_1 + p_3 = p_1^3 + 2p_1p_2 + p_3. \end{aligned}$$

This agrees with the result found in Example 1.17.

Example 1.34. Let D be the digraph (V, A), where $V = \{1, 2, 3\}$ and

$$A = \{(1,3),\ (2,1),\ (3,1),\ (3,2)\}.$$

Then, a straightforward computation using Theorem 1.31 shows that $U_D = p_1^3 - p_1p_2 + p_3$. (This example is due to Ira Gessel.)

The following two corollaries can be easily obtained from Theorem 1.31 (see Section 4 for their proofs):

Corollary 1.35. Let $D = (V, A)$ be a digraph. Then, U_D is a p-integral symmetric function (i.e., a symmetric function that can be written as a polynomial in $p_1, p_2, p_3, \ldots$). That is, we have $U_D \in \mathbb{Z}[p_1, p_2, p_3, \ldots]$.

Corollary 1.36. Let $D = (V, A)$ be a digraph. Assume that every D-cycle has odd length. Then,

$$U_D = \sum_{\sigma \in \mathfrak{S}_V\left(D,\overline{D}\right)} p_{\mathrm{type}\,\sigma} \in \mathbb{N}[p_1, p_2, p_3, \ldots].$$

1.8.3. The second main theorem: tournaments

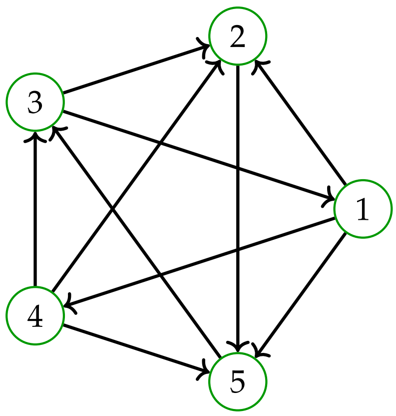
After we will have proved Theorem 1.31, we will use it to derive a simpler formula, which however is specific to tournaments. First, we recall the definition of a tournament:

Definition 1.37. A *tournament* means a digraph $D = (V, A)$ that satisfies the following two axioms:

- *Looplessness*: We have $(u, u) \notin A$ for any $u \in V$.

- *Tournament axiom*: For any two distinct vertices u and v of D, **exactly** one of the two pairs (u, v) and (v, u) is an arc of D.

Example 1.38. Neither the digraph D from Example 1.3, nor its complement $\overline{D}$, is a tournament. Here is a tournament:



.

We can now state our second main theorem (which we will prove in Section 3):

Theorem 1.39. Let $D = (V, A)$ be a tournament. For each $\sigma \in \mathfrak{S}_V$, let $\psi(\sigma)$ denote the number of nontrivial cycles of σ. Then,

$$U_D = \sum_{\substack{\sigma \in \mathfrak{S}_V(D); \\ \text{all cycles of } \sigma \text{ have odd length}}} 2^{\psi(\sigma)} p_{\operatorname{type} \sigma}.$$

Once this is proved, the following corollary will be easy to derive (see Section 4 for the details):

Corollary 1.40. Let $D = (V, A)$ be a tournament. Then,

$$U_D \in \mathbb{N}[p_1, 2p_3, 2p_5, 2p_7, \ldots] = \mathbb{N}[p_1, \ 2p_i \ \mid \ i > 1 \text{ is odd}].$$

(Here, $\mathbb{N}[p_1, 2p_3, 2p_5, 2p_7, \ldots]$ means the set of all values of the form $f(p_1, 2p_3, 2p_5, 2p_7, \ldots)$, where f is a polynomial in countably many indeterminates with coefficients in $\mathbb{N}$.)

1.8.4. The third main theorem: digraphs with no 2-cycles

A more general version of Theorem 1.39 is the following:

Theorem 1.41. Let $D = (V, A)$ be a digraph. Assume that there exist no two distinct vertices u and v of D such that both pairs (u, v) and (v, u) belong to A.

(a) Then, U_D is a p-positive symmetric function (i.e., a symmetric function that can be written as a polynomial in $p_1, p_2, p_3, \ldots$ with coefficients in $\mathbb{N}$). That is, we have $U_D \in \mathbb{N}[p_1, p_2, p_3, \ldots]$.

(b) A rotation-equivalence class γ of nonempty tuples of elements of V will be called *risky* if its length is even and it has the property that either γ or the reversal of γ is a D-cycle. Then,

$$U_D = \sum_{\substack{\sigma \in \mathfrak{S}_V\left(D, \overline{D}\right); \\ \text{no cycle of } \sigma \text{ is risky}}} p_{\operatorname{type}\sigma}.$$

We will prove this in Section 5. Note that Theorem 1.41 **(a)** generalizes [Chow96, Theorem 7].[5]

Remark 1.42. The converse of Theorem 1.41 **(a)** does not hold. Indeed, consider the digraph $D = (V, A)$ with $V = \{1, 2, 3, 4\}$ and

$$A = \{(1,2), (2,1), (2,3), (2,4), (3,4)\}.$$

Then, D does not satisfy the assumption of Theorem 1.41 (since the two distinct vertices 1 and 2 satisfy both $(1,2) \in A$ and $(2,1) \in A$), but the corresponding symmetric function U_D is p-positive (indeed, $U_D = p_1^4 + p_2 p_1^2 + p_3 p_1$). It would be interesting to know some more precise criteria for the p-positivity of U_D.

The next sections are devoted to the proofs of the above results. Afterwards, we will proceed with further properties of the Redei-Berge symmetric functions U_D (Section 8), applications to reproving Redei's and Berge's theorems (Section 6) and a (not very substantial) generalization (Section 9).

Remark on alternative versions

This paper also has a detailed version [GriSta23], which includes more details (and less handwaving) in some of the proofs (and some straightforward proofs that have

[5] To see how, one needs to observe that

1. any acyclic digraph D satisfies the assumption of Theorem 1.41;
2. the $\omega_x \Xi_D$ from [Chow96] equals our U_D in the case when D is acyclic.

The first of these two observations is obvious. The second follows from the equality (47) further below, combined with the fact that $\Xi_D = \Xi_D(x, 0)$ when D acyclic (since the y-variables do not actually appear in Ξ_D for lack of cycles), and the fact that $U_{\overline{D}} = \Xi_D(x, 0)$ (stated above in the equivalent form $U_D = \Xi_{\overline{D}}(x, 0)$).

been omitted from the present version).

2. Proof of Theorem 1.31

In the following, we will outline the proof of Theorem 1.31. We hope that the proof can still be simplified further.

2.1. Basic conventions

The following two conventions are popular in enumerative combinatorics, and we too will use them on occasion:

Convention 2.1. The symbol # shall mean "number". For instance, $(\# \text{ of subsets of } \{1,2,3\}) = 8$.

Convention 2.2. We shall use the *Iverson bracket notation*: For any logical statement $\mathcal{A}$, we let $[\mathcal{A}]$ denote the truth value of $\mathcal{A}$. This is the number $\begin{cases} 1, & \text{if } \mathcal{A} \text{ is true;} \\ 0, & \text{if } \mathcal{A} \text{ is false.} \end{cases}$

Our proof of Theorem 1.31 will rely on many lemmas. The first is a well-known cancellation lemma (see, e.g., [Grinbe21, Proposition 7.8.10]):

Lemma 2.3. Let B be a finite set. Then, $\sum\limits_{F \subseteq B} (-1)^{|F|} = [B = \varnothing]$.

2.2. Path covers and linear sets

We begin with some more notations:

Definition 2.4. Let V be a finite set.

(a) A *path* of V means a nonempty tuple of distinct elements of V.

(b) An element v is said to *belong* to a given tuple t if v is an entry of t.

(c) A *path cover* of V means a set of paths of V such that each $v \in V$ belongs to exactly one of these paths.

For example, $\{(1,4,3), (2,8), (5), (7,6)\}$ is a path cover of $[8]$. We stress once again the words "exactly one" in the definition of a path cover. Thus, the paths constituting a path cover are disjoint (i.e., have no entries in common). For instance, $\{(1,2), (2,3)\}$ is **not** a path cover of $[3]$.

In Definition 1.19 **(a)**, we have introduced the arc set of a path of V (and, more generally, of any nonempty tuple of elements of V). We now extend this to path covers in the obvious way:

Definition 2.5. Let V be a finite set.

(a) If C is a path cover of V, then the *arc set* of C is defined to be the subset

$$\bigcup_{v \in C} \operatorname{Arcs} v \qquad \text{of } V \times V.$$

This arc set will be denoted by $\operatorname{Arcs} C$.

(b) A subset F of $V \times V$ will be called *linear* if it is the arc set of some path cover of V.

For example, the path cover $\{(1,4,3), (2,8), (5), (7,6)\}$ of $[8]$ has arc set

$$\begin{aligned}
&\operatorname{Arcs}\{(1,4,3), (2,8), (5), (7,6)\} \\
&= \operatorname{Arcs}(1,4,3) \cup \operatorname{Arcs}(2,8) \cup \operatorname{Arcs}(5) \cup \operatorname{Arcs}(7,6) \\
&= \{(1,4), \ (4,3)\} \cup \{(2,8)\} \cup \varnothing \cup \{(7,6)\} \\
&= \{(1,4), \ (4,3), \ (2,8), \ (7,6)\}.
\end{aligned}$$

Thus, the latter set is linear (as a subset of $[8] \times [8]$).

Note that the notion of "path of V" depends on V alone, not on any digraph structure on V. Thus, if V is the vertex set of a digraph $D = (V, A)$, then a path of V is not the same as a D-path; in fact, the D-paths are precisely the paths v of V that satisfy $\operatorname{Arcs} v \subseteq A$.

We shall now see a few properties and characterizations of linear subsets of $V \times V$. Here is a first one, which will not be used in what follows but might help in visualizing the concept:

Proposition 2.6. Let V be a finite set. Let F be a subset of $V \times V$. Then, F is linear if and only if the digraph (V, F) has no cycles and no vertices with outdegree > 1 and no vertices with indegree > 1.

We omit the proof of this proposition, since we shall have no use for it.

The following is also easy to see:

Proposition 2.7. Let V be a finite set. Let F be a linear subset of $V \times V$. Then, any subset of F is linear as well.

Proof. It suffices to show that removing a single element e from a linear subset F of $V \times V$ yields a linear subset. But this follows from the fact that if we remove an arc f from a path, then the path breaks up into two smaller paths (the "part before f" and the "part after f"). $\square$

This quickly leads to the following alternative characterization of linear subsets:

Proposition 2.8. Let V be a finite set. Let F be a subset of $V \times V$. Then:

(a) If the subset F is not linear, then there exists no V-listing v satisfying $F \subseteq \operatorname{Arcs} v$.

(b) If $F = \operatorname{Arcs} C$ for some path cover C of V, then there are exactly $|C|!$ many V-listings v satisfying $F \subseteq \operatorname{Arcs} v$. (Note that $|C|$ is the number of paths in C.)

(c) The subset F is linear if and only if it is a subset of $\operatorname{Arcs} v$ for some V-listing v.

Proof. **(a)** It clearly suffices to prove the contrapositive: i.e., that if $F \subseteq \operatorname{Arcs} v$ for some V-listing v, then F is linear.

Let us prove this. Assume that $F \subseteq \operatorname{Arcs} v$ for some V-listing v. Consider this F. Then, $\operatorname{Arcs} v$ is linear (since $\operatorname{Arcs} v = \operatorname{Arcs} \{v\}$ for the path cover $\{v\}$), and thus Proposition 2.7 shows that F is also linear (since F is a subset of $\operatorname{Arcs} v$). This completes the proof of part **(a)**.

(b) Assume that $F = \operatorname{Arcs} C$ for some path cover C of V. Consider this C.

Then, each V-listing v satisfying $F \subseteq \operatorname{Arcs} v$ can be obtained by concatenating the paths in C in some order (and conversely, each such concatenation is a V-listing v satisfying $F \subseteq \operatorname{Arcs} v$). There are clearly $|C|!$ many such concatenations (since there are $|C|!$ many orders), and they all lead to different V-listings v (since the paths in C are disjoint and nonempty). Hence, there are exactly $|C|!$ many V-listings v satisfying $F \subseteq \operatorname{Arcs} v$. This proves Proposition 2.8 **(b)**.

(c) $\Longrightarrow$: This follows from part **(b)** (since $|C|! > 0$).

$\Longleftarrow$: This is just the contrapositive of part **(a)**. $\square$

Next, let us address a technical issue. We defined the notion of a "linear subset of $V \times V$" using path covers of V. When we say that a certain set is "linear", we are thus tacitly assuming that it is clear what the relevant set V is. This may cause an ambiguity: Sometimes, two different sets V_1 and V_2 can reasonably qualify as V, and we may have a subset F of $V_1 \times V_1$ that is also a subset of $V_2 \times V_2$. In that case, when we say that F is "linear", do we mean that F is linear as a subset of $V_1 \times V_1$ or as a subset of $V_2 \times V_2$? Fortunately, this does not matter (at least when V_1 is a subset of V_2), as the following proposition shows:

Proposition 2.9. Let V be a finite set. Let W be a subset of V. Let F be a subset of $W \times W$. Then, F is linear as a subset of $W \times W$ if and only if F is linear as a subset of $V \times V$.

Proof. $\Longrightarrow$: Assume that F is linear as a subset of $W \times W$. Thus, F is the arc set of some path cover C of W. If we add a trivial path (v) for each $v \in V \setminus W$ to this path cover C, then it becomes a path cover of V, but its arc set does not change (and thus remains F). Hence, F is the arc set of the resulting path cover of V. In other words, F is linear as a subset of $V \times V$.

$\Longleftarrow$: Assume that F is linear as a subset of $V \times V$. Thus, F is the arc set of some path cover C of V. Consider this C. For each $v \in V \setminus W$, there must be a path in C that contains v, and this path must be the trivial path (v) (since otherwise, this path would have at least one arc containing v, and this arc would then belong to $\operatorname{Arcs} C = F$; but this would contradict the fact that $F \subseteq W \times W$). Hence, the path cover C contains the trivial path (v) for each $v \in V \setminus W$. Removing all these trivial paths will turn C into a path cover of W, while leaving its arc set unchanged (so it remains F). Hence, F is the arc set of the resulting path cover of W. In other words, F is linear as a subset of $W \times W$. $\square$

We will also use the following fact:

Proposition 2.10. Let V be a finite set. Let $V_1, V_2, \ldots, V_k$ be several disjoint subsets of V such that $V = V_1 \cup V_2 \cup \cdots \cup V_k$. For each $i \in [k]$, let F_i be a subset of $V_i \times V_i$. Let $F = F_1 \cup F_2 \cup \cdots \cup F_k$. Then, the set F is linear (as a subset of $V \times V$) if and only if all the subsets F_i for $i \in [k]$ are linear.

Proof. This is straightforward. $\square$

2.3. The arrow set of a permutation

We will now see another way to obtain subsets of $V \times V$:

Definition 2.11. Let V be a finite set. Let σ be a permutation of V. Then, $\mathbf{A}_\sigma$ shall denote the subset

$$\{(v, \sigma(v)) \mid v \in V\} = \bigcup_{c \in \operatorname{Cycs} \sigma} \operatorname{CArcs} c$$

of $V \times V$.

Example 2.12. Let $V = \{1,2,3,4,5,6\}$, and let σ be the permutation of V that sends $1,2,3,4,5,6$ to $2,3,1,5,4,6$ (respectively). Then,

$$\operatorname{Cycs} \sigma = \{(1,2,3),\ (4,5),\ (6)\}$$

and

$$\begin{aligned} \mathbf{A}_\sigma &= \{(1,2),\ (2,3),\ (3,1),\ (4,5),\ (5,4),\ (6,6)\} \\ &= \underbrace{\operatorname{CArcs}(1,2,3)}_{=\{(1,2),\ (2,3),\ (3,1)\}} \cup \underbrace{\operatorname{CArcs}(4,5)}_{=\{(4,5),\ (5,4)\}} \cup \underbrace{\operatorname{CArcs}(6)}_{=\{(6,6)\}}. \end{aligned}$$

The following is a counterpart to Proposition 2.8 **(b)**:

Proposition 2.13. Let V be a finite set. Let F be a subset of $V \times V$. If $F = \operatorname{Arcs} C$ for some path cover C of V, then there are exactly $|C|!$ many permutations $\sigma \in \mathfrak{S}_V$ satisfying $F \subseteq \mathbf{A}_\sigma$. (Note that $|C|$ is the number of paths in C.)

Proof. Assume that $F = \operatorname{Arcs} C$ for some path cover C of V. Consider this C. We shall refer to the paths in C as "C-paths".

Let $k = |C|$. Let $s_1, s_2, \ldots, s_k$ be the starting points (i.e., first entries) of the C-paths, and let $t_1, t_2, \ldots, t_k$ be their respective ending points (i.e., last entries). We note that a permutation $\sigma \in \mathfrak{S}_V$ satisfies $F \subseteq \mathbf{A}_\sigma$ if and only if it has the property that $\sigma(v) = w$ whenever v and w are two consecutive entries of a C-path. Thus, the condition $F \subseteq \mathbf{A}_\sigma$ uniquely determines the value $\sigma(v)$ for each $v \in V \setminus \{t_1, t_2, \ldots, t_k\}$ (namely, $\sigma(v)$ has to be the next entry after v on the C-path that contains v), and uniquely determines the value $\sigma^{-1}(w)$ for each $w \in V \setminus \{s_1, s_2, \ldots, s_k\}$ (namely, $\sigma^{-1}(w)$ has to be the entry just before v on the C-path that contains v).

Hence, in order to construct a permutation $\sigma \in \mathfrak{S}_V$ satisfying $F \subseteq \mathbf{A}_\sigma$, we only need to specify the k values $\sigma(t_1), \sigma(t_2), \ldots, \sigma(t_k)$ (since all other values $\sigma(v)$ are already decided by the requirement $F \subseteq \mathbf{A}_\sigma$), and we must choose these k values from the set $\{s_1, s_2, \ldots, s_k\}$ (since all other elements of V have already been assigned preimages under σ by the requirement $F \subseteq \mathbf{A}_\sigma$). Thus, we must choose a bijection from the k-element set $\{t_1, t_2, \ldots, t_k\}$ to the k-element set $\{s_1, s_2, \ldots, s_k\}$. This can be done in $k!$ many ways, i.e., in $|C|!$ many ways (since $k = |C|$). Thus, there are exactly $|C|!$ many permutations $\sigma \in \mathfrak{S}_V$ satisfying $F \subseteq \mathbf{A}_\sigma$. $\square$

2.4. Counting hamps by inclusion-exclusion

Our next lemma will be about counting Hamiltonian paths – which we abbreviate as "hamps". Here is how they are defined:

Definition 2.14. Let D be a digraph. A *hamp* of D means a D-path that contains each vertex of D. (The word "hamp" is short for "Hamiltonian path".)

For a digraph $D = (V, A)$, there is an obvious connection between the linear subsets of A and the hamps of D: If v is a hamp of D, then $\operatorname{Arcs} v$ is a maximum-size linear subset of A (and this maximum size is $|V| - 1$ if V is nonempty). More interestingly, there is a far less obvious connection between the linear subsets of A and the hamps of the complement $\overline{D}$:

Lemma 2.15. Let $D = (V, A)$ be a digraph with $V \neq \varnothing$. Then,

$$\sum_{F \subseteq A \text{ is linear}} (-1)^{|F|} \cdot (\# \text{ of } \sigma \in \mathfrak{S}_V \text{ satisfying } F \subseteq \mathbf{A}_\sigma) = \left(\# \text{ of hamps of } \overline{D}\right).$$

(We are using Convention 2.1 here.)

Proof. We will use the Iverson bracket notation (as in Convention 2.2). We have

$$\begin{aligned}
\sum_{v \text{ is a } V\text{-listing}} \underbrace{\sum_{\substack{F\subseteq A;\\ F\subseteq \operatorname{Arcs} v}} (-1)^{|F|}}_{\substack{= \sum\limits_{F\subseteq A\cap \operatorname{Arcs} v} (-1)^{|F|} \\ =[A\cap \operatorname{Arcs} v=\varnothing] \\ \text{(by Lemma 2.3)}}} &= \sum_{v \text{ is a } V\text{-listing}} [A \cap \operatorname{Arcs} v = \varnothing] \\
&= (\text{\# of } V\text{-listings } v \text{ satisfying } A \cap \operatorname{Arcs} v = \varnothing) \\
&= \left(\text{\# of hamps of } \overline{D}\right)
\end{aligned}$$

(since the hamps of $\overline{D}$ are precisely the V-listings v such that $A \cap \operatorname{Arcs} v = \varnothing$). Thus,

$$\begin{aligned}
\left(\text{\# of hamps of } \overline{D}\right) &= \sum_{v \text{ is a } V\text{-listing}} \sum_{\substack{F\subseteq A;\\ F\subseteq \operatorname{Arcs} v}} (-1)^{|F|} \\
&= \sum_{F\subseteq A} \sum_{\substack{v \text{ is a } V\text{-listing};\\ F\subseteq \operatorname{Arcs} v}} (-1)^{|F|} \\
&= \sum_{F\subseteq A \text{ is linear}} \sum_{\substack{v \text{ is a } V\text{-listing};\\ F\subseteq \operatorname{Arcs} v}} (-1)^{|F|} \qquad (4)
\end{aligned}$$

(here, we have restricted the outer sum to only the linear subsets F of A, because if a subset F of A is not linear, then the inner sum $\sum\limits_{\substack{v \text{ is a } V\text{-listing};\\ F\subseteq \operatorname{Arcs} v}} (-1)^{|F|}$ is empty[6]).

Now, let F be a linear subset of A. Thus, $F = \operatorname{Arcs} C$ for some path cover C of V. Consider this C. Then, Proposition 2.13 yields that

$$(\text{\# of } \sigma \in \mathfrak{S}_V \text{ satisfying } F \subseteq \mathbf{A}_\sigma) = |C|!. \qquad (5)$$

On the other hand, Proposition 2.8 **(b)** yields that

$$(\text{\# of } V\text{-listings } v \text{ satisfying } F \subseteq \operatorname{Arcs} v) = |C|!.$$

Hence,

$$\begin{aligned}
\sum_{\substack{v \text{ is a } V\text{-listing};\\ F\subseteq \operatorname{Arcs} v}} (-1)^{|F|} &= \underbrace{(\text{\# of } V\text{-listings } v \text{ satisfying } F \subseteq \operatorname{Arcs} v)}_{\substack{=|C|! \\ =(\text{\# of } \sigma\in\mathfrak{S}_V \text{ satisfying } F\subseteq \mathbf{A}_\sigma) \\ \text{(by (5))}}} \cdot (-1)^{|F|} \\
&= (\text{\# of } \sigma \in \mathfrak{S}_V \text{ satisfying } F \subseteq \mathbf{A}_\sigma) \cdot (-1)^{|F|} \\
&= (-1)^{|F|} \cdot (\text{\# of } \sigma \in \mathfrak{S}_V \text{ satisfying } F \subseteq \mathbf{A}_\sigma). \qquad (6)
\end{aligned}$$

[6]by Proposition 2.8 **(a)**

Forget that we fixed F. We thus have proved (6) for each linear subset F of A. Now, (4) becomes

$$\begin{aligned}\left(\#\text{ of hamps of }\overline{D}\right) &= \sum_{F\subseteq A\text{ is linear}} \underbrace{\sum_{\substack{v\text{ is a }V\text{-listing;}\\ F\subseteq \operatorname{Arcs} v}} (-1)^{|F|}}_{\substack{=(-1)^{|F|}\cdot(\#\text{ of }\sigma\in\mathfrak{S}_V\text{ satisfying }F\subseteq\mathbf{A}_\sigma)\\ \text{(by (6))}}} \\ &= \sum_{F\subseteq A\text{ is linear}} (-1)^{|F|}\cdot\left(\#\text{ of }\sigma\in\mathfrak{S}_V\text{ satisfying }F\subseteq\mathbf{A}_\sigma\right).\end{aligned}$$

This proves Lemma 2.15. ☐

2.5. Level decomposition and maps f satisfying $f\circ\sigma = f$

This entire subsection is devoted to building up some language that will only ever be used in the proof of Lemma 2.31. All proofs are omitted, as they are straightforward exercises in understanding the underlying definitions. (They can be found in the detailed version [GriSta23], too.)

We shall study what happens when a function $f : V \to \mathbb{P}$ is introduced into a digraph $D = (V, A)$. The nonempty fibers of f (i.e., the sets $f^{-1}(j)$ for all $j \in f(V)$) partition the vertex set V, and this leads to a decomposition of D into subdigraphs. Let us introduce some notation for this, starting with the case of an arbitrary set V (we will later specialize to digraphs):

Definition 2.16. Let V be any set. Let $f : V \to \mathbb{P}$ be any map.

- **(a)** For each $v \in V$, we will refer to the number $f(v)$ as the *level* of v (with respect to f).
- **(b)** For each $j \in \mathbb{P}$, the subset $f^{-1}(j)$ of V shall be called the *j-th level set* of f.

Example 2.17. Let $V = \{1, 2, 3\}$. Let $f : V \to \mathbb{P}$ be given by $f(1) = 1$, $f(2) = 4$ and $f(3) = 1$. Then, the level sets of f are

$$f^{-1}(1) = \{1,3\}, \qquad f^{-1}(4) = \{2\}, \qquad \text{and}$$
$$f^{-1}(j) = \varnothing \text{ for all } j \in \mathbb{P}\setminus\{1,4\}.$$

Remark 2.18. Let V be any set. Let $f : V \to \mathbb{P}$ be any map. Let $j \in \mathbb{P}$. Then, the j-th level set $f^{-1}(j)$ is empty if and only if $j \notin f(V)$. Hence, the nonempty level sets of f correspond to the elements of $f(V)$.

Definition 2.19. Let $D = (V, A)$ be a digraph. Let $f : V \to \mathbb{P}$ be any map.

(a) For each $j \in \mathbb{P}$, we define a subset A_j of A by

$$\begin{aligned} A_j &:= \left\{ (u,v) \in A \ \mid \ u,v \in f^{-1}(j) \right\} && (7) \\ &= \{ (u,v) \in A \ \mid \ f(u) = f(v) = j \} && (8) \\ &= A \cap \left(f^{-1}(j) \times f^{-1}(j) \right). && (9) \end{aligned}$$

This set A_j is also a subset of $f^{-1}(j) \times f^{-1}(j)$, of course.

(b) We let A_f denote the subset

$$\{ (u,v) \in A \ \mid \ f(u) = f(v) \}$$

of A.

(c) For each $j \in \mathbb{P}$, we let D_j denote the digraph $\left(f^{-1}(j),\ A_j \right)$. This digraph D_j is the restriction of the digraph D to the subset $f^{-1}(j)$ (that is, the digraph obtained from D by removing all vertices that don't belong to $f^{-1}(j)$ and all arcs that contain any of these vertices).

This digraph D_j will be called the *j-th level subdigraph* of D with respect to f. (We should properly call it $D_{j,f}$ instead of D_j, but we will usually keep f fixed when we study it.)

Example 2.20. Let D be as in Example 1.3. Let $f : V \to \mathbb{P}$ be given by $f(1) = 1$, $f(2) = 4$ and $f(3) = 1$. Then,

$$A_1 = \{(3,3)\}, \qquad A_4 = \{(2,2)\},$$
$$A_j = \varnothing \text{ for all } j \in \mathbb{P} \setminus \{1,4\},$$

and

$$A_f = \{(3,3),\ (2,2)\}.$$

The level subdigraphs of D are the two digraphs

$$D_1 = (\{1,3\},\ \{(3,3)\}) \qquad \text{and} \qquad D_4 = (\{2\},\ \{(2,2)\})$$

(as well as the infinitely many empty digraphs D_j for all $j \in \mathbb{P} \setminus \{1,4\}$). Note that the arc $(3,3)$ of D is contained in D_1, and the arc $(2,2)$ is contained in D_4, but the arc $(1,2)$ is contained in none of the level subdigraphs (since its two endpoints 1 and 2 have different levels).

Remark 2.21. Let $D = (V, A)$ be a digraph. Let $f : V \to \mathbb{P}$ be any map. Let $j \in \mathbb{P}$. Then, the j-th level subdigraph D_j and its arc set A_j are empty if $j \notin f(V)$. (However, A_j can be empty even if j does belong to $f(V)$.)

In the following, the symbols "$\sqcup$" and "$\bigsqcup$" stand for unions of disjoint sets. Thus, for example, "$A_1 \sqcup A_2 \sqcup A_3 \sqcup \cdots$" will mean the union of some (pairwise) disjoint sets $A_1, A_2, A_3, \ldots$.

Proposition 2.22. Let V and J be two finite sets. Let V_j be a subset of V for each $j \in J$. Assume that the sets V_j for different $j \in J$ are disjoint. Let C_j be a path cover of V_j for each $j \in J$. Then:

(a) The sets C_j for different $j \in J$ are disjoint.

(b) Their union $\bigsqcup\limits_{j \in J} C_j$ is a path cover of $\bigsqcup\limits_{j \in J} V_j$, and its arc set is $\operatorname{Arcs}\left(\bigsqcup\limits_{j \in J} C_j\right) = \bigsqcup\limits_{j \in J} \operatorname{Arcs}\left(C_j\right)$.

Corollary 2.23. Let V and J be two finite sets. Let V_j be a subset of V for each $j \in J$. Assume that the sets V_j for different $j \in J$ are disjoint. For each $j \in J$, let F_j be a linear subset of $V_j \times V_j$. Then, the union $\bigcup\limits_{j \in J} F_j$ is a linear subset of $V \times V$.

Proposition 2.24. Let $D = (V, A)$ be a digraph. Let $f : V \to \mathbb{P}$ be any map. Then, the sets $A_1, A_2, A_3, \ldots$ are disjoint, and their union is

$$A_1 \sqcup A_2 \sqcup A_3 \sqcup \cdots = \bigsqcup_{j \in f(V)} A_j = A_f.$$

Let us now connect the level decomposition to linear sets:

Proposition 2.25. Let $D = (V, A)$ be a digraph. Let $f : V \to \mathbb{P}$ be any map. Let F be any set. Then:

(a) The set F is a linear subset of A_f if and only if F can be written as $F = \bigsqcup\limits_{j \in f(V)} F_j$, where each F_j is a linear subset of A_j.

(b) In this case, the subsets F_j are uniquely determined by F (namely, $F_j = F \cap A_j$ for each $j \in f(V)$).

Next, we return to studying permutations.

When a set V is a union of two disjoint subsets A and B, and we are given a permutation σ_A of A and a permutation σ_B of B, then we can combine these two permutations to obtain a permutation $\sigma_A \oplus \sigma_B$ of V: Namely, this latter permutation sends each $a \in A$ to $\sigma_A(a)$, and sends each $b \in B$ to $\sigma_B(b)$. That is, this permutation $\sigma_A \oplus \sigma_B$ is "acting as σ_A" on the subset A and "acting as σ_B" on the subset B.

The same construction can be performed when V is a union of more than two disjoint subsets (and we are given a permutation of each of these subsets). We will encounter this situation when a map $f : V \to \mathbb{P}$ subdivides the set V into its level sets $f^{-1}(1)$, $f^{-1}(2)$, $f^{-1}(3)$, ..., and we are given a permutation $\sigma_j \in \mathfrak{S}_{f^{-1}(j)}$ of each level set $f^{-1}(j)$ (to be more precise, we only need σ_j to be given when $j \in f(V)$, since the level set $f^{-1}(j)$ is empty otherwise). The permutation of V obtained by combining these permutations σ_j will then be denoted by $\bigoplus\limits_{j \in f(V)} \sigma_j$. Here is its explicit definition:

Definition 2.26. Let V be any set. Let $f : V \to \mathbb{P}$ be any map.

For each $j \in f(V)$, let $\sigma_j \in \mathfrak{S}_{f^{-1}(j)}$ be a permutation of the level set $f^{-1}(j)$.

Then, $\bigoplus\limits_{j \in f(V)} \sigma_j$ shall denote the permutation of V that sends each $v \in V$ to $\sigma_{f(v)}(v)$. This is the permutation that acts as σ_j on each level set $f^{-1}(j)$.

Proposition 2.27. Let V be any set. Let $f : V \to \mathbb{P}$ be any map. Let $\sigma \in \mathfrak{S}_V$ be any permutation. Then:

- **(a)** We have $f \circ \sigma = f$ if and only if σ can be written in the form $\sigma = \bigoplus\limits_{j \in f(V)} \sigma_j$, where $\sigma_j \in \mathfrak{S}_{f^{-1}(j)}$ for each $j \in f(V)$.
- **(b)** In this case, the permutations σ_j for all $j \in f(V)$ are uniquely determined by σ (namely, σ_j is the restriction of σ to the subset $f^{-1}(j)$ for each $j \in f(V)$).

Now, we recall the set $\mathbf{A}_\sigma$ defined in Definition 2.11 for any finite set V and any permutation σ of V.

Proposition 2.28. Let V be a finite set. Let $f : V \to \mathbb{P}$ be any map. Let $\sigma \in \mathfrak{S}_V$ be a permutation satisfying $f \circ \sigma = f$. Write σ in the form $\sigma = \bigoplus\limits_{j \in f(V)} \sigma_j$, where $\sigma_j \in \mathfrak{S}_{f^{-1}(j)}$ for each $j \in f(V)$. (This can be done, because of Proposition 2.27 **(a)**.) Then,

$$\mathbf{A}_\sigma = \bigsqcup_{j \in f(V)} \mathbf{A}_{\sigma_j}.$$

Next, we connect the above construction with the level subdigraphs of a digraph:

Proposition 2.29. Let $D = (V, A)$ be a digraph. Let $f : V \to \mathbb{P}$ be any map. Let $\sigma \in \mathfrak{S}_V$ be a permutation satisfying $f \circ \sigma = f$. Then,

$$\mathbf{A}_\sigma \cap A \subseteq A_f.$$

Our last result in this section is the following trivial yet complex-looking lemma, which will be used in the proof after it:

Lemma 2.30. Let $D = (V, A)$ be a digraph. Let $f : V \to \mathbb{P}$ be any map. Let $\sigma_j \in \mathfrak{S}_{f^{-1}(j)}$ be a permutation for each $j \in f(V)$. Let F_j be a subset of A_j for each $j \in f(V)$. Then, we have the following logical equivalence:

$$\left(\bigsqcup_{j \in f(V)} F_j \subseteq \bigsqcup_{j \in f(V)} \mathbf{A}_{\sigma_j} \right) \iff \left(F_j \subseteq \mathbf{A}_{\sigma_j} \text{ for each } j \in f(V) \right).$$

2.6. An alternating sum involving permutations σ with $f \circ \sigma = f$

Now, we come to a crucial lemma, which generalizes Lemma 2.15 to the case of a digraph $D = (V, A)$ "shattered" by a map $f : V \to \mathbb{P}$:

Lemma 2.31. Let $D = (V, A)$ be a digraph. Let $f : V \to \mathbb{P}$ be any map. For each $j \in \mathbb{P}$, we define a digraph D_j as in Definition 2.19 **(c)**. Then,

$$\sum_{\substack{\sigma \in \mathfrak{S}_V; \\ f \circ \sigma = f}} \sum_{\substack{F \subseteq \mathbf{A}_\sigma \cap A \\ \text{is linear}}} (-1)^{|F|} = \prod_{j \in f(V)} \left(\# \text{ of hamps of } \overline{D_j} \right).$$

Proof. We shall use the notations from Definition 2.16, Definition 2.19 and Definition 2.26. We recall that every $j \in \mathbb{P}$ satisfies $D_j = \left(f^{-1}(j),\ A_j \right)$ (by the definition of D_j).

Let $\sigma \in \mathfrak{S}_V$ be a permutation satisfying $f \circ \sigma = f$. Then, Proposition 2.29 yields $\mathbf{A}_\sigma \cap A \subseteq A_f$. Hence, $\mathbf{A}_\sigma \cap A = \mathbf{A}_\sigma \cap A_f$ (because $\underbrace{\mathbf{A}_\sigma}_{= \mathbf{A}_\sigma \cap \mathbf{A}_\sigma} \cap A = \mathbf{A}_\sigma \cap \underbrace{\mathbf{A}_\sigma \cap A}_{\subseteq A_f} \subseteq$

$\mathbf{A}_\sigma \cap A_f$ and conversely $\mathbf{A}_\sigma \cap \underbrace{A_f}_{\subseteq A} \subseteq \mathbf{A}_\sigma \cap A$). Thus,

$$\sum_{\substack{F \subseteq \mathbf{A}_\sigma \cap A \\ \text{is linear}}} (-1)^{|F|} = \sum_{\substack{F \subseteq \mathbf{A}_\sigma \cap A_f \\ \text{is linear}}} (-1)^{|F|} = \sum_{\substack{F \subseteq A_f \text{ is linear;} \\ F \subseteq \mathbf{A}_\sigma}} (-1)^{|F|} \tag{10}$$

(since a subset of $\mathbf{A}_\sigma \cap A_f$ is the same thing as a subset F of A_f that satisfies $F \subseteq \mathbf{A}_\sigma$).

Forget that we fixed σ. We thus have proved (10) for every $\sigma \in \mathfrak{S}_V$ satisfying $f \circ \sigma = f$.

Summing up the equality (10) over all permutations $\sigma \in \mathfrak{S}_V$ satisfying $f \circ \sigma = f$, we obtain

$$\begin{aligned}
&\sum_{\substack{\sigma \in \mathfrak{S}_V; \\ f \circ \sigma = f}} \sum_{\substack{F \subseteq \mathbf{A}_\sigma \cap A \\ \text{is linear}}} (-1)^{|F|} \\
&= \sum_{\substack{\sigma \in \mathfrak{S}_V; \\ f \circ \sigma = f}} \sum_{\substack{F \subseteq A_f \text{ is linear;} \\ F \subseteq \mathbf{A}_\sigma}} (-1)^{|F|} \\
&= \sum_{F \subseteq A_f \text{ is linear}} \underbrace{\sum_{\substack{\sigma \in \mathfrak{S}_V; \\ F \subseteq \mathbf{A}_\sigma; \\ f \circ \sigma = f}} (-1)^{|F|}}_{= (-1)^{|F|} \cdot (\text{\# of } \sigma \in \mathfrak{S}_V \text{ satisfying } F \subseteq \mathbf{A}_\sigma \text{ and } f \circ \sigma = f)} \\
&= \sum_{F \subseteq A_f \text{ is linear}} (-1)^{|F|} \cdot (\text{\# of } \sigma \in \mathfrak{S}_V \text{ satisfying } F \subseteq \mathbf{A}_\sigma \text{ and } f \circ \sigma = f) .
\end{aligned} \tag{11}$$

Now, a linear subset F of A_f is the same as a set F of the form $\bigsqcup_{j \in f(V)} F_j$, where each F_j is a linear subset of A_j (by Proposition 2.25 **(a)**); furthermore, if F is such a subset, then the latter subsets F_j satisfying $F = \bigsqcup_{j \in f(V)} F_j$ are uniquely determined by F (by Proposition 2.25 **(b)**). Hence, we can substitute $\bigsqcup_{j \in f(V)} F_j$ for F in the sum on the right hand side of (11). We thus obtain

$$\sum_{F\subseteq A_f \text{ is linear}} (-1)^{|F|} \cdot (\text{\# of } \sigma \in \mathfrak{S}_V \text{ satisfying } F \subseteq \mathbf{A}_\sigma \text{ and } f\circ\sigma = f)$$

$$= \sum_{\substack{\left(F_j\right)_{j\in f(V)} \text{ is a family} \\ \text{of linear subsets } F_j\subseteq A_j}} (-1)^{\left|\bigsqcup_{j\in f(V)} F_j\right|}$$

$$\cdot \left(\text{\# of } \sigma \in \mathfrak{S}_V \text{ satisfying } \bigsqcup_{j\in f(V)} F_j \subseteq \mathbf{A}_\sigma \text{ and } f\circ\sigma = f\right). \tag{12}$$

Furthermore, a permutation $\sigma \in \mathfrak{S}_V$ satisfies $f\circ\sigma = f$ if and only if it can be written in the form $\sigma = \bigoplus_{j\in f(V)} \sigma_j$, where $\sigma_j \in \mathfrak{S}_{f^{-1}(j)}$ for each $j \in f(V)$ (by Proposition 2.27 **(a)**). Moreover, if σ is written in this way, then the permutations σ_j are uniquely determined by σ (by Proposition 2.27 **(b)**), and we have $\mathbf{A}_\sigma = \bigsqcup_{j\in f(V)} \mathbf{A}_{\sigma_j}$ (by Proposition 2.28). Hence, for each family $\left(F_j\right)_{j\in f(V)}$ of linear subsets $F_j \subseteq A_j$, we have

$$\left(\text{\# of } \sigma \in \mathfrak{S}_V \text{ satisfying } \bigsqcup_{j\in f(V)} F_j \subseteq \mathbf{A}_\sigma \text{ and } f\circ\sigma = f\right)$$

$$= \left(\text{\# of families } \left(\sigma_j\right)_{j\in f(V)} \in \prod_{j\in f(V)} \mathfrak{S}_{f^{-1}(j)} \text{ satisfying } \bigsqcup_{j\in f(V)} F_j \subseteq \bigsqcup_{j\in f(V)} \mathbf{A}_{\sigma_j}\right)$$

$$= \left(\text{\# of families } \left(\sigma_j\right)_{j\in f(V)} \in \prod_{j\in f(V)} \mathfrak{S}_{f^{-1}(j)} \text{ satisfying } F_j \subseteq \mathbf{A}_{\sigma_j} \text{ for each } j\in f(V)\right)$$

$$\left(\begin{array}{c}\text{since the condition ``} \bigsqcup_{j\in f(V)} F_j \subseteq \bigsqcup_{j\in f(V)} \mathbf{A}_{\sigma_j}\text{''} \\ \text{is equivalent to ``} F_j \subseteq \mathbf{A}_{\sigma_j} \text{ for each } j\in f(V)\text{''} \\ \text{(by Lemma 2.30)}\end{array}\right)$$

$$= \prod_{j\in f(V)} \left(\text{\# of } \sigma_j \in \mathfrak{S}_{f^{-1}(j)} \text{ satisfying } F_j \subseteq \mathbf{A}_{\sigma_j}\right)$$

$$= \prod_{j\in f(V)} \left(\text{\# of } \sigma \in \mathfrak{S}_{f^{-1}(j)} \text{ satisfying } F_j \subseteq \mathbf{A}_{\sigma}\right) \tag{13}$$

$$\left(\text{here, we have renamed the index } \sigma_j \text{ as } \sigma\right).$$

Thus, (12) becomes

$$\sum_{F\subseteq A_f \text{ is linear}} (-1)^{|F|} \cdot (\text{\# of } \sigma \in \mathfrak{S}_V \text{ satisfying } F \subseteq \mathbf{A}_\sigma \text{ and } f\circ\sigma = f)$$

$$= \sum_{\substack{(F_j)_{j\in f(V)} \text{ is a family} \\ \text{of linear subsets } F_j\subseteq A_j}} \underbrace{(-1)^{\left|\bigsqcup_{j\in f(V)} F_j\right|}}_{\substack{=(-1)^{\sum_{j\in f(V)}|F_j|} \\ =\prod_{j\in f(V)} (-1)^{|F_j|}}}$$

$$\cdot \underbrace{\left(\text{\# of } \sigma \in \mathfrak{S}_V \text{ satisfying } \bigsqcup_{j\in f(V)} F_j \subseteq \mathbf{A}_\sigma \text{ and } f\circ\sigma = f\right)}_{\substack{=\prod_{j\in f(V)} \left(\text{\# of } \sigma\in\mathfrak{S}_{f^{-1}(j)} \text{ satisfying } F_j\subseteq\mathbf{A}_\sigma\right) \\ \text{(by (13))}}}$$

$$= \sum_{\substack{(F_j)_{j\in f(V)} \text{ is a family} \\ \text{of linear subsets } F_j\subseteq A_j}} \underbrace{\left(\prod_{j\in f(V)} (-1)^{|F_j|}\right) \cdot \prod_{j\in f(V)} \left(\text{\# of } \sigma \in \mathfrak{S}_{f^{-1}(j)} \text{ satisfying } F_j \subseteq \mathbf{A}_\sigma\right)}_{=\prod_{j\in f(V)} \left((-1)^{|F_j|}\cdot\left(\text{\# of } \sigma\in\mathfrak{S}_{f^{-1}(j)} \text{ satisfying } F_j\subseteq\mathbf{A}_\sigma\right)\right)}$$

$$= \sum_{\substack{(F_j)_{j\in f(V)} \text{ is a family} \\ \text{of linear subsets } F_j\subseteq A_j}} \prod_{j\in f(V)} \left((-1)^{|F_j|} \cdot \left(\text{\# of } \sigma \in \mathfrak{S}_{f^{-1}(j)} \text{ satisfying } F_j \subseteq \mathbf{A}_\sigma\right)\right)$$

$$= \prod_{j\in f(V)} \sum_{F_j\subseteq A_j \text{ is linear}} (-1)^{|F_j|} \cdot \left(\text{\# of } \sigma \in \mathfrak{S}_{f^{-1}(j)} \text{ satisfying } F_j \subseteq \mathbf{A}_\sigma\right)$$

$$\text{(by the product rule)}$$

$$= \prod_{j\in f(V)} \underbrace{\sum_{F\subseteq A_j \text{ is linear}} (-1)^{|F|} \cdot \left(\text{\# of } \sigma \in \mathfrak{S}_{f^{-1}(j)} \text{ satisfying } F \subseteq \mathbf{A}_\sigma\right)}_{\substack{=\left(\text{\# of hamps of } \overline{D_j}\right) \\ \text{(by Lemma 2.15, applied to } D_j=\left(f^{-1}(j),\ A_j\right) \text{ instead of } D=(V,A))}}$$

$$\left(\text{here, we have renamed the summation index } F_j \text{ as } F\right)$$

$$= \prod_{j\in f(V)} \left(\text{\# of hamps of } \overline{D_j}\right).$$

In view of this, we can rewrite (11) as

$$\sum_{\substack{\sigma\in\mathfrak{S}_V; \\ f\circ\sigma=f}} \sum_{\substack{F\subseteq\mathbf{A}_\sigma\cap A \\ \text{is linear}}} (-1)^{|F|} = \prod_{j\in f(V)} \left(\text{\# of hamps of } \overline{D_j}\right).$$

This proves Lemma 2.31. □

2.7. (f, D)-friendly V-listings

The following restatement of Lemma 2.31 will be useful for us:

Lemma 2.32. Let $D = (V, A)$ be a digraph. Let $f : V \to \mathbb{P}$ be any map. A V-listing $v = (v_1, v_2, \ldots, v_n)$ will be called (f, D)-*friendly* if it has the properties that $f(v_1) \leq f(v_2) \leq \cdots \leq f(v_n)$ and that

$$f(v_p) < f(v_{p+1}) \text{ for each } p \in [n-1] \text{ satisfying } (v_p, v_{p+1}) \in A. \tag{14}$$

Then,

$$\sum_{\substack{\sigma \in \mathfrak{S}_V; \\ f \circ \sigma = f}} \sum_{\substack{F \subseteq \mathbf{A}_\sigma \cap A \\ \text{is linear}}} (-1)^{|F|} = (\# \text{ of } (f, D)\text{-friendly } V\text{-listings}).$$

Proof. For each $j \in f(V)$, we define a digraph D_j as in Definition 2.19 **(c)**. This digraph D_j is the restriction of the digraph D to the subset $f^{-1}(j)$. In particular, its vertex set is $f^{-1}(j)$. In other words, its vertices are precisely those vertices of D that have level j (with respect to f).

Clearly, a V-listing $v = (v_1, v_2, \ldots, v_n)$ satisfies $f(v_1) \leq f(v_2) \leq \cdots \leq f(v_n)$ if and only if it lists the vertices of D in the order of increasing level, i.e., if it first lists the vertices of the smallest level, then the vertices of the second-smallest level, and so on.

In other words, a V-listing $v = (v_1, v_2, \ldots, v_n)$ satisfies $f(v_1) \leq f(v_2) \leq \cdots \leq f(v_n)$ if and only if it can be constructed by choosing an $f^{-1}(j)$-listing $v^{(j)}$ for each $j \in f(V)$ and concatenating all these $f^{-1}(j)$-listings $v^{(j)}$ in the order of increasing j. Moreover, if it can be constructed in this way, then its construction is unique (i.e., each $v^{(j)}$ is determined uniquely by v). Finally, for a V-listing v that is written as a concatenation of such $f^{-1}(j)$-listings $v^{(j)}$, the condition (14) is equivalent to the condition that each $v^{(j)}$ is a hamp of $\overline{D_j}$ (indeed, this is easiest to see by rewriting (14) in the contrapositive form "if $p \in [n-1]$ satisfies $f(v_p) = f(v_{p+1})$, then $(v_p, v_{p+1}) \notin A$"). Thus, a V-listing $v = (v_1, v_2, \ldots, v_n)$ satisfies both $f(v_1) \leq f(v_2) \leq \cdots \leq f(v_n)$ and (14) at the same time if and only if it can be constructed by choosing a hamp of $\overline{D_j}$ for each $j \in f(V)$ and concatenating all these hamps in the order of increasing j. In other words, a V-listing v is (f, D)-friendly if and only if it can be constructed in this way. Since this construction is unique, we thus have

$$(\# \text{ of } (f, D)\text{-friendly } V\text{-listings}) = \prod_{j \in f(V)} \left(\# \text{ of hamps of } \overline{D_j}\right).$$

Thus, Lemma 2.32 follows from Lemma 2.31. $\square$

2.8. A bit of Pólya counting

The following lemma is well-known, e.g., from the theory of Pólya enumeration:

Lemma 2.33. Let V be a finite set. Let $\sigma \in \mathfrak{S}_V$ be a permutation of V. Then,

$$\sum_{\substack{f:V\to\mathbb{P};\\ f\circ\sigma=f}} \prod_{v\in V} x_{f(v)} = p_{\operatorname{type}\sigma}.$$

Proof. Let $\gamma_1, \gamma_2, \ldots, \gamma_k$ be the cycles of σ, listed with no repetition. For each $i \in [k]$, let V_i be the set of entries of the cycle γ_i. Thus, $V = V_1 \sqcup V_2 \sqcup \cdots \sqcup V_k$. For each $i \in [k]$, the set V_i is the set of entries of a cycle of σ (namely, of γ_i), and thus can be written as $\left\{\sigma^j(v_i) \mid j \in \mathbb{N}\right\}$ for some $v_i \in V_i$.

Hence, a map $f : V \to \mathbb{P}$ satisfies $f \circ \sigma = f$ if and only if f is constant on each of the k sets $V_1, V_2, \ldots, V_k$. Hence, in order to construct a map $f : V \to \mathbb{P}$ that satisfies $f \circ \sigma = f$, we only need to choose the values $a_1, a_2, \ldots, a_k$ that it takes on these k sets (i.e., for each $i \in [k]$, we need to choose the value a_i that f takes on all elements of V_i).

Let us be more precise: For each k-tuple $(a_1, a_2, \ldots, a_k) \in \mathbb{P}^k$, there is a unique map $\Gamma(a_1, a_2, \ldots, a_k) : V \to \mathbb{P}$ that sends each element of V_1 to a_1, each element of V_2 to a_2, and so on (since $V = V_1 \sqcup V_2 \sqcup \cdots \sqcup V_k$). The latter map $\Gamma(a_1, a_2, \ldots, a_k)$ is a map $f : V \to \mathbb{P}$ that satisfies $f \circ \sigma = f$ (by the preceding paragraph, since it is constant on each of the k sets $V_1, V_2, \ldots, V_k$). Thus, we obtain a map

$$\begin{aligned} \Gamma : \mathbb{P}^k &\to \{f : V \to \mathbb{P} \mid f \circ \sigma = f\}, \\ (a_1, a_2, \ldots, a_k) &\mapsto \Gamma(a_1, a_2, \ldots, a_k). \end{aligned}$$

This map Γ is easily seen to be injective (since $V_1, V_2, \ldots, V_k$ are nonempty) and surjective (again by the previous paragraph). Hence, it is bijective. Thus, substituting

$\Gamma\left(a_1, a_2, \ldots, a_k\right)$ for f in the sum $\sum\limits_{\substack{f:V\to\mathbb{P};\\ f\circ\sigma=f}} \prod\limits_{v\in V} x_{f(v)}$, we obtain

$$\begin{aligned}
\sum_{\substack{f:V\to\mathbb{P};\\ f\circ\sigma=f}} \prod_{v\in V} x_{f(v)} &= \sum_{(a_1,a_2,\ldots,a_k)\in\mathbb{P}^k} \underbrace{\prod_{v\in V}}_{\substack{=\prod_{i=1}^{k}\prod_{v\in V_i} \\ \text{(since } V=V_1\sqcup V_2\sqcup\cdots\sqcup V_k\text{)}}} x_{(\Gamma(a_1,a_2,\ldots,a_k))(v)} \\
&= \sum_{(a_1,a_2,\ldots,a_k)\in\mathbb{P}^k} \prod_{i=1}^{k} \prod_{v\in V_i} \underbrace{x_{(\Gamma(a_1,a_2,\ldots,a_k))(v)}}_{\substack{=x_{a_i} \\ \text{(since the map } \Gamma(a_1,a_2,\ldots,a_k)\text{)} \\ \text{sends each element of } V_i \text{ to } a_i\text{)}}} \\
&= \sum_{(a_1,a_2,\ldots,a_k)\in\mathbb{P}^k} \prod_{i=1}^{k} \underbrace{\prod_{v\in V_i} x_{a_i}}_{=x_{a_i}^{|V_i|}} = \sum_{(a_1,a_2,\ldots,a_k)\in\mathbb{P}^k} \prod_{i=1}^{k} x_{a_i}^{|V_i|} \\
&= \prod_{i=1}^{k} \underbrace{\sum_{a\in\mathbb{P}} x_a^{|V_i|}}_{\substack{=x_1^{|V_i|}+x_2^{|V_i|}+x_3^{|V_i|}+\cdots \\ =p_{|V_i|} \\ \text{(by the definition of } p_{|V_i|}\text{)}}} \qquad \text{(by the product rule)} \\
&= \prod_{i=1}^{k} p_{|V_i|}. \qquad (15)
\end{aligned}$$

However, the permutation σ has cycles $\gamma_1, \gamma_2, \ldots, \gamma_k$, and their respective sets of entries are $V_1, V_2, \ldots, V_k$. Thus, the lengths of the cycles of σ are $|V_1|, |V_2|, \ldots, |V_k|$. But the entries of the partition type σ are precisely the lengths of the cycles of σ (by the definition of type σ), and thus must be $|V_1|, |V_2|, \ldots, |V_k|$ in some order (by the preceding sentence). Therefore, the partition type σ can be obtained from the k-tuple $(|V_1|, |V_2|, \ldots, |V_k|)$ by sorting the entries in weakly decreasing order. Hence,

$$p_{\operatorname{type}\sigma} = \prod_{i=1}^{k} p_{|V_i|}.$$

Comparing this with (15), we obtain

$$\sum_{\substack{f:V\to\mathbb{P};\\ f\circ\sigma=f}} \prod_{v\in V} x_{f(v)} = p_{\operatorname{type}\sigma}.$$

This proves Lemma 2.33. □

2.9. A final alternating sum

We need one more alternating-sum identity:

Proposition 2.34. Let $D = (V, A)$ be a digraph. Let $\sigma \in \mathfrak{S}_V$ be a permutation of V. Then,

$$\sum_{\substack{F \subseteq \mathbf{A}_\sigma \cap A \\ \text{is linear}}} (-1)^{|F|} = \begin{cases} (-1)^{\varphi(\sigma)}, & \text{if } \sigma \in \mathfrak{S}_V \left(D, \overline{D}\right); \\ 0, & \text{else,} \end{cases}$$

where we set

$$\varphi(\sigma) := \sum_{\substack{\gamma \in \operatorname{Cycs} \sigma; \\ \gamma \text{ is a } D\text{-cycle}}} (\ell(\gamma) - 1).$$

Proof. Let us first prove the proposition in the case when σ has only one cycle. That is, we shall first prove the following claim:

Claim 1: Assume that σ has a unique cycle γ. Then,

$$\sum_{\substack{F \subseteq \mathbf{A}_\sigma \cap A \\ \text{is linear}}} (-1)^{|F|} = \begin{cases} (-1)^{\ell(\gamma)-1}, & \text{if } \gamma \text{ is a } D\text{-cycle;} \\ 1, & \text{if } \gamma \text{ is a } \overline{D}\text{-cycle;} \\ 0, & \text{else.} \end{cases}$$

[*Proof of Claim 1:* We have assumed that σ has a unique cycle γ. Thus, $\mathbf{A}_\sigma = \operatorname{CArcs} \gamma$. Hence, each proper subset of $\mathbf{A}_\sigma$ is linear[7], but $\mathbf{A}_\sigma$ itself is not[8]. Furthermore, we have $|\mathbf{A}_\sigma| = |\operatorname{CArcs} \gamma| = \ell(\gamma)$.

The sum $\sum_{\substack{F \subseteq \mathbf{A}_\sigma \cap A \\ \text{is linear}}} (-1)^{|F|}$ depends on whether γ is a D-cycle, a $\overline{D}$-cycle or neither:

- If γ is a D-cycle, then all arcs in $\mathbf{A}_\sigma$ belong to A, and therefore we have $\mathbf{A}_\sigma \cap A = \mathbf{A}_\sigma$. Thus, in this case, we have

  $$\begin{aligned} &\{\text{linear subsets of } \mathbf{A}_\sigma \cap A\} \\ &= \{\text{linear subsets of } \mathbf{A}_\sigma\} = \{\text{proper subsets of } \mathbf{A}_\sigma\} \end{aligned}$$

  (since each proper subset of $\mathbf{A}_\sigma$ is linear, but $\mathbf{A}_\sigma$ itself is not). Hence, in this

[7] since the removal of any cyclic arc from a cycle turns the cycle into a path, and the removal of any further arcs will break this path into smaller paths

[8] since the digraph $(V, \mathbf{A}_\sigma)$ has the cycle γ

case, we have

$$\sum_{\substack{F\subseteq \mathbf{A}_\sigma\cap A \\ \text{is linear}}} (-1)^{|F|} = \sum_{\substack{F\subseteq \mathbf{A}_\sigma \\ \text{is a proper subset}}} (-1)^{|F|} = \underbrace{\sum_{F\subseteq \mathbf{A}_\sigma} (-1)^{|F|}}_{\substack{=0 \\ \text{(by Lemma 2.3,} \\ \text{since } \mathbf{A}_\sigma\neq\varnothing)}} - (-1)^{|\mathbf{A}_\sigma|}$$
$$= -(-1)^{|\mathbf{A}_\sigma|} = (-1)^{|\mathbf{A}_\sigma|-1}$$
$$= (-1)^{\ell(\gamma)-1} \qquad (\text{since } |\mathbf{A}_\sigma| = \ell(\gamma)).$$

- If γ is a $\overline{D}$-cycle, then no arcs in $\mathbf{A}_\sigma$ belong to A, and therefore we have $\mathbf{A}_\sigma \cap A = \varnothing$. Thus, in this case, we have

$$\{\text{linear subsets of } \mathbf{A}_\sigma \cap A\} = \{\text{linear subsets of } \varnothing\} = \{\varnothing\}.$$

  Hence, in this case, the sum $\sum_{\substack{F\subseteq \mathbf{A}_\sigma\cap A \\ \text{is linear}}} (-1)^{|F|}$ has only one addend, namely the addend corresponding to $F = \varnothing$. Thus, this sum equals 1 in this case.

- If γ is neither a D-cycle nor a $\overline{D}$-cycle, then we have $\mathbf{A}_\sigma \cap A \neq \varnothing$ (since $\mathbf{A}_\sigma \cap A = \varnothing$ would imply that $\mathbf{A}_\sigma \subseteq (V \times V) \setminus A$, whence $\operatorname{CArcs}\gamma = \mathbf{A}_\sigma \subseteq (V \times V) \setminus A$; but this would mean that γ is a $\overline{D}$-cycle) and $\mathbf{A}_\sigma \not\subseteq A$ (since $\mathbf{A}_\sigma \subseteq A$ would mean that γ is a D-cycle). Hence, in this case, any $F \subseteq \mathbf{A}_\sigma \cap A$ is a proper subset of $\mathbf{A}_\sigma$ (since $\mathbf{A}_\sigma \not\subseteq A$ shows that $\mathbf{A}_\sigma \cap A$ is a proper subset of $\mathbf{A}_\sigma$) and therefore linear (since each proper subset of $\mathbf{A}_\sigma$ is linear). Thus, in this case, we have

$$\sum_{\substack{F\subseteq \mathbf{A}_\sigma\cap A \\ \text{is linear}}} (-1)^{|F|} = \sum_{F\subseteq \mathbf{A}_\sigma\cap A} (-1)^{|F|} = [\mathbf{A}_\sigma \cap A = \varnothing] \qquad \text{(by Lemma 2.3)}$$
$$= 0 \qquad (\text{since } \mathbf{A}_\sigma \cap A \neq \varnothing).$$

Combining these three cases, we see that

$$\sum_{\substack{F\subseteq \mathbf{A}_\sigma\cap A \\ \text{is linear}}} (-1)^{|F|} = \begin{cases} (-1)^{\ell(\gamma)-1}, & \text{if } \gamma \text{ is a } D\text{-cycle;} \\ 1, & \text{if } \gamma \text{ is a } \overline{D}\text{-cycle;} \\ 0, & \text{else.} \end{cases}$$

This proves Claim 1.]

Let us now prove Proposition 2.34 in the general case. Let $\gamma_1, \gamma_2, \ldots, \gamma_k$ be the cycles of σ, listed with no repetition. Thus, these cycles $\gamma_1, \gamma_2, \ldots, \gamma_k$ are distinct, and we have $\operatorname{Cycs}\sigma = \{\gamma_1, \gamma_2, \ldots, \gamma_k\}$.

For each $i \in [k]$, let V_i be the set of entries of the cycle γ_i. Thus, $V = V_1 \sqcup V_2 \sqcup \cdots \sqcup V_k$.

Furthermore, for each $i \in [k]$, we let D_i be the digraph obtained from D by removing all vertices that are not in V_i, and we let A_i be the set of all arcs of D_i. Thus, $A_i = A \cap (V_i \times V_i)$ and $D_i = (V_i, A_i)$.

For each $i \in [k]$, let σ_i be the permutation of V_i obtained by restricting σ to V_i (this is well-defined, since V_i is the set of entries of a cycle of σ and thus preserved under σ). This permutation σ_i has a unique cycle, namely γ_i. Thus, for each $i \in [k]$, Claim 1 (applied to $D_i = (V_i, A_i)$ and σ_i and γ_i instead of $D = (V, A)$ and σ and γ) yields

$$\sum_{\substack{F \subseteq \mathbf{A}_{\sigma_i} \cap A_i \\ \text{is linear}}} (-1)^{|F|} = \begin{cases} (-1)^{\ell(\gamma_i)-1}, & \text{if } \gamma_i \text{ is a } D_i\text{-cycle;} \\ 1, & \text{if } \gamma_i \text{ is a } \overline{D_i}\text{-cycle;} \\ 0, & \text{else} \end{cases} = \begin{cases} (-1)^{\ell(\gamma_i)-1}, & \text{if } \gamma_i \text{ is a } D\text{-cycle;} \\ 1, & \text{if } \gamma_i \text{ is a } \overline{D}\text{-cycle;} \\ 0, & \text{else} \end{cases} \qquad (16)$$

(since the statement "γ_i is a D_i-cycle" is equivalent to "γ_i is a D-cycle", and the statement "γ_i is a $\overline{D_i}$-cycle" is equivalent to "γ_i is a $\overline{D}$-cycle").

It is easy to see that

$$\mathbf{A}_{\sigma_i} \cap A_i = \mathbf{A}_{\sigma_i} \cap A \qquad (17)$$

for each $i \in [k]$ [9]. Hence, we can rewrite (16) as follows: For each $i \in [k]$, we have

$$\sum_{\substack{F \subseteq \mathbf{A}_{\sigma_i} \cap A \\ \text{is linear}}} (-1)^{|F|} = \begin{cases} (-1)^{\ell(\gamma_i)-1}, & \text{if } \gamma_i \text{ is a } D\text{-cycle;} \\ 1, & \text{if } \gamma_i \text{ is a } \overline{D}\text{-cycle;} \\ 0, & \text{else.} \end{cases} \qquad (18)$$

The definition of $\varphi(\sigma)$ yields

$$\varphi(\sigma) = \sum_{\substack{\gamma \in \operatorname{Cycs} \sigma; \\ \gamma \text{ is a } D\text{-cycle}}} (\ell(\gamma) - 1) = \sum_{\substack{i \in [k]; \\ \gamma_i \text{ is a } D\text{-cycle}}} (\ell(\gamma_i) - 1)$$

[9] *Proof.* Let $i \in [k]$. Then, $\mathbf{A}_{\sigma_i} \subseteq V_i \times V_i$ (since σ_i is a permutation of V_i), so that $\mathbf{A}_{\sigma_i} \cap (V_i \times V_i) = \mathbf{A}_{\sigma_i}$. Therefore,

$$\mathbf{A}_{\sigma_i} \cap \underbrace{A_i}_{= A \cap (V_i \times V_i)} = \mathbf{A}_{\sigma_i} \cap A \cap (V_i \times V_i) = \underbrace{\mathbf{A}_{\sigma_i} \cap (V_i \times V_i)}_{= \mathbf{A}_{\sigma_i}} \cap A = \mathbf{A}_{\sigma_i} \cap A.$$

This proves (17).

(since $\gamma_1, \gamma_2, \ldots, \gamma_k$ are distinct, and $\operatorname{Cycs} \sigma = \{\gamma_1, \gamma_2, \ldots, \gamma_k\}$). Therefore,

$$(-1)^{\varphi(\sigma)} = (-1)^{\sum\limits_{\substack{i \in [k]; \\ \gamma_i \text{ is a } D\text{-cycle}}} (\ell(\gamma_i)-1)} = \prod_{\substack{i \in [k]; \\ \gamma_i \text{ is a } D\text{-cycle}}} (-1)^{\ell(\gamma_i)-1}. \tag{19}$$

However, it is easy to see that $\mathbf{A}_\sigma = \mathbf{A}_{\sigma_1} \sqcup \mathbf{A}_{\sigma_2} \sqcup \cdots \sqcup \mathbf{A}_{\sigma_k}$ (since $\sigma_1, \sigma_2, \ldots, \sigma_k$ are the restrictions of σ to the subsets $V_1, V_2, \ldots, V_k$, which cover V without overlap). Thus,

$$\begin{aligned}\mathbf{A}_\sigma \cap A &= \left(\mathbf{A}_{\sigma_1} \sqcup \mathbf{A}_{\sigma_2} \sqcup \cdots \sqcup \mathbf{A}_{\sigma_k}\right) \cap A \\ &= \left(\mathbf{A}_{\sigma_1} \cap A\right) \sqcup \left(\mathbf{A}_{\sigma_2} \cap A\right) \sqcup \cdots \sqcup \left(\mathbf{A}_{\sigma_k} \cap A\right).\end{aligned}$$

Hence, a subset F of $\mathbf{A}_\sigma \cap A$ is the same thing as a (necessarily disjoint) union $F_1 \sqcup F_2 \sqcup \cdots \sqcup F_k$ of subsets $F_i \subseteq \mathbf{A}_{\sigma_i} \cap A$ for all $i \in [k]$ [10]. Moreover, in this case, the subsets F_i for all $i \in [k]$ are uniquely determined by F (namely, we have $F_i = F \cap \mathbf{A}_{\sigma_i}$ for each $i \in [k]$). Finally, the former subset F is linear if and only if all the latter subsets F_i are linear[11]. Hence, we can substitute $F_1 \sqcup F_2 \sqcup \cdots \sqcup F_k$ for F in

[10]This sentence should be understood as follows:

1. A subset F of $\mathbf{A}_\sigma \cap A$ is the same thing as a union $F_1 \cup F_2 \cup \cdots \cup F_k$ of subsets $F_i \subseteq \mathbf{A}_{\sigma_i} \cap A$ for all $i \in [k]$.
2. Any union of the latter form is a disjoint union (thus can be written as $F_1 \sqcup F_2 \sqcup \cdots \sqcup F_k$).

[11]*Proof.* Let F be a subset of $\mathbf{A}_\sigma \cap A$, and let us assume that F is written as a disjoint union $F_1 \sqcup F_2 \sqcup \cdots \sqcup F_k$ of subsets $F_i \subseteq \mathbf{A}_{\sigma_i} \cap A$ for all $i \in [k]$. We must prove that F is linear if and only if all the subsets F_i are linear.

For each $i \in [k]$, we have $F_i \subseteq \mathbf{A}_{\sigma_i} \cap A \subseteq \mathbf{A}_{\sigma_i} \subseteq V_i \times V_i$ (because σ_i is a permutation of V_i). In other words, for each $i \in [k]$, the set F_i is a subset of $V_i \times V_i$. We have $F = F_1 \sqcup F_2 \sqcup \cdots \sqcup F_k = F_1 \cup F_2 \cup \cdots \cup F_k$. Moreover, the sets $V_1, V_2, \ldots, V_k$ are disjoint subsets of V and satisfy $V = V_1 \cup V_2 \cup \cdots \cup V_k$. Hence, Proposition 2.10 shows that the set F is linear if and only if all the subsets F_i for $i \in [k]$ are linear. This completes our proof.

the sum $\sum\limits_{\substack{F\subseteq \mathbf{A}_\sigma \cap A \\ \text{is linear}}} (-1)^{|F|}$. We thus obtain

$$\begin{aligned}
\sum_{\substack{F\subseteq \mathbf{A}_\sigma \cap A \\ \text{is linear}}} (-1)^{|F|} &= \sum_{\substack{(F_i)_{i\in[k]} \text{ is a family,} \\ \text{where each } F_i \text{ is a linear} \\ \text{subset of } \mathbf{A}_{\sigma_i}\cap A}} \underbrace{(-1)^{|F_1\sqcup F_2 \sqcup \cdots \sqcup F_k|}}_{=(-1)^{|F_1|+|F_2|+\cdots+|F_k|}=\prod_{i=1}^{k}(-1)^{|F_i|}} \\
&= \sum_{\substack{(F_i)_{i\in[k]} \text{ is a family,} \\ \text{where each } F_i \text{ is a linear} \\ \text{subset of } \mathbf{A}_{\sigma_i}\cap A}} \prod_{i=1}^{k} (-1)^{|F_i|} \\
&= \prod_{i=1}^{k} \sum_{\substack{F_i\subseteq \mathbf{A}_{\sigma_i} \cap A \\ \text{is linear}}} (-1)^{|F_i|} \qquad \text{(by the product rule)} \\
&= \prod_{i=1}^{k} \sum_{\substack{F\subseteq \mathbf{A}_{\sigma_i} \cap A \\ \text{is linear}}} (-1)^{|F|} \qquad \left(\begin{array}{c}\text{here, we have renamed the} \\ \text{summation index } F_i \text{ as } F\end{array}\right) \\
&= \prod_{i=1}^{k} \begin{cases} (-1)^{\ell(\gamma_i)-1}, & \text{if } \gamma_i \text{ is a } D\text{-cycle;} \\ 1, & \text{if } \gamma_i \text{ is a } \overline{D}\text{-cycle;} \\ 0, & \text{else} \end{cases} \qquad \text{(by (18))}.
\end{aligned}$$

The right hand side of this equality is clearly 0 unless each of the cycles $\gamma_1, \gamma_2, \ldots, \gamma_k$ is a D-cycle or a $\overline{D}$-cycle; otherwise, it equals

$$\prod_{i=1}^{k} \begin{cases} (-1)^{\ell(\gamma_i)-1}, & \text{if } \gamma_i \text{ is a } D\text{-cycle;} \\ 1, & \text{if } \gamma_i \text{ is a } \overline{D}\text{-cycle} \end{cases} = \prod_{\substack{i\in[k]; \\ \gamma_i \text{ is a } D\text{-cycle}}} (-1)^{\ell(\gamma_i)-1} = (-1)^{\varphi(\sigma)}$$

(by (19)). Hence, in either case, it equals

$$\begin{aligned}
&\begin{cases} (-1)^{\varphi(\sigma)}, & \text{if each of } \gamma_1, \gamma_2, \ldots, \gamma_k \text{ is a } D\text{-cycle or a } \overline{D}\text{-cycle;} \\ 0, & \text{else} \end{cases} \\
&= \begin{cases} (-1)^{\varphi(\sigma)}, & \text{if each cycle of } \sigma \text{ is a } D\text{-cycle or a } \overline{D}\text{-cycle;} \\ 0, & \text{else} \end{cases} \\
&\qquad \qquad \text{(since } \gamma_1, \gamma_2, \ldots, \gamma_k \text{ are the cycles of } \sigma\text{)} \\
&= \begin{cases} (-1)^{\varphi(\sigma)}, & \text{if } \sigma \in \mathfrak{S}_V\left(D, \overline{D}\right); \\ 0, & \text{else} \end{cases}
\end{aligned}$$

(by the definition of $\mathfrak{S}_V\left(D, \overline{D}\right)$). This proves Proposition 2.34. $\square$

2.10. A trivial lemma

We need one more trivial "data conversion" lemma:

Lemma 2.35. Let V be a finite set. Let $w = (w_1, w_2, \ldots, w_n)$ be a V-listing. Then, the map

$$\begin{aligned} \{\text{maps } f : V \to \mathbb{P}\} &\to \mathbb{P}^n, \\ f &\mapsto (f(w_1), f(w_2), \ldots, f(w_n)) \end{aligned}$$

is well-defined and is a bijection.

Proof. This is just saying that every map $f : V \to \mathbb{P}$ can be encoded by its list of values $(f(w_1), f(w_2), \ldots, f(w_n))$, and conversely, that any list $(i_1, i_2, \ldots, i_n) \in \mathbb{P}^n$ is the list of values of a unique map $f : V \to \mathbb{P}$. Both of these claims are clear, since $w_1, w_2, \ldots, w_n$ are the elements of V (listed with no repetitions). $\square$

2.11. The proof of Theorem 1.31

We are now ready to prove Theorem 1.31:

Proof of Theorem 1.31. Let $n = |V|$. Thus, the digraph $D = (V, A)$ has n vertices. Moreover, each V-listing w has n entries (since $|V| = n$), thus satisfies $w = (w_1, w_2, \ldots, w_n)$.

We will use a definition that we made back in Lemma 2.32: If $f : V \to \mathbb{P}$ is a map, and if $v = (v_1, v_2, \ldots, v_n)$ is a V-listing, then this V-listing v will be called *(f, D)-friendly* if it has the properties that $f(v_1) \leq f(v_2) \leq \cdots \leq f(v_n)$ and that

$$f(v_p) < f(v_{p+1}) \quad \text{for each } p \in [n-1] \text{ satisfying } (v_p, v_{p+1}) \in A.$$

The definition of U_D yields

$$U_D = \sum_{w \text{ is a } V\text{-listing}} L_{\mathrm{Des}(w,D),\ n}.$$

We shall now try to understand the addends in this sum better.

We fix a V-listing w. Then, w has n entries (since $|V| = n$), and thus satisfies $w = (w_1, w_2, \ldots, w_n)$. Moreover, the list $(w_1, w_2, \ldots, w_n) = w$ is a V-listing, i.e., consists of all elements of V and contains each of these elements exactly once. In other words, $(w_1, w_2, \ldots, w_n)$ is a list of all elements of V, with no repetitions. Hence, if we are given an element c_v of $\mathbb{Z}[[x_1, x_2, x_3, \ldots]]$ for each $v \in V$, then

$$\prod_{v \in V} c_v = c_{w_1} c_{w_2} \cdots c_{w_n}. \tag{20}$$

Thus, if $f : V \to \mathbb{P}$ is any map, then

$$\prod_{v \in V} x_{f(v)} = x_{f(w_1)} x_{f(w_2)} \cdots x_{f(w_n)} \tag{21}$$

(by (20), applied to $c_v = x_{f(v)}$).

However, the definition of $L_{\operatorname{Des}(w,D),\ n}$ yields

$$\begin{aligned} L_{\operatorname{Des}(w,D),\ n} &= \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_n; \\ i_p < i_{p+1} \text{ for each } p \in \operatorname{Des}(w,D)}} x_{i_1} x_{i_2} \cdots x_{i_n} \\ &= \sum_{\substack{(i_1,i_2,\ldots,i_n) \in \mathbb{P}^n; \\ i_1 \leq i_2 \leq \cdots \leq i_n; \\ i_p < i_{p+1} \text{ for each } p \in \operatorname{Des}(w,D)}} x_{i_1} x_{i_2} \cdots x_{i_n} \end{aligned} \tag{22}$$

(here, we have added the "$(i_1, i_2, \ldots, i_n) \in \mathbb{P}^n$" condition under the summation sign, since this condition is tacitly implied when we sum over $i_1 \leq i_2 \leq \cdots \leq i_n$).

We recall that $\operatorname{Des}(w, D)$ is defined as the set of all D-descents of w, but these D-descents are defined as the elements $i \in [n-1]$ satisfying $(w_i, w_{i+1}) \in A$. Hence, $\operatorname{Des}(w, D)$ is the set of all elements $i \in [n-1]$ satisfying $(w_i, w_{i+1}) \in A$. Thus, an element of $\operatorname{Des}(w, D)$ is the same thing as an element $i \in [n-1]$ satisfying $(w_i, w_{i+1}) \in A$. Renaming the variable i as p in this sentence, we obtain the following: An element of $\operatorname{Des}(w, D)$ is the same thing as an element $p \in [n-1]$ satisfying $(w_p, w_{p+1}) \in A$.

Lemma 2.35 yields that the map

$$\begin{aligned} \{\text{maps } f : V \to \mathbb{P}\} &\to \mathbb{P}^n, \\ f &\mapsto (f(w_1), f(w_2), \ldots, f(w_n)) \end{aligned}$$

is well-defined and is a bijection. Hence, we can substitute $(f(w_1), f(w_2), \ldots, f(w_n))$ for $(i_1, i_2, \ldots, i_n)$ in the sum on the right hand side of (22). We thus obtain

$$\begin{aligned} &\sum_{\substack{(i_1,i_2,\ldots,i_n) \in \mathbb{P}^n; \\ i_1 \leq i_2 \leq \cdots \leq i_n; \\ i_p < i_{p+1} \text{ for each } p \in \operatorname{Des}(w,D)}} x_{i_1} x_{i_2} \cdots x_{i_n} \\ &= \sum_{\substack{f : V \to \mathbb{P} \text{ is a map}; \\ f(w_1) \leq f(w_2) \leq \cdots \leq f(w_n); \\ f(w_p) < f(w_{p+1}) \text{ for each } p \in \operatorname{Des}(w,D)}} \underbrace{x_{f(w_1)} x_{f(w_2)} \cdots x_{f(w_n)}}_{\substack{= \prod_{v \in V} x_{f(v)} \\ \text{(by (21))}}} \\ &= \sum_{\substack{f : V \to \mathbb{P} \text{ is a map}; \\ f(w_1) \leq f(w_2) \leq \cdots \leq f(w_n); \\ f(w_p) < f(w_{p+1}) \text{ for each } p \in \operatorname{Des}(w,D)}} \prod_{v \in V} x_{f(v)} \\ &= \sum_{\substack{f : V \to \mathbb{P} \text{ is a map}; \\ f(w_1) \leq f(w_2) \leq \cdots \leq f(w_n); \\ f(w_p) < f(w_{p+1}) \text{ for each } p \in [n-1] \\ \text{satisfying } (w_p, w_{p+1}) \in A}} \prod_{v \in V} x_{f(v)} \end{aligned}$$

(here, we have replaced the condition "$p \in \operatorname{Des}(w, D)$" under the summation sign by the equivalent condition "$p \in [n-1]$ satisfying $(w_p, w_{p+1}) \in A$", because an element of $\operatorname{Des}(w, D)$ is the same thing as an element $p \in [n-1]$ satisfying $(w_p, w_{p+1}) \in A$). Thus, (22) becomes

$$L_{\operatorname{Des}(w,D),\ n} = \sum_{\substack{(i_1,i_2,\ldots,i_n)\in\mathbb{P}^n;\\ i_1\leq i_2\leq\cdots\leq i_n;\\ i_p<i_{p+1} \text{ for each } p\in\operatorname{Des}(w,D)}} x_{i_1}x_{i_2}\cdots x_{i_n}$$
$$= \sum_{\substack{f:V\to\mathbb{P} \text{ is a map;}\\ f(w_1)\leq f(w_2)\leq\cdots\leq f(w_n);\\ f(w_p)<f(w_{p+1}) \text{ for each } p\in[n-1]\\ \text{satisfying } (w_p,w_{p+1})\in A}} \prod_{v\in V} x_{f(v)}. \tag{23}$$

The sum on the right hand side of (23) ranges over all maps $f : V \to \mathbb{P}$ that satisfy the condition

$$\text{“}f(w_1) \leq f(w_2) \leq \cdots \leq f(w_n)\text{”}$$
$$\wedge\ \text{“}f(w_p) < f(w_{p+1}) \text{ for each } p \in [n-1] \text{ satisfying } (w_p, w_{p+1}) \in A\text{”}.$$

However, this condition is equivalent to the condition “the V-listing w is (f, D)-friendly” (because this is how the notion of “(f, D)-friendly” was defined). Therefore, we can replace the former condition by the latter condition under the summation sign on the right hand side of (23). Thus, we can rewrite (23) as follows:

$$L_{\operatorname{Des}(w,D),\ n} = \sum_{\substack{f:V\to\mathbb{P} \text{ is a map;}\\ \text{the } V\text{-listing } w \text{ is } (f,D)\text{-friendly}}} \prod_{v\in V} x_{f(v)}. \tag{24}$$

Now, forget that we fixed w. We thus have proved (24) for each V-listing w.

Now,

$$\begin{aligned}
U_D &= \sum_{w \text{ is a } V\text{-listing}} L_{\mathrm{Des}(w,D),\ n} \\
&= \underbrace{\sum_{w \text{ is a } V\text{-listing}} \ \sum_{\substack{f:V\to\mathbb{P} \text{ is a map;} \\ \text{the } V\text{-listing } w \text{ is } (f,D)\text{-friendly}}}}_{=\sum\limits_{f:V\to\mathbb{P}} \ \sum\limits_{w \text{ is an } (f,D)\text{-friendly } V\text{-listing}}} \prod_{v\in V} x_{f(v)} \qquad \text{(by (24))} \\
&= \sum_{f:V\to\mathbb{P}} \underbrace{\sum_{w \text{ is an } (f,D)\text{-friendly } V\text{-listing}} \ \prod_{v\in V} x_{f(v)}}_{=(\#\text{ of } (f,D)\text{-friendly } V\text{-listings})\cdot \prod\limits_{v\in V} x_{f(v)}} \\
&= \sum_{f:V\to\mathbb{P}} \underbrace{(\#\text{ of } (f,D)\text{-friendly } V\text{-listings})}_{\substack{=\sum\limits_{\substack{\sigma\in\mathfrak{S}_V; \\ f\circ\sigma=f}} \ \sum\limits_{\substack{F\subseteq \mathbf{A}_\sigma\cap A \\ \text{is linear}}} (-1)^{|F|} \\ \text{(by Lemma 2.32)}}} \cdot \prod_{v\in V} x_{f(v)} \\
&= \sum_{f:V\to\mathbb{P}} \ \sum_{\substack{\sigma\in\mathfrak{S}_V; \\ f\circ\sigma=f}} \ \sum_{\substack{F\subseteq \mathbf{A}_\sigma\cap A \\ \text{is linear}}} (-1)^{|F|} \cdot \prod_{v\in V} x_{f(v)} \\
&= \sum_{\sigma\in\mathfrak{S}_V} \underbrace{\sum_{\substack{F\subseteq \mathbf{A}_\sigma\cap A \\ \text{is linear}}} (-1)^{|F|}}_{\substack{=\begin{cases} (-1)^{\varphi(\sigma)}, & \text{if } \sigma\in\mathfrak{S}_V\left(D,\overline{D}\right); \\ 0, & \text{else} \end{cases} \\ \text{(by Proposition 2.34)}}} \ \underbrace{\sum_{\substack{f:V\to\mathbb{P}; \\ f\circ\sigma=f}} \ \prod_{v\in V} x_{f(v)}}_{\substack{=p_{\mathrm{type}\,\sigma} \\ \text{(by Lemma 2.33)}}} \\
&= \sum_{\sigma\in\mathfrak{S}_V} \begin{cases} (-1)^{\varphi(\sigma)}, & \text{if } \sigma\in\mathfrak{S}_V\left(D,\overline{D}\right); \\ 0, & \text{else} \end{cases} \quad p_{\mathrm{type}\,\sigma} \\
&= \sum_{\sigma\in\mathfrak{S}_V\left(D,\overline{D}\right)} (-1)^{\varphi(\sigma)}\, p_{\mathrm{type}\,\sigma}.
\end{aligned}$$

This proves Theorem 1.31. □

3. Proof of Theorem 1.39

Theorem 1.39 can be derived from Theorem 1.31 by combining some addends that have the same $p_{\mathrm{type}\,\sigma}$ factor. Depending on the respective $(-1)^{\varphi(\sigma)}$ factors, these addends either cancel each other out or combine to form a multiple of $p_{\mathrm{type}\,\sigma}$.

Proof of Theorem 1.39. We have assumed that D is a tournament. Hence, for any two distinct vertices u and v of D, we have the logical equivalences

$$((u,v) \text{ is an arc of } D) \iff \left((v,u) \text{ is an arc of } \overline{D}\right)$$

and

$$\left((u,v) \text{ is an arc of } \overline{D}\right) \iff \left((v,u) \text{ is an arc of } D\right).$$

Therefore, the reversal[12] of a nontrivial D-cycle is always a nontrivial $\overline{D}$-cycle, and vice versa.

We define a map $\Psi : \mathfrak{S}_V\left(D,\overline{D}\right) \to \mathfrak{S}_V\left(D\right)$ as follows: If $\sigma \in \mathfrak{S}_V\left(D\right)$, then we let $\Psi\left(\sigma\right)$ be the permutation obtained from σ by reversing each cycle of σ that is a nontrivial $\overline{D}$-cycle (i.e., replacing this cycle of σ by its reversal, i.e., replacing σ by σ^{-1} on all entries of this cycle)[13]. This map Ψ is well-defined (i.e., we really have $\Psi\left(\sigma\right) \in \mathfrak{S}_V\left(D\right)$ for each $\sigma \in \mathfrak{S}_V\left(D,\overline{D}\right)$), because as we just said, the reversal of a nontrivial $\overline{D}$-cycle is always a nontrivial D-cycle. Moreover, the map Ψ preserves the cycle type of a permutation – i.e., we have

$$\operatorname{type}\left(\Psi\left(\sigma\right)\right) = \operatorname{type}\sigma \tag{25}$$

for each $\sigma \in \mathfrak{S}_V\left(D,\overline{D}\right)$.

Now, Theorem 1.31 yields

$$\begin{aligned}
U_D &= \sum_{\sigma\in\mathfrak{S}_V\left(D,\overline{D}\right)} \left(-1\right)^{\varphi(\sigma)} \underbrace{p_{\operatorname{type}\sigma}}_{\substack{=p_{\operatorname{type}(\Psi(\sigma))} \\ \text{(by (25))}}} = \sum_{\sigma\in\mathfrak{S}_V\left(D,\overline{D}\right)} \left(-1\right)^{\varphi(\sigma)} p_{\operatorname{type}(\Psi(\sigma))} \\
&= \sum_{\tau\in\mathfrak{S}_V(D)} \sum_{\substack{\sigma\in\mathfrak{S}_V\left(D,\overline{D}\right); \\ \Psi(\sigma)=\tau}} \left(-1\right)^{\varphi(\sigma)} p_{\operatorname{type}\tau} \qquad \left(\begin{array}{c}\text{here, we have split up the sum} \\ \text{according to the value of } \Psi\left(\sigma\right)\end{array}\right) \\
&= \sum_{\tau\in\mathfrak{S}_V(D)} \left(\sum_{\substack{\sigma\in\mathfrak{S}_V\left(D,\overline{D}\right); \\ \Psi(\sigma)=\tau}} \left(-1\right)^{\varphi(\sigma)}\right) p_{\operatorname{type}\tau}.
\end{aligned} \tag{26}$$

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[12]See Definition 1.23 for the meanings of "reversal" and "nontrivial".

[13]Here is what this means in rigorous terms: We let $\Psi\left(\sigma\right)$ be the permutation of V defined by setting

$$\left(\Psi\left(\sigma\right)\right)\left(z\right) = \begin{cases} \sigma^{-1}\left(z\right), & \text{if } z \text{ is an entry of a cycle of } \sigma \text{ that is a nontrivial } \overline{D}\text{-cycle;} \\ \sigma\left(z\right), & \text{otherwise} \end{cases}$$
$$\text{for each } z \in V.$$

The cycles of this permutation $\Psi\left(\sigma\right)$ are precisely

- the reversals of those cycles of σ that are nontrivial $\overline{D}$-cycles, and
- the remaining cycles of σ.

Now, we claim that each $\tau \in \mathfrak{S}_V(D)$ satisfies

$$\sum_{\substack{\sigma \in \mathfrak{S}_V\left(D,\overline{D}\right);\\ \Psi(\sigma)=\tau}} (-1)^{\varphi(\sigma)} = \begin{cases} 2^{\psi(\tau)}, & \text{if all cycles of } \tau \text{ have odd length;} \\ 0, & \text{otherwise.} \end{cases} \qquad (27)$$

[*Proof of (27):* Let $\tau \in \mathfrak{S}_V(D)$. Then, τ has exactly $\psi(\tau)$ many nontrivial cycles (by the definition of $\psi(\tau)$), and all of these nontrivial cycles are D-cycles (by the definition of $\mathfrak{S}_V(D)$). The permutations $\sigma \in \mathfrak{S}_V\left(D,\overline{D}\right)$ that satisfy $\Psi(\sigma) = \tau$ can be obtained by choosing some of these nontrivial cycles and reversing them, which turns them into $\overline{D}$-cycles. This can be done in $2^{\psi(\tau)}$ many ways, since each of the $\psi(\tau)$ many nontrivial cycles can be either reversed or not. If all cycles of τ have odd length, then all $2^{\psi(\tau)}$ permutations σ obtained in this way will satisfy $(-1)^{\varphi(\sigma)} = 1$ (because $\varphi(\sigma) = \sum_{\substack{\gamma \in \operatorname{Cycs}\sigma;\\ \gamma \text{ is a } D\text{-cycle}}} \left(\underbrace{\ell(\gamma)}_{\text{odd}} - 1\right)$ will always be even); therefore, the sum $\sum_{\substack{\sigma \in \mathfrak{S}_V\left(D,\overline{D}\right);\\ \Psi(\sigma)=\tau}} (-1)^{\varphi(\sigma)}$ will be a sum of $2^{\psi(\tau)}$ many 1s and therefore simplify to $2^{\psi(\tau)}$. On the other hand, if not all cycles of τ have odd length, then there is at least one cycle δ of τ that has even length, and of course this cycle δ will be nontrivial (since a trivial cycle has odd length); thus, among the permutations $\sigma \in \mathfrak{S}_V\left(D,\overline{D}\right)$ that satisfy $\Psi(\sigma) = \tau$, there will be as many that have δ reversed as ones that have δ not reversed, and the parities of $\varphi(\sigma)$ for the former will be opposite from the parities of $\varphi(\sigma)$ for the latter; thus, the sum $\sum_{\substack{\sigma \in \mathfrak{S}_V\left(D,\overline{D}\right);\\ \Psi(\sigma)=\tau}} (-1)^{\varphi(\sigma)}$ will have equally many 1s and -1s among its addends, and therefore will simplify to 0. In either case, we obtain (27).]

Now, (26) becomes

$$U_D = \sum_{\tau \in \mathfrak{S}_V(D)} \underbrace{\left(\sum_{\substack{\sigma \in \mathfrak{S}_V\left(D, \overline{D}\right); \\ \Psi(\sigma) = \tau}} (-1)^{\varphi(\sigma)} \right)}_{\substack{= \begin{cases} 2^{\psi(\tau)}, & \text{if all cycles of } \tau \text{ have odd length;} \\ 0, & \text{otherwise} \end{cases} \\ \text{(by (27))}}} p_{\operatorname{type} \tau}$$

$$= \sum_{\tau \in \mathfrak{S}_V(D)} \begin{cases} 2^{\psi(\tau)}, & \text{if all cycles of } \tau \text{ have odd length;} \\ 0, & \text{otherwise} \end{cases} p_{\operatorname{type} \tau}$$

$$= \sum_{\substack{\tau \in \mathfrak{S}_V(D); \\ \text{all cycles of } \tau \text{ have odd length}}} 2^{\psi(\tau)} p_{\operatorname{type} \tau} = \sum_{\substack{\sigma \in \mathfrak{S}_V(D); \\ \text{all cycles of } \sigma \text{ have odd length}}} 2^{\psi(\sigma)} p_{\operatorname{type} \sigma}.$$

This proves Theorem 1.39. □

4. Proving the corollaries

Let us now quickly go through the proofs of the corollaries we stated after Theorem 1.31 and after Theorem 1.39:

Proof of Corollary 1.35. We let $\mathbb{N}[p_1, p_2, p_3, \ldots]$ denote the set of all polynomials in $p_1, p_2, p_3, \ldots$ with coefficients in $\mathbb{N}$.

For each integer partition λ, we have

$$p_\lambda \in \mathbb{N}[p_1, p_2, p_3, \ldots] \tag{28}$$

(by the definition of p_λ).

Theorem 1.31 yields

$$U_D = \sum_{\sigma \in \mathfrak{S}_V\left(D, \overline{D}\right)} (-1)^{\varphi(\sigma)} \underbrace{p_{\operatorname{type} \sigma}}_{\substack{\in \mathbb{N}[p_1, p_2, p_3, \ldots] \\ \text{(by (28))}}}$$

$$\in \sum_{\sigma \in \mathfrak{S}_V\left(D, \overline{D}\right)} (-1)^{\varphi(\sigma)} \mathbb{N}[p_1, p_2, p_3, \ldots] \subseteq \mathbb{Z}[p_1, p_2, p_3, \ldots].$$

This proves Corollary 1.35. □

Proof of Corollary 1.36. Let $2\mathbb{Z}$ denote the set of all even integers.

Let $\sigma \in \mathfrak{S}_V\left(D, \overline{D}\right)$. The definition of $\varphi(\sigma)$ in Theorem 1.31 yields

$$\varphi(\sigma) = \sum_{\substack{\gamma \in \operatorname{Cycs}\sigma; \\ \gamma \text{ is a } D\text{-cycle}}} \underbrace{(\ell(\gamma) - 1)}_{\substack{\in 2\mathbb{Z} \\ \text{(since } \ell(\gamma) \text{ is odd} \\ \text{(because every } D\text{-cycle} \\ \text{has odd length))}}} \in 2\mathbb{Z},$$

so that

$$(-1)^{\varphi(\sigma)} = 1. \tag{29}$$

Theorem 1.31 now yields

$$\begin{aligned} U_D &= \sum_{\sigma \in \mathfrak{S}_V\left(D, \overline{D}\right)} \underbrace{(-1)^{\varphi(\sigma)}}_{\substack{=1 \\ \text{(by (29))}}} p_{\operatorname{type}\sigma} \\ &= \sum_{\sigma \in \mathfrak{S}_V\left(D, \overline{D}\right)} \underbrace{p_{\operatorname{type}\sigma}}_{\substack{\in \mathbb{N}[p_1, p_2, p_3, \ldots] \\ \text{(by (28))}}} \\ &\in \sum_{\sigma \in \mathfrak{S}_V\left(D, \overline{D}\right)} \mathbb{N}\left[p_1, p_2, p_3, \ldots\right] \subseteq \mathbb{N}\left[p_1, p_2, p_3, \ldots\right]. \end{aligned}$$

This proves Corollary 1.36. □

Proof of Corollary 1.40. For each $\sigma \in \mathfrak{S}_V$, let $\psi(\sigma)$ denote the number of nontrivial cycles of σ.

Let $\sigma \in \mathfrak{S}_V(D)$ be a permutation whose all cycles have odd length. We shall show that $2^{\psi(\sigma)} p_{\operatorname{type}\sigma} \in \mathbb{N}\left[p_1, 2p_3, 2p_5, 2p_7, \ldots\right]$.

Indeed, let $k_1, k_2, \ldots, k_s$ be the lengths of all cycles of σ, listed in decreasing order. Then, the numbers $k_1, k_2, \ldots, k_s$ are odd (since all cycles of σ have odd length). Moreover, the definition of $\operatorname{type}\sigma$ yields $\operatorname{type}\sigma = (k_1, k_2, \ldots, k_s)$. Furthermore,

$$\begin{aligned} \psi(\sigma) &= (\#\text{ of nontrivial cycles of } \sigma) \\ &= (\#\text{ of cycles of } \sigma \text{ that have length } > 1) \\ &= (\#\text{ of } i \in [s] \text{ such that } k_i > 1) \\ &\qquad\qquad \left(\text{since the lengths of all cycles of } \sigma \text{ are } k_1, k_2, \ldots, k_s\right) \\ &= \sum_{i=1}^{s} [k_i > 1] \end{aligned}$$

(here, we are using the Iverson bracket notation), so that

$$2^{\psi(\sigma)} = 2^{\sum_{i=1}^{s} [k_i > 1]} = \prod_{i=1}^{s} 2^{[k_i > 1]}. \tag{30}$$

Now, recall that $\operatorname{type}\sigma = (k_1, k_2, \ldots, k_s)$. Hence, the definition of $p_{\operatorname{type}\sigma}$ yields

$$p_{\operatorname{type}\sigma} = p_{k_1} p_{k_2} \cdots p_{k_s} = \prod_{i=1}^{s} p_{k_i}. \tag{31}$$

Multiplying the equalities (30) and (31), we obtain

$$\begin{aligned} 2^{\psi(\sigma)} p_{\operatorname{type}\sigma} &= \left(\prod_{i=1}^{s} 2^{[k_i>1]}\right)\left(\prod_{i=1}^{s} p_{k_i}\right) = \prod_{i=1}^{s} \underbrace{\left(2^{[k_i>1]} p_{k_i}\right)}_{\substack{\in\{p_1,2p_3,2p_5,2p_7,\ldots\} \\ \text{(since } k_i \text{ is odd} \\ \text{(because } k_1,k_2,\ldots,k_s \text{ are odd))}}} \\ &= (\text{a product of } s \text{ elements of the set } \{p_1, 2p_3, 2p_5, 2p_7, \ldots\}) \\ &\in \mathbb{N}\left[p_1, 2p_3, 2p_5, 2p_7, \ldots\right]. \end{aligned} \tag{32}$$

Forget that we fixed σ. We thus have proved (32) for each permutation $\sigma \in \mathfrak{S}_V(D)$ whose all cycles have odd length. Now, Theorem 1.39 yields

$$U_D = \sum_{\substack{\sigma\in\mathfrak{S}_V(D); \\ \text{all cycles of } \sigma \text{ have odd length}}} \underbrace{2^{\psi(\sigma)} p_{\operatorname{type}\sigma}}_{\substack{\in\mathbb{N}[p_1,2p_3,2p_5,2p_7,\ldots] \\ \text{(by (32))}}} \in \mathbb{N}\left[p_1, 2p_3, 2p_5, 2p_7, \ldots\right].$$

This proves Corollary 1.40. □

5. Proof of Theorem 1.41

The proof of Theorem 1.41 is a slightly more complicated variant of our above proof of Theorem 1.39.

Proof of Theorem 1.41. **(b)** First, we attempt to gain a better understanding of risky cycles.

We start by noticing that the reversal of a risky rotation-equivalence class is again risky.

We have assumed that there exist no two distinct vertices u and v of D such that both pairs (u, v) and (v, u) belong to A. In other words, if (u, v) is an arc of D with $u \neq v$, then (v, u) is not an arc of D, and thus (v, u) must be an arc of $\overline{D}$.

Hence, if v is any D-cycle of length ≥ 2, then the reversal of v must be a $\overline{D}$-cycle, and thus cannot be a D-cycle. Therefore, in particular, if v is a risky rotation-equivalence class of tuples of elements of V, then either v or the reversal of v is a D-cycle (by the definition of "risky"), but not both at the same time.

Consequently, if v is a risky rotation-equivalence class of tuples of elements of V, then v and the reversal of v cannot be identical, i.e., we must have

$$v \neq \operatorname{rev} v. \tag{33}$$

We define a subset $\mathfrak{S}_V^\circ\left(D,\overline{D}\right)$ of $\mathfrak{S}_V\left(D,\overline{D}\right)$ by

$$\mathfrak{S}_V^\circ\left(D,\overline{D}\right) := \left\{\sigma \in \mathfrak{S}_V\left(D,\overline{D}\right) \mid \text{each risky cycle of } \sigma \text{ is a } D\text{-cycle}\right\}.$$

We define a map $\Gamma : \mathfrak{S}_V\left(D,\overline{D}\right) \to \mathfrak{S}_V^\circ\left(D,\overline{D}\right)$ as follows: If $\sigma \in \mathfrak{S}_V\left(D,\overline{D}\right)$, then we let $\Gamma(\sigma)$ be the permutation obtained from σ by reversing each risky cycle of σ that is not a D-cycle (i.e., replacing this cycle of σ by its reversal, i.e., replacing σ by σ^{-1} on all entries of this cycle). This map Γ is well-defined (i.e., we really have $\Gamma(\sigma) \in \mathfrak{S}_V^\circ\left(D,\overline{D}\right)$ for each $\sigma \in \mathfrak{S}_V\left(D,\overline{D}\right)$), because if a risky tuple is not a D-cycle, then its reversal is a D-cycle (by the definition of "risky"). Moreover, the map Γ preserves the cycle type of a permutation – i.e., we have

$$\operatorname{type}(\Gamma(\sigma)) = \operatorname{type}\sigma \tag{34}$$

for each $\sigma \in \mathfrak{S}_V\left(D,\overline{D}\right)$.

Now, Theorem 1.31 yields

$$\begin{aligned}
U_D &= \sum_{\sigma\in\mathfrak{S}_V\left(D,\overline{D}\right)} (-1)^{\varphi(\sigma)} \underbrace{p_{\operatorname{type}\sigma}}_{\substack{=p_{\operatorname{type}(\Gamma(\sigma))} \\ \text{(by (34))}}} = \sum_{\sigma\in\mathfrak{S}_V\left(D,\overline{D}\right)} (-1)^{\varphi(\sigma)} p_{\operatorname{type}(\Gamma(\sigma))} \\
&= \sum_{\tau\in\mathfrak{S}_V^\circ\left(D,\overline{D}\right)} \sum_{\substack{\sigma\in\mathfrak{S}_V\left(D,\overline{D}\right); \\ \Gamma(\sigma)=\tau}} (-1)^{\varphi(\sigma)} p_{\operatorname{type}\tau} \qquad \left(\begin{array}{c}\text{here, we have split up the sum} \\ \text{according to the value of } \Gamma(\sigma)\end{array}\right) \\
&= \sum_{\tau\in\mathfrak{S}_V^\circ\left(D,\overline{D}\right)} \left(\sum_{\substack{\sigma\in\mathfrak{S}_V\left(D,\overline{D}\right); \\ \Gamma(\sigma)=\tau}} (-1)^{\varphi(\sigma)}\right) p_{\operatorname{type}\tau}.
\end{aligned} \tag{35}$$

Now, we claim that each $\tau \in \mathfrak{S}_V^\circ\left(D,\overline{D}\right)$ satisfies

$$\sum_{\substack{\sigma\in\mathfrak{S}_V\left(D,\overline{D}\right); \\ \Gamma(\sigma)=\tau}} (-1)^{\varphi(\sigma)} = \begin{cases} (-1)^{\varphi(\tau)}, & \text{if no cycle of } \tau \text{ is risky;} \\ 0, & \text{otherwise.} \end{cases} \tag{36}$$

[*Proof of (36):* Let $\tau \in \mathfrak{S}_V^\circ\left(D,\overline{D}\right)$. Let $c_1, c_2, \ldots, c_k$ be the risky cycles of τ. All of these k risky cycles $c_1, c_2, \ldots, c_k$ are D-cycles (since $\tau \in \mathfrak{S}_V^\circ\left(D,\overline{D}\right)$). The permutations $\sigma \in \mathfrak{S}_V\left(D,\overline{D}\right)$ that satisfy $\Gamma(\sigma) = \tau$ can be obtained by choosing some of these k risky cycles $c_1, c_2, \ldots, c_k$ of τ and reversing them, which turns them into $\overline{D}$-cycles (because if v is any D-cycle of length ≥ 2, then the reversal of v must be a $\overline{D}$-cycle). This can be done in 2^k many ways, since each of the k risky cycles $c_1, c_2, \ldots, c_k$ can be either reversed or not[14]. The sum $\sum\limits_{\substack{\sigma\in\mathfrak{S}_V\left(D,\overline{D}\right); \\ \Gamma(\sigma)=\tau}} (-1)^{\varphi(\sigma)}$ thus has

[14]Fineprint: All of these k risky cycles are distinct from their reversals (by (33)). Thus, each of the 2^k possible choices of risky cycles to reverse leads to a different permutation $\sigma \in \mathfrak{S}_V\left(D,\overline{D}\right)$.

2^k many addends, and each of these addends corresponds to one way to decide which of the k risky cycles $c_1, c_2, \ldots, c_k$ to reverse and which not to reverse. If $k = 0$, then this sum therefore simplifies to $(-1)^{\varphi(\tau)}$. If, on the other hand, we have $k \neq 0$, then this sum equals 0 [15]. Combining the results from both of these cases, we obtain

$$\sum_{\substack{\sigma \in \mathfrak{S}_V(D,\overline{D});\\ \Gamma(\sigma)=\tau}} (-1)^{\varphi(\sigma)} = \begin{cases} (-1)^{\varphi(\tau)}, & \text{if } k = 0; \\ 0, & \text{otherwise} \end{cases}$$
$$= \begin{cases} (-1)^{\varphi(\tau)}, & \text{if no cycle of } \tau \text{ is risky;} \\ 0, & \text{otherwise.} \end{cases}$$

(since k is the number of risky cycles of τ). This proves (36).]

[15] *Proof.* Assume that $k \neq 0$. Thus, $k \geq 1$, so that the risky cycle c_1 exists. If $\sigma \in \mathfrak{S}_V\left(D, \overline{D}\right)$ is such that $\Gamma(\sigma) = \tau$, then either the cycle c_1 or its reversal (but not both) is a cycle of σ. Thus,

$$\sum_{\substack{\sigma \in \mathfrak{S}_V(D,\overline{D});\\ \Gamma(\sigma)=\tau}} (-1)^{\varphi(\sigma)} = \sum_{\substack{\sigma \in \mathfrak{S}_V(D,\overline{D});\\ \Gamma(\sigma)=\tau;\\ c_1 \text{ is a cycle of } \sigma}} (-1)^{\varphi(\sigma)} + \sum_{\substack{\sigma \in \mathfrak{S}_V(D,\overline{D});\\ \Gamma(\sigma)=\tau;\\ c_1 \text{ is not a cycle of } \sigma}} (-1)^{\varphi(\sigma)}. \tag{37}$$

The two sums on the right hand side of this equality have the same number of addends, and there is in fact a bijection between the addends of the former and those of the latter (given by replacing the cycle c_1 by its reversal or vice versa). Moreover, this bijection toggles the parity of the number $\varphi(\sigma)$ (that is, it changes this number from odd to even or vice versa), since $\varphi(\sigma)$ is defined to be the sum $\sum_{\substack{\gamma \in \operatorname{Cycs}\sigma;\\ \gamma \text{ is a } D\text{-cycle}}} (\ell(\gamma) - 1)$ (which contains the odd addend $\ell(c_1) - 1$ when c_1 is a cycle of σ, but does not contain this addend when c_1 is not a cycle of σ). Hence, this bijection flips the sign $(-1)^{\varphi(\sigma)}$. Therefore, the addends in the first sum on the right hand side of (37) cancel those in the second. Therefore, the two sums add up to 0. The equality (37) thus simplifies to $\sum_{\substack{\sigma \in \mathfrak{S}_V(D,\overline{D});\\ \Gamma(\sigma)=\tau}} (-1)^{\varphi(\sigma)} = 0$, qed.

Now, (35) becomes

$$U_D = \sum_{\tau \in \mathfrak{S}_V^\circ(D,\overline{D})} \underbrace{\left(\sum_{\substack{\sigma \in \mathfrak{S}_V(D,\overline{D});\\ \Gamma(\sigma)=\tau}} (-1)^{\varphi(\sigma)} \right)}_{\substack{= \begin{cases} (-1)^{\varphi(\tau)}, & \text{if no cycle of } \tau \text{ is risky;} \\ 0, & \text{otherwise} \end{cases} \\ \text{(by (36))}}} p_{\operatorname{type}\tau}$$

$$= \sum_{\tau \in \mathfrak{S}_V^\circ(D,\overline{D})} \begin{cases} (-1)^{\varphi(\tau)}, & \text{if no cycle of } \tau \text{ is risky;} \\ 0, & \text{otherwise} \end{cases} p_{\operatorname{type}\tau}$$

$$= \sum_{\substack{\tau \in \mathfrak{S}_V^\circ(D,\overline{D});\\ \text{no cycle of } \tau \text{ is risky}}} \underbrace{(-1)^{\varphi(\tau)}}_{\substack{=1 \\ \text{(since no cycle of } \tau \text{ is risky,} \\ \text{and thus it is easy to see} \\ \text{that } \varphi(\tau) \text{ is even)}}} p_{\operatorname{type}\tau}$$

$$= \sum_{\substack{\tau \in \mathfrak{S}_V^\circ(D,\overline{D});\\ \text{no cycle of } \tau \text{ is risky}}} p_{\operatorname{type}\tau} = \sum_{\substack{\tau \in \mathfrak{S}_V(D,\overline{D});\\ \text{no cycle of } \tau \text{ is risky}}} p_{\operatorname{type}\tau}$$

(since the permutations $\tau \in \mathfrak{S}_V^\circ\left(D,\overline{D}\right)$ that have no risky cycles are precisely the permutations $\tau \in \mathfrak{S}_V\left(D,\overline{D}\right)$ that have no risky cycles[16]). Renaming the summation index τ as σ on the right hand side, we obtain

$$U_D = \sum_{\substack{\sigma \in \mathfrak{S}_V(D,\overline{D});\\ \text{no cycle of } \sigma \text{ is risky}}} p_{\operatorname{type}\sigma}.$$

This proves Theorem 1.41 **(b)**.

(a) This follows trivially from part **(b)**, since $p_\lambda \in \mathbb{N}\left[p_1, p_2, p_3, \ldots\right]$ for each partition λ. □

6. Recovering Redei's and Berge's theorems

We shall now derive two well-known theorems from Theorem 1.31 and Theorem 1.39.

We recall Convention 2.1 and Definition 2.14. The two theorems we shall derive are the following:

[16] This follows trivially from the definition of $\mathfrak{S}_V^\circ\left(D,\overline{D}\right)$.

Theorem 6.1 (Rédei's Theorem)**.** Let D be a tournament. Then, the # of hamps of D is odd. Here, we agree to consider the empty list $()$ as a hamp of the empty tournament with 0 vertices.

Theorem 6.2 (Berge's Theorem)**.** Let D be a digraph. Then,

$$\left(\# \text{ of hamps of } \overline{D}\right) \equiv \left(\# \text{ of hamps of } D\right) \bmod 2.$$

Theorem 6.1 originates in Laszlo Rédei's 1933 paper [Redei33] (see [Moon13, proof of Theorem 14] for an English translation of his proof). Theorem 6.2 was found by Claude Berge (see [Berge76, §10.1, Theorem 1], [Berge91, §10.1, Theorem 1], [Tomesc85, solution to problem 7.8], [Lovasz07, Exercise 5.19] or [Grinbe17, Theorem 1.3.6] for his proof, and [Lass02, Corollaire 5.1] for another). Berge used Theorem 6.2 to give a new and simpler proof of Theorem 6.1 (see [Berge91, §10.2, Theorem 6] or [Lovasz07, Exercise 5.20] or [Grinbe17, Theorem 1.6.1]).

We can now give new proofs for both theorems. This will rely on the symmetric function U_D and also on a few simple tools:

We define $\zeta : \operatorname{QSym} \to \mathbb{Z}$ to be the evaluation homomorphism that sends each quasisymmetric function $f \in \operatorname{QSym}$ to its evaluation $f(1, 0, 0, 0, \ldots)$ (obtained by setting x_1 to be 1 and setting all other variables $x_2, x_3, x_4, \ldots$ to be 0). Note that ζ is a $\mathbb{Z}$-algebra homomorphism.[17] We shall show two simple lemmas:

Lemma 6.3. Let $n \in \mathbb{N}$. Let I be a subset of $[n-1]$. Then, $\zeta(L_{I,\, n}) = [I = \varnothing]$ (where we are using the Iverson bracket notation).

Proof. The definition of $L_{I,\, n}$ yields

$$L_{I,\, n} = \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_n; \\ i_p < i_{p+1} \text{ for each } p \in I}} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

When we apply ζ to the sum on the right hand side (i.e., substitute 1 for x_1 and substitute 0 for $x_2, x_3, x_4, \ldots$), any addend that contains at least one of the variables $x_2, x_3, x_4, \ldots$ becomes 0, whereas any addend that only contains copies of x_1 becomes 1. Hence, $\zeta(L_{I,\, n})$ is the number of addends that only contain copies of x_1. But this number is 1 if $I = \varnothing$ (namely, in this case, the addend for $(i_1, i_2, \ldots, i_n) = (1, 1, \ldots, 1)$ fits the bill), and is 0 if $I \neq \varnothing$ (because in this case, the condition "$i_p < i_{p+1}$ for each $p \in I$" forces at least one of the n numbers $i_1, i_2, \ldots, i_n$ in each addend $x_{i_1} x_{i_2} \cdots x_{i_n}$ to be larger than 1, and therefore each addend contains at least one of $x_2, x_3, x_4, \ldots$). Thus, altogether, this number is $[I = \varnothing]$. This proves Lemma 6.3. □

[17] We don't really need QSym here. We could just as well define ζ on the ring of bounded-degree power series (that is, of all power series $f \in \mathbb{Z}[[x_1, x_2, x_3, \ldots]]$ for which there exists an $N \in \mathbb{N}$ such that no monomial of degree $> N$ appears in f). However, we cannot define ζ on the whole ring $\mathbb{Z}[[x_1, x_2, x_3, \ldots]]$, since ζ would have to send $1 + x_1 + x_1^2 + x_1^3 + \cdots$ to $1 + 1 + 1^2 + 1^3 + \cdots$.

Lemma 6.4. Let λ be any partition. Then,

$$\zeta(p_\lambda) = 1.$$

Proof. Write the partition λ in the form $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, where the k entries $\lambda_1, \lambda_2, \ldots, \lambda_k$ are positive. Then, the definition of p_λ yields $p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_k}$. Hence,

$$\begin{aligned}
\zeta(p_\lambda) &= \zeta\left(p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_k}\right) = \zeta\left(p_{\lambda_1}\right) \zeta\left(p_{\lambda_2}\right) \cdots \zeta\left(p_{\lambda_k}\right) \\
&\qquad \text{(since } \zeta \text{ is a } \mathbb{Z}\text{-algebra homomorphism)} \\
&= \prod_{i=1}^{k} \zeta\left(p_{\lambda_i}\right). \qquad (38)
\end{aligned}$$

However, for each positive integer n, we have $p_n = x_1^n + x_2^n + x_3^n + \cdots$ (by the definition of p_n) and

$$\begin{aligned}
\zeta(p_n) &= p_n(1, 0, 0, 0, \ldots) \qquad \text{(by the definition of } \zeta\text{)} \\
&= \underbrace{1^n}_{=1} + \underbrace{0^n + 0^n + 0^n + \cdots}_{\substack{=0 \\ \text{(since } n \text{ is positive)}}} \qquad \left(\text{since } p_n = x_1^n + x_2^n + x_3^n + \cdots\right) \\
&= 1. \qquad (39)
\end{aligned}$$

Hence, (38) becomes

$$\zeta(p_\lambda) = \prod_{i=1}^{k} \underbrace{\zeta\left(p_{\lambda_i}\right)}_{\substack{=1 \\ \text{(by (39),} \\ \text{since } \lambda_i \text{ is positive)}}} = \prod_{i=1}^{k} 1 = 1.$$

This proves Lemma 6.4. □

Lemma 6.5. Let D be a digraph. Then,

$$\zeta(U_D) = \left(\text{\# of hamps of } \overline{D}\right).$$

Proof. Write D as $D = (V, A)$, and set $n := |V|$. Then, $\overline{D} = (V, (V \times V) \setminus A)$. Hence, a hamp of $\overline{D}$ is the same as a V-listing w such that each $i \in [n-1]$ satisfies $(w_i, w_{i+1}) \in (V \times V) \setminus A$. In other words, a hamp of $\overline{D}$ is the same as a V-listing w such that no $i \in [n-1]$ satisfies $(w_i, w_{i+1}) \in A$. In other words, a hamp of $\overline{D}$ is the same as a V-listing w that satisfies $\operatorname{Des}(w, D) = \varnothing$ (because $\operatorname{Des}(w, D)$ is defined to be the set of all $i \in [n-1]$ satisfying $(w_i, w_{i+1}) \in A$). Therefore,

$$\begin{aligned}
&\left(\text{\# of hamps of } \overline{D}\right) \\
&= \left(\text{\# of } V\text{-listings } w \text{ that satisfy } \operatorname{Des}(w, D) = \varnothing\right). \qquad (40)
\end{aligned}$$

The definition of U_D yields

$$U_D = \sum_{w \text{ is a } V\text{-listing}} L_{\operatorname{Des}(w,D),\ n}.$$

Hence,

$$\begin{aligned}
\zeta\left(U_D\right) &= \zeta\left(\sum_{w \text{ is a } V\text{-listing}} L_{\operatorname{Des}(w,D),\ n}\right) \\
&= \sum_{w \text{ is a } V\text{-listing}} \underbrace{\zeta\left(L_{\operatorname{Des}(w,D),\ n}\right)}_{\substack{=[\operatorname{Des}(w,D)=\varnothing] \\ \text{(by Lemma 6.3)}}} \qquad \text{(since the map } \zeta \text{ is } \mathbb{Z}\text{-linear)} \\
&= \sum_{w \text{ is a } V\text{-listing}} \left[\operatorname{Des}\left(w, D\right) = \varnothing\right] \\
&= \left(\#\text{ of } V\text{-listings } w \text{ that satisfy } \operatorname{Des}\left(w, D\right) = \varnothing\right) \\
&= \left(\#\text{ of hamps of } \overline{D}\right) \qquad \text{(by (40))}.
\end{aligned}$$

This proves Lemma 6.5. □

We can now state a formula for the # of hamps of $\overline{D}$ for any digraph D:

Theorem 6.6. Let $D = (V, A)$ be a digraph. Then:

(a) Set

$$\varphi\left(\sigma\right) := \sum_{\substack{\gamma \in \operatorname{Cycs}\sigma; \\ \gamma \text{ is a } D\text{-cycle}}} \left(\ell\left(\gamma\right) - 1\right) \qquad \text{for each } \sigma \in \mathfrak{S}_V.$$

Then,

$$\left(\#\text{ of hamps of } \overline{D}\right) = \sum_{\sigma \in \mathfrak{S}_V\left(D, \overline{D}\right)} \left(-1\right)^{\varphi(\sigma)}.$$

(b) We have $\left(\#\text{ of hamps of } \overline{D}\right) \equiv \left|\mathfrak{S}_V\left(D, \overline{D}\right)\right| \bmod 2$.

Proof. **(a)** Theorem 1.31 yields

$$U_D = \sum_{\sigma \in \mathfrak{S}_V\left(D, \overline{D}\right)} \left(-1\right)^{\varphi(\sigma)} p_{\operatorname{type}\sigma}.$$

Hence,

$$\begin{aligned}
\zeta\left(U_D\right) &= \zeta\left(\sum_{\sigma\in\mathfrak{S}_V\left(D,\overline{D}\right)} (-1)^{\varphi(\sigma)}\, p_{\operatorname{type}\sigma}\right) \\
&= \sum_{\sigma\in\mathfrak{S}_V\left(D,\overline{D}\right)} (-1)^{\varphi(\sigma)} \underbrace{\zeta\left(p_{\operatorname{type}\sigma}\right)}_{\substack{=1 \\ \text{(by Lemma 6.4,} \\ \text{applied to } \lambda=\operatorname{type}\sigma)}} \qquad \left(\text{since } \zeta \text{ is } \mathbb{Z}\text{-linear}\right) \\
&= \sum_{\sigma\in\mathfrak{S}_V\left(D,\overline{D}\right)} (-1)^{\varphi(\sigma)}.
\end{aligned}$$

However, Lemma 6.5 yields

$$\zeta\left(U_D\right) = \left(\text{\# of hamps of } \overline{D}\right).$$

Comparing these two equalities, we find

$$\left(\text{\# of hamps of } \overline{D}\right) = \sum_{\sigma\in\mathfrak{S}_V\left(D,\overline{D}\right)} (-1)^{\varphi(\sigma)}.$$

This proves Theorem 6.6 **(a)**.

(b) Theorem 6.6 **(a)** yields

$$\left(\text{\# of hamps of } \overline{D}\right) = \sum_{\sigma\in\mathfrak{S}_V\left(D,\overline{D}\right)} \underbrace{(-1)^{\varphi(\sigma)}}_{\substack{\equiv 1 \bmod 2 \\ \text{(since } (-1)^k \equiv 1 \bmod 2 \\ \text{for any } k\in\mathbb{Z})}} \equiv \sum_{\sigma\in\mathfrak{S}_V\left(D,\overline{D}\right)} 1 = \left|\mathfrak{S}_V\left(D,\overline{D}\right)\right| \bmod 2.$$

This proves Theorem 6.6 **(b)**. □

We are now ready to prove Rédei's and Berge's theorems:

Proof of Theorem 6.2. We have $\mathfrak{S}_V\left(\overline{D}, D\right) = \mathfrak{S}_V\left(D, \overline{D}\right)$ (since the digraphs D and $\overline{D}$ play symmetric roles in the definition of $\mathfrak{S}_V\left(D, \overline{D}\right)$). However, it is also easy to see (using the definition of the complement of a digraph) that $\overline{\overline{D}} = D$.

Theorem 6.6 **(b)** yields

$$\left(\text{\# of hamps of } \overline{D}\right) \equiv \left|\mathfrak{S}_V\left(D,\overline{D}\right)\right| \bmod 2. \tag{41}$$

However, Theorem 6.6 **(b)** (applied to $\overline{D}$ instead of D) yields

$$\left(\text{\# of hamps of } \overline{\overline{D}}\right) \equiv \left|\mathfrak{S}_V\left(\overline{D},\overline{\overline{D}}\right)\right| \bmod 2.$$

We can rewrite this as

$$\left(\text{\# of hamps of } D\right) \equiv \left|\mathfrak{S}_V\left(\overline{D},D\right)\right| \bmod 2$$

(since $\overline{\overline{D}} = D$). Hence,

$$\begin{aligned}
(\text{\# of hamps of } D) &\equiv \left|\mathfrak{S}_V\left(\overline{D}, D\right)\right| = \left|\mathfrak{S}_V\left(D, \overline{D}\right)\right| \qquad \left(\text{since } \mathfrak{S}_V\left(\overline{D}, D\right) = \mathfrak{S}_V\left(D, \overline{D}\right)\right) \\
&\equiv \left(\text{\# of hamps of } \overline{D}\right) \bmod 2 \qquad (\text{by (41)}).
\end{aligned}$$

This proves Theorem 6.2. ☐

Proof of Theorem 6.1. Write the tournament D as $D = (V, A)$. Set $n := |V|$.

For each $\sigma \in \mathfrak{S}_V$, let $\psi(\sigma)$ denote the number of nontrivial cycles of σ. Then, Theorem 1.39 yields

$$U_D = \sum_{\substack{\sigma \in \mathfrak{S}_V(D); \\ \text{all cycles of } \sigma \text{ have odd length}}} 2^{\psi(\sigma)} p_{\operatorname{type}\sigma}.$$

Hence,

$$\begin{aligned}
\zeta(U_D) &= \zeta\left(\sum_{\substack{\sigma \in \mathfrak{S}_V(D); \\ \text{all cycles of } \sigma \text{ have odd length}}} 2^{\psi(\sigma)} p_{\operatorname{type}\sigma}\right) \\
&= \sum_{\substack{\sigma \in \mathfrak{S}_V(D); \\ \text{all cycles of } \sigma \text{ have odd length}}} 2^{\psi(\sigma)} \underbrace{\zeta\left(p_{\operatorname{type}\sigma}\right)}_{\substack{=1 \\ \text{(by Lemma 6.4,} \\ \text{applied to } \lambda = \operatorname{type}\sigma)}} \qquad (\text{since } \zeta \text{ is } \mathbb{Z}\text{-linear}) \\
&= \sum_{\substack{\sigma \in \mathfrak{S}_V(D); \\ \text{all cycles of } \sigma \text{ have odd length}}} 2^{\psi(\sigma)} \qquad (42) \\
&= \underbrace{2^{\psi(\operatorname{id}_V)}}_{\substack{=1 \\ (\text{since } \psi(\operatorname{id}_V) = 0)}} + \sum_{\substack{\sigma \in \mathfrak{S}_V(D); \\ \text{all cycles of } \sigma \text{ have odd length;} \\ \sigma \neq \operatorname{id}_V}} \underbrace{2^{\psi(\sigma)}}_{\substack{\equiv 0 \bmod 2 \\ (\text{since } \psi(\sigma) \geq 1 \\ (\text{because } \sigma \neq \operatorname{id}_V \text{ shows} \\ \text{that } \sigma \text{ has at least} \\ \text{one nontrivial cycle}))}} \\
&\qquad \left(\begin{array}{c} \text{here, we have split off the addend for } \sigma = \operatorname{id}_V \\ \text{from the sum (since } \operatorname{id}_V \in \mathfrak{S}_V(D) \text{, and since} \\ \text{all cycles of } \operatorname{id}_V \text{ have odd length)} \end{array}\right) \\
&\equiv 1 + \underbrace{\sum_{\substack{\sigma \in \mathfrak{S}_V(D); \\ \text{all cycles of } \sigma \text{ have odd length;} \\ \sigma \neq \operatorname{id}_V}} 0}_{=0} = 1 \bmod 2.
\end{aligned}$$

In view of

$$\begin{aligned}
\zeta(U_D) &= \left(\text{\# of hamps of } \overline{D}\right) \qquad (\text{by Lemma 6.5}) \\
&\equiv (\text{\# of hamps of } D) \bmod 2 \qquad (\text{by Theorem 6.2}),
\end{aligned}$$

we can rewrite this as

$$(\text{\# of hamps of } D) \equiv 1 \bmod 2.$$

In other words, the # of hamps of D is odd. This proves Theorem 6.1. □

7. A modulo-4 improvement of Redei's theorem

We can extend Redei's theorem (Theorem 6.1) to a somewhat stronger result:

Theorem 7.1. Let D be a tournament. Then,

$$(\text{\# of hamps of } D) \equiv 1 + 2\,(\text{\# of nontrivial odd } D\text{-cycles}) \bmod 4.$$

Here:

- We agree to consider the empty list $()$ as a hamp of the empty tournament with 0 vertices (even though it is not a path).
- We say that a D-cycle is *odd* if its length is odd.
- We say that a D-cycle is *nontrivial* if its length is > 1. (This was already said in Definition 1.23 **(e)**.)

To prove this, we shall need a simple lemma:

Lemma 7.2. Let $D = (V, A)$ be a digraph. For each $\sigma \in \mathfrak{S}_V$, let $\psi(\sigma)$ denote the number of nontrivial cycles of σ. Let $\mathfrak{S}_V^{\text{odd}}(D)$ denote the set of all permutations $\sigma \in \mathfrak{S}_V(D)$ such that all cycles of σ have odd length. Then,

$$\begin{aligned}&\left(\text{\# of permutations } \sigma \in \mathfrak{S}_V^{\text{odd}}(D) \text{ satisfying } \psi(\sigma) = 1\right)\\ &= (\text{\# of nontrivial odd } D\text{-cycles}).\end{aligned}$$

(We are here using the same notations as in Theorem 7.1.)

Proof. If $\gamma = (a_1, a_2, \ldots, a_k)_\sim$ is any D-cycle (or, more generally, any cycle of the digraph $(V,\ V \times V)$), then $\operatorname{perm}\gamma$ shall denote the permutation of V that sends the elements $a_1, a_2, \ldots, a_{k-1}, a_k$ to $a_2, a_3, \ldots, a_k, a_1$ (respectively) while leaving all other elements of V unchanged. (This permutation $\operatorname{perm}\gamma$ is what is usually called "the cycle $(a_1, a_2, \ldots, a_k)$" in group theory.)

If γ is any nontrivial D-cycle, then the permutation $\operatorname{perm}\gamma$ belongs to $\mathfrak{S}_V(D)$ (since its only nontrivial cycle is γ, which is a D-cycle) and satisfies $\psi(\operatorname{perm}\gamma) = 1$ (by the definition of $\psi(\operatorname{perm}\gamma)$). Moreover, if γ is a nontrivial **odd** D-cycle, then this permutation $\operatorname{perm}\gamma$ furthermore has the property that all its cycles have odd length (since its only nontrivial cycle γ is odd, whereas its trivial cycles have length

1, which is also odd), i.e., belongs to $\mathfrak{S}_V^{\text{odd}}(D)$ (since we know that it belongs to $\mathfrak{S}_V(D)$). Thus, we obtain a map

$$\begin{aligned} &\text{from } \{\text{nontrivial odd } D\text{-cycles}\} \\ &\text{to } \left\{\text{permutations } \sigma \in \mathfrak{S}_V^{\text{odd}}(D) \text{ satisfying } \psi(\sigma) = 1\right\} \end{aligned}$$

which sends each nontrivial odd D-cycle γ to the permutation $\operatorname{perm}\gamma$. This map is furthermore injective (because any distinct nontrivial D-cycles γ and δ will always give rise to different permutations $\operatorname{perm}\gamma$ and $\operatorname{perm}\delta$) and surjective[18]. Thus, this map is bijective. Hence, the bijection principle yields

$$\begin{aligned} &(\text{\# of nontrivial odd } D\text{-cycles}) \\ &= \left(\text{\# of permutations } \sigma \in \mathfrak{S}_V^{\text{odd}}(D) \text{ satisfying } \psi(\sigma) = 1\right). \end{aligned}$$

This proves Lemma 7.2. □

We can now prove Theorem 7.1:

Proof of Theorem 7.1. We use the same notations as in Section 6. Write the tournament D as $D = (V, A)$.

For each $\sigma \in \mathfrak{S}_V$, let $\psi(\sigma)$ denote the number of nontrivial cycles of σ. Let $\mathfrak{S}_V^{\text{odd}}(D)$ denote the set of all permutations $\sigma \in \mathfrak{S}_V(D)$ such that all cycles of σ have odd length. Note that the identity permutation id_V belongs to $\mathfrak{S}_V^{\text{odd}}(D)$, since all its cycles are trivial.

[18]*Proof.* If $\sigma \in \mathfrak{S}_V(D)$ is a permutation satisfying $\psi(\sigma) = 1$, then $\sigma = \operatorname{perm}\gamma$ where γ is the unique nontrivial cycle of σ. Moreover, this cycle γ is a D-cycle (since $\sigma \in \mathfrak{S}_V(D)$). If we furthermore assume that $\sigma \in \mathfrak{S}_V^{\text{odd}}(D)$, then this cycle γ has odd length (since $\sigma \in \mathfrak{S}_V^{\text{odd}}(D)$ entails that all cycles of σ have odd length), i.e., is odd.

Then, from (42), we have

$$\begin{aligned}
\zeta\left(U_D\right) &= \sum_{\substack{\sigma\in\mathfrak{S}_V(D);\\ \text{all cycles of } \sigma \text{ have odd length}}} 2^{\psi(\sigma)} = \sum_{\sigma\in\mathfrak{S}_V^{\text{odd}}(D)} 2^{\psi(\sigma)} \\
&\qquad \left(\begin{array}{c}\text{since the permutations } \sigma\in\mathfrak{S}_V\left(D\right) \text{ such that all cycles} \\ \text{of } \sigma \text{ have odd length are precisely the elements of } \mathfrak{S}_V^{\text{odd}}\left(D\right)\end{array}\right) \\
&\equiv \sum_{\substack{\sigma\in\mathfrak{S}_V^{\text{odd}}(D);\\ \psi(\sigma)=0}} \underbrace{2^{\psi(\sigma)}}_{\substack{=1\\ (\text{since } \psi(\sigma)=0)}} + \sum_{\substack{\sigma\in\mathfrak{S}_V^{\text{odd}}(D);\\ \psi(\sigma)=1}} \underbrace{2^{\psi(\sigma)}}_{\substack{=2\\ (\text{since } \psi(\sigma)=1)}} + \sum_{\substack{\sigma\in\mathfrak{S}_V^{\text{odd}}(D);\\ \psi(\sigma)\geq 2}} \underbrace{2^{\psi(\sigma)}}_{\substack{\equiv 0 \bmod 4\\ (\text{since } \psi(\sigma)\geq 2)}} \\
&\qquad \left(\begin{array}{c}\text{here, we have split our sum according to} \\ \text{whether } \psi\left(\sigma\right) \text{ is 0 or 1 or } \geq 2\end{array}\right) \\
&\equiv \sum_{\substack{\sigma\in\mathfrak{S}_V^{\text{odd}}(D);\\ \psi(\sigma)=0}} 1 + \sum_{\substack{\sigma\in\mathfrak{S}_V^{\text{odd}}(D);\\ \psi(\sigma)=1}} 2 + \underbrace{\sum_{\substack{\sigma\in\mathfrak{S}_V^{\text{odd}}(D);\\ \psi(\sigma)\geq 2}} 0}_{=0} \\
&= \underbrace{\sum_{\substack{\sigma\in\mathfrak{S}_V^{\text{odd}}(D);\\ \psi(\sigma)=0}} 1}_{=\left(\#\text{ of permutations } \sigma\in\mathfrak{S}_V^{\text{odd}}(D) \text{ satisfying } \psi(\sigma)=0\right)\cdot 1} \\
&\qquad + \underbrace{\sum_{\substack{\sigma\in\mathfrak{S}_V^{\text{odd}}(D);\\ \psi(\sigma)=1}} 2}_{=\left(\#\text{ of permutations } \sigma\in\mathfrak{S}_V^{\text{odd}}(D) \text{ satisfying } \psi(\sigma)=1\right)\cdot 2} \\
&= \underbrace{\left(\#\text{ of permutations } \sigma\in\mathfrak{S}_V^{\text{odd}}\left(D\right) \text{ satisfying } \psi\left(\sigma\right)=0\right)}_{\substack{=1\\ (\text{since the only permutation } \sigma\in\mathfrak{S}_V^{\text{odd}}(D)\\ \text{satisfying } \psi(\sigma)=0 \text{ is the identity permutation})}} \cdot 1 \\
&\qquad + \underbrace{\left(\#\text{ of permutations } \sigma\in\mathfrak{S}_V^{\text{odd}}\left(D\right) \text{ satisfying } \psi\left(\sigma\right)=1\right)}_{\substack{=(\#\text{ of nontrivial odd } D\text{-cycles})\\ (\text{by Lemma 7.2})}} \cdot 2 \\
&= 1\cdot 1 + \left(\#\text{ of nontrivial odd } D\text{-cycles}\right)\cdot 2 \\
&= 1 + 2\left(\#\text{ of nontrivial odd } D\text{-cycles}\right) \bmod 4.
\end{aligned}$$

Comparing this with

$$\zeta\left(U_D\right) = \left(\#\text{ of hamps of } \overline{D}\right) \qquad (\text{by Lemma 6.5}),$$

we obtain

$$\begin{aligned}
&\left(\#\text{ of hamps of } \overline{D}\right) \\
&\equiv 1 + 2\left(\#\text{ of nontrivial odd } D\text{-cycles}\right) \bmod 4. \qquad (43)
\end{aligned}$$

However, recall that D is a tournament. Hence, the tournament axiom shows that a pair (u,v) of two distinct elements of V is an arc of D if and only if the pair (v,u) is not. In other words, a pair (u,v) of two distinct elements of V is an arc of D if and only if the pair (v,u) is an arc of $\overline{D}$. Thus, if $v=(v_1,v_2,\ldots,v_k)$ is a hamp of D, then its reversal $\operatorname{rev} v=(v_k,v_{k-1},\ldots,v_1)$ is a hamp of $\overline{D}$. Hence, we obtain a map

$$\{\text{hamps of } D\} \to \{\text{hamps of } \overline{D}\},$$
$$v \mapsto \operatorname{rev} v.$$

This map is furthermore easily seen to be injective and surjective. Hence, it is bijective. Thus, the bijection principle yields

$$\begin{aligned}(\#\text{ of hamps of } D) &= \left(\#\text{ of hamps of } \overline{D}\right)\\ &\equiv 1+2\,(\#\text{ of nontrivial odd } D\text{-cycles}) \bmod 4\end{aligned}$$

(by (43)). This proves Theorem 7.1. □

8. The antipode and the omega involution

Next, we will discuss how the Redei-Berge symmetric functions U_D interplay with two well-known involutions on the ring Λ: the omega involution ω and the antipode map S.

We shall not recall the standard definitions of these involutions ω and S (see, e.g., [GriRei20, §2.4]); however, we shall briefly state the few properties that will be used in what follows. Both the *omega involution* ω and the *antipode* S of Λ are endomorphisms of the $\mathbb{Z}$-algebra Λ; they satisfy the equalities

$$S(p_n) = -p_n \tag{44}$$

and

$$\omega(p_n) = (-1)^{n-1} p_n \tag{45}$$

for every positive integer n (see [GriRei20, Proposition 2.4.1 (i)] and [GriRei20, Proposition 2.4.3 (c)]). Moreover, if $f \in \Lambda$ is a homogeneous power series of degree n, then

$$S(f) = (-1)^n \omega(f) \tag{46}$$

(this is [GriRei20, Proposition 2.4.3 (e)]). We now claim the following theorem:

Theorem 8.1. Let $D=(V,A)$ be a digraph. Then,

$$\omega(U_D) = U_{\overline{D}}. \tag{47}$$

Furthermore, if $n := |V|$, then

$$S(U_D) = (-1)^n U_{\overline{D}}. \tag{48}$$

Proof. The definition of $\overline{D}$ yields that $\overline{\overline{D}} = D$. Hence, the definition of $\mathfrak{S}_V\left(D, \overline{D}\right)$ yields that $\mathfrak{S}_V\left(\overline{D}, D\right) = \mathfrak{S}_V\left(D, \overline{D}\right)$.

For each $\sigma \in \mathfrak{S}_V$, we set

$$\varphi\left(\sigma\right) := \sum_{\substack{\gamma\in\operatorname{Cycs}\sigma;\\ \gamma \text{ is a } D\text{-cycle}}} \left(\ell\left(\gamma\right)-1\right) \qquad \text{and} \qquad \overline{\varphi}\left(\sigma\right) := \sum_{\substack{\gamma\in\operatorname{Cycs}\sigma;\\ \gamma \text{ is a } \overline{D}\text{-cycle}}} \left(\ell\left(\gamma\right)-1\right).$$

Now, it is easy to see that

$$\omega\left(\left(-1\right)^{\varphi(\sigma)} p_{\operatorname{type}\sigma}\right) = \left(-1\right)^{\overline{\varphi}(\sigma)} p_{\operatorname{type}\sigma} \tag{49}$$

for each $\sigma \in \mathfrak{S}_V\left(D, \overline{D}\right)$.

[*Proof of (49):* Let $\sigma \in \mathfrak{S}_V\left(D, \overline{D}\right)$. Let $k_1, k_2, \ldots, k_s$ be the lengths of all cycles of σ, listed in decreasing order. Then, the definition of $\operatorname{type}\sigma$ yields $\operatorname{type}\sigma = \left(k_1, k_2, \ldots, k_s\right)$. Hence,

$$p_{\operatorname{type}\sigma} = p_{(k_1,k_2,\ldots,k_s)} = p_{k_1}p_{k_2}\cdots p_{k_s} = \prod_{\gamma\in\operatorname{Cycs}\sigma} p_{\ell(\gamma)} \tag{50}$$

(since $k_1, k_2, \ldots, k_s$ are the lengths of all cycles of σ). Hence,

$$\begin{aligned}
\omega\left(\left(-1\right)^{\varphi(\sigma)} p_{\operatorname{type}\sigma}\right) &= \omega\left(\left(-1\right)^{\varphi(\sigma)} \prod_{\gamma\in\operatorname{Cycs}\sigma} p_{\ell(\gamma)}\right) \\
&= \left(-1\right)^{\varphi(\sigma)} \prod_{\gamma\in\operatorname{Cycs}\sigma} \underbrace{\omega\left(p_{\ell(\gamma)}\right)}_{\substack{=(-1)^{\ell(\gamma)-1}p_{\ell(\gamma)}\\ \text{(by (45))}}} \qquad \left(\begin{array}{c}\text{since } \omega \text{ is a } \mathbb{Z}\text{-algebra}\\ \text{homomorphism}\end{array}\right) \\
&= \left(-1\right)^{\varphi(\sigma)} \prod_{\gamma\in\operatorname{Cycs}\sigma} \left(\left(-1\right)^{\ell(\gamma)-1} p_{\ell(\gamma)}\right) \\
&= \left(-1\right)^{\varphi(\sigma)} \underbrace{\left(\prod_{\gamma\in\operatorname{Cycs}\sigma} \left(-1\right)^{\ell(\gamma)-1}\right)}_{=(-1)^{\sum_{\gamma\in\operatorname{Cycs}\sigma}(\ell(\gamma)-1)}} \underbrace{\prod_{\gamma\in\operatorname{Cycs}\sigma} p_{\ell(\gamma)}}_{\substack{=p_{\operatorname{type}\sigma}\\ \text{(by (50))}}} \\
&= \left(-1\right)^{\varphi(\sigma)} \left(-1\right)^{\sum_{\gamma\in\operatorname{Cycs}\sigma}(\ell(\gamma)-1)} p_{\operatorname{type}\sigma}.
\end{aligned} \tag{51}$$

However, each $\gamma \in \operatorname{Cycs}\sigma$ is either a D-cycle or a $\overline{D}$-cycle (since $\sigma \in \mathfrak{S}_V\left(D, \overline{D}\right)$), but cannot be both at the same time (since D and $\overline{D}$ have no arcs in common). Thus,

$$\begin{aligned}
\sum_{\gamma\in\operatorname{Cycs}\sigma} \left(\ell\left(\gamma\right)-1\right) &= \underbrace{\sum_{\substack{\gamma\in\operatorname{Cycs}\sigma;\\ \gamma \text{ is a } D\text{-cycle}}} \left(\ell\left(\gamma\right)-1\right)}_{\substack{=\varphi(\sigma)\\ \text{(by the definition of } \varphi(\sigma))}} + \underbrace{\sum_{\substack{\gamma\in\operatorname{Cycs}\sigma;\\ \gamma \text{ is a } \overline{D}\text{-cycle}}} \left(\ell\left(\gamma\right)-1\right)}_{\substack{=\overline{\varphi}(\sigma)\\ \text{(by the definition of } \overline{\varphi}(\sigma))}} \\
&= \varphi\left(\sigma\right) + \overline{\varphi}\left(\sigma\right).
\end{aligned}$$

Thus, (51) rewrites as

$$\omega\left((-1)^{\varphi(\sigma)}\,p_{\operatorname{type}\sigma}\right)=\underbrace{(-1)^{\varphi(\sigma)}\,(-1)^{\varphi(\sigma)+\overline{\varphi}(\sigma)}}_{=(-1)^{\overline{\varphi}(\sigma)}}p_{\operatorname{type}\sigma}=(-1)^{\overline{\varphi}(\sigma)}\,p_{\operatorname{type}\sigma}.$$

This proves (49).]

Now, Theorem 1.31 yields

$$U_D=\sum_{\sigma\in\mathfrak{S}_V\left(D,\overline{D}\right)}(-1)^{\varphi(\sigma)}\,p_{\operatorname{type}\sigma}. \tag{52}$$

Also, Theorem 1.31 (applied to $\overline{D}$, $(V\times V)\setminus A$ and $\overline{\varphi}$ instead of D, A and φ) yields

$$\begin{aligned}
U_{\overline{D}}&=\sum_{\sigma\in\mathfrak{S}_V\left(\overline{D},D\right)}(-1)^{\overline{\varphi}(\sigma)}\,p_{\operatorname{type}\sigma}\\
&=\sum_{\sigma\in\mathfrak{S}_V\left(D,\overline{D}\right)}\underbrace{(-1)^{\overline{\varphi}(\sigma)}\,p_{\operatorname{type}\sigma}}_{\substack{=\omega\left((-1)^{\varphi(\sigma)}p_{\operatorname{type}\sigma}\right)\\ \text{(by (49))}}} &&\left(\text{since }\mathfrak{S}_V\left(\overline{D},D\right)=\mathfrak{S}_V\left(D,\overline{D}\right)\right)\\
&=\sum_{\sigma\in\mathfrak{S}_V\left(D,\overline{D}\right)}\omega\left((-1)^{\varphi(\sigma)}\,p_{\operatorname{type}\sigma}\right)\\
&=\omega\left(\underbrace{\sum_{\sigma\in\mathfrak{S}_V\left(D,\overline{D}\right)}(-1)^{\varphi(\sigma)}\,p_{\operatorname{type}\sigma}}_{\substack{=U_D\\ \text{(by (52))}}}\right) &&(\text{since }\omega\text{ is }\mathbb{Z}\text{-linear})\\
&=\omega\left(U_D\right).
\end{aligned}$$

This proves (47).

Now, let $n:=|V|$. Then, the definition of U_D easily yields that U_D is homogeneous of degree n. Hence, (46) (applied to $f=U_D$) yields

$$S\left(U_D\right)=(-1)^n\,\omega\left(U_D\right)=(-1)^n\,U_{\overline{D}}\qquad\text{(by (47))}.$$

Thus, (48) is proved. This completes the proof of Theorem 8.1. □

Theorem 8.1 can also be proved directly from the definition of U_D, using the formula for the antipode of a fundamental quasisymmetric function ([GriRei20, (5.2.7)]). Indeed, three different proofs of Theorem 8.1 (specifically, of (47)) are found in [Chow96] (where (47) appears as [Chow96, Corollary 2]), one of which is doing just this. A fourth proof can be found in [Wisema07, (6)].

We can use Theorem 8.1 to give a new proof of Berge's theorem (Theorem 6.2). For this purpose, we recall the $\mathbb{Z}$-algebra homomorphism ζ introduced in Section 6. We need another simple property of this ζ:

Lemma 8.2. Let $f \in \mathbb{Z}[p_1, p_2, p_3, \ldots]$. Then, $\zeta(\omega(f)) \equiv \zeta(f) \bmod 2$.

Proof. Let $\pi : \mathbb{Z} \to \mathbb{Z}/2$ be the projection map that sends each integer to its congruence class modulo 2. This π is a $\mathbb{Z}$-algebra homomorphism.

For each positive integer n, we have

$$\begin{aligned} \zeta(\omega(p_n)) &= \zeta\left((-1)^{n-1} p_n\right) && \text{(by (45))} \\ &= \underbrace{(-1)^{n-1}}_{\equiv 1 \bmod 2} \zeta(p_n) && \text{(since } \zeta \text{ is } \mathbb{Z}\text{-linear)} \\ &\equiv \zeta(p_n) \bmod 2 \end{aligned}$$

and thus

$$\pi(\zeta(\omega(p_n))) = \pi(\zeta(p_n))$$

(since two integers a and b satisfy $a \equiv b \bmod 2$ if and only if $\pi(a) = \pi(b)$). In other words, for each positive integer n, we have

$$(\pi \circ \zeta \circ \omega)(p_n) = (\pi \circ \zeta)(p_n).$$

In other words, the two maps $\pi \circ \zeta \circ \omega$ and $\pi \circ \zeta$ agree on each of the generators $p_1, p_2, p_3, \ldots$ of the $\mathbb{Z}$-algebra $\mathbb{Z}[p_1, p_2, p_3, \ldots]$. Since these two maps are $\mathbb{Z}$-algebra homomorphisms (because π, ζ and ω are $\mathbb{Z}$-algebra homomorphisms), this shows that these two maps agree on the entire $\mathbb{Z}$-algebra $\mathbb{Z}[p_1, p_2, p_3, \ldots]$. Hence, $(\pi \circ \zeta \circ \omega)(f) = (\pi \circ \zeta)(f)$. In other words, $\pi(\zeta(\omega(f))) = \pi(\zeta(f))$. In other words, $\zeta(\omega(f)) \equiv \zeta(f) \bmod 2$ (since two integers a and b satisfy $a \equiv b \bmod 2$ if and only if $\zeta(a) = \zeta(b)$). This proves Lemma 8.2. □

Second proof of Theorem 6.2. From (47), we obtain $\omega(U_D) = U_{\overline{D}}$.

Corollary 1.35 yields $U_D \in \mathbb{Z}[p_1, p_2, p_3, \ldots]$. Hence, Lemma 8.2 (applied to $f = U_D$) yields that

$$\zeta(\omega(U_D)) \equiv \zeta(U_D) \bmod 2.$$

In view of

$$\zeta(U_D) = \left(\text{\# of hamps of } \overline{D}\right) \qquad \text{(by Lemma 6.5)}$$

and

$$\begin{aligned} \zeta\left(\underbrace{\omega(U_D)}_{=U_{\overline{D}}}\right) &= \zeta\left(U_{\overline{D}}\right) = \left(\text{\# of hamps of } \overline{\overline{D}}\right) && \left(\begin{array}{c}\text{by Lemma 6.5,} \\ \text{applied to } \overline{D} \text{ instead of } D\end{array}\right) \\ &= (\text{\# of hamps of } D) && \left(\text{since } \overline{\overline{D}} = D\right), \end{aligned}$$

we can rewrite this as

$$(\text{\# of hamps of } D) \equiv \left(\text{\# of hamps of } \overline{D}\right) \bmod 2.$$

This proves Theorem 6.2 again. □

9. A multiparameter deformation

Let us now briefly discuss a multiparameter deformation of the Redei-Berge symmetric functions U_D, which replaces the digraph D by an arbitrary matrix.

We fix a commutative ring $\mathbf{k}$, which we shall now be using instead of $\mathbb{Z}$ as a base ring for our power series.

We fix an $n \in \mathbb{N}$, and a set V with n elements.

For any $a \in V \times V$, we fix an element $t_a \in \mathbf{k}$. (Thus, the family $\left(t_{(i,j)}\right)_{i,j \in V}$ of these elements can be viewed as a $V \times V$-matrix.)

For any $a \in V \times V$, we set $s_a := t_a + 1 \in \mathbf{k}$.

The following definition is inspired by a comment from Mike Zabrocki:

Definition 9.1. We define the *deformed Redei–Berge symmetric function* $\widetilde{U}_t$ to be the formal power series

$$\widetilde{U}_t = \sum_{\substack{w=(w_1,w_2,\ldots,w_n) \\ \text{is a } V\text{-listing}}} \ \sum_{i_1 \leq i_2 \leq \cdots \leq i_n} \left(\prod_{\substack{k \in [n-1]; \\ i_k = i_{k+1}}} s_{(w_k, w_{k+1})} \right) x_{i_1} x_{i_2} \cdots x_{i_n}$$
$$\in \mathbf{k}\left[\left[x_1, x_2, x_3, \ldots\right]\right].$$

For example, if $n = 2$ and $V = \{1, 2\}$, then

$$\begin{aligned}
\widetilde{U}_t &= \sum_{i_1 < i_2} x_{i_1} x_{i_2} + \sum_{i_1 = i_2} t_{(1,2)} x_{i_1} x_{i_2} + \sum_{i_1 < i_2} x_{i_1} x_{i_2} + \sum_{i_1 = i_2} t_{(2,1)} x_{i_1} x_{i_2} \\
&= \sum_{i<j} x_i x_j + \sum_i t_{(1,2)} x_i^2 + \sum_{i<j} x_i x_j + \sum_i t_{(2,1)} x_i^2 \\
&= p_1^2 + \left(s_{(1,2)} + s_{(2,1)} - 1\right) p_2 \\
&= p_1^2 + \left(t_{(1,2)} + t_{(2,1)} + 1\right) p_2
\end{aligned}$$

For a more complicated example, if $n = 3$ and $V = \{1, 2, 3\}$, then a longer computation shows that

$$\begin{aligned}\widetilde{U}_t = p_1^3 &+ \left(s_{(1,2)} + s_{(2,1)} + s_{(1,3)} + s_{(3,1)} + s_{(2,3)} + s_{(3,2)} - 3\right) p_2 p_1 \\ &+ \Big(s_{(1,2)}s_{(2,3)} + s_{(2,3)}s_{(3,1)} + s_{(3,1)}s_{(1,2)} \\ &\qquad + s_{(1,3)}s_{(3,2)} + s_{(3,2)}s_{(2,1)} + s_{(2,1)}t_{(1,3)} \\ &\qquad - s_{(1,2)} - s_{(2,1)} - s_{(1,3)} - s_{(3,1)} - s_{(2,3)} - s_{(3,2)} + 2\Big) p_3 \\ = p_1^3 &+ \left(t_{(1,2)} + t_{(2,1)} + t_{(1,3)} + t_{(3,1)} + t_{(2,3)} + t_{(3,2)} + 3\right) p_2 p_1 \\ &+ \Big(t_{(1,2)}t_{(2,3)} + t_{(2,3)}t_{(3,1)} + t_{(3,1)}t_{(1,2)} \\ &\qquad + t_{(1,3)}t_{(3,2)} + t_{(3,2)}t_{(2,1)} + t_{(2,1)}t_{(1,3)} \\ &\qquad + t_{(1,2)} + t_{(2,1)} + t_{(1,3)} + t_{(3,1)} + t_{(2,3)} + t_{(3,2)} + 2\Big) p_3.\end{aligned}$$

Why are we calling $\widetilde{U}_t$ a deformation of U_D ?

Example 9.2. Let $D = (V, A)$ be a digraph. Set $\mathbf{k} = \mathbb{Z}$, and let

$$t_a := \begin{cases} -1, & \text{if } a \in A; \\ 0, & \text{if } a \notin A \end{cases} \qquad \text{for each } a \in V \times V.$$

Then, $\widetilde{U}_t = U_D$, as can be seen by comparing the definitions.

All the above results leading up to Theorem 1.31 can be extended to this deformation, culminating in the following deformation of Theorem 1.31:

Theorem 9.3. We have

$$\widetilde{U}_t = \sum_{\sigma \in \mathfrak{S}_V} \left(\prod_{\gamma \text{ is a cycle of } \sigma} \left(\prod_{a \in \operatorname{CArcs} \gamma} s_a - \prod_{a \in \operatorname{CArcs} \gamma} t_a \right) \right) p_{\operatorname{type} \sigma}.$$

Alternatively, Theorem 9.3 can be deduced from Theorem 1.31 via the "multilinearity trick": View each t_a as an indeterminate, and argue that both sides in Theorem 9.3 are polynomials in degree ≤ 1 in these indeterminates (over the base ring $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$). Thus, in order to prove their equality, it suffices to prove that they are equal when each t_a is specialized to either 0 or -1. But this is precisely the claim of Theorem 1.31. (Thus, Theorem 9.3 is not essentially more general than Theorem 1.31.)

Theorem 9.3 shows that the $\widetilde{U}_t$ are p-integral symmetric functions (taking the t_a as "integers"). There do not seem to be any good opportunities for generalizing any of Theorem 1.39 and Theorem 1.41, however.

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