The Redei–Berge symmetric function of a directed graph

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detailed version

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Abstract. Let $D = (V, A)$ be a digraph with n vertices, where each arc $a \in A$ is a pair (u, v) of two vertices. We study the *Redei–Berge symmetric function* U_D, defined as the quasisymmetric function

$$\sum L_{\operatorname{Des}(w,D),\ n} \in \operatorname{QSym}.$$

Here, the sum ranges over all lists $w = (w_1, w_2, \ldots, w_n)$ that contain each vertex of D exactly once, and the corresponding addend is

$$L_{\operatorname{Des}(w,D),\ n} := \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_n; \\ i_p < i_{p+1} \text{ for each } p \text{ satisfying } (w_p, w_{p+1}) \in A}} x_{i_1} x_{i_2} \cdots x_{i_n}$$

(an instance of Gessel's fundamental quasisymmetric functions).

While U_D is a specialization of Chow's path-cycle symmetric function, which has been studied before, we prove some new formulas that express U_D in terms of the power-sum symmetric functions. We show that U_D is always p-integral, and furthermore is p-positive whenever D has no 2-cycles. When D is a tournament, U_D can be written as a polynomial in $p_1, 2p_3, 2p_5, 2p_7, \ldots$ with nonnegative integer coefficients. By specializing these results, we obtain the famous theorems of Redei and Berge on the number of Hamiltonian paths in digraphs and tournaments, as well as a modulo-4 refinement of Redei's theorem.

Keywords: directed graph, symmetric function, tournament, Hamiltonian path, power sum symmetric function.

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Contents

1. Definitions and the main theorems

We begin with introducing the notations, some of which come from [EC2sup22, Problem 120]. We use standard notations as defined, e.g., in [Stanle01, Chapter 7] and [GriRei20, Chapters 2 and 5].

1.1. Digraphs, V-listings and D-descents

We let $\mathbb{N} := \{0, 1, 2, \ldots\}$ and $\mathbb{P} := \{1, 2, 3, \ldots\}$. We set $[n] := \{1, 2, \ldots, n\}$ for each $n \in \mathbb{Z}$. (This set $[n]$ is empty if $n \leq 0$.)

The words "list" and "tuple" will be used interchangeably, and will always mean finite ordered tuples.

We shall next introduce some basic notations regarding digraphs (i.e., directed graphs):

Definition 1.1. A *digraph* means a pair (V, A), where V is a finite set and where A is a subset of $V \times V$. The elements of V are called the *vertices* of this digraph, and the elements of A are called the *arcs* of this digraph. For any further notations, we refer to standard literature (the definitions in [Grinbe17, §1.1-§1.2] should suffice) and common sense. (Our digraphs are allowed to have loops, but this has no effect on what follows.)

Definition 1.2. Let $D = (V, A)$ be a digraph. Then, the digraph $(V, \ (V \times V) \setminus A)$ will be denoted by $\overline{D}$ and called the *complement* of the digraph D. Its arcs will be called the *non-arcs* of D (since they are precisely the pairs $(u, v) \in V \times V$ that are not arcs of D).

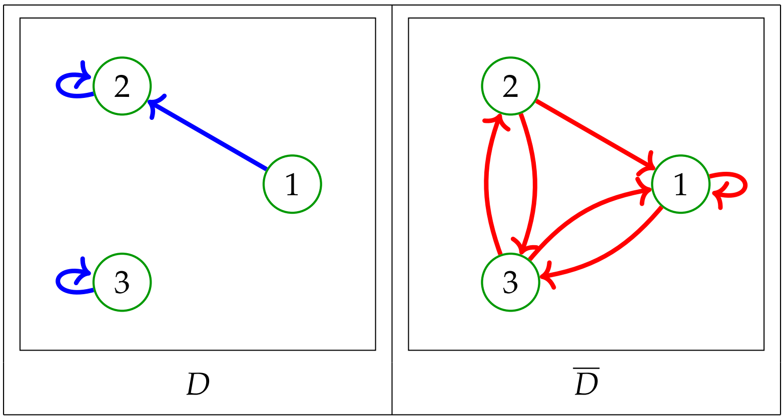
Example 1.3. If D is the digraph

$$(\{1, 2, 3\}, \ \{(1, 2), \ (2, 2), \ (3, 3)\}),$$

then its complement $\overline{D}$ is the digraph

$$(\{1, 2, 3\}, \ \{(1, 1), \ (1, 3), \ (2, 1), \ (2, 3), \ (3, 1), \ (3, 2)\}).$$

Here are the two digraphs, drawn side by side:



D $\overline{D}$

.

Definition 1.4. Let V be a finite set. A *V-listing* will mean a list of elements of V that contains each element of V exactly once.

For example, $(2,1,3)$ is a $\{1,2,3\}$-listing.

Of course, if V is a finite set, then there are exactly $|V|!$ many V-listings. They are in a canonical bijection with the bijective maps from $[|V|]$ to V, and in a non-canonical bijection with the permutations of V.

Convention 1.5. If w is any list (i.e., tuple), and if i is a positive integer, then w_i shall denote the i-th entry of w. (Thus, $w = (w_1, w_2, \ldots, w_k)$, where k is the length of w.)

Definition 1.6. Let $D = (V, A)$ be a digraph. Let $w = (w_1, w_2, \ldots, w_n)$ be a V-listing. Then:

(a) A *D-descent* of w means an $i \in [n-1]$ satisfying $(w_i, w_{i+1}) \in A$.

(b) We let $\operatorname{Des}(w, D)$ denote the set of all D-descents of w.

Example 1.7. Let D be the digraph D from Example 1.3, and let w be the V-listing $(3,1,2)$. Then, 2 is a D-descent of w (since $(w_2, w_3) = (1,2) \in A$), but 1 is not a D-descent of w (since $(w_1, w_2) = (3,1) \notin A$). Hence, $\operatorname{Des}(w, D) = \{2\}$.

Example 1.8. Let $n \in \mathbb{N}$, and let $V = [n]$. Let D be the digraph whose vertices are the elements of V and whose arcs are all the pairs $(i, j) \in [n]^2$ satisfying $i > j$. Let w be a V-listing. Then, the D-descents of w are exactly the descents of w in the usual sense (i.e., the numbers $i \in [n-1]$ satisfying $w_i > w_{i+1}$).

We note that D-descents for general digraphs D have already implicitly appeared in the work of Foata and Zeilberger [FoaZei96], which considers the number $\operatorname{maj}'_D w := \sum\limits_{i \in \operatorname{Des}(w,D)} i$ for each V-listing w. We would not be surprised if what follows can shed some new light on the results of [FoaZei96], but so far we have not found any deeper connections.

1.2. Quasisymmetric functions

Next, we introduce some notations from the theory of quasisymmetric functions (see, e.g., [Stanle01, §7.19] or [GriRei20, Chapter 5]):

Definition 1.9.

(a) A *composition* means a finite list of positive integers. If $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ is a composition, then the number k is called the *length* of α, whereas the number $\alpha_1 + \alpha_2 + \cdots + \alpha_k$ is called the *size* of α. If $n \in \mathbb{N}$, then a *composition of n* shall mean a composition having size n.

(b) A *partition* (or *integer partition*) means a composition that is weakly decreasing.

For example, $(2,5,3)$ is a composition of 10 that has length 3 and is not a partition (since $2 < 5$).

Definition 1.10. Let $n \in \mathbb{N}$. For any subset I of $[n-1]$, we let $\operatorname{comp}(I,n)$ be the list

$$(i_1 - i_0,\ i_2 - i_1,\ i_3 - i_2,\ \ldots,\ i_k - i_{k-1}),$$

where $i_0, i_1, \ldots, i_k$ are the elements of $\{0\} \cup I \cup \{n\}$ listed in strictly increasing order. This list $\operatorname{comp}(I,n)$ is a composition of n.

Example 1.11. If $n = 6$ and $I = \{2,3,5\}$, then $\operatorname{comp}(I,n) = (2,1,2,1)$.

Note that $\operatorname{comp}(I,n)$ is denoted by $\operatorname{co}(I)$ in [Stanle01, §7.19], but we prefer to make the dependence on n explicit here. In the notation of [GriRei20, Definition 5.1.10], the composition $\operatorname{comp}(I,n)$ is the preimage of I under the bijection $D : \operatorname{Comp}_n \to 2^{[n-1]}$.

For any $n \in \mathbb{N}$, the map

$$\begin{aligned} \{\text{subsets of } [n-1]\} &\to \{\text{compositions of } n\}, \\ I &\mapsto \operatorname{comp}(I,n) \end{aligned}$$

is a bijection.

Definition 1.12. Consider the ring $\mathbb{Z}[[x_1, x_2, x_3, \ldots]]$ of formal power series in countably many indeterminates $x_1, x_2, x_3, \ldots$. Two subrings of this ring $\mathbb{Z}[[x_1, x_2, x_3, \ldots]]$ are:

- the ring Λ of *symmetric functions* (defined, e.g., in [Stanle01, §7.1] or in [GriRei20, §2.1]);
- the ring QSym of *quasisymmetric functions* (defined, e.g., in [Stanle01, §7.19] or in [GriRei20, §5.1]).

We will not actually use any properties of these rings Λ and QSym anywhere except in Sections 8, 6 and 7 (and even there, only Λ will be used); thus, a reader unfamiliar with symmetric functions can read $\mathbb{Z}[[x_1, x_2, x_3, \ldots]]$ instead of Λ or QSym everywhere else.

Definition 1.13. Let α be a composition. Then, L_α will denote the *fundamental quasisymmetric function* corresponding to α. This is a formal power series in QSym, and is defined as follows: Let I be the unique subset of $[n-1]$ satisfying $\alpha = \operatorname{comp}(I, n)$. Then, we set

$$L_\alpha = \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_n; \\ i_p < i_{p+1} \text{ for each } p \in I}} x_{i_1} x_{i_2} \cdots x_{i_n} \in \mathrm{QSym}$$

(where the summation indices $i_1, i_2, \ldots, i_n$ range over $\mathbb{P}$).

See [Stanle01, §7.19] or [GriRei20, §5] for more about these fundamental quasisymmetric functions L_α (originally introduced by Ira Gessel)[1]. We will actually find it easier to index them not by the compositions α but rather by the corresponding subsets I of $[n-1]$. Thus, we define:

Definition 1.14. Let $n \in \mathbb{N}$, and let I be a subset of $[n-1]$. Then, we will use the notation $L_{I,n}$ for $L_{\operatorname{comp}(I,n)}$. Explicitly, we have

$$L_{I,n} = \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_n; \\ i_p < i_{p+1} \text{ for each } p \in I}} x_{i_1} x_{i_2} \cdots x_{i_n} \in \mathrm{QSym} \tag{1}$$

(where the summation indices $i_1, i_2, \ldots, i_n$ range over $\mathbb{P}$).

Example 1.15. If $n = 3$ and $I = \{2\}$, then

$$\begin{aligned} L_{I,n} = L_{\{2\},3} &= \sum_{\substack{i_1 \leq i_2 \leq i_3; \\ i_p < i_{p+1} \text{ for each } p \in \{2\}}} x_{i_1} x_{i_2} x_{i_3} = \sum_{i_1 \leq i_2 < i_3} x_{i_1} x_{i_2} x_{i_3} \\ &= x_1 x_1 x_2 + x_1 x_1 x_3 + \cdots + x_1 x_2 x_3 + x_1 x_2 x_4 + \cdots + \cdots + x_2 x_2 x_3 + \cdots . \end{aligned}$$

1.3. The Redei–Berge symmetric function

We are now ready to define the main protagonist of this paper:

Definition 1.16. Let $n \in \mathbb{N}$. Let $D = (V, A)$ be a digraph with n vertices. We define the *Redei–Berge symmetric function* U_D to be the quasisymmetric function

$$\sum_{w \text{ is a } V\text{-listing}} L_{\operatorname{Des}(w,D),\ n} \in \mathrm{QSym}.$$

[1] Note that the definition of L_α given in [GriRei20, Definition 5.2.4] differs from ours. However, it is equivalent to ours, since [GriRei20, Proposition 5.2.9] shows that the L_α defined in [GriRei20, Definition 5.2.4] satisfy the same formula that we used to define our L_α.

Example 1.17. Let D be the digraph D from Example 1.3. Then,

$$\begin{aligned}
U_D &= \sum_{w \text{ is a } V\text{-listing}} L_{\operatorname{Des}(w,D),\ 3} \\
&= L_{\operatorname{Des}((1,2,3),D),\ 3} + L_{\operatorname{Des}((1,3,2),D),\ 3} + L_{\operatorname{Des}((2,1,3),D),\ 3} \\
&\qquad + L_{\operatorname{Des}((2,3,1),D),\ 3} + L_{\operatorname{Des}((3,1,2),D),\ 3} + L_{\operatorname{Des}((3,2,1),D),\ 3} \\
&= L_{\{1\},\ 3} + L_{\varnothing,\ 3} + L_{\varnothing,\ 3} + L_{\varnothing,\ 3} + L_{\{2\},\ 3} + L_{\varnothing,\ 3} \\
&= 4 \cdot \underbrace{L_{\varnothing,\ 3}}_{= \sum_{i_1 \leq i_2 \leq i_3} x_{i_1} x_{i_2} x_{i_3}} + \underbrace{L_{\{1\},\ 3}}_{= \sum_{i_1 < i_2 \leq i_3} x_{i_1} x_{i_2} x_{i_3}} + \underbrace{L_{\{2\},\ 3}}_{= \sum_{i_1 \leq i_2 < i_3} x_{i_1} x_{i_2} x_{i_3}} \\
&= 4 \cdot \sum_{i_1 \leq i_2 \leq i_3} x_{i_1} x_{i_2} x_{i_3} + \sum_{i_1 < i_2 \leq i_3} x_{i_1} x_{i_2} x_{i_3} + \sum_{i_1 \leq i_2 < i_3} x_{i_1} x_{i_2} x_{i_3}.
\end{aligned}$$

From this expression, we can easily see that U_D is actually a symmetric function, and can be written (e.g.) as $p_1^3 + 2p_1p_2 + p_3$, where $p_k := x_1^k + x_2^k + x_3^k + \cdots$ is the k-th power-sum symmetric function for each $k \geq 1$.

The name "Redei–Berge symmetric function" for the power series U_D was chosen because (as we will soon see) it is actually a symmetric function and is related to two classical results of Redei and Berge on the number of Hamiltonian paths in digraphs. In [EC2sup22, Problem 120], it is called U_X, where X is what we call A (that is, the set of arcs of D); but we shall here put the entire digraph D into the subscript.

The Redei–Berge symmetric function U_D equals the quasisymmetric function $\Xi_{\overline{D}}(x, 0)$ from Chow's [Chow96].[2] It also is denoted by $\Pi_{\overline{D}}$ in [Wisema07].[3] Several properties of U_D have been shown in [Chow96] and in [Wisema07], and some of them will be reproved here for the sake of self-containedness and variety. However, our main results – Theorems 1.31, 1.39 and 1.41 – appear to be new.

Question 1.18. Can these results be extended to the more general functions $\Xi_D(x, y)$ from [Chow96]?

1.4. Arcs and cyclic arcs

The main results of this paper are explicit (albeit not, in general, subtraction-free) expansions of U_D in terms of the power-sum symmetric functions. To state these, we need some more notations. We shall soon define cycles of digraphs and cycles of permutations, and we will then connect the two notions. First, some auxiliary notations:

[2] Indeed, this equality follows immediately from [Chow96, Proposition 7], since the quasisymmetric function we call $L_{I,n}$ appears under the name of $Q_{I,n}$ in [Chow96], and since our $\operatorname{Des}(w, \overline{D})$ is what is called $S(w)$ in [Chow96].

[3] Indeed, comparing the definition of Π_D in [Wisema07, Definition 2.2] with the definition of Ξ_D in [Chow96, §2] shows that $\Pi_D = \Xi_D(x, 0)$. Thus, $\Pi_{\overline{D}} = \Xi_{\overline{D}}(x, 0) = U_D$ (as we already know).

Definition 1.19. Let V be a set. Let $v = (v_1, v_2, \ldots, v_k) \in V^k$ be a nonempty tuple of elements of V.

(a) We define a subset $\operatorname{Arcs} v$ of $V \times V$ by

$$\begin{aligned} \operatorname{Arcs} v := & \ \{(v_i, v_{i+1}) \mid i \in [k-1]\} \\ = & \ \{(v_1, v_2), (v_2, v_3), \ldots, (v_{k-1}, v_k)\} \\ \subseteq & \ V \times V. \end{aligned} \tag{2}$$

This subset $\operatorname{Arcs} v$ is called the *arc set* of the tuple v. Its elements (v_i, v_{i+1}) are called the *arcs* of v.

(b) We define a subset $\operatorname{CArcs} v$ of $V \times V$ by

$$\begin{aligned} \operatorname{CArcs} v := & \ \{(v_i, v_{i+1}) \mid i \in [k]\} \\ = & \ \{(v_1, v_2), (v_2, v_3), \ldots, (v_{k-1}, v_k), (v_k, v_1)\} \\ \subseteq & \ V \times V, \end{aligned} \tag{3}$$

where we set $v_{k+1} := v_1$. This subset $\operatorname{CArcs} v$ is called the *cyclic arc set* of the tuple v. Its elements (v_i, v_{i+1}) are called the *cyclic arcs* of v.

(c) The *reversal* of v is defined to be the tuple $(v_k, v_{k-1}, \ldots, v_1) \in V^k$.

Example 1.20. Let $V = \mathbb{N}$ and $v = (1, 4, 2, 6) \in V^4$. Then,

$$\begin{aligned} \operatorname{Arcs} v &= \{(1,4), (4,2), (2,6)\} \qquad \text{and} \\ \operatorname{CArcs} v &= \{(1,4), (4,2), (2,6), (6,1)\}. \end{aligned}$$

Note that if we cyclically rotate a nonempty tuple $v \in V^k$, then the set $\operatorname{CArcs} v$ remains unchanged: i.e., for any $(v_1, v_2, \ldots, v_k) \in V^k$, we have

$$\operatorname{CArcs}(v_1, v_2, \ldots, v_k) = \operatorname{CArcs}(v_2, v_3, \ldots, v_k, v_1).$$

1.5. Permutations and their cycles

Now, let us discuss permutations and their cycles. We start with some basic notations:

Definition 1.21. Let V be a finite set. Then, $\mathfrak{S}_V$ shall denote the symmetric group of V (that is, the group of all permutations of V).

Note that the order of this group is $|\mathfrak{S}_V| = |V|!$.

Definition 1.22. Let V be a set.

(a) Two tuples $v \in V^k$ and $w \in V^\ell$ of elements of V are said to be *rotation-equivalent* if w can be obtained from v by cyclic rotation, i.e., if $\ell = k$ and $w = (v_i, v_{i+1}, \ldots, v_k, v_1, v_2, \ldots, v_{i-1})$ for some $i \in [k]$.

(b) The relation "rotation-equivalent" is an equivalence relation on the set of all nonempty tuples of elements of V. Its equivalence classes are called the *rotation-equivalence classes*. In other words, the rotation-equivalence classes are the orbits of the operation

$$(a_1, a_2, \ldots, a_k) \mapsto (a_2, a_3, \ldots, a_k, a_1)$$

on the set of all nonempty tuples of elements of V.

(c) The rotation-equivalence class that contains a given nonempty tuple $v \in V^k$ will be denoted by $v_\sim$.

For instance, the tuple $(1,2,3,4)$ is rotation-equivalent to $(3,4,1,2)$, but not to $(4,3,2,1)$. Thus,

$$(1,2,3,4)_\sim = (3,4,1,2)_\sim \neq (4,3,2,1)_\sim .$$

Also,

$$(1,3,6)_\sim = \{(1,3,6)\,,\ (3,6,1)\,,\ (6,1,3)\}\,.$$

Definition 1.23. Let V be a set. Let γ be a rotation-equivalence class (of nonempty tuples of elements of V). Then:

(a) All tuples $v \in \gamma$ have the same length (i.e., number of entries). This length is denoted by $\ell(\gamma)$, and is called the *length* of γ. Thus, if $\gamma = v_\sim$ for some tuple $v \in V^k$, then $\ell(\gamma) = k$.

(b) All tuples $v \in \gamma$ have the same cyclic arc set $\operatorname{CArcs} v$ (since $\operatorname{CArcs} v$ remains unchanged if we cyclically rotate v). This cyclic arc set is denoted by $\operatorname{CArcs} \gamma$, and is called the *cyclic arc set* of γ. Thus, the cyclic arc set of a rotation-equivalence class $\gamma = (v_1, v_2, \ldots, v_k)_\sim$ is

$$\operatorname{CArcs} \gamma = \{(v_1, v_2)\,,\ (v_2, v_3)\,,\ \ldots,\ (v_{k-1}, v_k)\,,\ (v_k, v_1)\}\,.$$

(c) All tuples $v \in \gamma$ have the same entries (up to order). These entries are called the *entries* of γ. Thus, the entries of a rotation-equivalence class $\gamma = (v_1, v_2, \ldots, v_k)_\sim$ are $v_1, v_2, \ldots, v_k$.

(d) The reversals of all tuples $v \in \gamma$ are the elements of a single rotation-equivalence class $\operatorname{rev} \gamma$. This latter class will be called the *reversal* of γ. Thus, the reversal of a rotation-equivalence class $\gamma = (v_1, v_2, \ldots, v_k)_\sim$ is the rotation-equivalence class $(v_k, v_{k-1}, \ldots, v_1)_\sim$.

(e) We say that γ is *nontrivial* if $\ell(\gamma) > 1$.

For instance, the rotation-equivalence class $(3,1,4)_\sim$ has length 3, cyclic arc set $\{(3,1), (1,4), (4,3)\}$, and entries $3,1,4$. Its reversal is $(4,1,3)_\sim$, and it is nontrivial (since $\ell((3,1,4)_\sim) = 3 > 1$).

Definition 1.24. Let V be a finite set. Let $\sigma \in \mathfrak{S}_V$ be any permutation.

(a) The *cycles* of σ are the rotation-equivalence classes of the tuples of the form

$$\left(\sigma^0(i), \sigma^1(i), \ldots, \sigma^{k-1}(i)\right),$$

where i is some element of V, and where k is the smallest positive integer satisfying $\sigma^k(i) = i$.

For example, the permutation $w_0 \in \mathfrak{S}_{[7]}$ that sends each $i \in [7]$ to $8-i$ has cycles $(1,7)_\sim$, $(2,6)_\sim$, $(3,5)_\sim$ and $(4)_\sim$. (Note that we do allow a cycle to have length 1.)

(b) The *cycle type* of σ means the partition whose entries are the lengths of the cycles of σ. We denote this cycle type by $\operatorname{type}\sigma$. It is a partition of the number $|V|$.

(c) We let $\operatorname{Cycs}\sigma$ denote the set of all cycles of σ.

Example 1.25. Let $w_0 \in \mathfrak{S}_{[7]}$ be the permutation that sends each $i \in [7]$ to $8-i$. We have already seen that w_0 has cycles $(1,7)_\sim$, $(2,6)_\sim$, $(3,5)_\sim$ and $(4)_\sim$. Their respective lengths are $2,2,2,1$. Thus, the cycle type of w_0 is $\operatorname{type} w_0 = (2,2,2,1)$. We have $\operatorname{Cycs}\sigma = \{(1,7)_\sim, (2,6)_\sim, (3,5)_\sim, (4)_\sim\}$. The first three of the four cycles $(1,7)_\sim$, $(2,6)_\sim$, $(3,5)_\sim$ and $(4)_\sim$ are nontrivial.

1.6. D-paths and D-cycles

Next, we define paths and cycles in a digraph:

Definition 1.26. Let $D = (V, A)$ be a digraph.

(a) A *D-path* (or *path of D*) shall mean a nonempty tuple v of distinct elements of V such that $\operatorname{Arcs} v \subseteq A$.

(b) A *D-cycle* (or *cycle of D*) shall mean a rotation-equivalence class γ of nonempty tuples of distinct elements of V such that $\operatorname{CArcs}\gamma \subseteq A$.

We note that our notion of "cycle of D" differs slightly from the common one used in graph theory[4].

Example 1.27. Let D be the digraph D from Example 1.3. Then:

(a) The pair $(1,2)$ as well as the three 1-tuples (1), (2) and (3) are D-paths. The triple $(1,2,2)$ is not a D-path (even though it satisfies the "$\operatorname{Arcs} v \subseteq A$" condition), since its entries $1,2,2$ are not distinct. The triple $(1,2,3)$ is not a D-path, since $(2,3)$ is not an arc of D.

The triple $(2,3,1)$ is a $\overline{D}$-path (and there are several others).

(b) The only D-cycles are the rotation-equivalence classes $(2)_\sim$ and $(3)_\sim$. The $\overline{D}$-cycles are $(1)_\sim$, $(1,3)_\sim$, $(2,3)_\sim$ and $(2,1,3)_\sim$.

1.7. The sets $\mathfrak{S}_V(D)$ and $\mathfrak{S}_V\left(D,\overline{D}\right)$

Now, we can connect digraphs with permutations by comparing their cycles:

Definition 1.28. Let $D = (V, A)$ be a digraph. Then, we define

$$\mathfrak{S}_V(D) := \{\sigma \in \mathfrak{S}_V \mid \text{each nontrivial cycle of } \sigma \text{ is a } D\text{-cycle}\}$$

and

$$\mathfrak{S}_V\left(D,\overline{D}\right) := \left\{\sigma \in \mathfrak{S}_V \mid \text{each cycle of } \sigma \text{ is a } D\text{-cycle or a } \overline{D}\text{-cycle}\right\}.$$

Note that we could just as well replace "each cycle" by "each nontrivial cycle" in the definition of $\mathfrak{S}_V\left(D,\overline{D}\right)$, since a cycle of length 1 is always a D-cycle or a $\overline{D}$-cycle (depending on whether its only cyclic arc belongs to A or not). However, we could not replace "nontrivial cycle" by "cycle" in the definition of $\mathfrak{S}_V(D)$.

Example 1.29. Let D be the digraph D from Example 1.3. Let $V = \{1,2,3\}$ be its set of vertices. Then:

(a) We have $\mathfrak{S}_V(D) = \{\mathrm{id}_V\}$, since the only D-cycles have length 1.

(b) We have

$$\mathfrak{S}_V\left(D,\overline{D}\right) = \left\{\mathrm{id}_V,\ \mathrm{cyc}_{1,3},\ \mathrm{cyc}_{2,3},\ \mathrm{cyc}_{1,3,2}\right\},$$

where $\mathrm{cyc}_{i_1,i_2,\ldots,i_k}$ denotes the permutation that cyclically permutes the elements $i_1, i_2, \ldots, i_k$ while leaving all other elements of V unchanged.

[4] Namely, cycles in graph theory have their first vertex repeated at the end, whereas our cycles don't. However, this difference is purely notational: A cycle $(v_1, v_2, \ldots, v_k)_\sim$ in our sense corresponds to the cycle $(v_1, v_2, \ldots, v_k, v_1)$ in the graph-theorists' terminology.

1.8. Formulas for U_D

1.8.1. The power-sum symmetric functions

We now introduce some of the best-known (and easiest to define) symmetric functions:

Definition 1.30.

(a) For each positive integer n, we define the *power-sum symmetric function*

$$p_n := x_1^n + x_2^n + x_3^n + \cdots \in \Lambda.$$

(b) If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is a partition with k positive entries, then we set

$$p_\lambda := p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_k} \in \Lambda.$$

For instance, $p_{(2,2,1)} = p_2 p_2 p_1 = \left(x_1^2 + x_2^2 + x_3^2 + \cdots\right)^2 (x_1 + x_2 + x_3 + \cdots)$.

1.8.2. The first main theorem: general digraphs

We now state our first main theorem (which will be proved in Section 2):

Theorem 1.31. Let $D = (V, A)$ be a digraph. Set

$$\varphi(\sigma) := \sum_{\substack{\gamma \in \operatorname{Cycs} \sigma; \\ \gamma \text{ is a } D\text{-cycle}}} (\ell(\gamma) - 1) \qquad \text{for each } \sigma \in \mathfrak{S}_V.$$

Then,

$$U_D = \sum_{\sigma \in \mathfrak{S}_V\left(D, \overline{D}\right)} (-1)^{\varphi(\sigma)} p_{\operatorname{type} \sigma}.$$

Example 1.32. Let $V = \{1,2,3,4,5,6\}$ and $D = (V,\ V \times V)$. Let $\sigma \in \mathfrak{S}_V$ be the permutation whose cycles are $(1,3)_\sim$, $(2,4,5)_\sim$ and $(6)_\sim$. Then, every cycle of σ is a D-cycle, and the number $\varphi(\sigma)$ (as defined in Theorem 1.31) is

$$\begin{aligned} &(\ell((1,3)_\sim) - 1) + (\ell((2,4,5)_\sim) - 1) + (\ell((6)_\sim) - 1) \\ &= (2-1) + (3-1) + (1-1) = 3. \end{aligned}$$

Example 1.33. Let D be the digraph D from Example 1.3. Recall that $\mathfrak{S}_V\left(D,\overline{D}\right) = \left\{\operatorname{id}_V,\ \operatorname{cyc}_{1,3},\ \operatorname{cyc}_{2,3},\ \operatorname{cyc}_{1,3,2}\right\}$. Thus, Theorem 1.31 yields

$$\begin{aligned} U_D &= \underbrace{(-1)^{\varphi(\operatorname{id}_V)}}_{=(-1)^0=1} \underbrace{p_{\operatorname{type}(\operatorname{id}_V)}}_{=p_{(1,1,1)}=p_1^3} + \underbrace{(-1)^{\varphi\left(\operatorname{cyc}_{1,3}\right)}}_{=(-1)^0=1} \underbrace{p_{\operatorname{type}\left(\operatorname{cyc}_{1,3}\right)}}_{=p_{(2,1)}=p_2p_1} \\ &\qquad + \underbrace{(-1)^{\varphi\left(\operatorname{cyc}_{2,3}\right)}}_{=(-1)^0=1} \underbrace{p_{\operatorname{type}\left(\operatorname{cyc}_{2,3}\right)}}_{=p_{(2,1)}=p_2p_1} + \underbrace{(-1)^{\varphi\left(\operatorname{cyc}_{1,3,2}\right)}}_{=(-1)^0=1} \underbrace{p_{\operatorname{type}\left(\operatorname{cyc}_{1,3,2}\right)}}_{=p_{(3)}=p_3} \\ &= p_1^3 + p_2p_1 + p_2p_1 + p_3 = p_1^3 + 2p_1p_2 + p_3. \end{aligned}$$

This agrees with the result found in Example 1.17.

Example 1.34. Let D be the digraph (V, A), where $V = \{1, 2, 3\}$ and

$$A = \{(1,3),\ (2,1),\ (3,1),\ (3,2)\}.$$

Then, a straightforward computation using Theorem 1.31 shows that $U_D = p_1^3 - p_1p_2 + p_3$. (This example is due to Ira Gessel.)

The following two corollaries can be easily obtained from Theorem 1.31 (see Section 4 for their proofs):

Corollary 1.35. Let $D = (V, A)$ be a digraph. Then, U_D is a p-integral symmetric function (i.e., a symmetric function that can be written as a polynomial in $p_1, p_2, p_3, \ldots$). That is, we have $U_D \in \mathbb{Z}[p_1, p_2, p_3, \ldots]$.

Corollary 1.36. Let $D = (V, A)$ be a digraph. Assume that every D-cycle has odd length. Then,

$$U_D = \sum_{\sigma \in \mathfrak{S}_V\left(D,\overline{D}\right)} p_{\operatorname{type}\sigma} \in \mathbb{N}[p_1, p_2, p_3, \ldots].$$

1.8.3. The second main theorem: tournaments

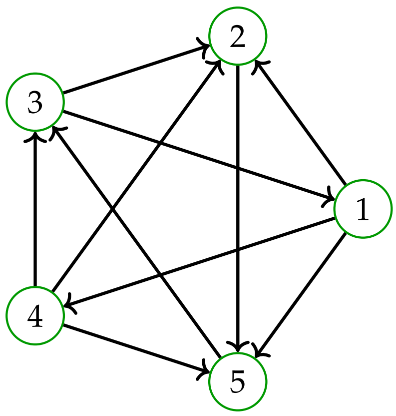
After we will have proved Theorem 1.31, we will use it to derive a simpler formula, which however is specific to tournaments. First, we recall the definition of a tournament:

Definition 1.37. A *tournament* means a digraph $D = (V, A)$ that satisfies the following two axioms:

- *Looplessness*: We have $(u, u) \notin A$ for any $u \in V$.

- *Tournament axiom*: For any two distinct vertices u and v of D, **exactly** one of the two pairs (u, v) and (v, u) is an arc of D.

Example 1.38. Neither the digraph D from Example 1.3, nor its complement $\overline{D}$, is a tournament. Here is a tournament:



.

We can now state our second main theorem (which we will prove in Section 3):

Theorem 1.39. Let $D = (V, A)$ be a tournament. For each $\sigma \in \mathfrak{S}_V$, let $\psi(\sigma)$ denote the number of nontrivial cycles of σ. Then,

$$U_D = \sum_{\substack{\sigma \in \mathfrak{S}_V(D); \\ \text{all cycles of } \sigma \text{ have odd length}}} 2^{\psi(\sigma)} p_{\operatorname{type} \sigma}.$$

Once this is proved, the following corollary will be easy to derive (see Section 4 for the details):

Corollary 1.40. Let $D = (V, A)$ be a tournament. Then,

$$U_D \in \mathbb{N}[p_1, 2p_3, 2p_5, 2p_7, \ldots] = \mathbb{N}[p_1,\ 2p_i \ \mid \ i > 1 \text{ is odd}].$$

(Here, $\mathbb{N}[p_1, 2p_3, 2p_5, 2p_7, \ldots]$ means the set of all values of the form $f(p_1, 2p_3, 2p_5, 2p_7, \ldots)$, where f is a polynomial in countably many indeterminates with coefficients in $\mathbb{N}$.)

1.8.4. The third main theorem: digraphs with no 2-cycles

A more general version of Theorem 1.39 is the following:

Theorem 1.41. Let $D = (V, A)$ be a digraph. Assume that there exist no two distinct vertices u and v of D such that both pairs (u, v) and (v, u) belong to A.

(a) Then, U_D is a p-positive symmetric function (i.e., a symmetric function that can be written as a polynomial in $p_1, p_2, p_3, \ldots$ with coefficients in $\mathbb{N}$). That is, we have $U_D \in \mathbb{N}[p_1, p_2, p_3, \ldots]$.

(b) A rotation-equivalence class γ of nonempty tuples of elements of V will be called *risky* if its length is even and it has the property that either γ or the reversal of γ is a D-cycle. Then,

$$U_D = \sum_{\substack{\sigma \in \mathfrak{S}_V(D, \overline{D});\\ \text{no cycle of } \sigma \text{ is risky}}} p_{\operatorname{type} \sigma}.$$

We will prove this in Section 5. Note that Theorem 1.41 **(a)** generalizes [Chow96, Theorem 7].[5]

Remark 1.42. The converse of Theorem 1.41 **(a)** does not hold. Indeed, consider the digraph $D = (V, A)$ with $V = \{1, 2, 3, 4\}$ and

$$A = \{(1, 2), (2, 1), (2, 3), (2, 4), (3, 4)\}.$$

Then, D does not satisfy the assumption of Theorem 1.41 (since the two distinct vertices 1 and 2 satisfy both $(1, 2) \in A$ and $(2, 1) \in A$), but the corresponding symmetric function U_D is p-positive (indeed, $U_D = p_1^4 + p_2 p_1^2 + p_3 p_1$). It would be interesting to know some more precise criteria for the p-positivity of U_D.

The next sections are devoted to the proofs of the above results. Afterwards, we will proceed with further properties of the Redei-Berge symmetric functions U_D (Section 8), applications to reproving Redei's and Berge's theorems (Section 6) and a (not very substantial) generalization (Section 9).

Remark on alternative versions

You are reading the detailed version of this paper. For the standard version (which is shorter by virtue of omitting some straightforward proofs and some details), see

[5] To see how, one needs to observe that

1. any acyclic digraph D satisfies the assumption of Theorem 1.41;
2. the $\omega_x \Xi_D$ from [Chow96] equals our U_D in the case when D is acyclic.

The first of these two observations is obvious. The second follows from the equality (84) further below, combined with the fact that $\Xi_D = \Xi_D(x, 0)$ when D acyclic (since the y-variables do not actually appear in Ξ_D for lack of cycles), and the fact that $U_{\overline{D}} = \Xi_D(x, 0)$ (stated above in the equivalent form $U_D = \Xi_{\overline{D}}(x, 0)$).

[GriSta23].

2. Proof of Theorem 1.31

In the following, we will outline the proof of Theorem 1.31. We hope that the proof can still be simplified further.

2.1. Basic conventions

The following two conventions are popular in enumerative combinatorics, and we too will use them on occasion:

Convention 2.1. The symbol # shall mean "number". For instance, $(\text{\# of subsets of } \{1,2,3\}) = 8$.

Convention 2.2. We shall use the *Iverson bracket notation*: For any logical statement $\mathcal{A}$, we let $[\mathcal{A}]$ denote the truth value of $\mathcal{A}$. This is the number $\begin{cases} 1, & \text{if } \mathcal{A} \text{ is true;} \\ 0, & \text{if } \mathcal{A} \text{ is false.} \end{cases}$

Our proof of Theorem 1.31 will rely on many lemmas. The first is a well-known cancellation lemma (see, e.g., [Grinbe21, Proposition 7.8.10]):

Lemma 2.3. Let B be a finite set. Then, $\sum\limits_{F \subseteq B} (-1)^{|F|} = [B = \varnothing]$.

2.2. Path covers and linear sets

We begin with some more notations:

Definition 2.4. Let V be a finite set.

(a) A *path* of V means a nonempty tuple of distinct elements of V.

(b) An element v is said to *belong* to a given tuple t if v is an entry of t.

(c) A *path cover* of V means a set of paths of V such that each $v \in V$ belongs to exactly one of these paths.

For example, $\{(1,4,3), (2,8), (5), (7,6)\}$ is a path cover of $[8]$. We stress once again the words "exactly one" in the definition of a path cover. Thus, the paths constituting a path cover are disjoint (i.e., have no entries in common). For instance, $\{(1,2), (2,3)\}$ is **not** a path cover of $[3]$.

In Definition 1.19 **(a)**, we have introduced the arc set of a path of V (and, more generally, of any nonempty tuple of elements of V). We now extend this to path covers in the obvious way:

Definition 2.5. Let V be a finite set.

(a) If C is a path cover of V, then the *arc set* of C is defined to be the subset

$$\bigcup_{v \in C} \operatorname{Arcs} v \qquad \text{of } V \times V.$$

This arc set will be denoted by $\operatorname{Arcs} C$.

(b) A subset F of $V \times V$ will be called *linear* if it is the arc set of some path cover of V.

For example, the path cover $\{(1,4,3),(2,8),(5),(7,6)\}$ of $[8]$ has arc set

$$\begin{aligned}
&\operatorname{Arcs}\{(1,4,3),(2,8),(5),(7,6)\} \\
&= \operatorname{Arcs}(1,4,3) \cup \operatorname{Arcs}(2,8) \cup \operatorname{Arcs}(5) \cup \operatorname{Arcs}(7,6) \\
&= \{(1,4),\ (4,3)\} \cup \{(2,8)\} \cup \varnothing \cup \{(7,6)\} \\
&= \{(1,4),\ (4,3),\ (2,8),\ (7,6)\}.
\end{aligned}$$

Thus, the latter set is linear (as a subset of $[8] \times [8]$).

Note that the notion of "path of V" depends on V alone, not on any digraph structure on V. Thus, if V is the vertex set of a digraph $D = (V, A)$, then a path of V is not the same as a D-path; in fact, the D-paths are precisely the paths v of V that satisfy $\operatorname{Arcs} v \subseteq A$.

We shall now see a few properties and characterizations of linear subsets of $V \times V$. Here is a first one, which will not be used in what follows but might help in visualizing the concept:

Proposition 2.6. Let V be a finite set. Let F be a subset of $V \times V$. Then, F is linear if and only if the digraph (V, F) has no cycles and no vertices with outdegree > 1 and no vertices with indegree > 1.

We omit the proof of this proposition, since we shall have no use for it.

The following is also easy to see:

Proposition 2.7. Let V be a finite set. Let F be a linear subset of $V \times V$. Then, any subset of F is linear as well.

Proof. It suffices to show that removing a single element e from a linear subset F of $V \times V$ yields a linear subset. But this is easy:

Let F be a linear subset of $V \times V$, and let e be an element of F. We must prove that $F \setminus \{e\}$ is again a linear subset.

The set F is linear, i.e., is the arc set of some path cover of V (by the definition of "linear"). In other words, we have $F = \operatorname{Arcs} C$ for some path cover C of V. Consider this path cover C.

We have $e \in F = \operatorname{Arcs} C$. Thus, e is an arc of some path $p \in C$. Consider this p.

If we remove an arc f from a path, then the path breaks up into two smaller paths (the "part before f" and the "part after f") [6]. Thus, if we remove the arc e from the path p, then this path p breaks up into the "part before e" and the "part after e". Let us denote these two parts by p' and p''. Let C' be the path cover of V obtained from C by breaking up the path p into its two parts p' and p'' (that is, let $C' := (C \setminus \{p\}) \cup \{p', p''\}$). Then, $\operatorname{Arcs}(C') = \underbrace{(\operatorname{Arcs} C)}_{=F} \setminus \{e\} = F \setminus \{e\}$. This shows that $F \setminus \{e\}$ is the arc set of a path cover of V (namely, of C'). In other words, $F \setminus \{e\}$ is linear. As explained above, this completes our proof of Proposition 2.7. $\square$

This quickly leads to the following alternative characterization of linear subsets:

Proposition 2.8. Let V be a finite set. Let F be a subset of $V \times V$. Then:

(a) If the subset F is not linear, then there exists no V-listing v satisfying $F \subseteq \operatorname{Arcs} v$.

(b) If $F = \operatorname{Arcs} C$ for some path cover C of V, then there are exactly $|C|!$ many V-listings v satisfying $F \subseteq \operatorname{Arcs} v$. (Note that $|C|$ is the number of paths in C.)

(c) The subset F is linear if and only if it is a subset of $\operatorname{Arcs} v$ for some V-listing v.

Proof. **(a)** We shall prove the contrapositive: i.e., that if there exists a V-listing v satisfying $F \subseteq \operatorname{Arcs} v$, then F is linear.

Indeed, assume that there exists a V-listing v satisfying $F \subseteq \operatorname{Arcs} v$. Consider this v. Then, v is a path of V that contains all elements of V. Hence, the 1-element set $\{v\}$ is a path cover of V. Its arc set $\operatorname{Arcs}\{v\}$ is therefore linear (by the definition of "linear"). In other words, the set $\operatorname{Arcs} v$ is linear (since $\operatorname{Arcs}\{v\} = \operatorname{Arcs} v$). Hence, Proposition 2.7 (applied to $\operatorname{Arcs} v$ instead of F) shows that any subset of $\operatorname{Arcs} v$ is linear as well. Thus, F is linear (since F is a subset of $\operatorname{Arcs} v$). This completes the proof of Proposition 2.8 **(a)**.

(b) Assume that $F = \operatorname{Arcs} C$ for some path cover C of V. Consider this C.

Then, each V-listing v satisfying $F \subseteq \operatorname{Arcs} v$ can be obtained by concatenating the

[6] To be specific: If the path is $(v_1, v_2, \ldots, v_k)$, and if the arc f is (v_i, v_{i+1}), then the resulting two smaller paths are $(v_1, v_2, \ldots, v_i)$ and $(v_{i+1}, v_{i+2}, \ldots, v_k)$.

paths in C in some order[7] (and conversely, each such concatenation is a V-listing v satisfying $F \subseteq \operatorname{Arcs} v$). There are clearly $|C|!$ many orders in which the paths in C can be concatenated, and they all lead to different concatenations (since the paths in C are disjoint and nonempty[8]), i.e., to different V-listings v. Hence, there are

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[7]*Proof.* This might appear intuitively clear, but let us give a proof nevertheless.

Let v be a V-listing satisfying $F \subseteq \operatorname{Arcs} v$. We must prove that v can be obtained by concatenating the paths in C in some order.

Let $p_1, p_2, \ldots, p_k$ be the paths in C (listed with no repetitions). Thus, we must prove that v is a concatenation of $p_1, p_2, \ldots, p_k$ in some order.

Note that the paths $p_1, p_2, \ldots, p_k$ are the distinct paths of the path cover C. Thus, they are disjoint, i.e., have no entries in common. Moreover, every element of V belongs to one of the paths $p_1, p_2, \ldots, p_k$ (since C is a path cover of V).

Note that v is a V-listing. Hence, each element of V appears exactly once in v. In particular, no entry appears more than once in v.

Fix $i \in [k]$. Write the path p_i as $p_i = (w_1, w_2, \ldots, w_\ell)$.

Let $j \in [\ell - 1]$. Then, in particular, w_j appears exactly once in v (since each element of V appears exactly once in v). Moreover, we have

$$\begin{aligned} \left(w_j, w_{j+1}\right) \in \operatorname{Arcs}\left(p_i\right) \subseteq \operatorname{Arcs} C \qquad & \left(\text{since } p_i \in C\right) \\ = F \subseteq \operatorname{Arcs} v. & \end{aligned}$$

Thus, the tuple v must have the form $\left(\ldots, w_j, w_{j+1}, \ldots\right)$ (where each "$\ldots$" stands for an arbitrary number of entries). In other words, w_{j+1} must be the next entry after w_j in the V-listing v (since w_j appears exactly once in v).

Forget that we fixed j. We thus have shown that for each $j \in [\ell - 1]$, the element w_{j+1} must be the next entry after w_j in the V-listing v. In other words, the entries $w_1, w_2, \ldots, w_\ell$ must appear in v in this order and as a contiguous block. In other words, the tuple $(w_1, w_2, \ldots, w_\ell)$ is a factor of the tuple v (where a "*factor*" of a tuple $\left(a_1, a_2, \ldots, a_g\right)$ means a contiguous block $\left(a_j, a_{j+1}, \ldots, a_{j+s}\right)$ of this tuple). In other words, the path p_i is a factor of the tuple v (since $p_i = (w_1, w_2, \ldots, w_\ell)$).

Forget that we fixed i. We thus have shown that for each $i \in [k]$, the path p_i is a factor of v. In other words, all k paths $p_1, p_2, \ldots, p_k$ are factors of v. These factors are nonempty (since paths are nonempty by definition) and do not overlap (since the paths $p_1, p_2, \ldots, p_k$ have no entries in common), and therefore appear in v in a well-defined order. In other words, there exists a permutation σ of $[k]$ such that the paths $p_{\sigma(1)}, p_{\sigma(2)}, \ldots, p_{\sigma(k)}$ appear as factors of v in this order (i.e., the factor $p_{\sigma(1)}$ appears before $p_{\sigma(2)}$, which in turn appears before $p_{\sigma(3)}$, and so on). Consider this σ. Note that every element of V belongs to one of the paths $p_{\sigma(1)}, p_{\sigma(2)}, \ldots, p_{\sigma(k)}$ (since every element of V belongs to one of the paths $p_1, p_2, \ldots, p_k$).

We claim that these k factors $p_{\sigma(1)}, p_{\sigma(2)}, \ldots, p_{\sigma(k)}$ cover the entire tuple v (that is, every entry of v belongs to one of these factors). In fact, if this was not the case, then some entry of v would lie outside all of these k factors $p_{\sigma(1)}, p_{\sigma(2)}, \ldots, p_{\sigma(k)}$; but then this same entry would also appear again inside one of these k factors (since every element of V belongs to one of the paths $p_{\sigma(1)}, p_{\sigma(2)}, \ldots, p_{\sigma(k)}$), and therefore would appear in v twice (once outside the k factors, and once again inside one of them), which would contradict the fact that no entry appears more than once in v. Hence, the k factors $p_{\sigma(1)}, p_{\sigma(2)}, \ldots, p_{\sigma(k)}$ cover the entire tuple v. Since these k factors do not overlap (because they are just a permutation of the k factors $p_1, p_2, \ldots, p_k$, which do not overlap), and since they appear in v in this order, we thus conclude that v is the concatenation of $p_{\sigma(1)}, p_{\sigma(2)}, \ldots, p_{\sigma(k)}$ in this order. Therefore, v is a concatenation of $p_1, p_2, \ldots, p_k$ in some order. This completes our proof.

[8]Here is this argument in more detail: If you concatenate the paths in C in some order, then the order in which you concatenate them can be reconstructed from the resulting concatenation,

exactly $|C|!$ many V-listings v satisfying $F \subseteq \operatorname{Arcs} v$. This proves Proposition 2.8 **(b)**.

(c) $\Longrightarrow$: Assume that F is linear. Thus, F is the arc set of a path cover of V. In other words, $F = \operatorname{Arcs} C$ for some path cover C of V. Consider this C. Proposition 2.8 **(b)** yields that there are exactly $|C|!$ many V-listings v satisfying $F \subseteq \operatorname{Arcs} v$. Hence, there is at least one such V-listing v (since $|C|! \geq 1$). Thus, F is a subset of $\operatorname{Arcs} v$ for some V-listing v (namely, the V-listing we just mentioned). This proves the "$\Longrightarrow$" direction of Proposition 2.8 **(c)**.

$\Longleftarrow$: Assume that F is a subset of $\operatorname{Arcs} v$ for some V-listing v. In other words, there exists a V-listing v satisfying $F \subseteq \operatorname{Arcs} v$. However, if the set F was not linear, then Proposition 2.8 **(a)** would yield that there exists no such V-listing; this would contradict the preceding sentence. Hence, the set F must be linear. This proves the "$\Longleftarrow$" direction of Proposition 2.8 **(c)**. $\square$

Next, let us address a technical issue. We defined the notion of a "linear subset of $V \times V$" using path covers of V. When we say that a certain set is "linear", we are thus tacitly assuming that it is clear what the relevant set V is. This may cause an ambiguity: Sometimes, two different sets V_1 and V_2 can reasonably qualify as V, and we may have a subset F of $V_1 \times V_1$ that is also a subset of $V_2 \times V_2$. In that case, when we say that F is "linear", do we mean that F is linear as a subset of $V_1 \times V_1$ or as a subset of $V_2 \times V_2$? Fortunately, this does not matter (at least when V_1 is a subset of V_2), as the following proposition shows:

Proposition 2.9. Let V be a finite set. Let W be a subset of V. Let F be a subset of $W \times W$. Then, F is linear as a subset of $W \times W$ if and only if F is linear as a subset of $V \times V$.

Proof. A path $(v_1, v_2, \ldots, v_k)$ of V will be called *trivial* if $k = 1$, and *nontrivial* otherwise. Clearly, if v is a trivial path, then $\operatorname{Arcs} v = \varnothing$. On the other hand, if v is a nontrivial path, then $\operatorname{Arcs} v \neq \varnothing$ (since a path cannot be empty).

Now, we shall prove the "$\Longrightarrow$" and "$\Longleftarrow$" directions of Proposition 2.9 separately:

$\Longrightarrow$: Assume that F is linear as a subset of $W \times W$. Thus, F is the arc set of some path cover of W. Let $C = \{c_1, c_2, \ldots, c_k\}$ be this path cover; thus, $F = \operatorname{Arcs} C$. Now, let $v_1, v_2, \ldots, v_\ell$ be the elements of $V \setminus W$ (each listed exactly once), and let us define a set

$$D := \{c_1, c_2, \ldots, c_k, (v_1), (v_2), \ldots, (v_\ell)\} = C \cup \{(v_1), (v_2), \ldots, (v_\ell)\}.$$

Then, D is a path cover of V (in fact, D is just the result of extending C to a path

since it is precisely the order in which the first entries of the paths appear in the concatenation. This relies on the fact that the paths are nonempty (so that these first entries exist) and disjoint (so that each of these first entries appears only once in the concatenation). Thus, different orders in which we can concatenate the paths lead to different resulting concatenations.

cover of V by inserting a trivial path (v) for each $v \in V \setminus W$). Furthermore,

$$\begin{aligned}
\operatorname{Arcs} D &= \operatorname{Arcs}\left(C \cup \{(v_1), (v_2), \ldots, (v_\ell)\}\right) \qquad \left(\text{since } D = C \cup \{(v_1), (v_2), \ldots, (v_\ell)\}\right) \\
&= (\operatorname{Arcs} C) \cup \underbrace{(\operatorname{Arcs}((v_1))) \cup (\operatorname{Arcs}((v_2))) \cup \cdots \cup (\operatorname{Arcs}((v_\ell)))}_{\substack{=\varnothing \\ (\text{since each trivial path } (v_i) \text{ satisfies } \operatorname{Arcs}((v_i))=\varnothing)}} \\
&= \operatorname{Arcs} C.
\end{aligned}$$

Hence, $F = \operatorname{Arcs} C = \operatorname{Arcs} D$. This shows that F is the arc set of some path cover of V (since D is a path cover of V). In other words, F is linear as a subset of $V \times V$. The "$\Longrightarrow$" direction of Proposition 2.9 is thus proved.

$\Longleftarrow$: Assume that F is linear as a subset of $V \times V$. Thus, F is the arc set of some path cover of V. Let $C = \{c_1, c_2, \ldots, c_k\}$ be this path cover; thus, $F = \operatorname{Arcs} C$. Now, let $v_1, v_2, \ldots, v_\ell$ be the elements of $V \setminus W$ (each listed exactly once). Thus, $V \setminus W = \{v_1, v_2, \ldots, v_\ell\}$.

Let $i \in [\ell]$. We shall prove that the trivial path (v_i) belongs to C.

Indeed, from $v_i \in V \setminus W$, we obtain $v_i \in V$ and $v_i \notin W$. The element v_i must clearly belong to some path in C (since C is a path cover of V). Let $p = (p_1, p_2, \ldots, p_r)$ be this path. Thus, $v_i = p_s$ for some $s \in [r]$. Consider this s.

Since the path p belongs to C, we have $\operatorname{Arcs} p \subseteq \operatorname{Arcs} C = F \subseteq W \times W$.

However, if we had $s < r$, then we would have $(p_s, p_{s+1}) \in \operatorname{Arcs} p \subseteq W \times W$, which would entail $p_s \in W$, which contradicts $p_s = v_i \notin W$. Thus, we cannot have $s < r$. Hence, $s = r$ (since $s \in [r]$).

Furthermore, if we had $s > 1$, then we would have $(p_{s-1}, p_s) \in \operatorname{Arcs} p \subseteq W \times W$, which would entail $p_s \in W$, which contradicts $p_s = v_i \notin W$. Thus, we cannot have $s > 1$. Hence, $s = 1$ (since $s \in [r]$). Therefore, $p_s = p_1$, so that $p_1 = p_s = v_i$.

Comparing $s = 1$ with $s = r$, we obtain $r = 1$, so that $(p_1, p_2, \ldots, p_r) = (p_1) = (v_i)$ (since $p_1 = v_i$). Thus, $p = (p_1, p_2, \ldots, p_r) = (v_i)$, so that $(v_i) = p \in C$. In other words, the trivial path (v_i) belongs to C.

Forget that we fixed i. We thus have shown that for each $i \in [\ell]$, the trivial path (v_i) belongs to C. In other words, all ℓ trivial paths $(v_1), (v_2), \ldots, (v_\ell)$ belong to C. Let us set $D := C \setminus \{(v_1), (v_2), \ldots, (v_\ell)\}$. Then,

$$C = D \cup \{(v_1), (v_2), \ldots, (v_\ell)\}$$

(since $(v_1), (v_2), \ldots, (v_\ell)$ belong to C), so that

$$\begin{aligned}
\operatorname{Arcs} C &= \operatorname{Arcs}\left(D \cup \{(v_1), (v_2), \ldots, (v_\ell)\}\right) \\
&= (\operatorname{Arcs} D) \cup \underbrace{(\operatorname{Arcs}((v_1))) \cup (\operatorname{Arcs}((v_2))) \cup \cdots \cup (\operatorname{Arcs}((v_\ell)))}_{\substack{=\varnothing \\ (\text{since each trivial path } (v_i) \text{ satisfies } \operatorname{Arcs}((v_i))=\varnothing)}} \\
&= \operatorname{Arcs} D.
\end{aligned}$$

In other words, $F = \operatorname{Arcs} D$ (since $F = \operatorname{Arcs} C$).

Recall that C is a path cover of V. Hence, each $v \in V$ belongs to exactly one path in C. Thus, it is easy to see that each path in D is a path of W [9]. Therefore, D is a set of paths of W. Furthermore, each $v \in W$ belongs to exactly one of these paths[10]. Hence, D is a path cover of W. Since $F = \operatorname{Arcs} D$, we thus conclude that F is the arc set of some path cover of W. In other words, F is linear as a subset of $W \times W$. The "$\Longleftarrow$" direction of Proposition 2.9 is thus proved. $\square$

We will also use the following fact:

Proposition 2.10. Let V be a finite set. Let $V_1, V_2, \ldots, V_k$ be several disjoint subsets of V such that $V = V_1 \cup V_2 \cup \cdots \cup V_k$. For each $i \in [k]$, let F_i be a subset of $V_i \times V_i$. Let $F = F_1 \cup F_2 \cup \cdots \cup F_k$. Then, the set F is linear (as a subset of $V \times V$) if and only if all the subsets F_i for $i \in [k]$ are linear.

Proof. $\Longrightarrow$: Assume that F is linear. Then, for each $i \in [k]$, the set F_i is a subset of F (since $F = F_1 \cup F_2 \cup \cdots \cup F_k \supseteq F_i$) and therefore is also linear[11] (by Proposition 2.7). Thus, the "$\Longrightarrow$" direction of Proposition 2.10 is proved.

$\Longleftarrow$: Assume that all the subsets F_i for $i \in [k]$ are linear. We must prove that F is linear.

Let $i \in [k]$. Then, F_i is a linear subset of $V_i \times V_i$. In other words, F_i is the arc set of some path cover of V_i. Let C_i be this path cover; thus, $F_i = \operatorname{Arcs}(C_i)$.

[9] *Proof.* Assume the contrary. Thus, some path in D is not a path of W. In other words, some path in D contains an element of $V \setminus W$ (since each path in D is a path of V). Let p be this path, and let v be this element. Thus, $p \in D$ and $v \in V \setminus W$, and v belongs to p. Note that $p \in D \subseteq C$ (by the definition of D).

However, $v \in V \setminus W = \{v_1, v_2, \ldots, v_\ell\}$. Thus, $v = v_i$ for some $i \in [\ell]$. Consider this i. Thus, v belongs to the trivial path (v_i) (since $v = v_i$). We have $(v_i) \in C$ (since all ℓ trivial paths $(v_1), (v_2), \ldots, (v_\ell)$ belong to C) and $(v_i) \notin D$ (since D was defined as $C \setminus \{(v_1), (v_2), \ldots, (v_\ell)\}$).

Recall that v belongs to exactly one path in C (since C is a path cover of V). Since v belongs to both paths (v_i) and p (both of which are paths in C, since $(v_i) \in C$ and $p \in C$), this entails that these two paths (v_i) and p must be identical. Hence, $(v_i) = p \in D$. But this contradicts $(v_i) \notin D$. This contradiction shows that our assumption was false, qed.

[10] *Proof.* Let $v \in W$. We must prove that v belongs to exactly one of the paths in D.

First of all, $v \in W \subseteq V$. Hence, v belongs to exactly one path in C (since C is a path cover of V).

On the other hand, we have $v \notin \{v_1, v_2, \ldots, v_\ell\}$ (because otherwise, we would have $v \in \{v_1, v_2, \ldots, v_\ell\} = V \setminus W$, which would entail $v \notin W$, but this would contradict $v \in W$). In other words, v is not one of the ℓ elements $v_1, v_2, \ldots, v_\ell$. In other words, v belongs to none of the trivial paths $(v_1), (v_2), \ldots, (v_\ell)$. In other words, v belongs to no paths in $\{(v_1), (v_2), \ldots, (v_\ell)\}$.

Now we know that:

- the element v belongs to exactly one path in C, but
- the element v belongs to no paths in $\{(v_1), (v_2), \ldots, (v_\ell)\}$.

Combining these two facts, we see that v must belong to exactly one path in $C \setminus \{(v_1), (v_2), \ldots, (v_\ell)\}$. In other words, v belongs to exactly one path in D (since $D = C \setminus \{(v_1), (v_2), \ldots, (v_\ell)\}$). This completes our proof.

[11] Here, we are tacitly using Proposition 2.9, which allows us to equivocate between "linear as a subset of $V \times V$" and "linear as a subset of $V_i \times V_i$".

Forget that we fixed i. Thus, for each $i \in [k]$, we have constructed a path cover C_i of V_i satisfying $F_i = \operatorname{Arcs}(C_i)$. It is easy to see that the union $C_1 \cup C_2 \cup \cdots \cup C_k$ of these path covers $C_1, C_2, \ldots, C_k$ is a path cover of $V_1 \cup V_2 \cup \cdots \cup V_k$ (since $V_1, V_2, \ldots, V_k$ are disjoint sets). In other words, $C_1 \cup C_2 \cup \cdots \cup C_k$ is a path cover of V (since $V = V_1 \cup V_2 \cup \cdots \cup V_k$). Moreover, the arc set of this path cover is

$$\begin{aligned}\operatorname{Arcs}(C_1 \cup C_2 \cup \cdots \cup C_k) &= (\operatorname{Arcs}(C_1)) \cup (\operatorname{Arcs}(C_2)) \cup \cdots \cup (\operatorname{Arcs}(C_k)) \\ &= F_1 \cup F_2 \cup \cdots \cup F_k \qquad (\text{since } \operatorname{Arcs}(C_i) = F_i \text{ for each } i \in [k]) \\ &= F.\end{aligned}$$

Hence, F is the arc set of a path cover of V (namely, of the path cover $C_1 \cup C_2 \cup \cdots \cup C_k$). In other words, F is linear. This proves the "$\Longleftarrow$" direction of Proposition 2.10. □

2.3. The arrow set of a permutation

We will now see another way to obtain subsets of $V \times V$:

Definition 2.11. Let V be a finite set. Let σ be a permutation of V. Then, $\mathbf{A}_\sigma$ shall denote the subset

$$\{(v, \sigma(v)) \mid v \in V\} = \bigcup_{c \in \operatorname{Cycs} \sigma} \operatorname{CArcs} c$$

of $V \times V$.

Example 2.12. Let $V = \{1, 2, 3, 4, 5, 6\}$, and let σ be the permutation of V that sends $1, 2, 3, 4, 5, 6$ to $2, 3, 1, 5, 4, 6$ (respectively). Then,

$$\operatorname{Cycs} \sigma = \{(1, 2, 3), \ (4, 5), \ (6)\}$$

and

$$\begin{aligned}\mathbf{A}_\sigma &= \{(1,2), \ (2,3), \ (3,1), \ (4,5), \ (5,4), \ (6,6)\} \\ &= \underbrace{\operatorname{CArcs}(1,2,3)}_{=\{(1,2), \ (2,3), \ (3,1)\}} \cup \underbrace{\operatorname{CArcs}(4,5)}_{=\{(4,5), \ (5,4)\}} \cup \underbrace{\operatorname{CArcs}(6)}_{=\{(6,6)\}}.\end{aligned}$$

The following is a counterpart to Proposition 2.8 **(b)**:

Proposition 2.13. Let V be a finite set. Let F be a subset of $V \times V$. If $F = \operatorname{Arcs} C$ for some path cover C of V, then there are exactly $|C|!$ many permutations $\sigma \in \mathfrak{S}_V$ satisfying $F \subseteq \mathbf{A}_\sigma$. (Note that $|C|$ is the number of paths in C.)

Proposition 2.13 is easily proved, but the proof is tricky to formalize due to its reliance on some enumerative ideas that are intuitively clear yet notationally intricate. To prepare for this proof, we begin with some basic enumerative results. First, a notation:

Definition 2.14. An *injection* shall mean an injective map.

(Of course, this is analogous to the concept of a *bijection*, which means a bijective map.)

Now, we can state our first enumerative result:[12]

Proposition 2.15. Let X, Y and Z be three finite sets such that $Y \subseteq X$. Let $f : Y \to Z$ be any injection. Then,

$$(\text{\# of injections } g : X \to Z \text{ such that } g\mid_Y = f) = \prod_{k=0}^{|X|-|Y|-1} (|Z| - |Y| - k).$$

Proof. We proceed by induction on $|X \setminus Y|$:

Base case: If $|X \setminus Y| = 0$, then the claim of Proposition 2.15 is easy to verify[13]. This completes the base case.

Induction step: Let $m \in \mathbb{N}$. Assume (as the induction hypothesis) that Proposition 2.15 holds for $|X \setminus Y| = m$. We must now prove that Proposition 2.15 holds for $|X \setminus Y| = m + 1$ as well.

[12]Recall Convention 2.1.

[13]*Proof.* Assume that $|X \setminus Y| = 0$. Then, $X \setminus Y = \varnothing$, so that $X \subseteq Y$. Combining this with $Y \subseteq X$, we obtain $Y = X$. Hence, for any injection $g : X \to Z$, we have $g\mid_Y = g\mid_X = g$. Thus,

$$\begin{aligned}
&\left(\text{\# of injections } g : \underbrace{X}_{=Y} \to Z \text{ such that } \underbrace{g\mid_Y}_{=g} = f\right) \\
&= (\text{\# of injections } g : Y \to Z \text{ such that } g = f) \\
&= 1
\end{aligned}$$

(since f itself is an injection $g : Y \to Z$ such that $g = f$, and clearly there are no other such injections). Comparing this with

$$\begin{aligned}
&\prod_{k=0}^{|X|-|Y|-1} (|Z| - |Y| - k) \\
&= \prod_{k=0}^{-1} (|Z| - |Y| - k) \qquad \left(\text{since } |X| - \left|\underbrace{Y}_{=X}\right| - 1 = |X| - |X| - 1 = -1\right) \\
&= (\text{empty product}) = 1,
\end{aligned}$$

we obtain

$$(\text{\# of injections } g : X \to Z \text{ such that } g\mid_Y = f) = \prod_{k=0}^{|X|-|Y|-1} (|Z| - |Y| - k).$$

Thus, Proposition 2.15 is proved under the assumption that $|X \setminus Y| = 0$.

So let X, Y and Z be three finite sets such that $Y \subseteq X$. Let $f : Y \to Z$ be any injection. Assume that $|X \setminus Y| = m + 1$. We must then prove that

$$(\text{\# of injections } g : X \to Z \text{ such that } g\mid_Y = f) = \prod_{k=0}^{|X|-|Y|-1} (|Z| - |Y| - k).$$

It is well-known that if A and B are two finite sets, and if $\varphi : A \to B$ is an injection, then $|\varphi(A)| = |A|$ (since an injection sends distinct elements to distinct elements, and thus all the $|A|$ distinct elements of A give rise to $|A|$ distinct elements of $\varphi(A)$). We can apply this to $A = Y$ and $B = Z$ and $\varphi = f$ (since $f : Y \to Z$ is an injection); thus we obtain $|f(Y)| = |Y|$.

From $Y \subseteq X$, we obtain $|X \setminus Y| = |X| - |Y|$, so that $|X| - |Y| = |X \setminus Y| = m + 1 \geq 1$ (since $m \geq 0$). Hence, $|X| - |Y| - 1 \geq 0$.

We have $|X \setminus Y| = m + 1 > m \geq 0$, so that the set $X \setminus Y$ is nonempty. In other words, the set $X \setminus Y$ has at least one element p. Consider this p. (We can choose p arbitrarily, but we then keep it fixed for the rest of this proof.)

Thus, $p \in X \setminus Y$. In other words, $p \in X$ and $p \notin Y$. Let Y' be the set $Y \cup \{p\}$. Thus,

$$Y' = Y \cup \{p\} \subseteq X \qquad (\text{since } Y \subseteq X \text{ and } p \in X).$$

Furthermore, from $Y' = Y \cup \{p\}$, we obtain

$$|Y'| = |Y \cup \{p\}| = |Y| + 1 \qquad (\text{since } p \notin Y).$$

Since $Y' \subseteq X$, we have

$$\begin{aligned} |X \setminus Y'| &= |X| - \underbrace{|Y'|}_{=|Y|+1} = |X| - (|Y| + 1) = \underbrace{|X| - |Y|}_{\substack{=|X \setminus Y| \\ (\text{since } Y \subseteq X)}} - 1 \\ &= \underbrace{|X \setminus Y|}_{=m+1} - 1 = (m+1) - 1 = m. \end{aligned}$$

From $Y' = Y \cup \{p\}$, we also obtain

$$Y' \setminus \{p\} = (Y \cup \{p\}) \setminus \{p\} = Y \qquad (\text{since } p \notin Y).$$

Also,

$$Y \subseteq Y \cup \{p\} = Y'.$$

If $y \in Y'$ is an element that satisfies $y \neq p$, then $y \in Y$ (since $y \in Y'$ and $y \neq p$ lead to $y \in Y' \setminus \{p\} = Y$), and therefore $f(y)$ is well-defined (since $f : Y \to Z$ is a map).

For any $z \in Z \setminus f(Y)$, we define a map $f_{p \to z} : Y' \to Z$ by setting

$$f_{p \to z}(y) = \begin{cases} z, & \text{if } y = p; \\ f(y), & \text{if } y \neq p \end{cases} \qquad \text{for each } y \in Y'.$$

This is well-defined, because if $y \in Y'$ is an element that satisfies $y \neq p$, then $f(y)$ is well-defined (as we saw in the previous paragraph).

Now we claim the following:

Claim 1: Let $z \in Z \setminus f(Y)$. Then, the map $f_{p \to z} : Y' \to Z$ is an injection.

Proof of Claim 1. Let u and v be two elements of Y' satisfying $f_{p \to z}(u) = f_{p \to z}(v)$. We shall show that $u = v$.

Indeed, we are in one of the following four cases:

Case 1: We have $u = p$ and $v = p$.

Case 2: We have $u = p$ but not $v = p$.

Case 3: We have $v = p$ but not $u = p$.

Case 4: We have neither $u = p$ nor $v = p$.

Let us first consider Case 1. In this case, we have $u = p$ and $v = p$. Hence, $u = p = v$. Thus, $u = v$ has been proved in Case 1.

Let us now consider Case 2. In this case, we have $u = p$ but not $v = p$. Hence, $v \neq p$ (since we do not have $v = p$). Combining $v \in Y'$ with $v \neq p$, we obtain $v \in Y' \setminus \{p\} = Y$.

The definition of $f_{p \to z}$ yields

$$f_{p \to z}(u) = \begin{cases} z, & \text{if } u = p; \\ f(u), & \text{if } u \neq p \end{cases} = z \qquad (\text{since } u = p),$$

so that

$$\begin{aligned} z = f_{p \to z}(u) &= f_{p \to z}(v) \\ &= \begin{cases} z, & \text{if } v = p; \\ f(v), & \text{if } v \neq p \end{cases} \qquad \left(\text{by the definition of } f_{p \to z}\right) \\ &= f(v) \qquad (\text{since } v \neq p) \\ &\in f(Y) \qquad (\text{since } v \in Y). \end{aligned}$$

However, from $z \in Z \setminus f(Y)$, we obtain $z \notin f(Y)$. This contradicts $z \in f(Y)$. Thus we have obtained a contradiction in Case 2. Hence, Case 2 cannot occur.

A similar argument (with the roles of u and v swapped) shows that Case 3 cannot occur.

Let us finally consider Case 4. In this case, we have neither $u = p$ nor $v = p$. In other words, we have $u \neq p$ and $v \neq p$. Combining $v \in Y'$ with $v \neq p$, we obtain $v \in Y' \setminus \{p\} = Y$. Similarly, $u \in Y$.

The definition of $f_{p \to z}$ yields

$$f_{p \to z}(u) = \begin{cases} z, & \text{if } u = p; \\ f(u), & \text{if } u \neq p \end{cases} = f(u) \qquad (\text{since } u \neq p).$$

Hence,

$$\begin{aligned} f(u) &= f_{p\to z}(u) = f_{p\to z}(v) \\ &= \begin{cases} z, & \text{if } v = p; \\ f(v), & \text{if } v \neq p \end{cases} \qquad \left(\text{by the definition of } f_{p\to z}\right) \\ &= f(v) \qquad (\text{since } v \neq p). \end{aligned}$$

However, the map f is an injection, i.e., is injective. Thus, if a and b are two elements of Y satisfying $f(a) = f(b)$, then $a = b$. Applying this to $a = u$ and $b = v$, we obtain $u = v$ (since $f(u) = f(v)$). Thus, we have proved $u = v$ in Case 4.

Let us summarize: We have proved that Cases 2 and 3 cannot occur; thus, we must actually be in one of the two Cases 1 and 4. But we have also proved that $u = v$ in each of the latter two Cases 1 and 4. Hence, $u = v$ always holds.

Forget that we fixed u and v. We thus have shown that if u and v are two elements of Y' satisfying $f_{p\to z}(u) = f_{p\to z}(v)$, then $u = v$. In other words, the map $f_{p\to z} : Y' \to Z$ is injective, i.e., an injection. This proves Claim 1. $\square$

> *Claim 2:* Let $g : X \to Z$ be any injection such that $g\mid_Y = f$. Then, $g(p) \in Z \setminus f(Y)$.

Proof of Claim 2. Clearly, $g(p) \in Z$. We shall now show that $g(p) \notin f(Y)$.

Indeed, assume the contrary. Thus, $g(p) \in f(Y)$. In other words, $g(p) = f(y)$ for some $y \in Y$. Consider this y. From $y \in Y$, we obtain $(g\mid_Y)(y) = g(y)$. Hence, $g(y) = \underbrace{(g\mid_Y)}_{=f}(y) = f(y) = g(p)$ (since $g(p) = f(y)$).

However, the map g is an injection, i.e., is injective. Hence, if a and b are two elements of X satisfying $g(a) = g(b)$, then $a = b$. Applying this to $a = y$ and $b = p$, we obtain $y = p$ (since $g(y) = g(p)$). Hence, $p = y \in Y$. But this contradicts $p \notin Y$. This contradiction shows that our assumption was false.

Hence, $g(p) \notin f(Y)$ is proved. Now, combining $g(p) \in Z$ with $g(p) \notin f(Y)$, we obtain $g(p) \in Z \setminus f(Y)$. This proves Claim 2. $\square$

> *Claim 3:* Let $g : X \to Z$ be any injection. Let $z \in Z \setminus f(Y)$. Then, the statement "$g\mid_Y = f$ and $g(p) = z$" is equivalent to the statement "$g\mid_{Y'} = f_{p\to z}$".

Proof of Claim 3. We must prove that these two statements are equivalent, i.e., that each of them implies the other.

Let us first show that the statement "$g\mid_Y = f$ and $g(p) = z$" implies the statement "$g\mid_{Y'} = f_{p\to z}$".

Proof that "$g\mid_Y = f$ and $g(p) = z$" implies "$g\mid_{Y'} = f_{p\to z}$": Assume that the statement "$g\mid_Y = f$ and $g(p) = z$" holds. We must prove that "$g\mid_{Y'} = f_{p\to z}$" holds as well.

Indeed, let $y \in Y'$. We shall prove the equality $g(y) = f_{p\to z}(y)$. This equality is easily proved in the case when $y = p$ [14]. Thus, for the rest of this proof, we WLOG assume that $y \neq p$. Combining $y \in Y'$ with $y \neq p$, we obtain $y \in Y' \setminus \{p\} = Y$. Hence, $(g\mid_Y)(y) = g(y)$. However, $g\mid_Y = f$ (since we assumed "$g\mid_Y = f$ and $g(p) = z$"). Thus, $\underbrace{(g\mid_Y)}_{=f}(y) = f(y)$. Furthermore, the definition of $f_{p\to z}$ yields

$$\begin{aligned} f_{p\to z}(y) &= \begin{cases} z, & \text{if } y = p; \\ f(y), & \text{if } y \neq p \end{cases} = f(y) \qquad (\text{since } y \neq p) \\ &= (g\mid_Y)(y) \qquad (\text{since } (g\mid_Y)(y) = f(y)) \\ &= g(y). \end{aligned}$$

In other words, $g(y) = f_{p\to z}(y)$. Thus, we have proved the equality $g(y) = f_{p\to z}(y)$.

Hence, $(g\mid_{Y'})(y) = g(y) = f_{p\to z}(y)$.

Forget that we fixed y. We thus have shown that $(g\mid_{Y'})(y) = f_{p\to z}(y)$ for each $y \in Y'$. In other words, $g\mid_{Y'} = f_{p\to z}$. We conclude that the statement "$g\mid_{Y'} = f_{p\to z}$" holds.

Thus, we have proved that the statement "$g\mid_Y = f$ and $g(p) = z$" implies the statement "$g\mid_{Y'} = f_{p\to z}$".

Let us now prove the reverse implication:

Proof that "$g\mid_{Y'} = f_{p\to z}$" implies "$g\mid_Y = f$ and $g(p) = z$": Assume that the statement "$g\mid_{Y'} = f_{p\to z}$" holds. We must prove that "$g\mid_Y = f$ and $g(p) = z$" holds as well.

Indeed, we have $g\mid_{Y'} = f_{p\to z}$ (since we assumed that "$g\mid_{Y'} = f_{p\to z}$" holds). Now, for each $y \in Y'$, we have

$$\begin{aligned} g(y) &= \underbrace{(g\mid_{Y'})}_{=f_{p\to z}}(y) \qquad (\text{since } y \in Y') \\ &= f_{p\to z}(y) \\ &= \begin{cases} z, & \text{if } y = p; \\ f(y), & \text{if } y \neq p \end{cases} \end{aligned} \tag{4}$$

(by the definition of $f_{p\to z}$).

[14] *Proof.* Assume that $y = p$. We must prove that $g(y) = f_{p\to z}(y)$.

Indeed, from $y = p$, we obtain $g(y) = g(p) = z$ (since we assumed "$g\mid_Y = f$ and $g(p) = z$"). On the other hand, the definition of $f_{p\to z}$ yields

$$f_{p\to z}(y) = \begin{cases} z, & \text{if } y = p; \\ f(y), & \text{if } y \neq p \end{cases} = z \qquad (\text{since } y = p).$$

Comparing this with $g(y) = z$, we obtain $g(y) = f_{p\to z}(y)$. Qed.

Hence, each $y \in Y$ satisfies

$$\begin{aligned} \left(g\mid_Y\right)(y) &= g(y) && (\text{since } y \in Y) \\ &= \begin{cases} z, & \text{if } y = p; \\ f(y), & \text{if } y \neq p \end{cases} && \left(\text{by (4) (since } y \in Y \subseteq Y')\right) \\ &= f(y) && (\text{since } y \neq p \text{ (because } y \in Y\text{, but } p \notin Y)). \end{aligned}$$

In other words, $g\mid_Y = f$. Furthermore, we have $p \in \{p\} \subseteq Y \cup \{p\} = Y'$. Hence, (4) (applied to $y = p$) yields

$$g(p) = \begin{cases} z, & \text{if } p = p; \\ f(p), & \text{if } p \neq p \end{cases} = z \qquad (\text{since } p = p).$$

Thus, we have shown that $g\mid_Y = f$ and $g(p) = z$. In other words, "$g\mid_Y = f$ and $g(p) = z$" holds. This completes the proof that the statement "$g\mid_{Y'} = f_{p\to z}$" implies "$g\mid_Y = f$ and $g(p) = z$".

Altogether, we have now shown that the statement "$g\mid_Y = f$ and $g(p) = z$" implies the statement "$g\mid_{Y'} = f_{p\to z}$", and vice versa. In other words, these two statements are equivalent. This proves Claim 3. ◻

Claim 4: Let $z \in Z \setminus f(Y)$. Then,

$$\begin{aligned} &(\text{\# of injections } g : X \to Z \text{ such that } g\mid_Y = f \text{ and } g(p) = z) \\ &= \prod_{k=1}^{|X|-|Y|-1} (|Z| - |Y| - k). \end{aligned}$$

Proof of Claim 4. Recall that $Y' \subseteq X$ and $|X \setminus Y'| = m$. Also, the map $f_{p\to z} : Y' \to Z$ is an injection (by Claim 1). However, our induction hypothesis says that Proposition 2.15 holds for $|X \setminus Y| = m$. Thus, we can apply Proposition 2.15 to Y' and $f_{p\to z}$ instead of Y and f. As a result, we obtain

$$\begin{aligned} &\left(\text{\# of injections } g : X \to Z \text{ such that } g\mid_{Y'} = f_{p\to z}\right) \\ &= \prod_{k=0}^{|X|-|Y'|-1} \left(|Z| - |Y'| - k\right) \\ &= \underbrace{\prod_{k=0}^{|X|-(|Y|+1)-1}}_{\substack{= \prod_{k=0}^{|X|-|Y|-1-1} \\ (\text{since } |X|-(|Y|+1)-1 = |X|-|Y|-1-1)}} \underbrace{(|Z| - (|Y|+1) - k)}_{=|Z|-|Y|-(k+1)} \qquad \left(\text{since } \left|Y'\right| = |Y| + 1\right) \\ &= \prod_{k=0}^{|X|-|Y|-1-1} (|Z| - |Y| - (k+1)) \\ &= \prod_{k=1}^{|X|-|Y|-1} (|Z| - |Y| - k) \end{aligned}$$

(here, we have substituted k for $k+1$ in the product).

However, for any injection $g : X \to Z$, the statement "$g\mid_Y = f$ and $g(p) = z$" is equivalent to the statement "$g\mid_{Y'} = f_{p\to z}$" (by Claim 3). Hence, we have

$$\begin{aligned}
&(\text{\# of injections } g : X \to Z \text{ such that } g\mid_Y = f \text{ and } g(p) = z) \\
&= \left(\text{\# of injections } g : X \to Z \text{ such that } g\mid_{Y'} = f_{p\to z}\right) \\
&= \prod_{k=1}^{|X|-|Y|-1} (|Z| - |Y| - k).
\end{aligned}$$

This proves Claim 4. □

We are now almost done. Since $f(Y)$ is a subset of Z, we have $|Z \setminus f(Y)| = |Z| - \underbrace{|f(Y)|}_{=|Y|} = |Z| - |Y|$.

However, if $g : X \to Z$ is any injection such that $g\mid_Y = f$, then $g(p) \in Z \setminus f(Y)$ (by Claim 2). Hence, in order to count the injections $g : X \to Z$ such that $g\mid_Y = f$, we can split them up according to the value of $g(p)$ as follows:

$$\begin{aligned}
&(\text{\# of injections } g : X \to Z \text{ such that } g\mid_Y = f) \\
&= \sum_{z \in Z\setminus f(Y)} \underbrace{(\text{\# of injections } g : X \to Z \text{ such that } g\mid_Y = f \text{ and } g(p) = z)}_{\substack{= \prod_{k=1}^{|X|-|Y|-1} (|Z|-|Y|-k) \\ \text{(by Claim 4)}}} \\
&= \sum_{z \in Z\setminus f(Y)} \prod_{k=1}^{|X|-|Y|-1} (|Z| - |Y| - k) = \underbrace{|Z \setminus f(Y)|}_{\substack{=|Z|-|Y| \\ =|Z|-|Y|-0}} \cdot \prod_{k=1}^{|X|-|Y|-1} (|Z| - |Y| - k) \\
&= (|Z| - |Y| - 0) \cdot \prod_{k=1}^{|X|-|Y|-1} (|Z| - |Y| - k).
\end{aligned}$$

Comparing this with

$$\prod_{k=0}^{|X|-|Y|-1} (|Z| - |Y| - k) = (|Z| - |Y| - 0) \cdot \prod_{k=1}^{|X|-|Y|-1} (|Z| - |Y| - k)$$
$$\left(\begin{array}{c}\text{here, we have split off the factor for } k = 0 \\ \text{from the product, since } |X| - |Y| - 1 \geq 0\end{array}\right),$$

we obtain

$$(\text{\# of injections } g : X \to Z \text{ such that } g\mid_Y = f) = \prod_{k=0}^{|X|-|Y|-1} (|Z| - |Y| - k).$$

In other words, the claim of Proposition 2.15 holds for our X, Y, Z and f.

This completes the induction step. Thus, we have proved Proposition 2.15 by induction. □

From Proposition 2.15, we can easily derive the following:

Corollary 2.16. Let X be a finite set, and let Y be a subset of X. Let $f : Y \to X$ be any injection. Then,

$$(\text{\# of permutations } \sigma \in \mathfrak{S}_X \text{ such that } \sigma\mid_Y = f) = (|X| - |Y|)!.$$

Proof. We have $Y \subseteq X$ (since Y is a subset of X) and thus $|Y| \leq |X|$. Hence, $|X| - |Y| \in \mathbb{N}$.

We recall the following basic fact about finite sets (one of the Pigeonhole Principles): If U and V are two finite sets having the same size (i.e., satisfying $|U| = |V|$), then any injective map from U to V is bijective.

Applying this to $U = X$ and $V = X$, we conclude that any injective map from X to X is bijective (since X and X are two finite sets having the same size). In other words, any injection from X to X is bijective (since an injection is the same as an injective map). Hence, any injection from X to X is a bijection from X to X. The converse of this claim is true as well (since any bijection is obviously an injection). Combining the preceding two sentences, we conclude that the injections from X to X are precisely the bijections from X to X. Therefore,

$$\begin{aligned}
&\{\text{injections from } X \text{ to } X\} \\
&= \{\text{bijections from } X \text{ to } X\} \\
&= \{\text{permutations of } X\} \qquad \left(\begin{array}{c}\text{since a permutation of } X \text{ is defined} \\ \text{as a bijection from } X \text{ to } X\end{array}\right) \\
&= \{\text{permutations } \sigma \in \mathfrak{S}_X\} \qquad (\text{by the definition of } \mathfrak{S}_X).
\end{aligned}$$

Thus,

$$\{\text{permutations } \sigma \in \mathfrak{S}_X\} = \{\text{injections from } X \text{ to } X\}.$$

In other words, the permutations $\sigma \in \mathfrak{S}_X$ are precisely the injections from X to X. Hence,

$$\begin{aligned}
&(\text{\# of permutations } \sigma \in \mathfrak{S}_X \text{ such that } \sigma\mid_Y = f) \\
&= (\text{\# of injections } \sigma \text{ from } X \text{ to } X \text{ such that } \sigma\mid_Y = f) \\
&= (\text{\# of injections } \sigma : X \to X \text{ such that } \sigma\mid_Y = f) \\
&= (\text{\# of injections } g : X \to X \text{ such that } g\mid_Y = f) \\
&\qquad\qquad (\text{here, we have renamed the index } \sigma \text{ as } g) \\
&= \prod_{k=0}^{|X|-|Y|-1} (|X| - |Y| - k) \qquad (\text{by Proposition 2.15, applied to } Z = X) \\
&= \prod_{i=1}^{|X|-|Y|} i \qquad (\text{here, we have substituted } i \text{ for } |X| - |Y| - k \text{ in the product}) \\
&= 1 \cdot 2 \cdot \cdots \cdot (|X| - |Y|) \\
&= (|X| - |Y|)! \qquad (\text{since } |X| - |Y| \in \mathbb{N}).
\end{aligned}$$

This proves Corollary 2.16. □

Next, we introduce some notations for paths:

Definition 2.17. Let V be a finite set. Let p be a path of V. Then:

(a) We let p_{last} denote the last entry of p. (This is well-defined, since p is a path, thus a nonempty tuple, and therefore has a last entry.)

(b) Let v be any entry of p distinct from p_{last}. Then, the tuple p contains v exactly once[15]. Furthermore, v is an entry of p, but is not the last entry of p (since v is distinct from p_{last}, which is the last entry of p). Hence, the tuple p has at least one entry coming after v. We let $\operatorname{next}(p, v)$ denote the next entry after v in the tuple p. (This is well-defined, since the tuple p contains v exactly once and has at least one entry coming after v.)

Example 2.18. Assume that $V = [10]$ and $p = (3, 4, 1, 6, 7)$. Then, $p_{\text{last}} = 7$ and $\operatorname{next}(p, 3) = 4$ and $\operatorname{next}(p, 4) = 1$ and $\operatorname{next}(p, 1) = 6$ and $\operatorname{next}(p, 6) = 7$.

Definition 2.19. Let V be a finite set. Let C be a path cover of V. Let $w \in V$.

Recall that C is a path cover of V. In other words, C is a set of paths of V such that each $v \in V$ belongs to exactly one of these paths (by the definition of a path cover). In particular, each $v \in V$ belongs to exactly one of the paths in C. Applying this to $v = w$, we conclude that w belongs to exactly one of the paths in C. In other words, there is exactly one path $p \in C$ such that w belongs to p. In other words, there is exactly one path $p \in C$ that contains w. We shall denote the latter path p by $\operatorname{path}(C, w)$.

Example 2.20. Assume that $V = [6]$ and $C = \{(1, 6, 4), (5), (2, 3)\}$. Then,

$$\begin{aligned}
&\operatorname{path}(C, 1) = \operatorname{path}(C, 6) = \operatorname{path}(C, 4) = (1, 6, 4); \\
&\operatorname{path}(C, 5) = (5); \\
&\operatorname{path}(C, 2) = \operatorname{path}(C, 3) = (2, 3).
\end{aligned}$$

The following lemma will help us reduce Proposition 2.13 to Corollary 2.16:

[15] *Proof.* Clearly, the tuple p contains v (since v is an entry of p). Furthermore, p is a path of V, that is, a nonempty tuple of distinct elements of V (by the definition of a path of V). Hence, in particular, the entries of p are distinct. In other words, p does not contain any entry more than once. Hence, p contains each entry of p exactly once. Thus, in particular, p contains v exactly once (since v is an entry of p).

Lemma 2.21. Let V be a finite set. Let C be a path cover of V.

Let $L = \{p_{\text{last}} \mid p \in C\}$. This is a subset of V.

Define a map $f : V \setminus L \to V$ as follows: Let $w \in V \setminus L$. Thus, $w \in V$ and $w \notin L$. Let q be the path $\operatorname{path}(C, w)$. Thus, q is the unique path $p \in C$ that contains w (by the definition of $\operatorname{path}(C, w)$). Hence, $q \in C$ is a path that contains w. We have $w \neq q_{\text{last}}$ (because otherwise, we would have

$$\begin{aligned} w = q_{\text{last}} &\in \{p_{\text{last}} \mid p \in C\} \qquad \text{(since } q \in C\text{)} \\ &= L, \end{aligned}$$

which would contradict $w \notin L$). Hence, w is an entry of q (since q contains w) that is distinct from q_{last} (since $w \neq q_{\text{last}}$). Thus, $\operatorname{next}(q, w)$ is well-defined (by Definition 2.17 **(b)**). We set $f(w) := \operatorname{next}(q, w)$.

Thus, we have defined a map $f : V \setminus L \to V$.

Now, we claim the following:

(a) This map f is an injection.

(b) Let $\sigma \in \mathfrak{S}_V$. Let $F = \operatorname{Arcs} C$. Then, we have $F \subseteq \mathbf{A}_\sigma$ if and only if $\sigma\mid_{V \setminus L} = f$.

Proof. We begin by showing a general property of the map f:

> *Claim 1:* For any $w \in V \setminus L$, we have $f(w) = \operatorname{next}(\operatorname{path}(C, w), w)$ and $\operatorname{path}(C, f(w)) = \operatorname{path}(C, w)$.

Proof of Claim 1. Let $w \in V \setminus L$. Thus, $w \in V$ and $w \notin L$. Let q be the path $\operatorname{path}(C, w)$. Thus, q is the unique path $p \in C$ that contains w (by the definition of $\operatorname{path}(C, w)$). Hence, $q \in C$ is a path that contains w. The definition of f yields $f(w) = \operatorname{next}(q, w)$. Since $q = \operatorname{path}(C, w)$, we can rewrite this as $f(w) = \operatorname{next}(\operatorname{path}(C, w), w)$.

However, $\operatorname{next}(q, w)$ is defined as the next entry after w in the tuple q. Hence, in particular, $\operatorname{next}(q, w)$ is an entry of q. In other words, $f(w)$ is an entry of q (since $f(w) = \operatorname{next}(q, w)$). In other words, the path q contains $f(w)$.

However, $\operatorname{path}(C, f(w))$ is defined as the unique path $p \in C$ that contains $f(w)$. Hence, if $p \in C$ is a path that contains $f(w)$, then $p = \operatorname{path}(C, f(w))$. Applying this to $p = q$, we obtain $q = \operatorname{path}(C, f(w))$ (since $q \in C$ is a path that contains $f(w)$). Comparing this with $q = \operatorname{path}(C, w)$, we find $\operatorname{path}(C, f(w)) = \operatorname{path}(C, w)$.

Thus, we have now shown that $f(w) = \operatorname{next}(\operatorname{path}(C, w), w)$ and $\operatorname{path}(C, f(w)) = \operatorname{path}(C, w)$. This proves Claim 1. □

We shall now prove the two parts of Lemma 2.21:

(a) Let u and v be two elements of $V \setminus L$ satisfying $f(u) = f(v)$. We shall prove that $u = v$.

Claim 1 (applied to $w = u$) yields $f(u) = \operatorname{next}(\operatorname{path}(C, u), u)$ and $\operatorname{path}(C, f(u)) = \operatorname{path}(C, u)$.

Claim 1 (applied to $w = v$) yields $f(v) = \operatorname{next}(\operatorname{path}(C, v), v)$ and $\operatorname{path}(C, f(v)) = \operatorname{path}(C, v)$.

From $\operatorname{path}(C, f(u)) = \operatorname{path}(C, u)$, we obtain

$$\operatorname{path}(C, u) = \operatorname{path}\left(C, \underbrace{f(u)}_{=f(v)}\right) = \operatorname{path}(C, f(v)) = \operatorname{path}(C, v).$$

Let us set $q := \operatorname{path}(C, u)$. Thus, $q = \operatorname{path}(C, u) = \operatorname{path}(C, v)$.

We have $q = \operatorname{path}(C, u)$. In other words, q is the unique path $p \in C$ that contains u (since $\operatorname{path}(C, u)$ is defined to be the unique path $p \in C$ that contains u). Hence, $q \in C$ is a path that contains u. The same argument (applied to v instead of u) shows that $q \in C$ is a path that contains v (since $q = \operatorname{path}(C, v)$).

In particular, q is a path of V. In other words, q is a nonempty tuple of distinct elements of V (by the definition of a path of V). Hence, in particular, the entries of q are distinct.

Write the path q as $q = (q_1, q_2, \ldots, q_k)$. Then, $u = q_i$ for some $i \in [k]$ (since q contains u). Similarly, $v = q_j$ for some $j \in [k]$ (since q contains v). Consider this i and this j.

But $\operatorname{next}(q, u)$ is the next entry after u in the tuple q (by the definition of $\operatorname{next}(q, u)$). In other words,

$$\operatorname{next}(q, u) = (\text{the next entry after } u \text{ in the tuple } q).$$

Now,

$$\begin{aligned} f(u) &= \operatorname{next}\left(\underbrace{\operatorname{path}(C, u)}_{=q}, u\right) = \operatorname{next}(q, u) \\ &= (\text{the next entry after } u \text{ in the tuple } q) \\ &= (\text{the next entry after } q_i \text{ in the tuple } q) \qquad (\text{since } u = q_i) \\ &= q_{i+1} \qquad \left(\begin{array}{c} \text{since } q = (q_1, q_2, \ldots, q_k)\text{, so that the entry } q_i \\ \text{is followed by } q_{i+1} \text{ in the tuple } q \end{array}\right). \end{aligned}$$

The same argument (applied to v and j instead of u and i) shows that $f(v) = q_{j+1}$ (since $\operatorname{path}(C, v) = q$ and $v = q_j$). Note that the equalities $f(u) = q_{i+1}$ and $f(v) = q_{j+1}$ show (in particular) that q_{i+1} and q_{j+1} are well-defined, i.e., that the elements $i+1$ and $j+1$ belong to $[k]$.

Now, from $f(u) = q_{i+1}$, we obtain

$$q_{i+1} = f(u) = f(v) = q_{j+1}.$$

But we know that the entries of q are distinct. In other words, $q_1, q_2, \ldots, q_k$ are distinct (since $q_1, q_2, \ldots, q_k$ are the entries of q (because $q = (q_1, q_2, \ldots, q_k)$)). In other words, if a and b are two elements of $[k]$ satisfying $q_a = q_b$, then $a = b$. Applying this to $a = i + 1$ and $b = j + 1$, we conclude that $i + 1 = j + 1$ (since $q_{i+1} = q_{j+1}$). Therefore, $i = j$. Hence, $q_i = q_j = v$ (since $v = q_j$), so that $u = q_i = v$.

Now, forget that we fixed u and v. We thus have shown that if u and v are two elements of $V \setminus L$ satisfying $f(u) = f(v)$, then $u = v$. In other words, the map f is injective. In other words, f is an injection. This proves Lemma 2.21 **(a)**.

(b) We must prove the equivalence $(F \subseteq \mathbf{A}_\sigma) \iff \left(\sigma \mid_{V \setminus L} = f\right)$. In order to do so, it clearly suffices to prove the two implications $(F \subseteq \mathbf{A}_\sigma) \implies \left(\sigma \mid_{V \setminus L} = f\right)$ and $(F \subseteq \mathbf{A}_\sigma) \impliedby \left(\sigma \mid_{V \setminus L} = f\right)$. Let us do so:

Proof of the implication $(F \subseteq \mathbf{A}_\sigma) \implies \left(\sigma \mid_{V \setminus L} = f\right)$. Assume that $F \subseteq \mathbf{A}_\sigma$ holds. We must show that $\sigma \mid_{V \setminus L} = f$ holds.

Indeed, let $w \in V \setminus L$. Thus, $f(w)$ is well-defined.

Let q be the path $\operatorname{path}(C, w)$. Thus, q is the unique path $p \in C$ that contains w (by the definition of $\operatorname{path}(C, w)$). Hence, $q \in C$ is a path that contains w.

Write the path q as $q = (q_1, q_2, \ldots, q_k)$. Then, $w = q_i$ for some $i \in [k]$ (since q contains w). Consider this i.

The definition of f yields

$$\begin{aligned} f(w) &= \operatorname{next}(q, w) \\ &= (\text{the next entry after } w \text{ in the tuple } q) \\ &\qquad \left(\begin{array}{c} \text{since } \operatorname{next}(q, w) \text{ is defined to be the} \\ \text{next entry after } w \text{ in the tuple } q \end{array}\right) \\ &= (\text{the next entry after } q_i \text{ in the tuple } q) \qquad (\text{since } w = q_i) \\ &= q_{i+1} \qquad \left(\begin{array}{c} \text{since } q = (q_1, q_2, \ldots, q_k), \text{ so that the entry } q_i \\ \text{is followed by } q_{i+1} \text{ in the tuple } q \end{array}\right). \end{aligned}$$

In particular, q_{i+1} is well-defined, so that $i + 1 \in [k]$. Hence, $i \in \{0, 1, \ldots, k-1\}$. Since i is positive, we thus conclude that $i \in [k-1]$.

Now,

$$\begin{aligned} F &= \operatorname{Arcs} C = \bigcup_{v \in C} \operatorname{Arcs} v \qquad (\text{by the definition of } \operatorname{Arcs} C) \\ &\supseteq \operatorname{Arcs} q \qquad \left(\begin{array}{c} \text{since } \operatorname{Arcs} q \text{ is one of the terms in the union } \bigcup_{v \in C} \operatorname{Arcs} v \\ (\text{because } q \in C) \end{array}\right) \\ &= \operatorname{Arcs}(q_1, q_2, \ldots, q_k) \qquad (\text{since } q = (q_1, q_2, \ldots, q_k)) \\ &= \{(q_1, q_2), (q_2, q_3), \ldots, (q_{k-1}, q_k)\} \qquad \left(\text{by (2), applied to } v = q \text{ and } v_j = q_j\right). \end{aligned}$$

However, from $w = q_i$ and $f(w) = q_{i+1}$, we obtain

$$\begin{aligned}
(w, f(w)) &= (q_i, q_{i+1}) \\
&\in \{(q_1, q_2), (q_2, q_3), \ldots, (q_{k-1}, q_k)\} \qquad (\text{since } i \in [k-1]) \\
&\subseteq F \qquad (\text{since } F \supseteq \{(q_1, q_2), (q_2, q_3), \ldots, (q_{k-1}, q_k)\}) \\
&\subseteq \mathbf{A}_\sigma \\
&= \{(v, \sigma(v)) \mid v \in V\} \qquad (\text{by the definition of } \mathbf{A}_\sigma).
\end{aligned}$$

In other words, $(w, f(w)) = (v, \sigma(v))$ for some $v \in V$. Consider this v. From $(w, f(w)) = (v, \sigma(v))$, we obtain $w = v$ and $f(w) = \sigma(v)$.

Now, $w \in V \setminus L$, so that $\left(\sigma\mid_{V\setminus L}\right)(w) = \sigma\left(\underbrace{w}_{=v}\right) = \sigma(v) = f(w)$ (since $f(w) = \sigma(v)$).

Forget that we fixed w. We thus have shown that $\left(\sigma\mid_{V\setminus L}\right)(w) = f(w)$ for each $w \in V \setminus L$. In other words, $\sigma\mid_{V\setminus L} = f$.

Altogether, we have now proved that $\sigma\mid_{V\setminus L} = f$ under the assumption that $F \subseteq \mathbf{A}_\sigma$. In other words, we have proved the implication $(F \subseteq \mathbf{A}_\sigma) \implies \left(\sigma\mid_{V\setminus L} = f\right)$. $\square$

Proof of the implication $(F \subseteq \mathbf{A}_\sigma) \impliedby \left(\sigma\mid_{V\setminus L} = f\right)$. Assume that $\sigma\mid_{V\setminus L} = f$ holds. We must show that $F \subseteq \mathbf{A}_\sigma$ holds.

Indeed, let $a \in F$. Then,

$$\begin{aligned}
a \in F = \operatorname{Arcs} C &= \bigcup_{v\in C} \operatorname{Arcs} v \qquad (\text{by the definition of } \operatorname{Arcs} C) \\
&= \bigcup_{q\in C} \operatorname{Arcs} q \qquad (\text{here, we have renamed the index } v \text{ as } q).
\end{aligned}$$

In other words, $a \in \operatorname{Arcs} q$ for some $q \in C$. Consider this q.

Recall that C is a path cover of V. In other words, C is a set of paths of V such that each $v \in V$ belongs to exactly one of these paths (by the definition of a path cover). Hence, C is a set of paths of V. Thus, q is a path of V (since $q \in C$).

Write the path q as $q = (q_1, q_2, \ldots, q_k)$. Thus,

$$\begin{aligned}
&\operatorname{Arcs} q \\
&= \operatorname{Arcs}(q_1, q_2, \ldots, q_k) \\
&= \{(q_1, q_2), (q_2, q_3), \ldots, (q_{k-1}, q_k)\} \qquad (\text{by (2), applied to } v = q \text{ and } v_j = q_j).
\end{aligned}$$

Hence,

$$a \in \operatorname{Arcs} q = \{(q_1, q_2), (q_2, q_3), \ldots, (q_{k-1}, q_k)\}.$$

In other words, $a = (q_i, q_{i+1})$ for some $i \in [k-1]$. Consider this i.

Clearly, q_i is an entry of q. In other words, the path q contains q_i.

Recall that $\operatorname{path}(C, q_i)$ is defined as the unique path $p \in C$ that contains q_i. Hence, if $p \in C$ is a path that contains q_i, then $p = \operatorname{path}(C, q_i)$. Applying this to $p = q$, we conclude that $q = \operatorname{path}(C, q_i)$ (since $q \in C$ is a path that contains q_i).

We shall next show that $q_i \notin L$.

Indeed, assume the contrary. Thus, $q_i \in L = \{p_{\text{last}} \mid p \in C\}$ (by the definition of L). In other words, $q_i = p_{\text{last}}$ for some $p \in C$. Consider this p. Clearly, p_{last} is the last entry of p (by the definition of p_{last}), and thus belongs to p. In other words, q_i belongs to p (since $q_i = p_{\text{last}}$). But q_i also belongs to q (since q_i is an entry of q).

Recall that C is a set of paths of V such that each $v \in V$ belongs to exactly one of these paths. Hence, in particular, each $v \in V$ belongs to exactly one of the paths in C. Applying this to $v = q_i$, we conclude that q_i belongs to exactly one of the paths in C. However, both p and q are paths in C. Thus, if the paths p and q were distinct, then q_i would belong to (at least) two distinct paths in C (since q_i belongs to both p and q), which would contradict the fact that q_i belongs to exactly one of the paths in C. Hence, the paths p and q cannot be distinct. In other words, $p = q$.

Thus,

$$\begin{aligned} p_{\text{last}} = q_{\text{last}} &= (\text{the last entry of } q) && (\text{by the definition of } q_{\text{last}}) \\ &= q_k && (\text{since } q = (q_1, q_2, \ldots, q_k)). \end{aligned}$$

In other words, $q_i = q_k$ (since $q_i = p_{\text{last}}$).

However, q is a path of V. In other words, q is a nonempty tuple of distinct elements of V (by the definition of a path of V). Hence, in particular, the entries of q are distinct. In other words, $q_1, q_2, \ldots, q_k$ are distinct (since $q_1, q_2, \ldots, q_k$ are the entries of q (because $q = (q_1, q_2, \ldots, q_k)$)). In other words, if b and c are two elements of $[k]$ satisfying $q_b = q_c$, then $b = c$. Applying this to $b = i$ and $c = k$, we conclude that $i = k$ (since $q_i = q_k$). Hence, $i = k \notin [k-1]$ (since $k > k-1$). But this contradicts $i \in [k-1]$.

This contradiction shows that our assumption was false. Hence, $q_i \notin L$ is proved.

Combining $q_i \in V$ with $q_i \notin L$, we obtain $q_i \in V \setminus L$. Hence, $f(q_i)$ is well-defined.

Now, Claim 1 (applied to $w = q_i$) yields $f(q_i) = \operatorname{next}(\operatorname{path}(C, q_i), q_i)$ and $\operatorname{path}(C, f(q_i)) = \operatorname{path}(C, q_i)$. Hence,

$$\begin{aligned} f(q_i) &= \operatorname{next}\left(\underbrace{\operatorname{path}(C, q_i)}_{\substack{=q \\ (\text{since } q = \operatorname{path}(C, q_i))}}, q_i\right) = \operatorname{next}(q, q_i) \\ &= (\text{the next entry after } q_i \text{ in the tuple } q) \\ &\qquad \left(\begin{array}{c} \text{since } \operatorname{next}(q, q_i) \text{ is defined to be the} \\ \text{next entry after } q_i \text{ in the tuple } q \end{array}\right) \\ &= q_{i+1} \qquad \left(\begin{array}{c} \text{since } q = (q_1, q_2, \ldots, q_k), \text{ so that the entry } q_i \\ \text{is followed by } q_{i+1} \text{ in the tuple } q \end{array}\right). \end{aligned}$$

However, we assumed that $\sigma\mid_{V\setminus L} = f$ holds. Thus, $\left(\sigma\mid_{V\setminus L}\right)(q_i) = f(q_i) = q_{i+1}$. Therefore,

$$q_{i+1} = \left(\sigma\mid_{V\setminus L}\right)(q_i) = \sigma(q_i).$$

Now,

$$\begin{aligned} a &= \left(q_i, \underbrace{q_{i+1}}_{=\sigma(q_i)}\right) = (q_i, \sigma(q_i)) \\ &\in \{(v, \sigma(v)) \mid v \in V\} \qquad (\text{since } q_i \in V) \\ &= \mathbf{A}_\sigma \qquad (\text{since } \mathbf{A}_\sigma \text{ is defined to be } \{(v, \sigma(v)) \mid v \in V\}). \end{aligned}$$

Forget that we fixed a. We thus have shown that $a \in \mathbf{A}_\sigma$ for each $a \in F$. In other words, $F \subseteq \mathbf{A}_\sigma$.

Altogether, we have now proved that $F \subseteq \mathbf{A}_\sigma$ under the assumption that $\sigma\mid_{V\setminus L} = f$. In other words, we have proved the implication $(F \subseteq \mathbf{A}_\sigma) \Longleftarrow \left(\sigma\mid_{V\setminus L} = f\right)$. □

We have now proved the two implications $(F \subseteq \mathbf{A}_\sigma) \Longrightarrow \left(\sigma\mid_{V\setminus L} = f\right)$ and $(F \subseteq \mathbf{A}_\sigma) \Longleftarrow \left(\sigma\mid_{V\setminus L} = f\right)$. Combining them, we obtain the equivalence

$$(F \subseteq \mathbf{A}_\sigma) \Longleftrightarrow \left(\sigma\mid_{V\setminus L} = f\right).$$

Thus, Lemma 2.21 **(b)** is proved. □

We are now ready to prove Proposition 2.13:

Proof of Proposition 2.13. Assume that $F = \operatorname{Arcs} C$ for some path cover C of V. Consider this path cover C. Define the set L and the map $f : V \setminus L \to V$ as in Lemma 2.21. Clearly, L is a subset of V. Hence, $|V \setminus L| = |V| - |L|$. Also, $V \setminus L$ is a subset of V.

It is now easy to show that $|L| = |C|$. Indeed:

Proof of $|L| = |C|$. Let p and q be two distinct paths in C. We shall show that $p_{\text{last}} \neq q_{\text{last}}$.

Indeed, assume the contrary. Thus, $p_{\text{last}} = q_{\text{last}}$.

Note that p_{last} is the last entry of p (by the definition of p_{last}). Hence, in particular, p_{last} belongs to p. Similarly, q_{last} belongs to q. In other words, p_{last} belongs to q (since $p_{\text{last}} = q_{\text{last}}$).

Now, we know that p_{last} belongs to both p and q. Since p and q are two distinct paths in C, we thus conclude that p_{last} belongs to (at least) two distinct paths in C.

However, C is a path cover of V. In other words, C is a set of paths of V such that each $v \in V$ belongs to exactly one of these paths (by the definition of a path cover). In particular, each $v \in V$ belongs to exactly one of the paths in C. Applying this to $v = p_{\text{last}}$, we conclude that p_{last} belongs to exactly one of the paths in C. But this contradicts the fact that p_{last} belongs to (at least) two distinct paths in C.

This contradiction shows that our assumption was false. Hence, $p_{\text{last}} \neq q_{\text{last}}$ is proved.

Forget that we fixed p and q. We thus have shown that if p and q are two distinct paths in C, then $p_{\text{last}} \neq q_{\text{last}}$. In other words, the elements p_{last} for all $p \in C$ are distinct. Hence, there are $|C|$ many such elements in total. In other words, $|\{p_{\text{last}} \mid p \in C\}| = |C|$. Since $L = \{p_{\text{last}} \mid p \in C\}$ (by the definition of L), we can rewrite this as $|L| = |C|$. Thus, $|L| = |C|$ is proved. $\square$

Now, Lemma 2.21 **(a)** yields that the map f is an injection. On the other hand, if $\sigma \in \mathfrak{S}_V$ is a permutation, then the statement "$F \subseteq \mathbf{A}_\sigma$" is equivalent to "$\sigma\mid_{V \setminus L} = f$" (by Lemma 2.21 **(b)**). Hence,

$$\begin{aligned}
&(\text{\# of permutations } \sigma \in \mathfrak{S}_V \text{ satisfying } F \subseteq \mathbf{A}_\sigma) \\
&= \left(\text{\# of permutations } \sigma \in \mathfrak{S}_V \text{ satisfying } \sigma\mid_{V\setminus L} = f\right) \\
&= \left(\text{\# of permutations } \sigma \in \mathfrak{S}_V \text{ such that } \sigma\mid_{V\setminus L} = f\right) \\
&= \left(|V| - \underbrace{|V \setminus L|}_{=|V|-|L|}\right)! \qquad \text{(by Corollary 2.16, applied to } X = V \text{ and } Y = V \setminus L) \\
&= \left(\underbrace{|V| - (|V| - |L|)}_{=|L|}\right)! = \underbrace{|L|}_{=|C|}! = |C|!.
\end{aligned}$$

In other words, there are exactly $|C|!$ many permutations $\sigma \in \mathfrak{S}_V$ satisfying $F \subseteq \mathbf{A}_\sigma$. This proves Proposition 2.13. $\square$

2.4. Counting hamps by inclusion-exclusion

Our next lemma will be about counting Hamiltonian paths – which we abbreviate as "hamps". Here is how they are defined:

Definition 2.22. Let D be a digraph. A *hamp* of D means a D-path that contains each vertex of D. (The word "hamp" is short for "Hamiltonian path".)

For a digraph $D = (V, A)$, there is an obvious connection between the linear subsets of A and the hamps of D: If v is a hamp of D, then $\operatorname{Arcs} v$ is a maximum-size linear subset of A (and this maximum size is $|V| - 1$ if V is nonempty). More interestingly, there is a far less obvious connection between the linear subsets of A and the hamps of the complement $\overline{D}$:

Lemma 2.23. Let $D = (V, A)$ be a digraph with $V \neq \varnothing$. Then,

$$\sum_{F \subseteq A \text{ is linear}} (-1)^{|F|} \cdot (\text{\# of } \sigma \in \mathfrak{S}_V \text{ satisfying } F \subseteq \mathbf{A}_\sigma) = \left(\text{\# of hamps of } \overline{D}\right).$$

(We are using Convention 2.1 here.)

Proof. The hamps of D are precisely the D-paths that contain each vertex of D. In other words, the hamps of D are precisely the D-paths that are V-listings (because a D-path contains each vertex of D if and only if it is a V-listing). In other words, the hamps of D are precisely the V-listings that are D-paths. In other words, the hamps of D are precisely the nonempty V-listings that are D-paths (since every V-listing is nonempty[16]). In other words, the hamps of D are precisely the V-listings v that satisfy $\operatorname{Arcs} v \subseteq A$ (since a nonempty V-listing v is a D-path if and only if it satisfies $\operatorname{Arcs} v \subseteq A$).

Applying the same reasoning to the digraph $\overline{D} = (V, (V \times V) \setminus A)$ instead of the digraph $D = (V, A)$, we obtain the following: The hamps of $\overline{D}$ are precisely the V-listings v that satisfy $\operatorname{Arcs} v \subseteq (V \times V) \setminus A$. In other words, the hamps of $\overline{D}$ are precisely the V-listings v that satisfy $A \cap \operatorname{Arcs} v = \varnothing$ (because if v is a V-listing, then $\operatorname{Arcs} v$ is a subset of $V \times V$, and thus the statement "$\operatorname{Arcs} v \subseteq (V \times V) \setminus A$" is equivalent to "$A \cap \operatorname{Arcs} v = \varnothing$"). Hence,

$$\begin{aligned} &\left(\text{\# of hamps of } \overline{D}\right) \\ &= (\text{\# of } V\text{-listings } v \text{ that satisfy } A \cap \operatorname{Arcs} v = \varnothing). \end{aligned} \tag{5}$$

We will use the Iverson bracket notation (as in Convention 2.2). We have

$$\begin{aligned} &\sum_{v \text{ is a } V\text{-listing}} \underbrace{\sum_{\substack{F \subseteq A; \\ F \subseteq \operatorname{Arcs} v}} (-1)^{|F|}}_{\substack{= \sum\limits_{F \subseteq A \cap \operatorname{Arcs} v} (-1)^{|F|} \\ = [A \cap \operatorname{Arcs} v = \varnothing] \\ \text{(by Lemma 2.3)}}} \\ &= \sum_{v \text{ is a } V\text{-listing}} [A \cap \operatorname{Arcs} v = \varnothing] \\ &= \sum_{\substack{v \text{ is a } V\text{-listing;} \\ A \cap \operatorname{Arcs} v = \varnothing}} \underbrace{[A \cap \operatorname{Arcs} v = \varnothing]}_{\substack{=1 \\ \text{(since } A \cap \operatorname{Arcs} v = \varnothing)}} + \sum_{\substack{v \text{ is a } V\text{-listing;} \\ \text{we don't have } A \cap \operatorname{Arcs} v = \varnothing}} \underbrace{[A \cap \operatorname{Arcs} v = \varnothing]}_{\substack{=0 \\ \text{(since we don't} \\ \text{have } A \cap \operatorname{Arcs} v = \varnothing)}} \\ &= \sum_{\substack{v \text{ is a } V\text{-listing;} \\ A \cap \operatorname{Arcs} v = \varnothing}} 1 + \underbrace{\sum_{\substack{v \text{ is a } V\text{-listing;} \\ \text{we don't have } A \cap \operatorname{Arcs} v = \varnothing}} 0}_{=0} = \sum_{\substack{v \text{ is a } V\text{-listing;} \\ A \cap \operatorname{Arcs} v = \varnothing}} 1 \\ &= (\text{\# of } V\text{-listings } v \text{ that satisfy } A \cap \operatorname{Arcs} v = \varnothing). \end{aligned}$$

[16] This is because $V \neq \varnothing$.

Comparing this with (5), we obtain

$$\begin{aligned}
&\left(\text{\# of hamps of } \overline{D}\right) \\
&= \sum_{v \text{ is a } V\text{-listing}} \ \sum_{\substack{F\subseteq A;\\ F\subseteq \operatorname{Arcs} v}} (-1)^{|F|} \\
&= \sum_{F\subseteq A} \ \sum_{\substack{v \text{ is a } V\text{-listing};\\ F\subseteq \operatorname{Arcs} v}} (-1)^{|F|} \\
&= \sum_{F\subseteq A \text{ is linear}} \ \sum_{\substack{v \text{ is a } V\text{-listing};\\ F\subseteq \operatorname{Arcs} v}} (-1)^{|F|} + \sum_{F\subseteq A \text{ is not linear}} \ \underbrace{\sum_{\substack{v \text{ is a } V\text{-listing};\\ F\subseteq \operatorname{Arcs} v}} (-1)^{|F|}}_{\substack{=0 \\ \text{(because Proposition 2.8 } \textbf{(a)} \\ \text{shows that this sum is empty)}}} \\
&= \sum_{F\subseteq A \text{ is linear}} \ \sum_{\substack{v \text{ is a } V\text{-listing};\\ F\subseteq \operatorname{Arcs} v}} (-1)^{|F|} + \underbrace{\sum_{F\subseteq A \text{ is not linear}} 0}_{=0} \\
&= \sum_{F\subseteq A \text{ is linear}} \ \sum_{\substack{v \text{ is a } V\text{-listing};\\ F\subseteq \operatorname{Arcs} v}} (-1)^{|F|}. \qquad (6)
\end{aligned}$$

Now, let F be a linear subset of A. Thus, $F = \operatorname{Arcs} C$ for some path cover C of V. Consider this C. Then, Proposition 2.13 yields that there are exactly $|C|!$ many permutations $\sigma \in \mathfrak{S}_V$ satisfying $F \subseteq \mathbf{A}_\sigma$. In other words, we have

$$(\text{\# of } \sigma \in \mathfrak{S}_V \text{ satisfying } F \subseteq \mathbf{A}_\sigma) = |C|!. \qquad (7)$$

On the other hand, Proposition 2.8 **(b)** yields that there are exactly $|C|!$ many V-listings v satisfying $F \subseteq \operatorname{Arcs} v$. In other words, we have

$$(\text{\# of } V\text{-listings } v \text{ satisfying } F \subseteq \operatorname{Arcs} v) = |C|!.$$

Comparing this with (7), we find

$$\begin{aligned}
&(\text{\# of } V\text{-listings } v \text{ satisfying } F \subseteq \operatorname{Arcs} v) \\
&= (\text{\# of } \sigma \in \mathfrak{S}_V \text{ satisfying } F \subseteq \mathbf{A}_\sigma). \qquad (8)
\end{aligned}$$

Hence,

$$\begin{aligned}
\sum_{\substack{v \text{ is a } V\text{-listing};\\ F\subseteq \operatorname{Arcs} v}} (-1)^{|F|} &= \underbrace{(\text{\# of } V\text{-listings } v \text{ satisfying } F \subseteq \operatorname{Arcs} v)}_{\substack{=(\text{\# of } \sigma\in\mathfrak{S}_V \text{ satisfying } F\subseteq \mathbf{A}_\sigma) \\ \text{(by (8))}}} \cdot (-1)^{|F|} \\
&= (\text{\# of } \sigma \in \mathfrak{S}_V \text{ satisfying } F \subseteq \mathbf{A}_\sigma) \cdot (-1)^{|F|} \\
&= (-1)^{|F|} \cdot (\text{\# of } \sigma \in \mathfrak{S}_V \text{ satisfying } F \subseteq \mathbf{A}_\sigma). \qquad (9)
\end{aligned}$$

Forget that we fixed F. We thus have proved (9) for each linear subset F of A. Now, (6) becomes

$$\left(\text{\# of hamps of } \overline{D}\right) = \sum_{F\subseteq A \text{ is linear}} \underbrace{\sum_{\substack{v \text{ is a } V\text{-listing;} \\ F\subseteq \operatorname{Arcs} v}} (-1)^{|F|}}_{\substack{=(-1)^{|F|}\cdot(\text{\# of } \sigma\in\mathfrak{S}_V \text{ satisfying } F\subseteq\mathbf{A}_\sigma) \\ \text{(by (9))}}}$$
$$= \sum_{F\subseteq A \text{ is linear}} (-1)^{|F|} \cdot (\text{\# of } \sigma \in \mathfrak{S}_V \text{ satisfying } F \subseteq \mathbf{A}_\sigma) .$$

This proves Lemma 2.23. □

2.5. Level decomposition and maps f satisfying $f \circ \sigma = f$

This entire subsection is devoted to building up some language that will only ever be used in the proof of Lemma 2.39. A reader familiar with combinatorial tropes should be able to skip all proofs in this subsection, along with many of the statements; nothing substantial is being done here, and all hindrances being surmounted are notational. We would not be surprised if the entire argument could be simplified or made slicker using some algebraic notions, but we have not been able to find such notions.

We shall study what happens when a function $f : V \to \mathbb{P}$ is introduced into a digraph $D = (V, A)$. The nonempty fibers of f (i.e., the sets $f^{-1}(j)$ for all $j \in f(V)$) partition the vertex set V, and this leads to a decomposition of D into subdigraphs. Let us introduce some notation for this, starting with the case of an arbitrary set V (we will later specialize to digraphs):

Definition 2.24. Let V be any set. Let $f : V \to \mathbb{P}$ be any map.

- **(a)** For each $v \in V$, we will refer to the number $f(v)$ as the *level* of v (with respect to f).
- **(b)** For each $j \in \mathbb{P}$, the subset $f^{-1}(j)$ of V shall be called the *j-th level set* of f.

Example 2.25. Let $V = \{1, 2, 3\}$. Let $f : V \to \mathbb{P}$ be given by $f(1) = 1$, $f(2) = 4$ and $f(3) = 1$. Then, the level sets of f are

$$f^{-1}(1) = \{1, 3\}, \qquad f^{-1}(4) = \{2\}, \qquad \text{and}$$
$$f^{-1}(j) = \varnothing \text{ for all } j \in \mathbb{P} \setminus \{1, 4\}.$$

Remark 2.26. Let V be any set. Let $f : V \to \mathbb{P}$ be any map. Let $j \in \mathbb{P}$. Then, the j-th level set $f^{-1}(j)$ is empty if and only if $j \notin f(V)$. Hence, the nonempty level sets of f correspond to the elements of $f(V)$.

Definition 2.27. Let $D = (V, A)$ be a digraph. Let $f : V \to \mathbb{P}$ be any map.

(a) For each $j \in \mathbb{P}$, we define a subset A_j of A by

$$\begin{aligned} A_j &:= \left\{ (u, v) \in A \mid u, v \in f^{-1}(j) \right\} & (10)\\ &= \{ (u, v) \in A \mid f(u) = f(v) = j \} & (11)\\ &= A \cap \left(f^{-1}(j) \times f^{-1}(j) \right). & (12) \end{aligned}$$

This set A_j is also a subset of $f^{-1}(j) \times f^{-1}(j)$, of course.

(b) We let A_f denote the subset

$$\{ (u, v) \in A \mid f(u) = f(v) \}$$

of A.

(c) For each $j \in \mathbb{P}$, we let D_j denote the digraph $\left(f^{-1}(j), A_j \right)$. This digraph D_j is the restriction of the digraph D to the subset $f^{-1}(j)$ (that is, the digraph obtained from D by removing all vertices that don't belong to $f^{-1}(j)$ and all arcs that contain any of these vertices).

This digraph D_j will be called the *j-th level subdigraph* of D with respect to f. (We should properly call it $D_{j,f}$ instead of D_j, but we will usually keep f fixed when we study it.)

Example 2.28. Let D be as in Example 1.3. Let $f : V \to \mathbb{P}$ be given by $f(1) = 1$, $f(2) = 4$ and $f(3) = 1$. Then,

$$\begin{aligned} A_1 &= \{(3,3)\}, \qquad A_4 = \{(2,2)\}, \\ A_j &= \varnothing \text{ for all } j \in \mathbb{P} \setminus \{1,4\}, \end{aligned}$$

and

$$A_f = \{(3,3), (2,2)\}.$$

The level subdigraphs of D are the two digraphs

$$D_1 = (\{1,3\}, \{(3,3)\}) \qquad \text{and} \qquad D_4 = (\{2\}, \{(2,2)\})$$

(as well as the infinitely many empty digraphs D_j for all $j \in \mathbb{P} \setminus \{1,4\}$). Note that the arc $(3,3)$ of D is contained in D_1, and the arc $(2,2)$ is contained in D_4, but the arc $(1,2)$ is contained in none of the level subdigraphs (since its two endpoints 1 and 2 have different levels).

Remark 2.29. Let $D = (V, A)$ be a digraph. Let $f : V \to \mathbb{P}$ be any map. Let $j \in \mathbb{P}$. Then, the j-th level subdigraph D_j and its arc set A_j are empty if $j \notin f(V)$. (However, A_j can be empty even if j does belong to $f(V)$.)

In the following, the symbols "$\sqcup$" and "$\bigsqcup$" stand for unions of disjoint sets. Thus, for example, "$A_1 \sqcup A_2 \sqcup A_3 \sqcup \cdots$" will mean the union of some (pairwise) disjoint sets $A_1, A_2, A_3, \ldots$.

Proposition 2.30. Let V and J be two finite sets. Let V_j be a subset of V for each $j \in J$. Assume that the sets V_j for different $j \in J$ are disjoint. Let C_j be a path cover of V_j for each $j \in J$. Then:

(a) The sets C_j for different $j \in J$ are disjoint.

(b) Their union $\bigsqcup\limits_{j \in J} C_j$ is a path cover of $\bigsqcup\limits_{j \in J} V_j$, and its arc set is $\operatorname{Arcs}\left(\bigsqcup\limits_{j \in J} C_j\right) = \bigsqcup\limits_{j \in J} \operatorname{Arcs}\left(C_j\right)$.

Proof. **(a)** It suffices to show that if A and B are two disjoint finite sets, then any path cover of A is disjoint from any path cover of B. But this is clear, since the elements of a path cover of A are paths of A, whereas the elements of a path cover of B are paths of B, and clearly a path of A cannot be a path of B (since A and B are disjoint).

(b) This is obvious from the definitions of path covers and arc sets. □

Corollary 2.31. Let V and J be two finite sets. Let V_j be a subset of V for each $j \in J$. Assume that the sets V_j for different $j \in J$ are disjoint. For each $j \in J$, let F_j be a linear subset of $V_j \times V_j$. Then, the union $\bigcup\limits_{j \in J} F_j$ is a linear subset of $V \times V$.

Proof. Let W be the union $\bigcup\limits_{j \in J} V_j$. This union $W = \bigcup\limits_{j \in J} V_j$ is a subset of V (since V_j is a subset of V for each $j \in J$).

The sets V_j for different $j \in J$ are disjoint (by assumption). Thus, their union $\bigcup\limits_{j \in J} V_j$ is a disjoint union. In other words, $\bigcup\limits_{j \in J} V_j = \bigsqcup\limits_{j \in J} V_j$. In other words, $W = \bigsqcup\limits_{j \in J} V_j$ (since $W = \bigcup\limits_{j \in J} V_j$).

For each $j \in J$, the set F_j is a linear subset of $V_j \times V_j$ (by assumption), and thus is the arc set of some path cover C_j of V_j (by the definition of "linear"). In other words, for each $j \in J$, there exists a path cover C_j of V_j such that

$$F_j = \operatorname{Arcs}\left(C_j\right). \tag{13}$$

Consider these path covers C_j.

Proposition 2.30 **(a)** shows that these path covers C_j for different $j \in J$ are disjoint. Hence, their union $\bigcup\limits_{j\in J} C_j$ is a disjoint union. In other words, $\bigcup\limits_{j\in J} C_j = \bigsqcup\limits_{j\in J} C_j$.

Proposition 2.30 **(b)** shows that their union $\bigsqcup\limits_{j\in J} C_j$ is a path cover of $\bigsqcup\limits_{j\in J} V_j$, and its arc set is $\operatorname{Arcs}\left(\bigsqcup\limits_{j\in J} C_j\right) = \bigsqcup\limits_{j\in J} \operatorname{Arcs}\left(C_j\right)$.

In particular, $\bigsqcup\limits_{j\in J} C_j$ is a path cover of $\bigsqcup\limits_{j\in J} V_j = W$. The arc set of this path cover is

$$\operatorname{Arcs}\left(\bigsqcup_{j\in J} C_j\right) = \bigsqcup_{j\in J} \underbrace{\operatorname{Arcs}\left(C_j\right)}_{\substack{=F_j \\ \text{(by (13))}}} = \bigsqcup_{j\in J} F_j = \bigcup_{j\in J} F_j.$$

Hence, $\bigcup\limits_{j\in J} F_j$ is the arc set of some path cover of W (namely, of the path cover $\bigsqcup\limits_{j\in J} C_j$). In other words, F is linear as a subset of $W \times W$ (by the definition of "linear"). Therefore, F is linear as a subset of $V \times W$ (by Proposition 2.9). This proves Corollary 2.31. □

Proposition 2.32. Let $D = (V, A)$ be a digraph. Let $f : V \to \mathbb{P}$ be any map. Then, the sets $A_1, A_2, A_3, \ldots$ are disjoint, and their union is

$$A_1 \sqcup A_2 \sqcup A_3 \sqcup \cdots = \bigsqcup_{j\in f(V)} A_j = A_f.$$

Proof. The sets

$$A_j = \{(u,v) \in A \mid f(u) = f(v) = j\} \tag{14}$$

for different $j \in \mathbb{P}$ are clearly disjoint, because a pair $(u, v) \in A$ cannot satisfy $f(u) = f(v) = j$ for two different values of j at the same time. In other words, the sets $A_1, A_2, A_3, \ldots$ are disjoint. Hence, their union is

$$\begin{aligned} A_1 \sqcup A_2 \sqcup A_3 \sqcup \cdots &= \bigsqcup_{j\in\mathbb{P}} A_j \\ &= \bigsqcup_{j\in\mathbb{P}} \{(u,v) \in A \mid f(u) = f(v) = j\} \qquad \text{(by (14))} \\ &= \{(u,v) \in A \mid f(u) = f(v) = j \text{ for some } j \in \mathbb{P}\} \\ &= \{(u,v) \in A \mid f(u) = f(v)\} \\ &= A_f \qquad \left(\text{by the definition of } A_f\right). \end{aligned}$$

It remains to observe that $A_1 \sqcup A_2 \sqcup A_3 \sqcup \cdots = \bigsqcup\limits_{j\in f(V)} A_j$ (since A_j is empty whenever $j \notin f(V)$). □

Let us now connect the level decomposition to linear sets:

Proposition 2.33. Let $D = (V, A)$ be a digraph. Let $f : V \to \mathbb{P}$ be any map. Let F be any set. Then:

(a) The set F is a linear subset of A_f if and only if F can be written as $F = \bigsqcup_{j \in f(V)} F_j$, where each F_j is a linear subset of A_j.

(b) In this case, the subsets F_j are uniquely determined by F (namely, $F_j = F \cap A_j$ for each $j \in f(V)$).

Proof. **(a)** $\Longleftarrow$: Assume that F can be written as $F = \bigsqcup_{j \in f(V)} F_j$, where each F_j is a linear subset of A_j. Consider these linear subsets F_j.

Each F_j is a linear subset of A_j (by assumption) and therefore is a linear subset of $f^{-1}(j) \times f^{-1}(j)$ as well (since $A_j \subseteq f^{-1}(j) \times f^{-1}(j)$). The level sets $f^{-1}(j)$ for different j's are disjoint. Thus, Corollary 2.31 (applied to $J = f(V)$ and $V_j = f^{-1}(j)$) shows that the union $\bigcup_{j \in f(V)} F_j$ is a linear subset of $V \times V$. In other words, F is a linear subset of $V \times V$ (since $F = \bigsqcup_{j \in f(V)} F_j = \bigcup_{j \in f(V)} F_j$). Furthermore,

$$F = \bigsqcup_{j \in f(V)} \underbrace{F_j}_{\substack{\subseteq A_j \\ \text{(since } F_j \text{ is a linear subset of } A_j \\ \text{(by assumption))}}} \subseteq \bigsqcup_{j \in f(V)} A_j = A_f$$

(by Proposition 2.32). Hence, F is a subset of A_f. This shows that F is a linear subset of A_f (since F is linear). This proves the "$\Longleftarrow$" direction of Proposition 2.33 **(a)**.

$\Longrightarrow$: Assume that F is a linear subset of A_f. In particular, F is linear. Thus, F is the arc set of a path cover C of V. Consider this C. Thus, $F = \operatorname{Arcs} C$.

We say that a path of V is *level* if all entries of this path have the same level (with respect to f). If p is a level path of V, then the *level* of p will mean the level of each entry of p.

We claim that each path in C is level. Indeed, let $v = (v_1, v_2, \ldots, v_k)$ be a path in C. Then, the pairs (v_1, v_2), (v_2, v_3), $\ldots$, (v_{k-1}, v_k) are arcs of v, thus belong to $\operatorname{Arcs} v$, therefore belong to $\operatorname{Arcs} C$ (since v is a path in C, and thus we have $\operatorname{Arcs} v \subseteq \operatorname{Arcs} C$). In other words, each $i \in [k-1]$ satisfies $(v_i, v_{i+1}) \in \operatorname{Arcs} C$. Hence, each $i \in [k-1]$ satisfies $(v_i, v_{i+1}) \in \operatorname{Arcs} C = F \subseteq A_f$ and therefore $f(v_i) = f(v_{i+1})$ (by the definition of A_f). In other words, $f(v_1) = f(v_2) = \cdots = f(v_k)$. In other words, all entries $v_1, v_2, \ldots, v_k$ of v have the same level. In other words, v is level. Forget now that we fixed v. We thus have shown that each path $v = (v_1, v_2, \ldots, v_k)$ in C is level. In other words, each path in C is level.

Hence, each path in C has a (unique) level. Set

$$C_j := \{\text{all paths of level } j \text{ in } C\} \qquad \text{for each } j \in \mathbb{P}.$$

Then, the sets $C_1, C_2, C_3, \ldots$ are disjoint (since a path cannot have two different levels simultaneously), and their union $C_1 \sqcup C_2 \sqcup C_3 \sqcup \cdots$ is C (since each path in C has a level). In particular, $C = C_1 \sqcup C_2 \sqcup C_3 \sqcup \cdots$.

Let $j \in \mathbb{P}$. Then, C_j is a path cover of $f^{-1}(j)$ [17]. Hence, $\operatorname{Arcs}(C_j)$ is a linear subset of $f^{-1}(j) \times f^{-1}(j)$ (by the definition of "linear"). Furthermore, $C_j \subseteq C$ (by the definition of C_j) and therefore $\operatorname{Arcs}(C_j) \subseteq \operatorname{Arcs} C = F \subseteq A_f \subseteq A$. Combining this with $\operatorname{Arcs}(C_j) \subseteq f^{-1}(j) \times f^{-1}(j)$, we obtain

$$\begin{aligned}\operatorname{Arcs}(C_j) &\subseteq A \cap \left(f^{-1}(j) \times f^{-1}(j)\right) \\ &= A_j \qquad \left(\text{since } A_j \text{ is defined to be } A \cap \left(f^{-1}(j) \times f^{-1}(j)\right)\right).\end{aligned}$$

Thus, $\operatorname{Arcs}(C_j)$ is a linear subset of A_j (since $\operatorname{Arcs}(C_j)$ is linear).

Forget that we fixed j. We thus have shown that $\operatorname{Arcs}(C_j)$ is a linear subset of A_j for each $j \in \mathbb{P}$.

In other words, the sets $\operatorname{Arcs}(C_1)$, $\operatorname{Arcs}(C_2)$, $\operatorname{Arcs}(C_3)$, $\ldots$ are subsets of the sets $A_1, A_2, A_3, \ldots$, respectively. Since the latter sets $A_1, A_2, A_3, \ldots$ are disjoint (by Proposition 2.32), we thus conclude that their subsets $\operatorname{Arcs}(C_1)$, $\operatorname{Arcs}(C_2)$, $\operatorname{Arcs}(C_3)$, $\ldots$ are disjoint as well.

Moreover, each positive integer $j \notin f(V)$ satisfies $\operatorname{Arcs}(C_j) = \varnothing$ [18].

---

[17] *Proof.* Let $p \in C_j$ be a path. Then, p is a path of level j in C (since $p \in C_j = \{\text{all paths of level } j \text{ in } C\}$). Therefore, all entries of p have level j. In other words, all entries of p belong to $f^{-1}(j)$. Hence, p is a path of $f^{-1}(j)$ (not just a path of V).

Forget that we fixed p. Thus, we have shown that each path $p \in C_j$ is a path of $f^{-1}(j)$. In other words, C_j is a set of paths of $f^{-1}(j)$.

Any element $v \in f^{-1}(j)$ belongs to V, and therefore must belong to a unique path in C (since C is a path cover of V). This latter path must have level j (since v has level j) and therefore belong to C_j (by the definition of C_j). Hence, we conclude that any element $v \in f^{-1}(j)$ belongs to a unique path in C_j. This shows that C_j is a path cover of $f^{-1}(j)$ (since C_j is a set of paths of $f^{-1}(j)$).

[18] *Proof.* Let j be a positive integer such that $j \notin f(V)$. Then, $f^{-1}(j) = \varnothing$. However, we have shown above that $\operatorname{Arcs}(C_j)$ is a linear subset of A_j. Hence,

$$\begin{aligned}\operatorname{Arcs}(C_j) &\subseteq A_j = A \cap \left(f^{-1}(j) \times f^{-1}(j)\right) \qquad (\text{by the definition of } A_j) \\ &\subseteq \underbrace{f^{-1}(j)}_{=\varnothing} \times \underbrace{f^{-1}(j)}_{=\varnothing} = \varnothing \times \varnothing = \varnothing,\end{aligned}$$

so that $\operatorname{Arcs}(C_j) = \varnothing$.

From $C = C_1 \sqcup C_2 \sqcup C_3 \sqcup \cdots = \bigsqcup_{j \in \mathbb{P}} C_j = \bigcup_{j \in \mathbb{P}} C_j$, we obtain

$$\begin{aligned}
\operatorname{Arcs} C &= \operatorname{Arcs}\left(\bigcup_{j\in\mathbb{P}} C_j\right) = \bigcup_{j\in\mathbb{P}} \operatorname{Arcs}\left(C_j\right) \\
&= \bigsqcup_{j\in\mathbb{P}} \operatorname{Arcs}\left(C_j\right) \qquad \left(\begin{array}{c}\text{since the}\\ \text{sets } \operatorname{Arcs}\left(C_1\right),\ \operatorname{Arcs}\left(C_2\right),\ \operatorname{Arcs}\left(C_3\right),\ \ldots\\ \text{are disjoint}\end{array}\right) \\
&= \bigsqcup_{j\in f(V)} \operatorname{Arcs}\left(C_j\right) \qquad (15)
\end{aligned}$$

(since each $j \notin f(V)$ satisfies $\operatorname{Arcs}\left(C_j\right) = \varnothing$).

Thus, $\operatorname{Arcs} C$ can be written as $\bigsqcup_{j\in f(V)} F_j$, where each F_j is a linear subset of A_j (since $\operatorname{Arcs}\left(C_j\right)$ is a linear subset of A_j for each $j \in \mathbb{P}$). In other words, F can be written in this way (since $F = \operatorname{Arcs} C$). This proves the "$\Longrightarrow$" direction of Proposition 2.33 **(a)**.

(b) Assume that F is written as $F = \bigsqcup_{j\in f(V)} F_j$, where each F_j is a linear subset of A_j. We must show that $F_j = F \cap A_j$ for each $j \in f(V)$.

Indeed, we have $F = \bigsqcup_{j\in f(V)} F_j = \bigsqcup_{i\in f(V)} F_i$.

Now, let $j \in f(V)$. Then, F_j is a subset of F (since $F = \bigsqcup_{i\in f(V)} F_i$) and also a subset of A_j (by definition of F_j). In other words, F_j is a subset of both F and A_j. Thus, F_j is a subset of the intersection $F \cap A_j$ as well. Let us now show that $F \cap A_j$ is a subset of F_j.

Indeed, let $\alpha \in F \cap A_j$. Then, $\alpha \in F \cap A_j \subseteq F = \bigsqcup_{i\in f(V)} F_i$, so that $\alpha \in F_i$ for some $i \in f(V)$. Consider this i. Then, $\alpha \in F_i \subseteq A_i$ (by the definition of F_i). However, $\alpha \in F \cap A_j \subseteq A_j$. Thus, the element α belongs to both sets A_i and A_j. Therefore, the sets A_i and A_j are not disjoint. However, Proposition 2.32 shows that the sets $A_1, A_2, A_3, \ldots$ are disjoint. The only way to reconcile the previous two sentences is when $i = j$.

Thus, we obtain $i = j$. Hence, $\alpha \in F_j$ (since $\alpha \in F_i$).

Forget that we fixed α. We thus have shown that $\alpha \in F_j$ for each $\alpha \in F \cap A_j$. In other words, $F \cap A_j \subseteq F_j$. Since F_j is (in turn) a subset of $F \cap A_j$, we thus conclude that $F_j = F \cap A_j$. This completes the proof of Proposition 2.33 **(b)**. $\square$

Next, we return to studying permutations.

When a set V is a union of two disjoint subsets A and B, and we are given a permutation σ_A of A and a permutation σ_B of B, then we can combine these two permutations to obtain a permutation $\sigma_A \oplus \sigma_B$ of V: Namely, this latter permutation sends each $a \in A$ to $\sigma_A(a)$, and sends each $b \in B$ to $\sigma_B(b)$. That is, this permutation $\sigma_A \oplus \sigma_B$ is "acting as σ_A" on the subset A and "acting as σ_B" on the subset B.

The same construction can be performed when V is a union of more than two disjoint subsets (and we are given a permutation of each of these subsets). We will encounter this situation when a map $f : V \to \mathbb{P}$ subdivides the set V into its level sets $f^{-1}(1)$, $f^{-1}(2)$, $f^{-1}(3)$, ..., and we are given a permutation $\sigma_j \in \mathfrak{S}_{f^{-1}(j)}$ of each level set $f^{-1}(j)$ (to be more precise, we only need σ_j to be given when $j \in f(V)$, since the level set $f^{-1}(j)$ is empty otherwise). The permutation of V obtained by combining these permutations σ_j will then be denoted by $\bigoplus_{j \in f(V)} \sigma_j$. Here is its explicit definition:

Definition 2.34. Let V be any set. Let $f : V \to \mathbb{P}$ be any map.

For each $j \in f(V)$, let $\sigma_j \in \mathfrak{S}_{f^{-1}(j)}$ be a permutation of the level set $f^{-1}(j)$.

Then, $\bigoplus_{j \in f(V)} \sigma_j$ shall denote the permutation of V that sends each $v \in V$ to $\sigma_{f(v)}(v)$. This is the permutation that acts as σ_j on each level set $f^{-1}(j)$.

Proposition 2.35. Let V be any set. Let $f : V \to \mathbb{P}$ be any map. Let $\sigma \in \mathfrak{S}_V$ be any permutation. Then:

- **(a)** We have $f \circ \sigma = f$ if and only if σ can be written in the form $\sigma = \bigoplus_{j \in f(V)} \sigma_j$, where $\sigma_j \in \mathfrak{S}_{f^{-1}(j)}$ for each $j \in f(V)$.
- **(b)** In this case, the permutations σ_j for all $j \in f(V)$ are uniquely determined by σ (namely, σ_j is the restriction of σ to the subset $f^{-1}(j)$ for each $j \in f(V)$).

Proof. **(a)** $\Longrightarrow$: Assume that $f \circ \sigma = f$.

Let $j \in f(V)$. Let $v \in f^{-1}(j)$. Then, $f(v) = j$. However, $f(\sigma(v)) = \underbrace{(f \circ \sigma)}_{=f}(v) = f(v) = j$, so that $\sigma(v) \in f^{-1}(j)$.

Forget that we fixed v. We thus have shown that $\sigma(v) \in f^{-1}(j)$ for each $v \in f^{-1}(j)$. Hence, the map

$$\begin{aligned} f^{-1}(j) &\to f^{-1}(j), \\ v &\mapsto \sigma(v) \end{aligned}$$

is well-defined. Let us denote this map by σ_j. This map σ_j is the restriction of σ to the subset $f^{-1}(j)$ of V.

The map σ is a permutation, thus has an inverse σ^{-1}. From $f \circ \sigma = f$, we obtain $\underbrace{f}_{=f \circ \sigma} \circ \sigma^{-1} = f \circ \underbrace{\sigma \circ \sigma^{-1}}_{=\mathrm{id}} = f$. Hence, just as we have constructed a map $\sigma_j : f^{-1}(j) \to f^{-1}(j)$ by restricting the map σ to $f^{-1}(j)$, we can likewise construct a map $\left(\sigma^{-1}\right)_j : f^{-1}(j) \to f^{-1}(j)$ by restricting the map σ^{-1} to $f^{-1}(j)$. These two maps σ_j and $\left(\sigma^{-1}\right)_j$ are mutually inverse (since they are restrictions of the mutually inverse maps σ and σ^{-1}). Hence, the map σ_j is invertible, i.e., is a permutation of $f^{-1}(j)$. In other words, $\sigma_j \in \mathfrak{S}_{f^{-1}(j)}$.

Forget now that we fixed j. Thus, for each $j \in f(V)$, we have constructed a permutation $\sigma_j \in \mathfrak{S}_{f^{-1}(j)}$ by restricting the map σ to $f^{-1}(j)$. These permutations clearly satisfy $\sigma = \bigoplus_{j \in f(V)} \sigma_j$ (since $V = \bigsqcup_{j \in f(V)} f^{-1}(j)$). This proves the "$\Longrightarrow$" direction of Proposition 2.35 **(a)**.

$\Longleftarrow$: Assume that σ can be written in the form $\sigma = \bigoplus_{j \in f(V)} \sigma_j$, where $\sigma_j \in \mathfrak{S}_{f^{-1}(j)}$ for each $j \in f(V)$.

Let $v \in V$. Let $i = f(v)$. Thus, $i \in f(V)$. From $\sigma = \bigoplus_{j \in f(V)} \sigma_j$, we obtain

$$\begin{aligned}
\sigma(v) &= \sigma_{f(v)}(v) && \left(\text{by the definition of } \bigoplus_{j \in f(V)} \sigma_j\right) \\
&= \sigma_i(v) && (\text{since } f(v) = i) \\
&\in f^{-1}(i) && \left(\text{since } \sigma_i \in \mathfrak{S}_{f^{-1}(i)} \text{ is a map from } f^{-1}(i) \text{ to } f^{-1}(i)\right).
\end{aligned}$$

In other words, $f(\sigma(v)) = i$. Hence, $(f \circ \sigma)(v) = f(\sigma(v)) = i = f(v)$.

Forget that we fixed v. We thus have shown that $(f \circ \sigma)(v) = f(v)$ for each $v \in V$. In other words, $f \circ \sigma = f$. We thus have proved the "$\Longleftarrow$" direction of Proposition 2.35 **(a)**.

(b) This is obvious. $\square$

Now, we recall the set $\mathbf{A}_\sigma$ defined in Definition 2.11 for any finite set V and any permutation σ of V.

Proposition 2.36. Let V be a finite set. Let $f : V \to \mathbb{P}$ be any map. Let $\sigma \in \mathfrak{S}_V$ be a permutation satisfying $f \circ \sigma = f$. Write σ in the form $\sigma = \bigoplus_{j \in f(V)} \sigma_j$, where $\sigma_j \in \mathfrak{S}_{f^{-1}(j)}$ for each $j \in f(V)$. (This can be done, because of Proposition 2.35 **(a)**.) Then,

$$\mathbf{A}_\sigma = \bigsqcup_{j \in f(V)} \mathbf{A}_{\sigma_j}.$$

Proof. We have $\sigma = \bigoplus_{j \in f(V)} \sigma_j$. Thus, for each $j \in f(V)$ and each $v \in f^{-1}(j)$, we have

$$\sigma(v) = \sigma_{f(v)}(v) = \sigma_j(v) \tag{16}$$

(since $f(v) = j$ (because $v \in f^{-1}(j)$)).

It is easy to see that the sets $\mathbf{A}_{\sigma_j}$ for different $j \in f(V)$ are disjoint[19]. Hence, the union of these sets is a disjoint union. That is,

$$\bigcup_{j\in f(V)} \mathbf{A}_{\sigma_j} = \bigsqcup_{j\in f(V)} \mathbf{A}_{\sigma_j}.$$

The definition of $\mathbf{A}_\sigma$ yields

$$\begin{aligned}
\mathbf{A}_\sigma &= \{(v,\sigma(v)) \mid v \in V\} \\
&= \bigcup_{j\in f(V)} \underbrace{\{(v,\sigma(v)) \mid v \in V \text{ and } f(v) = j\}}_{\substack{=\left\{(v,\sigma(v)) \mid v\in f^{-1}(j)\right\} \\ \text{(since the elements } v\in V \text{ satisfying } f(v)=j \\ \text{are precisely the elements of } f^{-1}(j))}} \\
&\qquad \left(\text{since each } v \in V \text{ satisfies } f(v) = j \text{ for some } j \in f(V)\right) \\
&= \bigcup_{j\in f(V)} \left\{\left(v, \underbrace{\sigma(v)}_{\substack{=\sigma_j(v) \\ \text{(by (16))}}}\right) \mid v \in f^{-1}(j)\right\} \\
&= \bigcup_{j\in f(V)} \underbrace{\left\{\left(v,\sigma_j(v)\right) \mid v \in f^{-1}(j)\right\}}_{\substack{=\mathbf{A}_{\sigma_j} \\ \text{(since } \mathbf{A}_{\sigma_j} \text{ is defined as } \left\{\left(v,\sigma_j(v)\right) \mid v\in f^{-1}(j)\right\})}} \\
&= \bigcup_{j\in f(V)} \mathbf{A}_{\sigma_j} = \bigsqcup_{j\in f(V)} \mathbf{A}_{\sigma_j}.
\end{aligned}$$

This proves Proposition 2.36. □

Next, we connect the above construction with the level subdigraphs of a digraph:

[19] *Proof.* Let i and j be two distinct elements of $f(V)$. We must prove that $\mathbf{A}_{\sigma_i}$ and $\mathbf{A}_{\sigma_j}$ are disjoint.

Indeed, the sets $f^{-1}(i)$ and $f^{-1}(j)$ are disjoint (since i and j are distinct). In other words, $f^{-1}(i) \cap f^{-1}(j) = \varnothing$. However, $\mathbf{A}_{\sigma_j} \subseteq f^{-1}(j) \times f^{-1}(j)$ (since σ_j is a permutation of $f^{-1}(j)$) and $\mathbf{A}_{\sigma_i} \subseteq f^{-1}(i) \times f^{-1}(i)$ (likewise). Hence,

$$\begin{aligned}
\underbrace{\mathbf{A}_{\sigma_i}}_{\subseteq f^{-1}(i)\times f^{-1}(i)} \cap \underbrace{\mathbf{A}_{\sigma_j}}_{\subseteq f^{-1}(j)\times f^{-1}(j)} &\subseteq \left(f^{-1}(i) \times f^{-1}(i)\right) \cap \left(f^{-1}(j) \times f^{-1}(j)\right) \\
&= \underbrace{\left(f^{-1}(i) \cap f^{-1}(j)\right)}_{=\varnothing} \times \underbrace{\left(f^{-1}(i) \cap f^{-1}(j)\right)}_{=\varnothing} \\
&= \varnothing \times \varnothing = \varnothing.
\end{aligned}$$

Hence, $\mathbf{A}_{\sigma_i} \cap \mathbf{A}_{\sigma_j} = \varnothing$. In other words, $\mathbf{A}_{\sigma_i}$ and $\mathbf{A}_{\sigma_j}$ are disjoint. This completes our proof.

Proposition 2.37. Let $D = (V, A)$ be a digraph. Let $f : V \to \mathbb{P}$ be any map. Let $\sigma \in \mathfrak{S}_V$ be a permutation satisfying $f \circ \sigma = f$. Then,

$$\mathbf{A}_\sigma \cap A \subseteq A_f.$$

Proof. Let $\alpha \in \mathbf{A}_\sigma \cap A$. Thus, $\alpha \in \mathbf{A}_\sigma$ and $\alpha \in A$. In particular, $\alpha \in \mathbf{A}_\sigma = \{(v, \sigma(v)) \mid v \in V\}$ (by the definition of $\mathbf{A}_\sigma$). Hence, $\alpha = (v, \sigma(v))$ for some $v \in V$. Consider this v. We have $f(\sigma(v)) = \underbrace{(f \circ \sigma)}_{=f}(v) = f(v)$. In other words, $f(v) = f(\sigma(v))$. Hence, $(v, \sigma(v)) \in A_f$ (by the definition of A_f, since $(v, \sigma(v)) = \alpha \in A$). In other words, $\alpha \in A_f$ (since $\alpha = (v, \sigma(v))$).

Forget that we fixed α. We thus have proved that $\alpha \in A_f$ for each $\alpha \in \mathbf{A}_\sigma \cap A$. In other words, $\mathbf{A}_\sigma \cap A \subseteq A_f$. □

Our last result in this section is the following trivial yet complex-looking lemma, which will be used in the proof after it:

Lemma 2.38. Let $D = (V, A)$ be a digraph. Let $f : V \to \mathbb{P}$ be any map. Let $\sigma_j \in \mathfrak{S}_{f^{-1}(j)}$ be a permutation for each $j \in f(V)$. Let F_j be a subset of A_j for each $j \in f(V)$. Then, we have the following logical equivalence:

$$\left(\bigsqcup_{j \in f(V)} F_j \subseteq \bigsqcup_{j \in f(V)} \mathbf{A}_{\sigma_j} \right) \iff \left(F_j \subseteq \mathbf{A}_{\sigma_j} \text{ for each } j \in f(V) \right).$$

Proof. The sets $f^{-1}(j)$ for different $j \in f(V)$ are clearly disjoint. Hence, the sets $f^{-1}(j) \times f^{-1}(j)$ for different $j \in f(V)$ are disjoint as well[20].

For each $j \in f(V)$, we have

$$\begin{aligned} F_j &\subseteq A_j \qquad \left(\text{by the definition of } F_j\right) \\ &= A \cap \left(f^{-1}(j) \times f^{-1}(j)\right) \qquad \left(\text{by the definition of } A_j\right) \\ &\subseteq f^{-1}(j) \times f^{-1}(j). \end{aligned} \tag{17}$$

[20] *Proof.* Let r and s be two distinct elements of $f(V)$. We must prove that the sets $f^{-1}(r) \times f^{-1}(r)$ and $f^{-1}(s) \times f^{-1}(s)$ are disjoint.

Indeed, r and s are distinct. Hence, $f^{-1}(r) \cap f^{-1}(s) = \varnothing$ (since the sets $f^{-1}(j)$ for different $j \in f(V)$ are disjoint). Now,

$$\begin{aligned} \left(f^{-1}(r) \times f^{-1}(r)\right) \cap \left(f^{-1}(s) \times f^{-1}(s)\right) &= \underbrace{\left(f^{-1}(r) \cap f^{-1}(s)\right)}_{=\varnothing} \times \underbrace{\left(f^{-1}(r) \cap f^{-1}(s)\right)}_{=\varnothing} \\ &= \varnothing \times \varnothing = \varnothing. \end{aligned}$$

In other words, the sets $f^{-1}(r) \times f^{-1}(r)$ and $f^{-1}(s) \times f^{-1}(s)$ are disjoint. Qed.

In other words, for each $j \in f(V)$, the set F_j is a subset of $f^{-1}(j) \times f^{-1}(j)$. Hence, the sets F_j for different $j \in f(V)$ are disjoint[21]. The disjoint union $\bigsqcup_{j \in f(V)} F_j$ thus is well-defined.

For each $j \in f(V)$, the set $\mathbf{A}_{\sigma_j}$ is a subset of $f^{-1}(j) \times f^{-1}(j)$ (since σ_j is a permutation of $f^{-1}(j)$). In other words, for each $j \in f(V)$, we have

$$\mathbf{A}_{\sigma_j} \subseteq f^{-1}(j) \times f^{-1}(j). \tag{18}$$

Hence, the sets $\mathbf{A}_{\sigma_j}$ for different $j \in f(V)$ are disjoint[22]. The disjoint union $\bigsqcup_{j \in f(V)} \mathbf{A}_{\sigma_j}$ thus is well-defined.

Our goal is to prove the equivalence

$$\left(\bigsqcup_{j \in f(V)} F_j \subseteq \bigsqcup_{j \in f(V)} \mathbf{A}_{\sigma_j} \right) \iff \left(F_j \subseteq \mathbf{A}_{\sigma_j} \text{ for each } j \in f(V) \right).$$

The "$\Longleftarrow$" direction of this equivalence is obvious. Thus, we only need to prove the "$\Longrightarrow$" direction.

Let us do this. We assume that $\bigsqcup_{j \in f(V)} F_j \subseteq \bigsqcup_{j \in f(V)} \mathbf{A}_{\sigma_j}$. We must prove that $F_j \subseteq \mathbf{A}_{\sigma_j}$ for each $j \in f(V)$.

[21] *Proof.* Let r and s be two distinct elements of $f(V)$. We must prove that the sets F_r and F_s are disjoint.

Recall that the sets $f^{-1}(j) \times f^{-1}(j)$ for different $j \in f(V)$ are disjoint. Hence, the sets $f^{-1}(r) \times f^{-1}(r)$ and $f^{-1}(s) \times f^{-1}(s)$ are disjoint (since r and s are distinct elements of $f(V)$). In other words, $\left(f^{-1}(r) \times f^{-1}(r)\right) \cap \left(f^{-1}(s) \times f^{-1}(s)\right) = \varnothing$.

We have

$$\underbrace{F_r}_{\substack{\subseteq f^{-1}(r) \times f^{-1}(r) \\ \text{(by (17),} \\ \text{applied to } j=r)}} \cap \underbrace{F_s}_{\substack{\subseteq f^{-1}(s) \times f^{-1}(s) \\ \text{(by (17),} \\ \text{applied to } j=s)}} \subseteq \left(f^{-1}(r) \times f^{-1}(r)\right) \cap \left(f^{-1}(s) \times f^{-1}(s)\right) = \varnothing.$$

Hence, $F_r \cap F_s = \varnothing$. In other words, the sets F_r and F_s are disjoint. Qed.

[22] *Proof.* Let r and s be two distinct elements of $f(V)$. We must prove that the sets $\mathbf{A}_{\sigma_r}$ and $\mathbf{A}_{\sigma_s}$ are disjoint.

Recall that the sets $f^{-1}(j) \times f^{-1}(j)$ for different $j \in f(V)$ are disjoint. Hence, the sets $f^{-1}(r) \times f^{-1}(r)$ and $f^{-1}(s) \times f^{-1}(s)$ are disjoint (since r and s are distinct elements of $f(V)$). In other words, $\left(f^{-1}(r) \times f^{-1}(r)\right) \cap \left(f^{-1}(s) \times f^{-1}(s)\right) = \varnothing$.

We have

$$\underbrace{\mathbf{A}_{\sigma_r}}_{\substack{\subseteq f^{-1}(r) \times f^{-1}(r) \\ \text{(by (18),} \\ \text{applied to } j=r)}} \cap \underbrace{\mathbf{A}_{\sigma_s}}_{\substack{\subseteq f^{-1}(s) \times f^{-1}(s) \\ \text{(by (18),} \\ \text{applied to } j=s)}} \subseteq \left(f^{-1}(r) \times f^{-1}(r)\right) \cap \left(f^{-1}(s) \times f^{-1}(s)\right) = \varnothing.$$

Hence, $\mathbf{A}_{\sigma_r} \cap \mathbf{A}_{\sigma_s} = \varnothing$. In other words, the sets $\mathbf{A}_{\sigma_r}$ and $\mathbf{A}_{\sigma_s}$ are disjoint. Qed.

Let $i \in f(V)$. Let $\alpha \in F_i$. Then,

$$\alpha \in F_i \subseteq \bigsqcup_{j \in f(V)} F_j \subseteq \bigsqcup_{j \in f(V)} \mathbf{A}_{\sigma_j}.$$

In other words, $\alpha \in \mathbf{A}_{\sigma_k}$ for some $k \in f(V)$. Consider this k. The set F_i is a subset of $f^{-1}(i) \times f^{-1}(i)$ (because for each $j \in f(V)$, the set F_j is a subset of $f^{-1}(j) \times f^{-1}(j)$). Thus, $F_i \subseteq f^{-1}(i) \times f^{-1}(i)$, so that $\alpha \in F_i \subseteq f^{-1}(i) \times f^{-1}(i)$.

However, $\mathbf{A}_{\sigma_k}$ is a subset of $f^{-1}(k) \times f^{-1}(k)$ (because for each $j \in f(V)$, the set $\mathbf{A}_{\sigma_j}$ is a subset of $f^{-1}(j) \times f^{-1}(j)$). In other words, $\mathbf{A}_{\sigma_k} \subseteq f^{-1}(k) \times f^{-1}(k)$. Hence, $\alpha \in \mathbf{A}_{\sigma_k} \subseteq f^{-1}(k) \times f^{-1}(k)$.

Thus, the two sets $f^{-1}(i) \times f^{-1}(i)$ and $f^{-1}(k) \times f^{-1}(k)$ both contain the element α. However, if we had $i \neq k$, then these two sets would be disjoint (since the sets $f^{-1}(j) \times f^{-1}(j)$ for different $j \in f(V)$ are disjoint), which would contradict the previous sentence. Thus, we have $i = k$. Hence, we can rewrite $\alpha \in \mathbf{A}_{\sigma_k}$ (which we know to be true) as $\alpha \in \mathbf{A}_{\sigma_i}$.

Forget that we fixed α. We thus have shown that $\alpha \in \mathbf{A}_{\sigma_i}$ for each $\alpha \in F_i$. In other words, $F_i \subseteq \mathbf{A}_{\sigma_i}$.

Forget that we fixed i. We thus have proved that $F_i \subseteq \mathbf{A}_{\sigma_i}$ for each $i \in f(V)$. In other words, $F_j \subseteq \mathbf{A}_{\sigma_j}$ for each $j \in f(V)$. This proves the "$\Longrightarrow$" direction of the above equivalence. Thus, the proof of Lemma 2.38 is complete. □

2.6. An alternating sum involving permutations σ with $f \circ \sigma = f$

Now, we come to a crucial lemma, which generalizes Lemma 2.23 to the case of a digraph $D = (V, A)$ "shattered" by a map $f : V \to \mathbb{P}$:

Lemma 2.39. Let $D = (V, A)$ be a digraph. Let $f : V \to \mathbb{P}$ be any map. For each $j \in \mathbb{P}$, we define a digraph D_j as in Definition 2.27 **(c)**. Then,

$$\sum_{\substack{\sigma \in \mathfrak{S}_V; \\ f \circ \sigma = f}} \ \sum_{\substack{F \subseteq \mathbf{A}_\sigma \cap A \\ \text{is linear}}} (-1)^{|F|} = \prod_{j \in f(V)} \left(\text{\# of hamps of } \overline{D_j}\right).$$

Proof. We shall use the notations from Definition 2.24, Definition 2.27 and Definition 2.34. We recall that every $j \in \mathbb{P}$ satisfies $D_j = \left(f^{-1}(j),\ A_j\right)$ (by definition of D_j).

Let $\sigma \in \mathfrak{S}_V$ be a permutation satisfying $f \circ \sigma = f$. Then, Proposition 2.37 yields $\mathbf{A}_\sigma \cap A \subseteq A_f$. Hence, $\mathbf{A}_\sigma \cap A = \mathbf{A}_\sigma \cap A_f$ (because $\underbrace{\mathbf{A}_\sigma}_{= \mathbf{A}_\sigma \cap \mathbf{A}_\sigma} \cap A = \mathbf{A}_\sigma \cap \underbrace{\mathbf{A}_\sigma \cap A}_{\subseteq A_f} \subseteq$

$\mathbf{A}_\sigma \cap A_f$ and conversely $\mathbf{A}_\sigma \cap \underbrace{A_f}_{\subseteq A} \subseteq \mathbf{A}_\sigma \cap A$). Hence,

$$\sum_{\substack{F \subseteq \mathbf{A}_\sigma \cap A \\ \text{is linear}}} (-1)^{|F|} = \sum_{\substack{F \subseteq \mathbf{A}_\sigma \cap A_f \\ \text{is linear}}} (-1)^{|F|} = \sum_{\substack{F \subseteq A_f \text{ is linear;} \\ F \subseteq \mathbf{A}_\sigma}} (-1)^{|F|} \qquad (19)$$

(since a subset of $\mathbf{A}_\sigma \cap A_f$ is the same thing as a subset F of A_f that satisfies $F \subseteq \mathbf{A}_\sigma$).

Forget that we fixed σ. We thus have proved (19) for every $\sigma \in \mathfrak{S}_V$ satisfying $f \circ \sigma = f$.

Summing up the equality (19) over all permutations $\sigma \in \mathfrak{S}_V$ satisfying $f \circ \sigma = f$, we obtain

$$\begin{aligned} & \sum_{\substack{\sigma \in \mathfrak{S}_V; \\ f \circ \sigma = f}} \sum_{\substack{F \subseteq \mathbf{A}_\sigma \cap A \\ \text{is linear}}} (-1)^{|F|} \\ &= \sum_{\substack{\sigma \in \mathfrak{S}_V; \\ f \circ \sigma = f}} \sum_{\substack{F \subseteq A_f \text{ is linear;} \\ F \subseteq \mathbf{A}_\sigma}} (-1)^{|F|} \\ &= \sum_{F \subseteq A_f \text{ is linear}} \underbrace{\sum_{\substack{\sigma \in \mathfrak{S}_V; \\ F \subseteq \mathbf{A}_\sigma; \\ f \circ \sigma = f}} (-1)^{|F|}}_{= (-1)^{|F|} \cdot (\text{\# of } \sigma \in \mathfrak{S}_V \text{ satisfying } F \subseteq \mathbf{A}_\sigma \text{ and } f \circ \sigma = f)} \\ &= \sum_{F \subseteq A_f \text{ is linear}} (-1)^{|F|} \cdot (\text{\# of } \sigma \in \mathfrak{S}_V \text{ satisfying } F \subseteq \mathbf{A}_\sigma \text{ and } f \circ \sigma = f) . \end{aligned} \qquad (20)$$

Now, we observe the following: If F_j is a linear subset of A_j for each $j \in f(V)$, then the disjoint union $\bigsqcup_{j \in f(V)} F_j$ is well-defined[23], and is a linear subset of A_f (by the "$\Longleftarrow$" direction of Proposition 2.33 **(a)**, applied to $F = \bigsqcup_{j \in f(V)} F_j$). Hence, the map

from $\left\{ \text{families } (F_j)_{j \in f(V)}, \text{ where each } F_j \text{ is a linear subset of } A_j \right\}$
to $\left\{ \text{linear subsets of } A_f \right\}$
that sends each family $(F_j)_{j \in f(V)}$ to $\bigsqcup_{j \in f(V)} F_j$

[23] *Proof.* Let F_j be a linear subset of A_j for each $j \in f(V)$. The sets A_j for different $j \in f(V)$ are disjoint (since Proposition 2.32 yields that the sets $A_1, A_2, A_3, \ldots$ are disjoint). Hence, their subsets F_j must be disjoint as well (since F_j is a subset of A_j for each $j \in f(V)$). Thus, the disjoint union $\bigsqcup_{j \in f(V)} F_j$ is well-defined.

is well-defined. Moreover, this map is injective (since Proposition 2.33 **(b)** shows that the sets F_j are uniquely determined by their union $\bigsqcup_{j\in f(V)} F_j$) and surjective (by the "$\Longrightarrow$" direction of Proposition 2.33 **(a)**). Thus, it is bijective. Hence, we can substitute $\bigsqcup_{j\in f(V)} F_j$ for F in the sum on the right hand side of (20). We thus obtain

$$\sum_{F\subseteq A_f \text{ is linear}} (-1)^{|F|} \cdot (\text{\# of } \sigma \in \mathfrak{S}_V \text{ satisfying } F \subseteq \mathbf{A}_\sigma \text{ and } f\circ\sigma = f)$$

$$= \sum_{\substack{(F_j)_{j\in f(V)} \text{ is a family} \\ \text{of linear subsets } F_j\subseteq A_j}} (-1)^{\left|\bigsqcup_{j\in f(V)} F_j\right|} \cdot \left(\text{\# of } \sigma \in \mathfrak{S}_V \text{ satisfying } \bigsqcup_{j\in f(V)} F_j \subseteq \mathbf{A}_\sigma \text{ and } f\circ\sigma = f\right). \tag{21}$$

Now, fix a family $(F_j)_{j\in f(V)}$ of linear subsets $F_j \subseteq A_j$.

A permutation $\sigma \in \mathfrak{S}_V$ satisfies $f\circ\sigma = f$ if and only if it can be written in the form $\sigma = \bigoplus_{j\in f(V)} \sigma_j$, where $\sigma_j \in \mathfrak{S}_{f^{-1}(j)}$ for each $j \in f(V)$ (by Proposition 2.35 **(a)**). Moreover, if σ is written in this way, then we have $\mathbf{A}_\sigma = \bigsqcup_{j\in f(V)} \mathbf{A}_{\sigma_j}$ (by Proposition 2.36).

Hence, if $(\sigma_j)_{j\in f(V)} \in \prod_{j\in f(V)} \mathfrak{S}_{f^{-1}(j)}$ is a family of permutations (i.e., if we are given a permutation $\sigma_j \in \mathfrak{S}_{f^{-1}(j)}$ for each $j \in f(V)$) satisfying $\bigsqcup_{j\in f(V)} F_j \subseteq \bigsqcup_{j\in f(V)} \mathbf{A}_{\sigma_j}$, then $\bigoplus_{j\in f(V)} \sigma_j$ is a permutation $\sigma \in \mathfrak{S}_V$ satisfying $\bigsqcup_{j\in f(V)} F_j \subseteq \mathbf{A}_\sigma$ and $f\circ\sigma = f$ [24].

[24] *Proof.* Let $(\sigma_j)_{j\in f(V)} \in \prod_{j\in f(V)} \mathfrak{S}_{f^{-1}(j)}$ be a family of permutations satisfying $\bigsqcup_{j\in f(V)} F_j \subseteq \bigsqcup_{j\in f(V)} \mathbf{A}_{\sigma_j}$.

Set $\sigma = \bigoplus_{j\in f(V)} \sigma_j$. Then, we must prove that $\bigsqcup_{j\in f(V)} F_j \subseteq \mathbf{A}_\sigma$ and $f\circ\sigma = f$.

However, this is easy: The "$\Longleftarrow$" direction of Proposition 2.35 **(a)** yields $f\circ\sigma = f$ (since $\sigma = \bigoplus_{j\in f(V)} \sigma_j$ with $\sigma_j \in \mathfrak{S}_{f^{-1}(j)}$ for each $j \in f(V)$). Thus, Proposition 2.36 yields $\mathbf{A}_\sigma = \bigsqcup_{j\in f(V)} \mathbf{A}_{\sigma_j}$. Hence, $\bigsqcup_{j\in f(V)} F_j \subseteq \bigsqcup_{j\in f(V)} \mathbf{A}_{\sigma_j} = \mathbf{A}_\sigma$. Thus, both $\bigsqcup_{j\in f(V)} F_j \subseteq \mathbf{A}_\sigma$ and $f\circ\sigma = f$ are proved.

Thus, the map

$$\text{from } \left\{ \text{families } \left(\sigma_j\right)_{j\in f(V)} \in \prod_{j\in f(V)} \mathfrak{S}_{f^{-1}(j)} \text{ satisfying } \bigsqcup_{j\in f(V)} F_j \subseteq \bigsqcup_{j\in f(V)} \mathbf{A}_{\sigma_j} \right\}$$
$$\text{to } \left\{ \sigma \in \mathfrak{S}_V \text{ satisfying } \bigsqcup_{j\in f(V)} F_j \subseteq \mathbf{A}_\sigma \text{ and } f \circ \sigma = f \right\}$$
$$\text{that sends each family } \left(\sigma_j\right)_{j\in f(V)} \text{ to } \bigoplus_{j\in f(V)} \sigma_j$$

is well-defined. This map is furthermore surjective (this follows easily from the "$\Longrightarrow$" direction of Proposition 2.35 **(a)**[25]) and injective (since Proposition 2.35 **(b)** shows that the permutations σ_j are uniquely determined by σ when $\sigma = \bigoplus_{j\in f(V)} \sigma_j$). Thus, this map is bijective.

[25]*Proof.* Let $\sigma \in \mathfrak{S}_V$ be a permutation satisfying $\bigsqcup_{j\in f(V)} F_j \subseteq \mathbf{A}_\sigma$ and $f \circ \sigma = f$. We must prove that σ is an image of some family $\left(\sigma_j\right)_{j\in f(V)} \in \prod_{j\in f(V)} \mathfrak{S}_{f^{-1}(j)}$ under the map we just constructed. In other words, we must prove that there exists a family $\left(\sigma_j\right)_{j\in f(V)} \in \prod_{j\in f(V)} \mathfrak{S}_{f^{-1}(j)}$ satisfying $\bigsqcup_{j\in f(V)} F_j \subseteq \bigsqcup_{j\in f(V)} \mathbf{A}_{\sigma_j}$ such that $\sigma = \bigoplus_{j\in f(V)} \sigma_j$.

However, the "$\Longrightarrow$" direction of Proposition 2.35 **(a)** yields that σ can be written in the form $\sigma = \bigoplus_{j\in f(V)} \sigma_j$, where $\sigma_j \in \mathfrak{S}_{f^{-1}(j)}$ for each $j \in f(V)$. In other words, there exists a family $\left(\sigma_j\right)_{j\in f(V)} \in \prod_{j\in f(V)} \mathfrak{S}_{f^{-1}(j)}$ such that $\sigma = \bigoplus_{j\in f(V)} \sigma_j$. This family $\left(\sigma_j\right)_{j\in f(V)}$ furthermore satisfies $\bigsqcup_{j\in f(V)} F_j \subseteq \mathbf{A}_\sigma = \bigsqcup_{j\in f(V)} \mathbf{A}_{\sigma_j}$ (by Proposition 2.36). Thus, we have proved that there exists a family $\left(\sigma_j\right)_{j\in f(V)} \in \prod_{j\in f(V)} \mathfrak{S}_{f^{-1}(j)}$ satisfying $\bigsqcup_{j\in f(V)} F_j \subseteq \bigsqcup_{j\in f(V)} \mathbf{A}_{\sigma_j}$ such that $\sigma = \bigoplus_{j\in f(V)} \sigma_j$.

Hence, by the bijection principle, we have

$$\begin{aligned}
&\left(\text{\# of } \sigma \in \mathfrak{S}_V \text{ satisfying } \bigsqcup_{j\in f(V)} F_j \subseteq \mathbf{A}_\sigma \text{ and } f\circ\sigma = f\right) \\
&= \left(\text{\# of families } \left(\sigma_j\right)_{j\in f(V)} \in \prod_{j\in f(V)} \mathfrak{S}_{f^{-1}(j)} \text{ satisfying } \bigsqcup_{j\in f(V)} F_j \subseteq \bigsqcup_{j\in f(V)} \mathbf{A}_{\sigma_j}\right) \\
&= \left(\text{\# of families } \left(\sigma_j\right)_{j\in f(V)} \in \prod_{j\in f(V)} \mathfrak{S}_{f^{-1}(j)} \text{ satisfying } F_j \subseteq \mathbf{A}_{\sigma_j} \text{ for each } j \in f(V)\right) \\
&\qquad \left(\begin{array}{c}\text{since the condition ``} \bigsqcup_{j\in f(V)} F_j \subseteq \bigsqcup_{j\in f(V)} \mathbf{A}_{\sigma_j}\text{''} \\ \text{is equivalent to ``}F_j \subseteq \mathbf{A}_{\sigma_j} \text{ for each } j\in f(V)\text{''} \\ \text{(by Lemma 2.38)}\end{array}\right) \\
&= \prod_{j\in f(V)} \left(\text{\# of } \sigma_j \in \mathfrak{S}_{f^{-1}(j)} \text{ satisfying } F_j \subseteq \mathbf{A}_{\sigma_j}\right) \qquad \text{(by the product rule)} \\
&= \prod_{j\in f(V)} \left(\text{\# of } \sigma \in \mathfrak{S}_{f^{-1}(j)} \text{ satisfying } F_j \subseteq \mathbf{A}_{\sigma}\right) \qquad (22) \\
&\qquad \left(\text{here, we have renamed the index } \sigma_j \text{ as } \sigma\right).
\end{aligned}$$

Also, we have $\left|\bigsqcup_{j\in f(V)} F_j\right| = \sum_{j\in f(V)} \left|F_j\right|$ (by the sum rule), and thus

$$(-1)^{\left|\bigsqcup_{j\in f(V)} F_j\right|} = (-1)^{\sum_{j\in f(V)} \left|F_j\right|} = \prod_{j\in f(V)} (-1)^{\left|F_j\right|}. \qquad (23)$$

Multiplying the equalities (23) and (22), we obtain

$$\begin{aligned}
&(-1)^{\left|\bigsqcup_{j\in f(V)} F_j\right|} \cdot \left(\text{\# of } \sigma \in \mathfrak{S}_V \text{ satisfying } \bigsqcup_{j\in f(V)} F_j \subseteq \mathbf{A}_\sigma \text{ and } f\circ\sigma = f\right) \\
&= \left(\prod_{j\in f(V)} (-1)^{\left|F_j\right|}\right) \cdot \prod_{j\in f(V)} \left(\text{\# of } \sigma \in \mathfrak{S}_{f^{-1}(j)} \text{ satisfying } F_j \subseteq \mathbf{A}_\sigma\right) \\
&= \prod_{j\in f(V)} \left((-1)^{\left|F_j\right|} \cdot \left(\text{\# of } \sigma \in \mathfrak{S}_{f^{-1}(j)} \text{ satisfying } F_j \subseteq \mathbf{A}_\sigma\right)\right). \qquad (24)
\end{aligned}$$

Forget that we fixed $\left(F_j\right)_{j\in f(V)}$. We thus have proved the equality (24) for any family $\left(F_j\right)_{j\in f(V)}$ of linear subsets $F_j \subseteq A_j$.

Now, (21) becomes

$$\begin{aligned}
&\sum_{F\subseteq A_f \text{ is linear}} (-1)^{|F|} \cdot (\text{\# of } \sigma \in \mathfrak{S}_V \text{ satisfying } F \subseteq \mathbf{A}_\sigma \text{ and } f\circ\sigma = f) \\
&= \sum_{\substack{(F_j)_{j\in f(V)} \text{ is a family} \\ \text{of linear subsets } F_j \subseteq A_j}} \underbrace{\cdot (-1)^{\left|\bigsqcup_{j\in f(V)} F_j\right|} \cdot \left(\text{\# of } \sigma \in \mathfrak{S}_V \text{ satisfying } \bigsqcup_{j\in f(V)} F_j \subseteq \mathbf{A}_\sigma \text{ and } f\circ\sigma = f\right)}_{\substack{= \prod_{j\in f(V)} \left((-1)^{\left|F_j\right|} \cdot \left(\text{\# of } \sigma\in\mathfrak{S}_{f^{-1}(j)} \text{ satisfying } F_j\subseteq\mathbf{A}_\sigma\right)\right) \\ \text{(by (24))}}} \\
&= \sum_{\substack{(F_j)_{j\in f(V)} \text{ is a family} \\ \text{of linear subsets } F_j \subseteq A_j}} \prod_{j\in f(V)} \left((-1)^{\left|F_j\right|} \cdot \left(\text{\# of } \sigma \in \mathfrak{S}_{f^{-1}(j)} \text{ satisfying } F_j \subseteq \mathbf{A}_\sigma\right)\right) \\
&= \prod_{j\in f(V)} \sum_{F_j\subseteq A_j \text{ is linear}} (-1)^{\left|F_j\right|} \cdot \left(\text{\# of } \sigma \in \mathfrak{S}_{f^{-1}(j)} \text{ satisfying } F_j \subseteq \mathbf{A}_\sigma\right) \\
&\qquad \text{(by the product rule)} \\
&= \prod_{j\in f(V)} \underbrace{\sum_{F\subseteq A_j \text{ is linear}} (-1)^{|F|} \cdot \left(\text{\# of } \sigma \in \mathfrak{S}_{f^{-1}(j)} \text{ satisfying } F \subseteq \mathbf{A}_\sigma\right)}_{\substack{=\left(\text{\# of hamps of } \overline{D_j}\right) \\ \text{(by Lemma 2.23, applied to } D_j=\left(f^{-1}(j),\ A_j\right) \text{ instead of } D=(V,A) \\ \text{(since } f^{-1}(j)\neq\varnothing \text{ (because } j\in f(V))))}} \\
&\qquad \left(\text{here, we have renamed the summation index } F_j \text{ as } F\right) \\
&= \prod_{j\in f(V)} \left(\text{\# of hamps of } \overline{D_j}\right).
\end{aligned} \tag{25}$$

Now, (20) becomes

$$\begin{aligned}
&\sum_{\substack{\sigma\in\mathfrak{S}_V; \\ f\circ\sigma=f}} \sum_{\substack{F\subseteq\mathbf{A}_\sigma\cap A \\ \text{is linear}}} (-1)^{|F|} \\
&= \sum_{F\subseteq A_f \text{ is linear}} (-1)^{|F|} \cdot (\text{\# of } \sigma \in \mathfrak{S}_V \text{ satisfying } F \subseteq \mathbf{A}_\sigma \text{ and } f\circ\sigma = f) \\
&= \prod_{j\in f(V)} \left(\text{\# of hamps of } \overline{D_j}\right) \qquad \text{(by (25))}.
\end{aligned}$$

This proves Lemma 2.39. ☐

2.7. (f, D)-friendly V-listings

The following restatement of Lemma 2.39 will be useful for us:

Lemma 2.40. Let $D = (V, A)$ be a digraph. Let $f : V \to \mathbb{P}$ be any map. A V-listing $v = (v_1, v_2, \ldots, v_n)$ will be called *(f, D)-friendly* if it has the properties that $f(v_1) \leq f(v_2) \leq \cdots \leq f(v_n)$ and that

$$f(v_p) < f(v_{p+1}) \text{ for each } p \in [n-1] \text{ satisfying } (v_p, v_{p+1}) \in A. \qquad (26)$$

Then,

$$\sum_{\substack{\sigma \in \mathfrak{S}_V; \\ f \circ \sigma = f}} \sum_{\substack{F \subseteq \mathbf{A}_\sigma \cap A \\ \text{is linear}}} (-1)^{|F|} = (\# \text{ of } (f, D)\text{-friendly } V\text{-listings}).$$

Proof. For each $j \in f(V)$, we define a digraph D_j as in Definition 2.27 **(c)**. The vertex set of this digraph D_j is $f^{-1}(j)$. In other words, its vertices are precisely those vertices of D that have level j (with respect to f).

Clearly, a V-listing $v = (v_1, v_2, \ldots, v_n)$ satisfies $f(v_1) \leq f(v_2) \leq \cdots \leq f(v_n)$ if and only if it lists the vertices of D in the order of increasing level, i.e., if it first lists the vertices of the smallest level, then the vertices of the second-smallest level, and so on.

If $(a_j)_{j \in f(V)}$ is a family of (finite) lists (one list a_j for each $j \in f(V)$), then $\bigotimes_{j \in f(V)} a_j$ shall denote the concatenation of these lists a_j in the order of increasing j (that is, the list starting with the entries of a_j for the smallest $j \in f(V)$, then continuing with the entries of a_j for the second-smallest $j \in f(V)$, and so on). For instance, if $f(V) = \{2, 3, 5\}$ and $a_2 = (u, v)$ and $a_3 = (x, y, z)$ and $a_5 = (p)$, then $\bigotimes_{j \in f(V)} a_j = (u, v, x, y, z, p)$. The lists a_j are called the *factors* of the concatenation $\bigotimes_{j \in f(V)} a_j$.

Now, we shall prove five claims:

Claim 1: Let $\left(v^{(j)}\right)_{j \in f(V)}$ be a family of lists, where each $v^{(j)}$ is an $f^{-1}(j)$-listing. Write the concatenation $\bigotimes_{j \in f(V)} v^{(j)}$ in the form

$$\bigotimes_{j \in f(V)} v^{(j)} = v = (v_1, v_2, \ldots, v_n).$$

Then, v is a V-listing and satisfies $f(v_1) \leq f(v_2) \leq \cdots \leq f(v_n)$.

[*Proof of Claim 1:* For each $j \in f(V)$, the list $v^{(j)}$ is an $f^{-1}(j)$-listing, thus a list of elements of $f^{-1}(j)$, therefore a list of elements of V (since $f^{-1}(j) \subseteq V$). Hence, the concatenation $\bigotimes_{j \in f(V)} v^{(j)}$ of these lists $v^{(j)}$ is a list of elements of V as well. In other words, v is a list of elements of V (since $\bigotimes_{j \in f(V)} v^{(j)} = v$).

Moreover, each element of V is contained exactly once in v [26]. Hence, v is a V-listing (since v is a list of elements of V).

It remains to show that $f(v_1) \leq f(v_2) \leq \cdots \leq f(v_n)$.

We fix $p \in [n-1]$. We shall show that $f(v_p) \leq f(v_{p+1})$.

Indeed, assume the contrary. Thus, $f(v_p) > f(v_{p+1})$. Set $\alpha = f(v_p)$ and $\beta = f(v_{p+1})$. Thus, $\alpha = f(v_p) > f(v_{p+1}) = \beta$. Moreover, $\alpha = f(v_p) \in f(V)$ and $\beta = f(v_{p+1}) \in f(V)$. Furthermore, from $f(v_p) = \alpha$, we see that v_p is an element of $f^{-1}(\alpha)$. From $f(v_{p+1}) = \beta$, we see that v_{p+1} is an element of $f^{-1}(\beta)$.

We recall that v is a V-listing. Thus, each entry of v appears only once in v.

Recall that for each $j \in f(V)$, the list $v^{(j)}$ is an $f^{-1}(j)$-listing. Hence, in particular, $v^{(\alpha)}$ is an $f^{-1}(\alpha)$-listing. Hence, each element of $f^{-1}(\alpha)$ appears in $v^{(\alpha)}$. Thus, in particular, v_p appears in $v^{(\alpha)}$ (since v_p is an element of $f^{-1}(\alpha)$). The same argument (applied to $p+1$ and β instead of p and α) shows that v_{p+1} appears in $v^{(\beta)}$.

However, $\beta < \alpha$ (since $\alpha > \beta$). Thus, in the concatenation $\bigotimes_{j \in f(V)} v^{(j)}$, the factor $v^{(\beta)}$ appears to the left of the factor $v^{(\alpha)}$ (since this concatenation is concatenating the lists $v^{(j)}$ in the order of increasing j). Hence, in particular, in this concatenation $\bigotimes_{j \in f(V)} v^{(j)}$, the entry v_{p+1} appears to the left of the entry v_p (since v_{p+1} appears in $v^{(\beta)}$, whereas v_p appears in $v^{(\alpha)}$). In other words, in the list v, the entry v_{p+1} appears to the left of the entry v_p (since $v = \bigotimes_{j \in f(V)} v^{(j)}$). On the other hand, it is clear that the entry v_{p+1} appears to the right of the entry v_p in the list v (since $v = (v_1, v_2, \ldots, v_n)$ and $p+1 > p$).

Thus, we have shown that in the list v, the entry v_{p+1} appears both to the left and to the right of the entry v_p. Clearly, this is only possible if one of these entries

[26] *Proof.* Let p be an arbitrary element of V. We must then show that p is contained exactly once in v.

Indeed, let $i = f(p)$. Then, p is an element of $f^{-1}(i)$. Also, $i = f(p) \in f(V)$.

Moreover, the list $v^{(i)}$ is an $f^{-1}(i)$-listing (because for each $j \in f(V)$, the list $v^{(j)}$ is an $f^{-1}(j)$-listing). Hence, this list $v^{(i)}$ contains each element of $f^{-1}(i)$ exactly once. In particular, this shows that $v^{(i)}$ contains p exactly once (since p is an element of $f^{-1}(i)$). In other words, p appears exactly once in the list $v^{(i)}$.

Now, let $j \in f(V)$ be distinct from i. Then, $j \neq i = f(p)$, so that $f(p) \neq j$. Therefore, p is not an element of $f^{-1}(j)$. However, the list $v^{(j)}$ is an $f^{-1}(j)$-listing (by assumption), and thus is a list of elements of $f^{-1}(j)$. Hence, this list $v^{(j)}$ does not contain p (since p is not an element of $f^{-1}(j)$). In other words, p does not appear in this list $v^{(j)}$.

Forget that we fixed j. We thus have shown that if $j \in f(V)$ is distinct from i, then p does not appear in the list $v^{(j)}$.

Altogether, we now know that p appears exactly once in the list $v^{(i)}$ but does not appear in the list $v^{(j)}$ for any $j \in f(V)$ that is distinct from i. In other words, p appears exactly once in the list $v^{(i)}$ and does not appear in any other list $v^{(j)}$ with $j \neq i$. Consequently, p appears exactly once in the concatenation $\bigotimes_{j \in f(V)} v^{(j)}$. In other words, p appears exactly once in v (since $v = \bigotimes_{j \in f(V)} v^{(j)}$). In other words, p is contained exactly once in v. Qed.

appears more than once in the list v. We thus conclude that one of these entries appears more than once in the list v. However, this contradicts the fact that each entry of v appears only once in v.

This contradiction shows that our assumption was false. Hence, $f\left(v_p\right) \leq f\left(v_{p+1}\right)$ is proved.

Forget that we fixed p. We thus have shown that $f\left(v_p\right) \leq f\left(v_{p+1}\right)$ for each $p \in [n-1]$. In other words, $f\left(v_1\right) \leq f\left(v_2\right) \leq \cdots \leq f\left(v_n\right)$. This completes the proof of Claim 1.]

> *Claim 2:* Let $\left(v^{(j)}\right)_{j\in f(V)}$ be a family of lists, where each $v^{(j)}$ is an $f^{-1}(j)$-listing. Then, this family $\left(v^{(j)}\right)_{j\in f(V)}$ can be uniquely reconstructed from the concatenation $\bigotimes\limits_{j\in f(V)} v^{(j)}$.

[*Proof of Claim 2:* Fix $i \in f(V)$. Then, $v^{(i)}$ is an $f^{-1}(i)$-listing (since each $v^{(j)}$ is an $f^{-1}(j)$-listing). This $f^{-1}(i)$-listing $v^{(i)}$ is a factor of the concatenation $\bigotimes\limits_{j\in f(V)} v^{(j)}$. This factor $v^{(i)}$ consists entirely of elements of $f^{-1}(i)$ (since it is an $f^{-1}(i)$-listing), whereas all the other factors $v^{(j)}$ of the concatenation $\bigotimes\limits_{j\in f(V)} v^{(j)}$ contain no elements of $f^{-1}(i)$ whatsoever[27]. Thus, we can reconstruct $v^{(i)}$ from $\bigotimes\limits_{j\in f(V)} v^{(j)}$ by removing all entries that don't belong to $f^{-1}(i)$ (since this removal preserves the factor $v^{(i)}$ but makes all the other factors $v^{(j)}$ disappear).

Forget that we fixed i. Thus, we have shown that for each $i \in f(V)$, we can reconstruct $v^{(i)}$ from $\bigotimes\limits_{j\in f(V)} v^{(j)}$. In other words, we can reconstruct the family $\left(v^{(i)}\right)_{i\in f(V)}$ from $\bigotimes\limits_{j\in f(V)} v^{(j)}$. In other words, we can reconstruct the family $\left(v^{(j)}\right)_{j\in f(V)}$ from $\bigotimes\limits_{j\in f(V)} v^{(j)}$ (since $\left(v^{(i)}\right)_{i\in f(V)} = \left(v^{(j)}\right)_{j\in f(V)}$). This proves Claim 2.]

[27] *Proof.* We must prove that if $j \in f(V)$ is distinct from i, then $v^{(j)}$ contains no elements of $f^{-1}(i)$ whatsoever.

So let $j \in f(V)$ be distinct from i. We must prove that $v^{(j)}$ contains no elements of $f^{-1}(i)$ whatsoever.

Assume the contrary. Thus, $v^{(j)}$ contains some element of $f^{-1}(i)$. Let p be this element. Then, $f(p) = i$ (since p is an element of $f^{-1}(i)$). However, $v^{(j)}$ is an $f^{-1}(j)$-listing (by assumption), and thus is a list of elements of $f^{-1}(j)$. Hence, each entry of $v^{(j)}$ belongs to $f^{-1}(j)$. Since p is an entry of $v^{(j)}$ (because $v^{(j)}$ contains p), we thus conclude that p belongs to $f^{-1}(j)$. In other words, $f(p) = j$. Therefore, $j = f(p) = i$, which contradicts the fact that j is distinct from i.

This contradiction shows that our assumption was false. Hence, we have shown that $v^{(j)}$ contains no elements of $f^{-1}(i)$ whatsoever.

Claim 3: Let $\left(v^{(j)}\right)_{j\in f(V)}$ be a family of lists, where each $v^{(j)}$ is a hamp of $\overline{D_j}$. Then, the concatenation $\bigotimes_{j\in f(V)} v^{(j)}$ is an (f,D)-friendly V-listing.

[*Proof of Claim 3:* Let us write the concatenation $\bigotimes_{j\in f(V)} v^{(j)}$ in the form

$$\bigotimes_{j\in f(V)} v^{(j)} = v = (v_1, v_2, \ldots, v_n).$$

For each $j \in f(V)$, the list $v^{(j)}$ is a hamp of $\overline{D_j}$ and thus an $f^{-1}(j)$-listing[28]. Hence, Claim 1 shows that v is a V-listing and satisfies $f(v_1) \leq f(v_2) \leq \cdots \leq f(v_n)$.

We shall now show that this V-listing v is (f,D)-friendly. Indeed, as we just proved, it satisfies $f(v_1) \leq f(v_2) \leq \cdots \leq f(v_n)$. In order to prove that it is (f,D)-friendly, it thus suffices to show that it satisfies

$$f\left(v_p\right) < f\left(v_{p+1}\right) \text{ for each } p \in [n-1] \text{ satisfying } \left(v_p, v_{p+1}\right) \in A.$$

So let us do this. Let $p \in [n-1]$ be such that $\left(v_p, v_{p+1}\right) \in A$. We must prove that $f\left(v_p\right) < f\left(v_{p+1}\right)$.

Assume the contrary. Thus, $f\left(v_p\right) \geq f\left(v_{p+1}\right)$. Combining this with $f\left(v_p\right) \leq f\left(v_{p+1}\right)$ (which follows from $f(v_1) \leq f(v_2) \leq \cdots \leq f(v_n)$), we obtain $f\left(v_p\right) = f\left(v_{p+1}\right)$. Set $i = f\left(v_p\right)$. Then, $i = f\left(v_p\right) \in f(V)$ and $i = f\left(v_p\right) = f\left(v_{p+1}\right)$. We have $v_p \in f^{-1}(i)$ (since $f\left(v_p\right) = i$) and $v_{p+1} \in f^{-1}(i)$ (since $f\left(v_{p+1}\right) = i$). In other words, v_p and v_{p+1} belong to $f^{-1}(i)$.

In the above proof of Claim 2, we noticed the following: The factor $v^{(i)}$ of the concatenation $\bigotimes_{j\in f(V)} v^{(j)}$ consists entirely of elements of $f^{-1}(i)$, whereas all the

[28] *Proof.* Let $j \in f(V)$. Then, the list $v^{(j)}$ is a hamp of $\overline{D_j}$. In other words, this list $v^{(j)}$ is a $\overline{D_j}$-path that contains each vertex of $\overline{D_j}$ (by the definition of a "hamp").

We recall that the digraph D_j was defined to be $\left(f^{-1}(j),\ A_j\right)$. Hence, its complement $\overline{D_j}$ is $\left(f^{-1}(j),\ \left(f^{-1}(j) \times f^{-1}(j)\right) \setminus A_j\right)$ (by the definition of the complement of a digraph). Thus, the vertices of $\overline{D_j}$ are the elements of $f^{-1}(j)$, whereas the arcs of $\overline{D_j}$ are the elements of $\left(f^{-1}(j) \times f^{-1}(j)\right) \setminus A_j$.

Now, recall that the list $v^{(j)}$ is a $\overline{D_j}$-path. In other words, $v^{(j)}$ is a nonempty tuple of distinct elements of $f^{-1}(j)$ such that $\operatorname{Arcs}\left(v^{(j)}\right) \subseteq \left(f^{-1}(j) \times f^{-1}(j)\right) \setminus A_j$ (by the definition of a "$\overline{D_j}$-path", since $\overline{D_j} = \left(f^{-1}(j),\ \left(f^{-1}(j) \times f^{-1}(j)\right) \setminus A_j\right)$). Thus, in particular, $v^{(j)}$ is a tuple of distinct elements of $f^{-1}(j)$. Hence, $v^{(j)}$ contains no entry more than once.

Furthermore, recall that the tuple $v^{(j)}$ contains each vertex of $\overline{D_j}$. In other words, $v^{(j)}$ contains each element of $f^{-1}(j)$ (since the vertices of $\overline{D_j}$ are the elements of $f^{-1}(j)$). Hence, $v^{(j)}$ contains each element of $f^{-1}(j)$ exactly once (since $v^{(j)}$ contains no entry more than once).

Thus, we know that $v^{(j)}$ is a list of elements of $f^{-1}(j)$ that contains each element of $f^{-1}(j)$ exactly once. In other words, $v^{(j)}$ is an $f^{-1}(j)$-listing. Qed.

other factors $v^{(j)}$ of this concatenation contain no elements of $f^{-1}(i)$ whatsoever. Hence, any entry of the concatenation $\bigotimes_{j\in f(V)} v^{(j)}$ that belongs to $f^{-1}(i)$ must appear only in the $v^{(i)}$ factor of this concatenation.

However, v_p and v_{p+1} are consecutive entries of the concatenation $\bigotimes_{j\in f(V)} v^{(j)}$ (since $\bigotimes_{j\in f(V)} v^{(j)} = v = (v_1, v_2, \ldots, v_n)$). Since these two entries v_p and v_{p+1} belong to $f^{-1}(i)$, we thus conclude that v_p and v_{p+1} appear only in the $v^{(i)}$ factor of this concatenation (since any entry of the concatenation $\bigotimes_{j\in f(V)} v^{(j)}$ that belongs to $f^{-1}(i)$ must appear only in the $v^{(i)}$ factor of this concatenation). Therefore, v_p and v_{p+1} are two consecutive entries of $v^{(i)}$ (since v_p and v_{p+1} are consecutive entries of the concatenation $\bigotimes_{j\in f(V)} v^{(j)}$). In other words, the list $v^{(i)}$ has the form $(\ldots, v_p, v_{p+1}, \ldots)$ (where each "..." stands for some number of entries). Therefore, the pair (v_p, v_{p+1}) is an arc of $v^{(i)}$. In other words, $(v_p, v_{p+1}) \in \operatorname{Arcs}\left(v^{(i)}\right)$.

However, $v^{(i)}$ is a hamp of $\overline{D_i}$ (since each $v^{(j)}$ is a hamp of $\overline{D_j}$). In other words, $v^{(i)}$ is a $\overline{D_i}$-path that contains each vertex of $\overline{D_i}$ (by the definition of a "hamp").

We recall that the digraph D_i was defined to be $(f^{-1}(i), A_i)$. Hence, its complement $\overline{D_i}$ is $(f^{-1}(i), (f^{-1}(i) \times f^{-1}(i)) \setminus A_i)$ (by the definition of the complement of a digraph). Since $v^{(i)}$ is a $\overline{D_i}$-path, we thus have $\operatorname{Arcs}\left(v^{(i)}\right) \subseteq (f^{-1}(i) \times f^{-1}(i)) \setminus A_i$ (by the definition of a "$\overline{D_i}$-path").

Hence, $(v_p, v_{p+1}) \in \operatorname{Arcs}\left(v^{(i)}\right) \subseteq (f^{-1}(i) \times f^{-1}(i)) \setminus A_i$. In other words, $(v_p, v_{p+1}) \in f^{-1}(i) \times f^{-1}(i)$ and $(v_p, v_{p+1}) \notin A_i$.

Combining $(v_p, v_{p+1}) \in A$ with $(v_p, v_{p+1}) \in f^{-1}(i) \times f^{-1}(i)$, we obtain

$$(v_p, v_{p+1}) \in A \cap \left(f^{-1}(i) \times f^{-1}(i)\right) = A_i$$

(since (12) (applied to $j = i$) yields $A_i = A \cap (f^{-1}(i) \times f^{-1}(i))$). But this contradicts $(v_p, v_{p+1}) \notin A_i$. This contradiction shows that our assumption was false. Hence, we have shown that $f(v_p) < f(v_{p+1})$.

Forget that we fixed p. We thus have proved that

$$f(v_p) < f(v_{p+1}) \text{ for each } p \in [n-1] \text{ satisfying } (v_p, v_{p+1}) \in A.$$

Since we furthermore know that $f(v_1) \leq f(v_2) \leq \cdots \leq f(v_n)$, we thus conclude that the V-listing $v = (v_1, v_2, \ldots, v_n)$ is (f, D)-friendly (by the definition of "(f, D)-friendly"). Hence, v is an (f, D)-friendly V-listing. In other words, $\bigotimes_{j\in f(V)} v^{(j)}$ is an (f, D)-friendly V-listing (since $v = \bigotimes_{j\in f(V)} v^{(j)}$). This proves Claim 3.]

Claim 4: Let v be an (f, D)-friendly V-listing. Then, v can be written in the form $v = \bigotimes_{j \in f(V)} v^{(j)}$, where $v^{(j)}$ is a hamp of $\overline{D_j}$ for each $j \in f(V)$.

[*Proof of Claim 4:* Let $j_1, j_2, \ldots, j_q$ be the elements of $f(V)$, listed in increasing order (so that $j_1 < j_2 < \cdots < j_q$). Thus, $j_1, j_2, \ldots, j_q$ are all the levels (with respect to f) that a vertex of D can have, listed in increasing order.

Write the V-listing v in the form $v = (v_1, v_2, \ldots, v_n)$. As we assumed, this V-listing v is (f, D)-friendly. In other words, it has the properties that $f(v_1) \leq f(v_2) \leq \cdots \leq f(v_n)$ and that

$$f\left(v_p\right) < f\left(v_{p+1}\right) \text{ for each } p \in [n-1] \text{ satisfying } \left(v_p, v_{p+1}\right) \in A \tag{27}$$

(by the definition of "(f, D)-friendly").

In particular, we have $f(v_1) \leq f(v_2) \leq \cdots \leq f(v_n)$. In other words, the entries of the V-listing v appear in v in the order of increasing level. In other words, the V-listing v first lists the vertices of the smallest level, then the vertices of the second-smallest level, and so on. Since v is a V-listing (i.e., contains each element of V exactly once), we can restate this as follows: The V-listing v first lists each vertex of the smallest level exactly once, then lists each vertex of the second-smallest level exactly once, and so on. In other words, the V-listing v first lists each vertex of level j_1 exactly once, then lists each vertex of level j_2 exactly once, and so on (since $j_1, j_2, \ldots, j_q$ are all the levels that a vertex of D can have, listed in increasing order). In other words, the V-listing v first lists each element of $f^{-1}(j_1)$ exactly once, then lists each element of $f^{-1}(j_2)$ exactly once, and so on. In other words, the V-listing v can be written as a concatenation of an $f^{-1}(j_1)$-listing, an $f^{-1}(j_2)$-listing, and so on (in this order).

In other words, v can be written as a concatenation $\bigotimes_{j \in f(V)} v^{(j)}$, where $v^{(j)}$ is an $f^{-1}(j)$-listing for each $j \in f(V)$ (since $j_1, j_2, \ldots, j_q$ are the elements of $f(V)$, listed in increasing order).

Let us write v in this way. Thus, $v^{(j)}$ is an $f^{-1}(j)$-listing for each $j \in f(V)$, and we have $v = \bigotimes_{j \in f(V)} v^{(j)}$.

We shall now show that $v^{(j)}$ is a hamp of $\overline{D_j}$ for each $j \in f(V)$.

Indeed, let $i \in f(V)$ be arbitrary. We recall that the digraph D_i was defined to be $\left(f^{-1}(i),\ A_i\right)$. Hence, its complement $\overline{D_i}$ is $\left(f^{-1}(i),\ \left(f^{-1}(i) \times f^{-1}(i)\right) \setminus A_i\right)$ (by the definition of the complement of a digraph). In particular, the vertices of $\overline{D_i}$ are the elements of $f^{-1}(i)$.

Note that $v^{(i)}$ is an $f^{-1}(i)$-listing (since $v^{(j)}$ is an $f^{-1}(j)$-listing for each $j \in f(V)$). Thus, $v^{(i)}$ is a list of all elements of $f^{-1}(i)$. In particular, all entries of the list $v^{(i)}$ belong to $f^{-1}(i)$. Note that the set $f^{-1}(i)$ is nonempty (since $i \in f(V)$), so that any $f^{-1}(i)$-listing must also be nonempty. Hence, $v^{(i)}$ is nonempty (since $v^{(i)}$ is an $f^{-1}(i)$-listing). Furthermore, $v^{(i)}$ is a tuple of distinct elements of $f^{-1}(i)$ (since $v^{(i)}$ is an $f^{-1}(i)$-listing).

Clearly, $v^{(i)}$ is a factor of the concatenation $\bigotimes_{j\in f(V)} v^{(j)}$. Thus, $v^{(i)}$ is a contiguous block of the list $\bigotimes_{j\in f(V)} v^{(j)}$. In other words, $v^{(i)}$ is a contiguous block of the list v (since $v = \bigotimes_{j\in f(V)} v^{(j)}$). In other words, $v^{(i)} = (v_k, v_{k+1}, \ldots, v_\ell)$ for some two elements k and ℓ of $[n]$ (since $v = (v_1, v_2, \ldots, v_n)$). Consider these k and ℓ. From $v^{(i)} = (v_k, v_{k+1}, \ldots, v_\ell)$, we obtain

$$\begin{aligned}
\operatorname{Arcs}\left(v^{(i)}\right) &= \operatorname{Arcs}\left((v_k, v_{k+1}, \ldots, v_\ell)\right) \\
&= \left\{(v_k, v_{k+1}),\ (v_{k+1}, v_{k+2}),\ \ldots,\ (v_{\ell-1}, v_\ell)\right\} \\
&= \left\{\left(v_p, v_{p+1}\right) \mid p \in \{k, k+1, \ldots, \ell-1\}\right\}.
\end{aligned} \tag{28}$$

Now, let $p \in \{k, k+1, \ldots, \ell-1\}$. We shall show that $\left(v_p, v_{p+1}\right) \in \left(f^{-1}(i) \times f^{-1}(i)\right) \setminus A_i$.

Indeed, $p \in \{k, k+1, \ldots, \ell-1\} \subseteq \{1, 2, \ldots, n-1\}$ (since $k \geq 1$ and $\underbrace{\ell}_{\leq n} -1 \leq n-1$). In other words, $p \in [n-1]$ (since $[n-1] = \{1, 2, \ldots, n-1\}$).

Also, from $p \in \{k, k+1, \ldots, \ell-1\}$, we see that both p and $p+1$ belong to the set $\{k, k+1, \ldots, \ell\}$. Hence, both v_p and v_{p+1} are entries of the list $(v_k, v_{k+1}, \ldots, v_\ell)$. In other words, both v_p and v_{p+1} are entries of the list $v^{(i)}$ (since $v^{(i)} = (v_k, v_{k+1}, \ldots, v_\ell)$). Hence, both v_p and v_{p+1} belong to $f^{-1}(i)$ (since all entries of the list $v^{(i)}$ belong to $f^{-1}(i)$). Therefore, $\left(v_p, v_{p+1}\right) \in f^{-1}(i) \times f^{-1}(i)$.

Now, we shall show that $\left(v_p, v_{p+1}\right) \notin A_i$. Indeed, assume the contrary. Thus,

$$\left(v_p, v_{p+1}\right) \in A_i = A \cap \left(f^{-1}(i) \times f^{-1}(i)\right) \qquad \text{(by (12), applied to } j = i\text{)}$$
$$\subseteq A$$

and therefore $f\left(v_p\right) < f\left(v_{p+1}\right)$ (by (27)). However, $f\left(v_p\right) = i$ (since v_p belongs to $f^{-1}(i)$) and $f\left(v_{p+1}\right) = i$ (since v_{p+1} belongs to $f^{-1}(i)$), so that $f\left(v_p\right) = i = f\left(v_{p+1}\right)$. This contradicts $f\left(v_p\right) < f\left(v_{p+1}\right)$. This contradiction shows that our assumption was false. Hence, $\left(v_p, v_{p+1}\right) \notin A_i$ is proved.

Combining $\left(v_p, v_{p+1}\right) \in f^{-1}(i) \times f^{-1}(i)$ with $\left(v_p, v_{p+1}\right) \notin A_i$, we obtain $\left(v_p, v_{p+1}\right) \in \left(f^{-1}(i) \times f^{-1}(i)\right) \setminus A_i$.

Forget that we fixed p. We thus have proved that $\left(v_p, v_{p+1}\right) \in \left(f^{-1}(i) \times f^{-1}(i)\right) \setminus A_i$ for each $p \in \{k, k+1, \ldots, \ell-1\}$. In other words,

$$\left\{\left(v_p, v_{p+1}\right) \mid p \in \{k, k+1, \ldots, \ell-1\}\right\} \subseteq \left(f^{-1}(i) \times f^{-1}(i)\right) \setminus A_i.$$

In view of (28), we can rewrite this as

$$\operatorname{Arcs}\left(v^{(i)}\right) \subseteq \left(f^{-1}(i) \times f^{-1}(i)\right) \setminus A_i.$$

Now, we know that $v^{(i)}$ is a nonempty tuple of distinct elements of $f^{-1}(i)$ and has the property that $\operatorname{Arcs}\left(v^{(i)}\right) \subseteq \left(f^{-1}(i) \times f^{-1}(i)\right) \setminus A_i$. In other words, $v^{(i)}$ is a $\overline{D_i}$-path (by the definition of a "$\overline{D_i}$-path", since the digraph $\overline{D_i}$ is $\left(f^{-1}(i), \left(f^{-1}(i) \times f^{-1}(i)\right) \setminus A_i\right)$). This $\overline{D_i}$-path $v^{(i)}$ furthermore contains each element of $f^{-1}(i)$ (since it is an $f^{-1}(i)$-listing). In other words, this $\overline{D_i}$-path $v^{(i)}$ contains each vertex of $\overline{D_i}$ (since the vertices of $\overline{D_i}$ are the elements of $f^{-1}(i)$).

In other words, $v^{(i)}$ is a hamp of $\overline{D_i}$ (by the definition of a hamp).

Forget that we fixed i. We thus have shown that $v^{(i)}$ is a hamp of $\overline{D_i}$ for each $i \in f(V)$. Renaming the variable i as j in this sentence, we obtain the following: $v^{(j)}$ is a hamp of $\overline{D_j}$ for each $j \in f(V)$.

We have thus written v in the form $v = \bigotimes\limits_{j \in f(V)} v^{(j)}$, where $v^{(j)}$ is a hamp of $\overline{D_j}$ for each $j \in f(V)$. This shows that v can be written in this form. Claim 4 is thus proven.]

> *Claim 5:* Let $\left(v^{(j)}\right)_{j \in f(V)}$ be a family of lists, where each $v^{(j)}$ is a hamp of $\overline{D_j}$. Then, this family $\left(v^{(j)}\right)_{j \in f(V)}$ can be uniquely reconstructed from the concatenation $\bigotimes\limits_{j \in f(V)} v^{(j)}$.

[*Proof of Claim 5:* For each $j \in f(V)$, the list $v^{(j)}$ is a hamp of $\overline{D_j}$ and thus an $f^{-1}(j)$-listing[29]. Hence, Claim 2 shows that the family $\left(v^{(j)}\right)_{j \in f(V)}$ can be uniquely reconstructed from the concatenation $\bigotimes\limits_{j \in f(V)} v^{(j)}$. This proves Claim 5.]

Now, if $\left(v^{(j)}\right)_{j \in f(V)} \in \prod\limits_{j \in f(V)} \left\{\text{hamps of } \overline{D_j}\right\}$ is any family (i.e., if $\left(v^{(j)}\right)_{j \in f(V)}$ is any family of lists such that each $v^{(j)}$ is a hamp of $\overline{D_j}$), then the concatenation $\bigotimes\limits_{j \in f(V)} v^{(j)}$ is an (f, D)-friendly V-listing (by Claim 3). Hence, the map

$$\prod_{j \in f(V)} \left\{\text{hamps of } \overline{D_j}\right\} \to \{(f, D)\text{-friendly } V\text{-listings}\},$$
$$\left(v^{(j)}\right)_{j \in f(V)} \mapsto \bigotimes_{j \in f(V)} v^{(j)}$$

is well-defined. This map is furthermore injective (since Claim 5 shows that a family $\left(v^{(j)}\right)_{j \in f(V)} \in \prod\limits_{j \in f(V)} \left\{\text{hamps of } \overline{D_j}\right\}$ can be uniquely reconstructed from

[29] This can be shown just as in the proof of Claim 3.

the concatenation $\bigotimes_{j\in f(V)} v^{(j)}$) and surjective (since Claim 4 says that any (f,D)-friendly V-listing v can be written in the form $v = \bigotimes_{j\in f(V)} v^{(j)}$ for some family $\left(v^{(j)}\right)_{j\in f(V)} \in \prod_{j\in f(V)} \left\{\text{hamps of } \overline{D_j}\right\}$). Therefore, this map is bijective. The bijection principle thus yields

$$\begin{aligned}
&\left|\left\{(f,D)\text{-friendly } V\text{-listings}\right\}\right| \\
&= \left|\prod_{j\in f(V)} \left\{\text{hamps of } \overline{D_j}\right\}\right| = \prod_{j\in f(V)} \underbrace{\left|\left\{\text{hamps of } \overline{D_j}\right\}\right|}_{=\left(\#\text{ of hamps of } \overline{D_j}\right)} \\
&= \prod_{j\in f(V)} \left(\#\text{ of hamps of } \overline{D_j}\right).
\end{aligned}$$

However, Lemma 2.39 yields

$$\sum_{\substack{\sigma\in\mathfrak{S}_V;\\ f\circ\sigma=f}} \ \sum_{\substack{F\subseteq\mathbf{A}_\sigma\cap A\\ \text{is linear}}} (-1)^{|F|} = \prod_{j\in f(V)} \left(\#\text{ of hamps of } \overline{D_j}\right).$$

Comparing these two equalities, we obtain

$$\begin{aligned}
\sum_{\substack{\sigma\in\mathfrak{S}_V;\\ f\circ\sigma=f}} \ \sum_{\substack{F\subseteq\mathbf{A}_\sigma\cap A\\ \text{is linear}}} (-1)^{|F|} &= \left|\left\{(f,D)\text{-friendly } V\text{-listings}\right\}\right| \\
&= \left(\#\text{ of } (f,D)\text{-friendly } V\text{-listings}\right).
\end{aligned}$$

This proves Lemma 2.40. □

2.8. A bit of Pólya counting

The following lemma is well-known, e.g., from the theory of Pólya enumeration:

Lemma 2.41. Let V be a finite set. Let $\sigma \in \mathfrak{S}_V$ be a permutation of V. Then,

$$\sum_{\substack{f:V\to\mathbb{P};\\ f\circ\sigma=f}} \prod_{v\in V} x_{f(v)} = p_{\operatorname{type}\sigma}.$$

Proof. Let $\gamma_1, \gamma_2, \ldots, \gamma_k$ be the cycles of σ, listed with no repetition[30]. For each $i \in [k]$, let V_i be the set of entries of the cycle γ_i. We shall now collect some basic properties of these cycles γ_i and the corresponding sets V_i:

[30] Keep in mind that a cycle is a rotation-equivalence class. Thus, "listed with no repetition" means that no two of $\gamma_1, \gamma_2, \ldots, \gamma_k$ are the same rotation-equivalence class. For example, if γ_1 is $(1,2)_\sim$, then γ_2 cannot be $(2,1)_\sim$.

Claim 1: Let $v \in V$. Then, there exists a unique $i \in [k]$ such that $v \in V_i$.

[*Proof of Claim 1:* We know that σ is a permutation of V. Hence, each element of V belongs to exactly one cycle of σ. In particular, v belongs to exactly one cycle of σ. In other words, there exists exactly one cycle of σ such that v is an entry of this cycle. In other words, there exists a unique $i \in [k]$ such that v is an entry of γ_i (since $\gamma_1, \gamma_2, \ldots, \gamma_k$ are the cycles of σ, listed with no repetition). In other words, there exists a unique $i \in [k]$ such that $v \in V_i$ (since the statement "$v \in V_i$" is equivalent to the statement "v is an entry of γ_i" [31]). This proves Claim 1.]

Claim 2: Let $i \in [k]$. Then:

(a) We have $\sigma(V_i) \subseteq V_i$.

(b) There exists an element $v_i \in V_i$ such that

$$V_i = \left\{ \sigma^j(v_i) \mid j \in \mathbb{N} \right\}.$$

[*Proof of Claim 2:* It is well-known (from Definition 1.24 **(a)**) that each cycle of σ has the form

$$\left(\sigma^0(w), \sigma^1(w), \sigma^2(w), \ldots, \sigma^{q-1}(w) \right),$$

where w is an element of V and where q is the smallest positive integer satisfying $\sigma^q(w) = w$. Thus, in particular, γ_i has this form (since γ_i is a cycle of σ). In other words, there exists an element w of V such that

$$\gamma_i = \left(\sigma^0(w), \sigma^1(w), \sigma^2(w), \ldots, \sigma^{q-1}(w) \right),$$

where q is the smallest positive integer satisfying $\sigma^q(w) = w$. Let us consider this w and this q.

We have $\gamma_i = \left(\sigma^0(w), \sigma^1(w), \sigma^2(w), \ldots, \sigma^{q-1}(w) \right)$. Thus, the entries of the cycle γ_i are $\sigma^0(w), \sigma^1(w), \sigma^2(w), \ldots, \sigma^{q-1}(w)$.

The set V_i was defined as the set of entries of the cycle γ_i. Thus,

$$V_i = \left\{ \sigma^0(w), \sigma^1(w), \sigma^2(w), \ldots, \sigma^{q-1}(w) \right\}$$

(since the entries of the cycle γ_i are $\sigma^0(w), \sigma^1(w), \sigma^2(w), \ldots, \sigma^{q-1}(w)$). Since q is a positive integer, we have $q - 1 \in \mathbb{N}$ and thus $0 \in \{0, 1, \ldots, q-1\}$. Hence,

$$\sigma^0(w) \in \left\{ \sigma^0(w), \sigma^1(w), \sigma^2(w), \ldots, \sigma^{q-1}(w) \right\} = V_i.$$

In other words, $w \in V_i$ (since $\underbrace{\sigma^0}_{=\mathrm{id}}(w) = \mathrm{id}(w) = w$).

(a) Let $g \in V_i$. We shall prove that $\sigma(g) \in V_i$.

[31] because V_i is the set of entries of γ_i

Indeed, $g \in V_i = \left\{\sigma^0 (w), \sigma^1 (w), \sigma^2 (w), \ldots, \sigma^{q-1} (w)\right\}$. In other words, $g = \sigma^r (w)$ for some $r \in \{0, 1, \ldots, q-1\}$. Consider this r. Applying the map σ to both sides of $g = \sigma^r (w)$, we obtain $\sigma (g) = \sigma (\sigma^r (w)) = \underbrace{(\sigma \circ \sigma^r)}_{=\sigma^{r+1}} (w) = \sigma^{r+1} (w)$.

We are in one of the following two cases:

Case 1: We have $r \neq q - 1$.

Case 2: We have $r = q - 1$.

Let us first consider Case 1. In this case, we have $r \neq q - 1$. Combining this with $r \in \{0, 1, \ldots, q-1\}$, we obtain $r \in \{0, 1, \ldots, q-1\} \setminus \{q-1\} = \{0, 1, \ldots, q-2\}$. Hence, $r + 1 \in \{1, 2, \ldots, q-1\} \subseteq \{0, 1, \ldots, q-1\}$. Thus,

$$\sigma^{r+1} (w) \in \left\{\sigma^0 (w), \sigma^1 (w), \sigma^2 (w), \ldots, \sigma^{q-1} (w)\right\} = V_i.$$

Now, $\sigma (g) = \sigma^{r+1} (w) \in V_i$. Hence, $\sigma (g) \in V_i$ is proved in Case 1.

Let us next consider Case 2. In this case, we have $r = q - 1$. Hence, $r + 1 = q$. Now, $\sigma (g) = \sigma^{r+1} (w) = \sigma^q (w)$ (since $r + 1 = q$), so that $\sigma (g) = \sigma^q (w) = w \in V_i$. Hence, $\sigma (g) \in V_i$ is proved in Case 2.

We have now proved $\sigma (g) \in V_i$ in both Cases 1 and 2. Hence, $\sigma (g) \in V_i$ always holds.

Forget that we fixed g. We thus have shown that $\sigma (g) \in V_i$ for each $g \in V_i$. In other words, $\sigma (V_i) \subseteq V_i$. This proves Claim 2 **(a)**.

(b) We shall first show that $\sigma^j (w) \in V_i$ for each $j \in \mathbb{N}$.

Indeed, we shall prove this by induction on j:

Base case: Our claim $\sigma^j (w) \in V_i$ holds for $j = 0$, since $\sigma^0 (w) \in V_i$.

Induction step: Let $s \in \mathbb{N}$. Assume (as the induction hypothesis) that $\sigma^j (w) \in V_i$ holds for $j = s$. We must prove that $\sigma^j (w) \in V_i$ holds for $j = s + 1$.

We have assumed that $\sigma^j (w) \in V_i$ holds for $j = s$. In other words, $\sigma^s (w) \in V_i$. Now,

$$\underbrace{\sigma^{s+1}}_{=\sigma \circ \sigma^s} (w) = (\sigma \circ \sigma^s) (w) = \sigma \left(\underbrace{\sigma^s (w)}_{\in V_i} \right) \in \sigma (V_i) \subseteq V_i$$

(by Claim 2 **(a)**). In other words, $\sigma^j (w) \in V_i$ holds for $j = s + 1$. This completes the induction step.

Thus, we have proved that $\sigma^j (w) \in V_i$ for each $j \in \mathbb{N}$. In other words,

$$\left\{\sigma^j (w) \mid j \in \mathbb{N}\right\} \subseteq V_i.$$

Combining this with

$$\begin{aligned} V_i &= \left\{\sigma^0 (w), \sigma^1 (w), \sigma^2 (w), \ldots, \sigma^{q-1} (w)\right\} \\ &= \left\{\sigma^j (w) \mid j \in \{0, 1, \ldots, q-1\}\right\} \\ &\subseteq \left\{\sigma^j (w) \mid j \in \mathbb{N}\right\} \qquad (\text{since } \{0, 1, \ldots, q-1\} \subseteq \mathbb{N}), \end{aligned}$$

we obtain $V_i = \left\{\sigma^j(w) \mid j \in \mathbb{N}\right\}$. Since we know that $w \in V_i$, we can thus conclude that there exists an element $v_i \in V_i$ such that $V_i = \left\{\sigma^j(v_i) \mid j \in \mathbb{N}\right\}$ (namely, $v_i = w$). This proves Claim 2 **(b)**.]

Claim 3: The entries of the partition $\operatorname{type}\sigma$ are the numbers $|V_1|, |V_2|, \ldots, |V_k|$ in some order.

[*Proof of Claim 3:* For each $i \in [k]$, we have

$$(\text{the length of the cycle } \gamma_i) = |V_i| \tag{29}$$

[32]. However, we defined $\operatorname{type}\sigma$ to be the partition whose entries are the lengths of the cycles of σ. Thus,

$$\begin{aligned}
&(\text{the entries of } \operatorname{type}\sigma)\\
&= (\text{the lengths of the cycles of } \sigma)\\
&\qquad\qquad\qquad (\text{where we disregard the order of the entries})\\
&= (\text{the lengths of the cycles } \gamma_1, \gamma_2, \ldots, \gamma_k)\\
&\qquad\qquad\qquad \left(\begin{array}{c}\text{since the cycles of } \sigma \text{ are } \gamma_1, \gamma_2, \ldots, \gamma_k\\ \text{(listed without repetition)}\end{array}\right)\\
&= (\text{the numbers } |V_1|, |V_2|, \ldots, |V_k|) \qquad (\text{by (29)}).
\end{aligned}$$

In other words, the entries of $\operatorname{type}\sigma$ are the numbers $|V_1|, |V_2|, \ldots, |V_k|$ in some order. This proves Claim 3.]

Now, we introduce two crucial pieces of notation:

- For each $v \in V$, we let $\operatorname{ind} v$ denote the unique $i \in [k]$ such that $v \in V_i$. (This is well-defined, since Claim 1 shows that there indeed exists a unique $i \in [k]$ such that $v \in V_i$.)

[32] *Proof.* Let $i \in [k]$. Then, V_i is the set of entries of γ_i (by the definition of V_i). Hence, the elements of V_i are the entries of γ_i.

The cycle γ_i of σ is a rotation-equivalence class of tuples of distinct elements (since any cycle of any permutation is such a class). Hence, its entries are distinct.

Now,

$$\begin{aligned}
|V_i| &= (\text{the number of distinct elements of } V_i)\\
&= (\text{the number of distinct entries of } \gamma_i)\\
&\qquad\qquad (\text{since the elements of } V_i \text{ are the entries of } \gamma_i)\\
&= (\text{the number of entries of } \gamma_i)\\
&\qquad\qquad (\text{since the entries of } \gamma_i \text{ are distinct})\\
&= (\text{the length of the cycle } \gamma_i).
\end{aligned}$$

Therefore, $(\text{the length of the cycle } \gamma_i) = |V_i|$.

- For any k-tuple $(a_1, a_2, \ldots, a_k) \in \mathbb{P}^k$, we define a map
$$g[a_1, a_2, \ldots, a_k] : V \to \mathbb{P}$$
by setting
$$((g[a_1, a_2, \ldots, a_k])(v) := a_{\operatorname{ind} v} \qquad \text{for each } v \in V).$$

We observe the following:

Claim 4: Let $j \in [k]$, and let $v \in V_j$. Then, $\operatorname{ind} v = j$.

[*Proof of Claim 4:* Recall that $\operatorname{ind} v$ is defined as the unique $i \in [k]$ such that $v \in V_i$. Hence, if some $i \in [k]$ satisfies $v \in V_i$, then $\operatorname{ind} v = i$. Applying this to $i = j$, we obtain $\operatorname{ind} v = j$ (since $v \in V_j$). This proves Claim 4.]

Claim 5: For any k-tuple $(a_1, a_2, \ldots, a_k) \in \mathbb{P}^k$, we have $g[a_1, a_2, \ldots, a_k] \in \{f : V \to \mathbb{P} \mid f \circ \sigma = f\}$.

[*Proof of Claim 5:* Let $(a_1, a_2, \ldots, a_k) \in \mathbb{P}^k$ be a k-tuple. Then, $g[a_1, a_2, \ldots, a_k]$ is a map from V to $\mathbb{P}$. We shall now show that $(g[a_1, a_2, \ldots, a_k]) \circ \sigma = g[a_1, a_2, \ldots, a_k]$.

Indeed, let $v \in V$ be arbitrary. Recall that $\operatorname{ind} v$ is defined as the unique $i \in [k]$ such that $v \in V_i$. Hence, $\operatorname{ind} v \in [k]$ and $v \in V_{\operatorname{ind} v}$. Thus, Claim 2 **(a)** (applied to $i = \operatorname{ind} v$) yields $\sigma(V_{\operatorname{ind} v}) \subseteq V_{\operatorname{ind} v}$. From $v \in V_{\operatorname{ind} v}$, we obtain $\sigma(v) \in \sigma(V_{\operatorname{ind} v}) \subseteq V_{\operatorname{ind} v}$.

Therefore, Claim 4 (applied to $\operatorname{ind} v$ and $\sigma(v)$ instead of j and v) yields $\operatorname{ind}(\sigma(v)) = \operatorname{ind} v$ (since $\sigma(v) \in V_{\operatorname{ind} v}$).

The definition of $g[a_1, a_2, \ldots, a_k]$ yields
$$(g[a_1, a_2, \ldots, a_k])(v) = a_{\operatorname{ind} v} \qquad \text{and}$$
$$(g[a_1, a_2, \ldots, a_k])(\sigma(v)) = a_{\operatorname{ind}(\sigma(v))} = a_{\operatorname{ind} v} \qquad (\text{since } \operatorname{ind}(\sigma(v)) = \operatorname{ind} v).$$
Comparing these two equalities, we obtain
$$(g[a_1, a_2, \ldots, a_k])(v) = (g[a_1, a_2, \ldots, a_k])(\sigma(v)) = ((g[a_1, a_2, \ldots, a_k]) \circ \sigma)(v).$$

Forget that we fixed v. We thus have proved that $(g[a_1, a_2, \ldots, a_k])(v) = ((g[a_1, a_2, \ldots, a_k]) \circ \sigma)(v)$ for each $v \in V$. In other words, $g[a_1, a_2, \ldots, a_k] = (g[a_1, a_2, \ldots, a_k]) \circ \sigma$. In other words, $(g[a_1, a_2, \ldots, a_k]) \circ \sigma = g[a_1, a_2, \ldots, a_k]$.

Thus, $g[a_1, a_2, \ldots, a_k]$ is a map $f : V \to \mathbb{P}$ satisfying $f \circ \sigma = f$. In other words, $g[a_1, a_2, \ldots, a_k] \in \{f : V \to \mathbb{P} \mid f \circ \sigma = f\}$. This proves Claim 5.]

Claim 5 allows us to define a map
$$\Gamma : \mathbb{P}^k \to \{f : V \to \mathbb{P} \mid f \circ \sigma = f\},$$
$$(a_1, a_2, \ldots, a_k) \mapsto g[a_1, a_2, \ldots, a_k].$$

Consider this map Γ. Next, we claim:

Claim 6: The map Γ is injective.

[*Proof of Claim 6:* Let $(a_1, a_2, \ldots, a_k)$ and $(b_1, b_2, \ldots, b_k)$ be two elements of $\mathbb{P}^k$ satisfying $\Gamma(a_1, a_2, \ldots, a_k) = \Gamma(b_1, b_2, \ldots, b_k)$. We shall show that $(a_1, a_2, \ldots, a_k) = (b_1, b_2, \ldots, b_k)$.

Indeed, let us fix $i \in [k]$. Claim 2 **(b)** shows that there exists an element $v_i \in V_i$ such that

$$V_i = \left\{\sigma^j(v_i) \mid j \in \mathbb{N}\right\}.$$

Consider this v_i. Claim 4 (applied to $j = i$ and $v = v_i$) yields $\operatorname{ind}(v_i) = i$ (since $v_i \in V_i$). The definition of $g[a_1, a_2, \ldots, a_k]$ yields

$$(g[a_1, a_2, \ldots, a_k])(v_i) = a_{\operatorname{ind}(v_i)} = a_i \qquad (\text{since } \operatorname{ind}(v_i) = i).$$

The definition of Γ yields $\Gamma(a_1, a_2, \ldots, a_k) = g[a_1, a_2, \ldots, a_k]$. Thus,

$$\underbrace{(\Gamma(a_1, a_2, \ldots, a_k))}_{=g[a_1,a_2,\ldots,a_k]}(v_i) = (g[a_1, a_2, \ldots, a_k])(v_i) = a_i. \tag{30}$$

The same argument (applied to $(b_1, b_2, \ldots, b_k)$ instead of $(a_1, a_2, \ldots, a_k)$) yields

$$(\Gamma(b_1, b_2, \ldots, b_k))(v_i) = b_i. \tag{31}$$

However, (30) yields

$$a_i = \underbrace{(\Gamma(a_1, a_2, \ldots, a_k))}_{=\Gamma(b_1,b_2,\ldots,b_k)}(v_i) = (\Gamma(b_1, b_2, \ldots, b_k))(v_i) = b_i \qquad (\text{by (31)}).$$

Forget that we fixed i. We thus have proved that $a_i = b_i$ for each $i \in [k]$. In other words, $(a_1, a_2, \ldots, a_k) = (b_1, b_2, \ldots, b_k)$.

Forget that we fixed $(a_1, a_2, \ldots, a_k)$ and $(b_1, b_2, \ldots, b_k)$. We thus have shown that if $(a_1, a_2, \ldots, a_k)$ and $(b_1, b_2, \ldots, b_k)$ are two elements of $\mathbb{P}^k$ satisfying $\Gamma(a_1, a_2, \ldots, a_k) = \Gamma(b_1, b_2, \ldots, b_k)$, then $(a_1, a_2, \ldots, a_k) = (b_1, b_2, \ldots, b_k)$. In other words, the map Γ is injective. This proves Claim 6.]

Claim 7: The map Γ is surjective.

[*Proof of Claim 7:* Let $h \in \{f : V \to \mathbb{P} \mid f \circ \sigma = f\}$. We shall construct a k-tuple $(a_1, a_2, \ldots, a_k) \in \mathbb{P}^k$ such that $h = \Gamma(a_1, a_2, \ldots, a_k)$.

Indeed, we have $h \in \{f : V \to \mathbb{P} \mid f \circ \sigma = f\}$. In other words, h is a map $f : V \to \mathbb{P}$ satisfying $f \circ \sigma = f$. In other words, h is a map from V to $\mathbb{P}$ and satisfies $h \circ \sigma = h$. Hence,

$$h \circ \sigma^j = h \tag{32}$$

for each $j \in \mathbb{N}$ [33].

[33] *Proof of (32):* We prove (32) by induction on j:

For each $i \in [k]$, there exists an element $v_i \in V_i$ such that

$$V_i = \left\{ \sigma^j \left(v_i \right) \ \mid \ j \in \mathbb{N} \right\} \tag{33}$$

(by Claim 2 **(b)**). Consider such a v_i for each $i \in [k]$. For each $i \in [k]$, we set

$$a_i := h \left(v_i \right).$$

Thus, we have defined k elements $a_1, a_2, \ldots, a_k \in \mathbb{P}$. Hence, $\left(a_1, a_2, \ldots, a_k \right) \in \mathbb{P}^k$.

Now, we shall show that $h = \Gamma \left(a_1, a_2, \ldots, a_k \right)$.

Indeed, let $v \in V$ be arbitrary. Recall that $\operatorname{ind} v$ is defined as the unique $i \in [k]$ such that $v \in V_i$. Hence, $\operatorname{ind} v \in [k]$ and $v \in V_{\operatorname{ind} v}$. Therefore,

$$v \in V_{\operatorname{ind} v} = \left\{ \sigma^j \left(v_{\operatorname{ind} v} \right) \ \mid \ j \in \mathbb{N} \right\}$$

(by (33), applied to $i = \operatorname{ind} v$). In other words, $v = \sigma^j \left(v_{\operatorname{ind} v} \right)$ for some $j \in \mathbb{N}$. Consider this j. Now,

$$h \left(\underbrace{v}_{= \sigma^j \left(v_{\operatorname{ind} v} \right)} \right) = h \left(\sigma^j \left(v_{\operatorname{ind} v} \right) \right) = \underbrace{\left(h \circ \sigma^j \right)}_{\substack{= h \\ \text{(by (32))}}} \left(v_{\operatorname{ind} v} \right) = h \left(v_{\operatorname{ind} v} \right).$$

Comparing this with

$$\begin{aligned} \underbrace{\left(\Gamma \left(a_1, a_2, \ldots, a_k \right) \right)}_{\substack{= g[a_1, a_2, \ldots, a_k] \\ \text{(by the definition of } \Gamma)}} (v) &= \left(g \left[a_1, a_2, \ldots, a_k \right] \right) (v) \\ &= a_{\operatorname{ind} v} \qquad \left(\text{by the definition of } g \left[a_1, a_2, \ldots, a_k \right] \right) \\ &= h \left(v_{\operatorname{ind} v} \right) \qquad \left(\text{by the definition of } a_{\operatorname{ind} v} \right), \end{aligned}$$

we obtain

$$h (v) = \left(\Gamma \left(a_1, a_2, \ldots, a_k \right) \right) (v).$$

Forget that we fixed v. We thus have proved that $h(v) = \left(\Gamma \left(a_1, a_2, \ldots, a_k \right) \right) (v)$ for each $v \in V$. In other words, $h = \Gamma \left(a_1, a_2, \ldots, a_k \right)$. Hence, $h \in \Gamma \left(\mathbb{P}^k \right)$ (since $\left(a_1, a_2, \ldots, a_k \right) \in \mathbb{P}^k$).

Base case: We have $h \circ \underbrace{\sigma^0}_{= \mathrm{id}} = h \circ \mathrm{id} = h$. In other words, (32) holds for $j = 0$.

Induction step: Let $i \in \mathbb{N}$. Assume (as the induction hypothesis) that (32) holds for $j = i$. We must prove that (32) holds for $j = i + 1$.

We have assumed that (32) holds for $j = i$. In other words, $h \circ \sigma^i = h$. Now,

$$h \circ \underbrace{\sigma^{i+1}}_{= \sigma \circ \sigma^i} = \underbrace{h \circ \sigma}_{= h} \circ \sigma^i = h \circ \sigma^i = h.$$

In other words, (32) holds for $j = i + 1$. This completes the induction step. Thus, we have proved (32) by induction.

Forget that we fixed h. We thus have proved that $h \in \Gamma\left(\mathbb{P}^k\right)$ for each $h \in \{f : V \to \mathbb{P} \mid f \circ \sigma = f\}$. In other words, $\{f : V \to \mathbb{P} \mid f \circ \sigma = f\} \subseteq \Gamma\left(\mathbb{P}^k\right)$. In other words, the map Γ is surjective. This proves Claim 7.]

Claim 8: Let $(a_1, a_2, \ldots, a_k) \in \mathbb{P}^k$. Then,

$$\prod_{v \in V} x_{(\Gamma(a_1,a_2,\ldots,a_k))(v)} = \prod_{i=1}^{k} x_{a_i}^{|V_i|}.$$

[*Proof of Claim 8:* For each $v \in V$, we have

$$\begin{aligned}
\underbrace{(\Gamma(a_1, a_2, \ldots, a_k))}_{\substack{=g[a_1,a_2,\ldots,a_k] \\ \text{(by the definition of } \Gamma)}}(v) &= (g[a_1, a_2, \ldots, a_k])(v) \\
&= a_{\operatorname{ind} v} \qquad \text{(by the definition of } g[a_1, a_2, \ldots, a_k]) .
\end{aligned}$$

Thus, for each $v \in V$, we have

$$x_{(\Gamma(a_1,a_2,\ldots,a_k))(v)} = x_{a_{\operatorname{ind} v}}.$$

Multiplying these equalities for all $v \in V$, we obtain

$$\begin{aligned}
\prod_{v \in V} x_{(\Gamma(a_1,a_2,\ldots,a_k))(v)} = \prod_{v \in V} x_{a_{\operatorname{ind} v}} &= \prod_{j=1}^{k} \prod_{\substack{v \in V; \\ \operatorname{ind} v = j}} \underbrace{x_{a_{\operatorname{ind} v}}}_{\substack{=x_{a_j} \\ \text{(since } \operatorname{ind} v = j)}} \\
&\qquad \left(\begin{array}{c}\text{here, we have split the product} \\ \text{according to the value of } \operatorname{ind} v\end{array}\right) \\
&= \prod_{j=1}^{k} \prod_{\substack{v \in V; \\ \operatorname{ind} v = j}} x_{a_j}.
\end{aligned} \tag{34}$$

Now, fix $j \in [k]$. Then, $\{v \in V \mid \operatorname{ind} v = j\} = V_j$ [34]. Hence, the product sign "$\prod\limits_{\substack{v \in V; \\ \operatorname{ind} v = j}}$" can be rewritten as "$\prod\limits_{v \in V_j}$". Thus, in particular,

$$\prod_{\substack{v \in V; \\ \operatorname{ind} v = j}} x_{a_j} = \prod_{v \in V_j} x_{a_j} = x_{a_j}^{|V_j|}. \tag{35}$$

[34] *Proof.* If $v \in V_j$, then $v \in V$ (since V_j is a subset of V) and $\operatorname{ind} v = j$ (by Claim 4). Thus, every element of V_j is an element $v \in V$ satisfying $\operatorname{ind} v = j$. In other words, $V_j \subseteq \{v \in V \mid \operatorname{ind} v = j\}$.

Let $v \in V$ satisfy $\operatorname{ind} v = j$. Recall that $\operatorname{ind} v$ is defined as the unique $i \in [k]$ such that $v \in V_i$. Hence, $\operatorname{ind} v \in [k]$ and $v \in V_{\operatorname{ind} v}$. Thus, $v \in V_{\operatorname{ind} v} = V_j$ (since $\operatorname{ind} v = j$). In other words, v belongs to V_j.

Forget that we fixed v. We thus have shown that every $v \in V$ satisfying $\operatorname{ind} v = j$ must belong to V_j. In other words, $\{v \in V \mid \operatorname{ind} v = j\} \subseteq V_j$. Combining this with $V_j \subseteq \{v \in V \mid \operatorname{ind} v = j\}$, we obtain $\{v \in V \mid \operatorname{ind} v = j\} = V_j$.

Now, forget that we fixed j. We thus have proved (35) for each $j \in [k]$. Now, (34) becomes

$$\prod_{v \in V} x_{(\Gamma(a_1,a_2,\ldots,a_k))(v)} = \prod_{j=1}^{k} \underbrace{\prod_{\substack{v \in V; \\ \operatorname{ind} v = j}} x_{a_j}}_{\substack{= x_{a_j}^{\left|V_j\right|} \\ \text{(by (35))}}} = \prod_{j=1}^{k} x_{a_j}^{\left|V_j\right|} = \prod_{i=1}^{k} x_{a_i}^{|V_i|}$$

(here, we have renamed the index j as i in the product). This proves Claim 8.]

Now, our proof is almost complete. The map $\Gamma : \mathbb{P}^k \to \{f : V \to \mathbb{P} \mid f \circ \sigma = f\}$ is injective (by Claim 6) and surjective (by Claim 7); thus, it is bijective. In other words, Γ is a bijection. Hence, we can substitute $\Gamma(a_1, a_2, \ldots, a_k)$ for f in the sum $\sum\limits_{\substack{f:V \to \mathbb{P}; \\ f \circ \sigma = f}} \prod\limits_{v \in V} x_{f(v)}$. We thus obtain

$$\begin{aligned} \sum_{\substack{f:V \to \mathbb{P}; \\ f \circ \sigma = f}} \prod_{v \in V} x_{f(v)} &= \sum_{(a_1,a_2,\ldots,a_k) \in \mathbb{P}^k} \underbrace{\prod_{v \in V} x_{(\Gamma(a_1,a_2,\ldots,a_k))(v)}}_{\substack{= \prod_{i=1}^{k} x_{a_i}^{|V_i|} \\ \text{(by Claim 8)}}} \\ &= \sum_{(a_1,a_2,\ldots,a_k) \in \mathbb{P}^k} \prod_{i=1}^{k} x_{a_i}^{|V_i|}. \end{aligned} \tag{36}$$

On the other hand, Claim 3 shows that the entries of the partition $\operatorname{type} \sigma$ are the numbers $|V_1|, |V_2|, \ldots, |V_k|$ in some order. In other words, there exists a permutation τ of $[k]$ such that $\operatorname{type} \sigma = \left(\left|V_{\tau(1)}\right|, \left|V_{\tau(2)}\right|, \ldots, \left|V_{\tau(k)}\right|\right)$. Consider this τ. The map $\tau : [k] \to [k]$ is a bijection (since it is a permutation of $[k]$). From

$\operatorname{type}\sigma = \left(\left|V_{\tau(1)}\right|, \left|V_{\tau(2)}\right|, \ldots, \left|V_{\tau(k)}\right|\right)$, we obtain

$$\begin{aligned}
p_{\operatorname{type}\sigma} &= p_{\left(\left|V_{\tau(1)}\right|, \left|V_{\tau(2)}\right|, \ldots, \left|V_{\tau(k)}\right|\right)} \\
&= p_{\left|V_{\tau(1)}\right|} p_{\left|V_{\tau(2)}\right|} \cdots p_{\left|V_{\tau(k)}\right|} \qquad \left(\text{by the definition of } p_{\left(\left|V_{\tau(1)}\right|, \left|V_{\tau(2)}\right|, \ldots, \left|V_{\tau(k)}\right|\right)}\right) \\
&= \prod_{i \in [k]} p_{\left|V_{\tau(i)}\right|} \\
&= \underbrace{\prod_{i \in [k]}}_{= \prod_{i=1}^{k}} \underbrace{p_{\left|V_i\right|}}_{\substack{= x_1^{\left|V_i\right|} + x_2^{\left|V_i\right|} + x_3^{\left|V_i\right|} + \cdots \\ \text{(by the definition of } p_{\left|V_i\right|}\text{)}}} \qquad \left(\begin{array}{c}\text{here, we have substituted } i \text{ for } \tau(i) \text{ in the} \\ \text{product, since } \tau : [k] \to [k] \text{ is a bijection}\end{array}\right) \\
&= \prod_{i=1}^{k} \underbrace{\left(x_1^{\left|V_i\right|} + x_2^{\left|V_i\right|} + x_3^{\left|V_i\right|} + \cdots\right)}_{= \sum_{a \in \mathbb{P}} x_a^{\left|V_i\right|}} = \prod_{i=1}^{k} \sum_{a \in \mathbb{P}} x_a^{\left|V_i\right|} \\
&= \sum_{(a_1, a_2, \ldots, a_k) \in \mathbb{P}^k} \prod_{i=1}^{k} x_{a_i}^{\left|V_i\right|} \qquad \text{(by the product rule)}.
\end{aligned}$$

Comparing this with (36), we obtain

$$\sum_{\substack{f : V \to \mathbb{P}; \\ f \circ \sigma = f}} \prod_{v \in V} x_{f(v)} = p_{\operatorname{type}\sigma}.$$

Thus, Lemma 2.41 is proven. □

2.9. A final alternating sum

We need one more alternating-sum identity:

Proposition 2.42. Let $D = (V, A)$ be a digraph. Let $\sigma \in \mathfrak{S}_V$ be a permutation of V. Then,

$$\sum_{\substack{F \subseteq \mathbf{A}_\sigma \cap A \\ \text{is linear}}} (-1)^{|F|} = \begin{cases} (-1)^{\varphi(\sigma)}, & \text{if } \sigma \in \mathfrak{S}_V\left(D, \overline{D}\right); \\ 0, & \text{else,} \end{cases}$$

where we set

$$\varphi(\sigma) := \sum_{\substack{\gamma \in \operatorname{Cycs}\sigma; \\ \gamma \text{ is a } D\text{-cycle}}} (\ell(\gamma) - 1).$$

Our proof of this proposition requires several auxiliary results. We begin by proving some lemmas on the linearity of certain sets:

Lemma 2.43. Let V be a finite set. Let p be a path of V. Then, $\operatorname{Arcs} p$ is a linear subset of $V \times V$.

Proof. Recall that a path of V means a nonempty tuple of distinct elements of V. Hence, p is a nonempty tuple of distinct elements of V (since p is a path of V).

Let W be the set of all entries of p. Then, W is a subset of V (since p is a tuple of elements of V). Moreover, the entries of p are precisely the elements of W (since W is the set of all entries of p). Thus, all entries of p belong to W. Hence, p is a tuple of elements of W. Thus, p is a nonempty tuple of distinct elements of W (since p is a nonempty tuple of distinct elements of V). In other words, p is a path of W (by the definition of a "path of W").

Each $v \in W$ is an element of W. In other words, each $v \in W$ is an entry of p (since the entries of p are precisely the elements of W). In other words, each $v \in W$ belongs to the path p.

Now, we claim that the 1-element set $\{p\}$ is a path cover of W. Indeed, $\{p\}$ is clearly a set of paths of W (since p is a path of W) and has the property that each $v \in W$ belongs to exactly one of these paths (because each $v \in W$ belongs to the path p). In other words, $\{p\}$ is a path cover of W (by the definition of a "path cover"). The arc set $\operatorname{Arcs}\{p\}$ of this path cover is

$$\begin{aligned} \operatorname{Arcs}\{p\} &= \bigcup_{v \in \{p\}} \operatorname{Arcs} v \qquad \text{(by the definition of } \operatorname{Arcs}\{p\}) \\ &= \operatorname{Arcs} p. \end{aligned}$$

Thus, $\operatorname{Arcs} p$ is the arc set of some path cover of W (namely, of the path cover $\{p\}$).

It is furthermore easy to see that $\operatorname{Arcs} p$ is a subset of $W \times W$ [35].

Now, recall that a subset F of $W \times W$ is said to be linear if it is the arc set of some path cover of W (by the definition of "linear"). Hence, the subset $\operatorname{Arcs} p$ of $W \times W$ is linear (since it is the arc set of some path cover of W).

However, Proposition 2.9 (applied to $F = \operatorname{Arcs} p$) shows that $\operatorname{Arcs} p$ is linear as a subset of $W \times W$ if and only if $\operatorname{Arcs} p$ is linear as a subset of $V \times V$. Thus, $\operatorname{Arcs} p$ is linear as a subset of $V \times V$ (since $\operatorname{Arcs} p$ is linear as a subset of $W \times W$). This proves Lemma 2.43. □

[35] *Proof.* Write the path p as $(v_1, v_2, \ldots, v_k)$. Then, the entries of p are $v_1, v_2, \ldots, v_k$. However, we know that all entries of p belong to W. In other words, all of $v_1, v_2, \ldots, v_k$ belong to W (since the entries of p are $v_1, v_2, \ldots, v_k$). Hence, the pairs $(v_1, v_2), (v_2, v_3), \ldots, (v_{k-1}, v_k)$ belong to $W \times W$.

However, $p = (v_1, v_2, \ldots, v_k)$. Thus,

$$\begin{aligned} \operatorname{Arcs} p &= \operatorname{Arcs}(v_1, v_2, \ldots, v_k) \\ &= \{(v_1, v_2), (v_2, v_3), \ldots, (v_{k-1}, v_k)\} \qquad \text{(by the definition of } \operatorname{Arcs}(v_1, v_2, \ldots, v_k)) \\ &\subseteq W \times W \qquad \text{(since the pairs } (v_1, v_2), (v_2, v_3), \ldots, (v_{k-1}, v_k) \text{ belong to } W \times W). \end{aligned}$$

In other words, $\operatorname{Arcs} p$ is a subset of $W \times W$.

Lemma 2.44. Let V be a finite set. Let $\sigma \in \mathfrak{S}_V$ be a permutation of V. Let γ be a cycle of σ. Let $a \in \operatorname{CArcs}\gamma$. Then, there exists a tuple $w = (w_1, w_2, \ldots, w_k) \in \gamma$ such that $a = (w_k, w_1)$.

Proof. The cycle γ is a cycle of σ, and thus is a rotation-equivalence class of nonempty tuples of distinct elements of V (since any cycle of σ is such a class). Hence, we can write γ in the form $\gamma = v_\sim$, where v is a nonempty tuple of distinct elements of V. Consider this v.

Write this tuple v as $v = (v_1, v_2, \ldots, v_k)$. Then, $k \geq 1$ (since v is nonempty), and the entries $v_1, v_2, \ldots, v_k$ of v are distinct (since v is a tuple of distinct elements of V). Let us set $v_{k+1} := v_1$. We have

$$\begin{aligned} \operatorname{CArcs}\gamma &= \operatorname{CArcs}(v_\sim) && (\text{since } \gamma = v_\sim) \\ &= \operatorname{CArcs} v && (\text{by the definition of } \operatorname{CArcs}(v_\sim)\text{, since } v \in v_\sim) \\ &= \{(v_i, v_{i+1}) \mid i \in [k]\} && (37) \end{aligned}$$

(by the definition of $\operatorname{CArcs} v$, since $v = (v_1, v_2, \ldots, v_k)$ and $v_{k+1} = v_1$). Thus,

$$\begin{aligned} \operatorname{CArcs}\gamma &= \{(v_i, v_{i+1}) \mid i \in [k]\} \\ &= \{(v_1, v_2), (v_2, v_3), \ldots, (v_k, v_{k+1})\}. \end{aligned} \tag{38}$$

We have

$$a \in \operatorname{CArcs}\gamma = \{(v_i, v_{i+1}) \mid i \in [k]\} \qquad (\text{by (37)}).$$

In other words, there exists some $i \in [k]$ such that $a = (v_i, v_{i+1})$. Consider this i.

We now define a k-tuple $w \in V^k$ by

$$w := (v_{i+1}, v_{i+2}, \ldots, v_k, v_1, v_2, \ldots, v_i).$$

Thus, w can be obtained from v by a cyclic rotation (specifically, by cyclically rotating v a total of i steps to the left). Hence, w is rotation-equivalent to v. Thus, w belongs to the same rotation-equivalence class as v. In other words, $w \in v_\sim$ (since $v_\sim$ is the rotation-equivalence class to which v belongs). In other words, $w \in \gamma$ (since $\gamma = v_\sim$). In other words, $(w_1, w_2, \ldots, w_k) \in \gamma$ (since $w = (w_1, w_2, \ldots, w_k)$).

Let us write the k-tuple $w \in V^k$ as $w = (w_1, w_2, \ldots, w_k)$. Then, it is easy to see that $v_i = w_k$ [36] and $v_{i+1} = w_1$ [37]. Now,

$$a = \left(\underbrace{v_i}_{=w_k}, \underbrace{v_{i+1}}_{=w_1} \right) = (w_k, w_1).$$

We have thus found a tuple $w = (w_1, w_2, \ldots, w_k) \in \gamma$ such that $a = (w_k, w_1)$. Hence, such a tuple exists. This proves Lemma 2.44. $\square$

Lemma 2.45. Let V be a finite set. Let $\sigma \in \mathfrak{S}_V$ be a permutation of V. Let γ be a cycle of σ. Let $C = \operatorname{CArcs} \gamma$. Then:

(a) The subset C of $V \times V$ is not linear.

(b) Every proper subset of C is linear.

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[36] *Proof.* We have $w = (v_{i+1}, v_{i+2}, \ldots, v_k, v_1, v_2, \ldots, v_i)$. Thus,

$$\begin{aligned}
&(\text{the last entry of the } k\text{-tuple } w) \\
&= (\text{the last entry of the } k\text{-tuple } (v_{i+1}, v_{i+2}, \ldots, v_k, v_1, v_2, \ldots, v_i)) \\
&= v_i \qquad (\text{since } i \in [k]).
\end{aligned}$$

Hence,

$$v_i = (\text{the last entry of the } k\text{-tuple } w) = w_k$$

(since $w = (w_1, w_2, \ldots, w_k)$).

[37] *Proof.* We are in one of the following two cases:

Case 1: We have $i \neq k$.

Case 2: We have $i = k$.

Let us first consider Case 1. In this case, we have $i \neq k$. Combining $i \in [k]$ with $i \neq k$, we find $i \in [k] \setminus \{k\} = [k-1]$. However, we have $w = (v_{i+1}, v_{i+2}, \ldots, v_k, v_1, v_2, \ldots, v_i)$. Thus,

$$\begin{aligned}
&(\text{the first entry of the } k\text{-tuple } w) \\
&= (\text{the first entry of the } k\text{-tuple } (v_{i+1}, v_{i+2}, \ldots, v_k, v_1, v_2, \ldots, v_i)) \\
&= v_{i+1} \qquad (\text{since } i \in [k-1]).
\end{aligned}$$

Hence, $v_{i+1} = (\text{the first entry of the } k\text{-tuple } w) = w_1$ (since $w = (w_1, w_2, \ldots, w_k)$). Thus, $v_{i+1} = w_1$ is proved in Case 1.

Let us now consider Case 2. In this case, we have $i = k$. Hence, $v_{i+1} = v_{k+1} = v_1$. However, we have

$$\begin{aligned}
w &= (v_{i+1}, v_{i+2}, \ldots, v_k, v_1, v_2, \ldots, v_i) \\
&= (v_{k+1}, v_{k+2}, \ldots, v_k, v_1, v_2, \ldots, v_k) \qquad (\text{since } i = k) \\
&= (v_1, v_2, \ldots, v_k).
\end{aligned}$$

Hence,

$$(\text{the first entry of the } k\text{-tuple } w) = v_1 = v_{i+1} \qquad (\text{since } v_{i+1} = v_1).$$

Thus, $v_{i+1} = (\text{the first entry of the } k\text{-tuple } w) = w_1$ (since $w = (w_1, w_2, \ldots, w_k)$). Thus, $v_{i+1} = w_1$ is proved in Case 2.

We have now proved $v_{i+1} = w_1$ in both Cases 1 and 2. Thus, $v_{i+1} = w_1$ always holds.

Proof of Lemma 2.45. The cycle γ is a cycle of σ, and thus is a rotation-equivalence class of nonempty tuples of distinct elements of V (since any cycle of σ is such a class). Hence, we can write γ in the form $\gamma = v_{\sim}$, where v is a nonempty tuple of distinct elements of V. Consider this v.

Write the tuple v as $v = (v_1, v_2, \ldots, v_k)$. Then, $k \geq 1$ (since v is nonempty), and the entries $v_1, v_2, \ldots, v_k$ of v are distinct (since v is a tuple of distinct elements of V). Let us set $v_{k+1} := v_1$. We have

$$\begin{aligned} C &= \operatorname{CArcs} \gamma = \operatorname{CArcs}(v_{\sim}) \qquad (\text{since } \gamma = v_{\sim}) \\ &= \operatorname{CArcs} v \qquad (\text{by the definition of } \operatorname{CArcs}(v_{\sim}), \text{ since } v \in v_{\sim}) \\ &= \{(v_i, v_{i+1}) \mid i \in [k]\} \end{aligned} \tag{39}$$

(by the definition of $\operatorname{CArcs} v$, since $v = (v_1, v_2, \ldots, v_k)$ and $v_{k+1} = v_1$).

(a) Assume the contrary. Thus, C is linear. In other words, C is the arc set of some path cover of V (by the definition of "linear"). Let P be this path cover. Thus, C is the arc set of P. In other words, $C = \operatorname{Arcs} P$.

We have $C = \operatorname{Arcs} P = \bigcup_{q \in P} \operatorname{Arcs} q$ (by the definition of $\operatorname{Arcs} P$).

Now, $1 \in [k]$ (since $k \geq 1$) and therefore

$$\begin{aligned} (v_1, v_2) &\in \{(v_i, v_{i+1}) \mid i \in [k]\} = C \qquad (\text{by (39)}) \\ &= \bigcup_{q \in P} \operatorname{Arcs} q. \end{aligned}$$

In other words, there exists a path $q \in P$ such that $(v_1, v_2) \in \operatorname{Arcs} q$. Consider such a path q, and denote it by w. Thus, $w \in P$ and $(v_1, v_2) \in \operatorname{Arcs} w$.

Let us write the path w as $w = (w_1, w_2, \ldots, w_\ell)$. Thus, the definition of $\operatorname{Arcs} w$ yields

$$\operatorname{Arcs} w = \{(w_1, w_2), (w_2, w_3), \ldots, (w_{\ell-1}, w_\ell)\}.$$

Hence, $(v_1, v_2) \in \operatorname{Arcs} w = \{(w_1, w_2), (w_2, w_3), \ldots, (w_{\ell-1}, w_\ell)\}$. In other words, there exists a $p \in [\ell - 1]$ such that $(v_1, v_2) = (w_p, w_{p+1})$. Consider this p.

From $(v_1, v_2) = (w_p, w_{p+1})$, we obtain $v_1 = w_p$ and $v_2 = w_{p+1}$.

Note that $w_1, w_2, \ldots, w_\ell$ are the entries of w (because $w = (w_1, w_2, \ldots, w_\ell)$).

We know that w is a path of V (since $w \in P$, but P is a path cover of V), thus a nonempty tuple of distinct elements of V (since a path of V is defined to be a nonempty tuple of distinct elements of V). Hence, the entries of w are distinct. In other words, $w_1, w_2, \ldots, w_\ell$ are distinct (since $w_1, w_2, \ldots, w_\ell$ are the entries of w).

We now claim the following:

Claim 1: For each $i \in [k+1]$, we have $p - 1 + i \in [\ell]$ and $v_i = w_{p-1+i}$.

[*Proof of Claim 1:* We proceed by induction on i:

Base case: We have $p - 1 + 1 = p \in [\ell - 1] \subseteq [\ell]$ and $v_1 = w_p = w_{p-1+1}$ (since $p = p - 1 + 1$). In other words, Claim 1 holds for $i = 1$.

Induction step: Let $j \in [k]$. Assume (as the induction hypothesis) that Claim 1 holds for $i = j$. We must prove that Claim 1 holds for $i = j + 1$ as well.

We have assumed that Claim 1 holds for $i = j$. In other words, we have $p - 1 + j \in [\ell]$ and $v_j = w_{p-1+j}$.

We have $j \in [k]$. Thus,

$$\left(v_j, v_{j+1}\right) \in \{(v_i, v_{i+1}) \mid i \in [k]\} = \bigcup_{q \in P} \operatorname{Arcs} q.$$

Hence, there exists a path $q \in P$ such that $\left(v_j, v_{j+1}\right) \in \operatorname{Arcs} q$. Consider this q. From $\left(v_j, v_{j+1}\right) \in \operatorname{Arcs} q$, it follows that v_j and v_{j+1} are two entries of the path q (since both entries of any pair in $\operatorname{Arcs} q$ are entries of q). Hence, in particular, v_j is an entry of q. Also, from $v_j = w_{p-1+j}$, it follows that v_j is an entry of the path w (since $w_1, w_2, \ldots, w_\ell$ are the entries of w).

Now, v_j is both an entry of q and an entry of w. In other words, v_j belongs to the paths q and w.

Recall that P is a path cover of V. Hence, each $v \in V$ belongs to exactly one of the paths in P. In particular, v_j belongs to exactly one of the paths in P.

However, we know that v_j belongs to the paths q and w. If these paths q and w were distinct, then this would mean that v_j belongs to at least two of the paths in P (since both q and w are paths in P); but this would contradict the fact that v_j belongs to exactly one of the paths in P. Hence, the paths q and w cannot be distinct.

In other words, $q = w$. However,

$$\left(v_j, v_{j+1}\right) \in \operatorname{Arcs} \underbrace{q}_{=w} = \operatorname{Arcs} w = \{(w_1, w_2), (w_2, w_3), \ldots, (w_{\ell-1}, w_\ell)\}.$$

In other words, there exists some $z \in [\ell - 1]$ such that $\left(v_j, v_{j+1}\right) = (w_z, w_{z+1})$. Consider this z.

From $\left(v_j, v_{j+1}\right) = (w_z, w_{z+1})$, we obtain $v_j = w_z$ and $v_{j+1} = w_{z+1}$. Comparing $v_j = w_z$ with $v_j = w_{p-1+j}$, we obtain $w_z = w_{p-1+j}$. This entails $z = p - 1 + j$ (since $w_1, w_2, \ldots, w_\ell$ are distinct). Hence, $p - 1 + j = z \in [\ell - 1]$, so that $p - 1 + j \leq \ell - 1$. Adding 1 to both sides of this inequality, we obtain $p + j \leq \ell$. In other words, $p - 1 + (j + 1) \leq \ell$ (since $p - 1 + (j + 1) = p + j$). Thus, $p - 1 + (j + 1) \in [\ell]$. Moreover, we now know that $v_{j+1} = w_{z+1} = w_{p-1+(j+1)}$ (since $\underbrace{z}_{=p-1+j} +1 = p - 1 + j + 1 = p - 1 + (j + 1)$).

Altogether, we have shown that $p - 1 + (j + 1) \in [\ell]$ and $v_{j+1} = w_{p-1+(j+1)}$. In other words, Claim 1 holds for $i = j + 1$. This completes the induction step. Thus, Claim 1 is proven.]

Now, $k + 1 \in [k + 1]$ (since $k + 1 \geq k \geq 1$). Hence, we can apply Claim 1 to $i = k + 1$, and thus obtain $p - 1 + (k + 1) \in [\ell]$ and $v_{k+1} = w_{p-1+(k+1)}$. In other words, $p + k \in [\ell]$ and $v_{k+1} = w_{p+k}$ (since $p - 1 + (k + 1) = p + k$).

However, $v_{k+1} = v_1 = w_p$. Hence, $w_p = v_{k+1} = w_{p+k}$. This entails $p = p + k$ (since $w_1, w_2, \ldots, w_\ell$ are distinct). Hence, $k = 0$, which contradicts $k \geq 1 > 0$. This contradiction shows that our assumption was false. Thus, Lemma 2.45 **(a)** is proved.

(b) Let D be a proper subset of C. We must show that D is linear.

We have $C \setminus D \neq \varnothing$ (since D is a proper subset of C). Hence, there exists some $a \in C \setminus D$. Consider this a.

We have $a \in C \setminus D$. In other words, $a \in C$ and $a \notin D$.

We have $a \in C = \operatorname{CArcs} \gamma$. Hence, Lemma 2.44 shows that there exists a tuple $w = (w_1, w_2, \ldots, w_k) \in \gamma$ such that $a = (w_k, w_1)$. Consider this tuple $w = (w_1, w_2, \ldots, w_k)$. The entries of w are $w_1, w_2, \ldots, w_k$ (since $w = (w_1, w_2, \ldots, w_k)$).

The tuple w belongs to γ (since $w \in \gamma$), but γ is a rotation-equivalence class of nonempty tuples of distinct elements of V. Hence, w is a nonempty tuple of distinct elements of V. Hence, w is a path of V (by the definition of a "path of V").

In particular, w is a tuple of distinct elements of V. In other words, the entries of w are distinct. In other words, $w_1, w_2, \ldots, w_k$ are distinct (since the entries of w are $w_1, w_2, \ldots, w_k$). The arc set of this path w is

$$\operatorname{Arcs} w = \{(w_1, w_2), \ (w_2, w_3), \ \ldots, \ (w_{k-1}, w_k)\} \tag{40}$$

(by (2), applied to $w = (w_1, w_2, \ldots, w_k)$ instead of $v = (v_1, v_2, \ldots, v_k)$).

We have $a = (w_k, w_1)$, thus $(w_k, w_1) = a$.

We know that γ is a rotation-equivalence class that contains w (since $w \in \gamma$). Hence, γ is the rotation-equivalence class of w. In other words, $\gamma = w_\sim$. Hence,

$$\begin{aligned}
\operatorname{CArcs} \gamma &= \operatorname{CArcs} (w_\sim) \\
&= \operatorname{CArcs} w \qquad \left(\text{by the definition of } \operatorname{CArcs} (w_\sim), \text{ since } w \in w_\sim\right) \\
&= \{(w_1, w_2), \ (w_2, w_3), \ \ldots, \ (w_{k-1}, w_k), \ (w_k, w_1)\}
\end{aligned}$$

(by (3), applied to $w = (w_1, w_2, \ldots, w_k)$ instead of $v = (v_1, v_2, \ldots, v_k)$). Hence,

$$\begin{aligned}
\operatorname{CArcs} \gamma &= \{(w_1, w_2), \ (w_2, w_3), \ \ldots, \ (w_{k-1}, w_k), \ (w_k, w_1)\} \\
&= \underbrace{\{(w_1, w_2), \ (w_2, w_3), \ \ldots, \ (w_{k-1}, w_k)\}}_{\substack{=\operatorname{Arcs} w \\ \text{(by (40))}}} \cup \left\{ \underbrace{(w_k, w_1)}_{=a} \right\} \\
&= (\operatorname{Arcs} w) \cup \{a\}.
\end{aligned}$$

The k pairs $(w_1, w_2), \ (w_2, w_3), \ \ldots, \ (w_{k-1}, w_k), \ (w_k, w_1)$ are distinct (since their first entries $w_1, w_2, \ldots, w_k$ are distinct). Hence, in particular, the last of these k pairs is not among the remaining $k - 1$ pairs. In other words,

$$(w_k, w_1) \notin \{(w_1, w_2), \ (w_2, w_3), \ \ldots, \ (w_{k-1}, w_k)\} = \operatorname{Arcs} w$$

(by (40)). Thus,

$$a = (w_k, w_1) \notin \operatorname{Arcs} w.$$

Now,

$$\underbrace{C}_{\substack{=\operatorname{CArcs}\gamma \\ =(\operatorname{Arcs} w)\cup\{a\}}} \setminus \{a\} = ((\operatorname{Arcs} w) \cup \{a\}) \setminus \{a\} = (\operatorname{Arcs} w) \setminus \{a\}$$

$$= \operatorname{Arcs} w \qquad (\text{since } a \notin \operatorname{Arcs} w).$$

However, D is a subset of C that does not contain a (since $a \notin D$). In other words, D is a subset of $C \setminus \{a\}$. In other words, D is a subset of $\operatorname{Arcs} w$ (since $C \setminus \{a\} = \operatorname{Arcs} w$).

However, w is a path of V. Hence, Lemma 2.43 (applied to $p = w$) shows that $\operatorname{Arcs} w$ is a linear subset of $V \times V$. Thus, Proposition 2.7 (applied to $F = \operatorname{Arcs} w$) shows that any subset of $\operatorname{Arcs} w$ is linear as well. Thus, D is linear (since D is a subset of $\operatorname{Arcs} w$). This completes the proof of Lemma 2.45 **(b)**. □

Lemma 2.46. Let V be a finite set. Let $\sigma \in \mathfrak{S}_V$ be a permutation of V. Let γ be a cycle of σ. Then, $|\operatorname{CArcs}\gamma| = \ell(\gamma) \geq 1$.

Proof. The cycle γ is a cycle of σ, and thus is a rotation-equivalence class of nonempty tuples of distinct elements of V (since any cycle of σ is such a class). Hence, we can write γ in the form $\gamma = v_\sim$, where v is a nonempty tuple of distinct elements of V. Consider this v.

Write the tuple v as $v = (v_1, v_2, \ldots, v_k)$. Then, the entries $v_1, v_2, \ldots, v_k$ of v are distinct (since v is a tuple of distinct elements of V). We have $\ell(v) = k$ (since $v = (v_1, v_2, \ldots, v_k)$ is a k-tuple) and $\ell(v) \geq 1$ (since v is nonempty).

From $\gamma = v_\sim$, we obtain $\ell(\gamma) = \ell(v_\sim) = \ell(v)$ (by Definition 1.23 **(a)**). Hence, $\ell(\gamma) = \ell(v) = k$ and $\ell(\gamma) = \ell(v) \geq 1$.

On the other hand, from $\gamma = v_\sim$, we obtain

$$\begin{aligned} \operatorname{CArcs}\gamma &= \operatorname{CArcs}(v_\sim) \\ &= \operatorname{CArcs} v \qquad (\text{by the definition of } \operatorname{CArcs}(v_\sim), \text{ since } v \in v_\sim) \\ &= \{(v_1, v_2), (v_2, v_3), \ldots, (v_{k-1}, v_k), (v_k, v_1)\} \end{aligned}$$

(by (3), since $v = (v_1, v_2, \ldots, v_k)$).

However, the k pairs $(v_1, v_2), (v_2, v_3), \ldots, (v_{k-1}, v_k), (v_k, v_1)$ are distinct (since their first entries $v_1, v_2, \ldots, v_k$ are distinct). Hence, the set $\{(v_1, v_2), (v_2, v_3), \ldots, (v_{k-1}, v_k), (v_k, v_1)\}$ of these k pairs has size k. In other words,

$$|\{(v_1, v_2), (v_2, v_3), \ldots, (v_{k-1}, v_k), (v_k, v_1)\}| = k.$$

This rewrites as $|\operatorname{CArcs}\gamma| = \ell(\gamma)$ (since we have $\ell(\gamma) = k$ and $\operatorname{CArcs}\gamma = \{(v_1, v_2), (v_2, v_3), \ldots, (v_{k-1}, v_k), (v_k, v_1)\}$). Hence, $|\operatorname{CArcs}\gamma| = \ell(\gamma) \geq 1$. Thus, Lemma 2.46 is proved. □

Lemma 2.47. Let $D = (V, A)$ be a digraph. Let $\sigma \in \mathfrak{S}_V$ be a permutation of V. Let γ be a cycle of σ. Let $C = \operatorname{CArcs} \gamma$. Then:

(a) If γ is a D-cycle, then $\sum\limits_{\substack{F \subseteq C \cap A \\ \text{is linear}}} (-1)^{|F|} = (-1)^{\ell(\gamma)-1}$.

(b) If γ is a $\overline{D}$-cycle, then $\sum\limits_{\substack{F \subseteq C \cap A \\ \text{is linear}}} (-1)^{|F|} = 1$.

(c) If γ is neither a D-cycle nor a $\overline{D}$-cycle, then $\sum\limits_{\substack{F \subseteq C \cap A \\ \text{is linear}}} (-1)^{|F|} = 0$.

Proof. We know that γ is a cycle of σ, thus a rotation-equivalence class of nonempty tuples of distinct elements of V (since any cycle of σ is such a rotation-equivalence class).

Lemma 2.46 yields $|\operatorname{CArcs} \gamma| = \ell(\gamma) \geq 1$. From $C = \operatorname{CArcs} \gamma$, we obtain $|C| = |\operatorname{CArcs} \gamma| = \ell(\gamma) \geq 1 > 0$, so that the set C is nonempty. In other words, $C \neq \varnothing$. Also, $C = \operatorname{CArcs} \gamma \subseteq V \times V$.

(a) Assume that γ is a D-cycle. Thus, $\operatorname{CArcs} \gamma \subseteq A$ (by the definition of a D-cycle). In other words, $C \subseteq A$ (since $C = \operatorname{CArcs} \gamma$). Therefore, $C \cap A = C$. Hence,

$$\sum_{\substack{F \subseteq C \cap A \\ \text{is linear}}} (-1)^{|F|} = \sum_{\substack{F \subseteq C \\ \text{is linear}}} (-1)^{|F|} = \sum_{\substack{F \text{ is a linear} \\ \text{subset of } C}} (-1)^{|F|}. \tag{41}$$

On the other hand, Lemma 2.3 (applied to $B = C$) yields $\sum\limits_{F \subseteq C} (-1)^{|F|} = [C = \varnothing] = 0$ (since $C \neq \varnothing$). Hence,

$$0 = \sum_{F \subseteq C} (-1)^{|F|} = (-1)^{|C|} + \sum_{\substack{F \subseteq C; \\ F \neq C}} (-1)^{|F|}$$

(here, we have split off the addend for $F = C$ from the sum). Therefore,

$$\sum_{\substack{F \subseteq C; \\ F \neq C}} (-1)^{|F|} = -(-1)^{|C|} = (-1)^{|C|-1} = (-1)^{\ell(\gamma)-1} \tag{42}$$

(since $|C| = \ell(\gamma)$).

However, from Lemma 2.45, we easily obtain

$$\{F \subseteq C \mid F \neq C\} = \{\text{linear subsets of } C\}$$

[38]. Hence, the summation sign " $\sum\limits_{\substack{F\subseteq C;\\ F\neq C}}$ " can be rewritten as " $\sum\limits_{\substack{F \text{ is a linear}\\ \text{subset of } C}}$ ". Thus,

$$\sum_{\substack{F\subseteq C;\\ F\neq C}} (-1)^{|F|} = \sum_{\substack{F \text{ is a linear}\\ \text{subset of } C}} (-1)^{|F|}.$$

Comparing this with (41), we find

$$\sum_{\substack{F\subseteq C\cap A\\ \text{is linear}}} (-1)^{|F|} = \sum_{\substack{F\subseteq C;\\ F\neq C}} (-1)^{|F|} = (-1)^{\ell(\gamma)-1} \qquad \text{(by (42))}.$$

This proves Lemma 2.47 **(a)**.

(b) Assume that γ is a $\overline{D}$-cycle. Thus, $\operatorname{CArcs}\gamma \subseteq (V\times V)\setminus A$ (by the definition of a $\overline{D}$-cycle, since $\overline{D} = (V,\ (V\times V)\setminus A)$). In other words, $C \subseteq (V\times V)\setminus A$ (since $C = \operatorname{CArcs}\gamma$). Therefore, $C\cap A = \varnothing$ [39]. Hence,

$$\sum_{\substack{F\subseteq C\cap A\\ \text{is linear}}} (-1)^{|F|} = \sum_{\substack{F\subseteq \varnothing\\ \text{is linear}}} (-1)^{|F|}. \tag{43}$$

[38] *Proof.* We shall first prove that $\{F\subseteq C \mid F\neq C\} \subseteq \{\text{linear subsets of } C\}$.

Indeed, let $G \in \{F\subseteq C \mid F\neq C\}$. Thus, G is a subset $F\subseteq C$ satisfying $F\neq C$. In other words, G is a subset of C such that $G\neq C$. In other words, G is a proper subset of C. Hence, G is linear (since Lemma 2.45 **(b)** shows that every proper subset of C is linear). Therefore, G is a linear subset of C. Hence, $G\in\{\text{linear subsets of } C\}$.

Forget that we fixed G. We thus have proved that each $G \in \{F\subseteq C \mid F\neq C\}$ satisfies $G\in\{\text{linear subsets of } C\}$. In other words, we have

$$\{F\subseteq C \mid F\neq C\} \subseteq \{\text{linear subsets of } C\}.$$

Let us now prove the reverse inclusion.

Indeed, let $H\in\{\text{linear subsets of } C\}$. Thus, H is a linear subset of C. Hence, H is a subset of C, so that $H\subseteq C$. The set H is linear, whereas the set C is not (by Lemma 2.45 **(a)**). Thus, H cannot be identical with C. In other words, $H\neq C$. Combining $H\subseteq C$ with $H\neq C$, we see that H is a subset F of C satisfying $F\neq C$. In other words, $H\in\{F\subseteq C \mid F\neq C\}$.

Forget that we fixed H. We thus have proved that each $H\in\{\text{linear subsets of } C\}$ satisfies $H\in\{F\subseteq C \mid F\neq C\}$. In other words, we have

$$\{\text{linear subsets of } C\} \subseteq \{F\subseteq C \mid F\neq C\}.$$

Combining this with

$$\{F\subseteq C \mid F\neq C\} \subseteq \{\text{linear subsets of } C\},$$

we obtain

$$\{F\subseteq C \mid F\neq C\} = \{\text{linear subsets of } C\}.$$

[39] *Proof.* Let $c\in C\cap A$. Thus, $c\in C\cap A\subseteq C\subseteq (V\times V)\setminus A$, so that $c\notin A$. But this contradicts $c\in C\cap A\subseteq A$.

Forget that we fixed c. We thus have obtained a contradiction for each $c\in C\cap A$. Hence, there is no $c\in C\cap A$. In other words, $C\cap A=\varnothing$.

However, the set $\varnothing$ is linear (as a subset of $V \times V$) [40]. Thus, the only linear subset of $\varnothing$ is $\varnothing$ itself (since $\varnothing$ is a linear subset of $\varnothing$, and is clearly the only subset of $\varnothing$). Therefore, the sum $\sum\limits_{\substack{F \subseteq \varnothing \\ \text{is linear}}} (-1)^{|F|}$ has only one addend, namely the addend for $F = \varnothing$. Thus, this sum rewrites as follows:

$$\sum_{\substack{F \subseteq \varnothing \\ \text{is linear}}} (-1)^{|F|} = (-1)^{|\varnothing|} = (-1)^0 \qquad (\text{since } |\varnothing| = 0)$$
$$= 1.$$

Hence, (43) rewrites as

$$\sum_{\substack{F \subseteq C \cap A \\ \text{is linear}}} (-1)^{|F|} = 1.$$

This proves Lemma 2.47 **(b)**.

(c) Assume that γ is neither a D-cycle nor a $\overline{D}$-cycle. Hence, $C \not\subseteq A$ [41]. Hence, $C \cap A \neq C$. However, $C \cap A$ is clearly a subset of C. Thus, $C \cap A$ is a proper subset of C (since $C \cap A \neq C$).

Furthermore, $C \cap A \neq \varnothing$ [42].

Now, from Lemma 2.45, we easily obtain

$$\{F \subseteq C \cap A\} = \{\text{linear subsets of } C \cap A\}$$

[40] *Proof.* We have $\varnothing \subseteq C$ and $\varnothing \neq C$ (since $C \neq \varnothing$). Hence, $\varnothing$ is a proper subset of C. However, Lemma 2.45 **(b)** shows that every proper subset of C is linear. Thus, $\varnothing$ is linear (since $\varnothing$ is a proper subset of C).

[41] *Proof.* Assume the contrary. Thus, $C \subseteq A$. In other words, $\operatorname{CArcs}\gamma \subseteq A$ (since $C = \operatorname{CArcs}\gamma$).

Recall that γ is a rotation-equivalence class of nonempty tuples of distinct elements of V. Hence, from $\operatorname{CArcs}\gamma \subseteq A$, we conclude that γ is a D-cycle (by the definition of a D-cycle). This contradicts the fact that γ is not a D-cycle. This contradiction shows that our assumption was false, qed.

[42] *Proof.* Assume the contrary. Thus, $C \cap A = \varnothing$. However, we have $X \setminus Y = X \setminus (X \cap Y)$ for any two sets X and Y. Applying this to $X = C$ and $Y = A$, we obtain $C \setminus A = C \setminus \underbrace{(C \cap A)}_{=\varnothing} = C \setminus \varnothing = C$.

Therefore, $C = \underbrace{C}_{\subseteq V \times V} \setminus A \subseteq (V \times V) \setminus A$. In other words, $\operatorname{CArcs}\gamma \subseteq (V \times V) \setminus A$ (since $C = \operatorname{CArcs}\gamma$).

Recall that γ is a rotation-equivalence class of nonempty tuples of distinct elements of V. Hence, from $\operatorname{CArcs}\gamma \subseteq (V \times V) \setminus A$, we conclude that γ is a $\overline{D}$-cycle (by the definition of a $\overline{D}$-cycle, since $\overline{D} = (V, (V \times V) \setminus A)$). This contradicts the fact that γ is not a $\overline{D}$-cycle. This contradiction shows that our assumption was false, qed.

[43]. Hence, the summation sign " $\sum\limits_{F\subseteq C\cap A}$ " can be rewritten as " $\sum\limits_{\substack{F \text{ is a linear} \\ \text{subset of } C\cap A}}$ ". Thus,

$$\sum_{F\subseteq C\cap A} (-1)^{|F|} = \sum_{\substack{F \text{ is a linear} \\ \text{subset of } C\cap A}} (-1)^{|F|} = \sum_{\substack{F\subseteq C\cap A \\ \text{is linear}}} (-1)^{|F|}.$$

Therefore,

$$\begin{aligned} \sum_{\substack{F\subseteq C\cap A \\ \text{is linear}}} (-1)^{|F|} &= \sum_{F\subseteq C\cap A} (-1)^{|F|} \\ &= [C\cap A = \varnothing] \qquad \text{(by Lemma 2.3, applied to } B = C\cap A\text{)} \\ &= 0 \qquad (\text{since } C\cap A \neq \varnothing). \end{aligned}$$

This proves Lemma 2.47 **(c)**. □

Before we state the next lemma, let us again recall that the symbols "$\sqcup$" and "$\bigsqcup$" stand for unions of disjoint sets.

Lemma 2.48. Let $D = (V, A)$ be a digraph. Let $\sigma \in \mathfrak{S}_V$ be a permutation of V. Let $\gamma_1, \gamma_2, \ldots, \gamma_k$ be the cycles of σ, listed with no repetition[44]. For each $i \in [k]$, let $C_i := \operatorname{CArcs}(\gamma_i)$.

Let F be any set. Then:

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[43] *Proof.* We shall first prove that $\{F \subseteq C\cap A\} \subseteq \{\text{linear subsets of } C\cap A\}$.

Indeed, let $G \in \{F \subseteq C\cap A\}$. Thus, G is a subset of $C\cap A$. Hence, G is a proper subset of C (since G is a subset of $C\cap A$, but $C\cap A$ is a proper subset of C). Hence, G is linear (since Lemma 2.45 **(b)** shows that every proper subset of C is linear). Therefore, G is a linear subset of $C\cap A$. Hence, $G \in \{\text{linear subsets of } C\cap A\}$.

Forget that we fixed G. We thus have proved that each $G \in \{F \subseteq C\cap A\}$ satisfies $G \in \{\text{linear subsets of } C\cap A\}$. In other words, we have

$$\{F \subseteq C\cap A\} \subseteq \{\text{linear subsets of } C\cap A\}.$$

Let us now prove the reverse inclusion.

Indeed, let $H \in \{\text{linear subsets of } C\cap A\}$. Thus, H is a linear subset of $C\cap A$. Hence, H is a subset of $C\cap A$, so that $H \subseteq C\cap A$. In other words, $H \in \{F \subseteq C\cap A\}$.

Forget that we fixed H. We thus have proved that each $H \in \{\text{linear subsets of } C\cap A\}$ satisfies $H \in \{F \subseteq C\cap A\}$. In other words, we have

$$\{\text{linear subsets of } C\cap A\} \subseteq \{F \subseteq C\cap A\}.$$

Combining this with

$$\{F \subseteq C\cap A\} \subseteq \{\text{linear subsets of } C\cap A\},$$

we obtain

$$\{F \subseteq C\cap A\} = \{\text{linear subsets of } C\cap A\}.$$

(a) The set F is a linear subset of $\mathbf{A}_\sigma \cap A$ if and only if F can be written as $F = \bigsqcup_{j \in [k]} F_j$, where each F_j is a linear subset of $C_j \cap A$.

(b) In this case, the subsets F_j are uniquely determined by F (namely, $F_j = F \cap C_j$ for each $j \in [k]$).

Proof. Recall that $\gamma_1, \gamma_2, \ldots, \gamma_k$ are the cycles of σ, listed with no repetition. Thus, these cycles $\gamma_1, \gamma_2, \ldots, \gamma_k$ are distinct, and we have $\operatorname{Cycs} \sigma = \{\gamma_1, \gamma_2, \ldots, \gamma_k\}$ (since $\operatorname{Cycs} \sigma$ is defined to be the set of all cycles of σ).

We know that σ is a permutation of V. Hence, each element of V belongs to exactly one cycle of σ. In other words, each element of V belongs to exactly one of the cycles $\gamma_1, \gamma_2, \ldots, \gamma_k$ (since $\gamma_1, \gamma_2, \ldots, \gamma_k$ are the cycles of σ, listed with no repetition).

For each $i \in [k]$, let V_i be the set of all entries of the cycle γ_i. Thus, $V_1, V_2, \ldots, V_k$ are k subsets of V. These k subsets $V_1, V_2, \ldots, V_k$ are furthermore disjoint[45]. In other words, the sets V_j for different $j \in [k]$ are disjoint.

For any given $i \in [k]$ and any given $v \in V$, we have the following logical equivalence:

$$(v \in V_i) \iff (v \text{ is an entry of the cycle } \gamma_i) \tag{44}$$

(since V_i is the set of all entries of the cycle γ_i).

---

[44]Keep in mind that a cycle is a rotation-equivalence class. Thus, "listed with no repetition" means that no two of $\gamma_1, \gamma_2, \ldots, \gamma_k$ are the same rotation-equivalence class. For example, if γ_1 is $(1,2)_\sim$, then γ_2 cannot be $(2,1)_\sim$.

[45]*Proof.* Let i and j be two distinct elements of $[k]$. We shall show that $V_i \cap V_j = \varnothing$.

Indeed, assume the contrary. Thus, $V_i \cap V_j \neq \varnothing$. Hence, there exists some element $v \in V_i \cap V_j$. Consider this v.

We have $v \in V_i \cap V_j \subseteq V_i$. In other words, v is an entry of the cycle γ_i (since V_i was defined as the set of all entries of the cycle γ_i).

We have $v \in V_i \cap V_j \subseteq V_j$. In other words, v is an entry of the cycle γ_j (since V_j was defined as the set of all entries of the cycle γ_j).

The two cycles γ_i and γ_j have the entry v in common (since v is an entry of the cycle γ_i, and since v is an entry of the cycle γ_j).

However, the cycles $\gamma_1, \gamma_2, \ldots, \gamma_k$ are distinct. Hence, the two cycles γ_i and γ_j are distinct (since i and j are distinct). However, two distinct cycles of σ cannot have any entries in common (by the basic properties of the cycles of a permutation). Thus, γ_i and γ_j have no entries in common (since γ_i and γ_j are two distinct cycles of σ). This contradicts the fact that the two cycles γ_i and γ_j have the entry v in common. This contradiction shows that our assumption was false. In other words, we have $V_i \cap V_j = \varnothing$.

Forget that we fixed i and j. We thus have proved that $V_i \cap V_j = \varnothing$ whenever i and j are two distinct elements of $[k]$. In other words, the subsets $V_1, V_2, \ldots, V_k$ are disjoint.

We have $V = V_1 \cup V_2 \cup \cdots \cup V_k$ [46]. Thus,

$$V = V_1 \cup V_2 \cup \cdots \cup V_k = \bigcup_{j \in [k]} V_j = \bigsqcup_{j \in [k]} V_j$$

(since the sets V_j for different $j \in [k]$ are disjoint).

For each $i \in [k]$, we have $C_i \subseteq V_i \times V_i$ [47]. Renaming the variable i as j in this sentence, we obtain the following: For each $j \in [k]$, we have

$$C_j \subseteq V_j \times V_j. \tag{45}$$

Hence, the sets $C_1, C_2, \ldots, C_k$ are disjoint[48].

---

[46] *Proof.* Let $v \in V$. Then, v belongs to exactly one of the cycles $\gamma_1, \gamma_2, \ldots, \gamma_k$ (since each element of V belongs to exactly one of the cycles $\gamma_1, \gamma_2, \ldots, \gamma_k$). In other words, there is exactly one $i \in [k]$ such that v belongs to γ_i. Consider this i. Now, v belongs to γ_i. In other words, v is an entry of the cycle γ_i. Hence, $v \in V_i$ (by (44)). Thus, $v \in V_i \subseteq V_1 \cup V_2 \cup \cdots \cup V_k$ (since V_i is one of the terms in the union $V_1 \cup V_2 \cup \cdots \cup V_k$).

Forget that we fixed v. We thus have shown that $v \in V_1 \cup V_2 \cup \cdots \cup V_k$ for each $v \in V$. In other words, $V \subseteq V_1 \cup V_2 \cup \cdots \cup V_k$. Combining this with $V_1 \cup V_2 \cup \cdots \cup V_k \subseteq V$ (which is obvious, since $V_1, V_2, \ldots, V_k$ are subsets of V), we obtain $V = V_1 \cup V_2 \cup \cdots \cup V_k$.

[47] *Proof.* Let $i \in [k]$. Then, $C_i = \operatorname{CArcs}(\gamma_i)$ (by the definition of C_i). Write the cycle γ_i in the form $\gamma_i = (a_1, a_2, \ldots, a_p)_\sim$. Thus, the entries of the cycle γ_i are $a_1, a_2, \ldots, a_p$. Hence, $V_i = \{a_1, a_2, \ldots, a_p\}$ (since V_i was defined as the set of all entries of the cycle γ_i). Therefore, $a_1, a_2, \ldots, a_p$ are elements of V_i. Hence, the pairs $(a_1, a_2), (a_2, a_3), \ldots, (a_{p-1}, a_p), (a_p, a_1)$ are elements of $V_i \times V_i$ (since all their entries belong to V_i).

However, from $\gamma_i = (a_1, a_2, \ldots, a_p)_\sim$, we obtain

$$\begin{aligned}
\operatorname{CArcs}(\gamma_i) &= \operatorname{CArcs}\left((a_1, a_2, \ldots, a_p)_\sim\right) \\
&= \operatorname{CArcs}(a_1, a_2, \ldots, a_p) \qquad \left(\begin{array}{c}\text{since Definition 1.23 \textbf{(b)} yields} \\ \text{that } \operatorname{CArcs}(w_\sim) = \operatorname{CArcs} w \text{ for each } w \in V^p\end{array}\right) \\
&= \{(a_1, a_2), (a_2, a_3), \ldots, (a_{p-1}, a_p), (a_p, a_1)\} \\
&\qquad \left(\begin{array}{c}\text{by (3), applied to } a = (a_1, a_2, \ldots, a_p) \\ \text{instead of } v = (v_1, v_2, \ldots, v_k)\end{array}\right) \\
&\subseteq V_i \times V_i
\end{aligned}$$

(since the pairs $(a_1, a_2), (a_2, a_3), \ldots, (a_{p-1}, a_p), (a_p, a_1)$ are elements of $V_i \times V_i$). Therefore,

$$C_i = \operatorname{CArcs}(\gamma_i) \subseteq V_i \times V_i,$$

qed.

[48] *Proof.* Let i and j be two distinct elements of $[k]$. We shall show that $C_i \cap C_j = \varnothing$.

Indeed, i and j are distinct, and thus we have $V_i \cap V_j = \varnothing$ (since the k subsets $V_1, V_2, \ldots, V_k$ are disjoint). However, $C_j \subseteq V_j \times V_j$ (by (45)) and $C_i \subseteq V_i \times V_i$ (similarly). Hence,

$$\underbrace{C_i}_{\subseteq V_i \times V_i} \cap \underbrace{C_j}_{\subseteq V_j \times V_j} \subseteq (V_i \times V_i) \cap (V_j \times V_j) = \underbrace{(V_i \cap V_j)}_{= \varnothing} \times \underbrace{(V_i \cap V_j)}_{= \varnothing} = \varnothing \times \varnothing = \varnothing.$$

Therefore, $C_i \cap C_j = \varnothing$.

Forget that we fixed i and j. We thus have shown that $C_i \cap C_j = \varnothing$ whenever i and j are two distinct elements of $[k]$. In other words, the sets $C_1, C_2, \ldots, C_k$ are disjoint.

The definition of $\mathbf{A}_\sigma$ yields

$$\begin{aligned}
\mathbf{A}_\sigma &= \bigcup_{c \in \operatorname{Cycs}\sigma} \operatorname{CArcs} c \\
&= \left(\operatorname{CArcs}\left(\gamma_1\right)\right) \cup \left(\operatorname{CArcs}\left(\gamma_2\right)\right) \cup \cdots \cup \left(\operatorname{CArcs}\left(\gamma_k\right)\right) \\
&\qquad \left(\text{since } \operatorname{Cycs}\sigma = \left\{\gamma_1, \gamma_2, \ldots, \gamma_k\right\}\right) \\
&= \bigcup_{j \in [k]} \underbrace{\operatorname{CArcs}\left(\gamma_j\right)}_{\substack{=C_j \\ \text{(since } C_j \text{ was defined} \\ \text{to be } \operatorname{CArcs}(\gamma_j))}} \\
&= \bigcup_{j \in [k]} C_j. \qquad (46)
\end{aligned}$$

(a) We must prove that F is a linear subset of $\mathbf{A}_\sigma \cap A$ if and only if F can be written as $F = \bigsqcup_{j \in [k]} F_j$, where each F_j is a linear subset of $C_j \cap A$.

We shall prove the "$\Longleftarrow$" and "$\Longrightarrow$" directions of this equivalence separately:

$\Longleftarrow$: Assume that F can be written as $F = \bigsqcup_{j \in [k]} F_j$, where each F_j is a linear subset of $C_j \cap A$. Consider these subsets F_j.

Let $j \in [k]$. Then, F_j is a linear subset of $C_j \cap A$ (according to the preceding paragraph). Thus, $F_j \subseteq C_j \cap A \subseteq C_j \subseteq V_j \times V_j$ (by (45)). Therefore, F_j is a linear subset of $V_j \times V_j$ (since F_j is linear).

Forget that we fixed j. We thus have shown that for each $j \in [k]$, the set F_j is a linear subset of $V_j \times V_j$. Hence, Corollary 2.31 (applied to $J = [k]$) shows that the union $\bigcup_{j \in J} F_j$ is a linear subset of $V \times V$. In other words, F is a linear subset of $V \times V$ (since $F = \bigsqcup_{j \in [k]} F_j = \bigcup_{j \in J} F_j$).

Finally,

$$\begin{aligned}
F &= \bigsqcup_{j \in [k]} F_j = \bigcup_{j \in [k]} \underbrace{F_j}_{\substack{\subseteq C_j \cap A \\ \text{(since } F_j \text{ is a linear subset of } C_j \cap A \\ \text{(by assumption))}}} \subseteq \bigcup_{j \in [k]} \left(C_j \cap A\right) \\
&= \underbrace{\left(\bigcup_{j \in [k]} C_j\right)}_{\substack{=\mathbf{A}_\sigma \\ \text{(by (46))}}} \cap A = \mathbf{A}_\sigma \cap A.
\end{aligned}$$

Hence, F is a subset of $\mathbf{A}_\sigma \cap A$. This shows that F is a linear subset of $\mathbf{A}_\sigma \cap A$ (since F is linear). This proves the "$\Longleftarrow$" direction of Lemma 2.48 **(a)**.

$\Longrightarrow$: Assume that F is a linear subset of $\mathbf{A}_\sigma \cap A$. Thus,

$$\begin{aligned} F \subseteq \mathbf{A}_\sigma \cap A \subseteq \mathbf{A}_\sigma &= \bigcup_{j\in[k]} C_j \qquad \text{(by (46))} \\ &= C_1 \cup C_2 \cup \cdots \cup C_k. \end{aligned}$$

For each $j \in [k]$, let us set $G_j := F \cap C_j$. Thus,

$$\begin{aligned} G_1 \cup G_2 \cup \cdots \cup G_k &= (F \cap C_1) \cup (F \cap C_2) \cup \cdots \cup (F \cap C_k) \\ &= F \cap (C_1 \cup C_2 \cup \cdots \cup C_k) \end{aligned}$$

(since $(X \cap Y_1) \cup (X \cap Y_2) \cup \cdots \cup (X \cap Y_n) = X \cap (Y_1 \cup Y_2 \cup \cdots \cup Y_n)$ for any sets $X, Y_1, Y_2, \ldots, Y_n$). Hence,

$$G_1 \cup G_2 \cup \cdots \cup G_k = F \cap (C_1 \cup C_2 \cup \cdots \cup C_k) = F$$

(since $F \subseteq C_1 \cup C_2 \cup \cdots \cup C_k$).

Now, for each $j \in [k]$, the set G_j is a linear subset of $C_j \cap A$ [49]. Moreover, the sets $G_1, G_2, \ldots, G_k$ are disjoint[50]. Thus, the disjoint union $G_1 \sqcup G_2 \sqcup \cdots \sqcup G_k = \bigsqcup_{j\in[k]} G_j$ is well-defined. This disjoint union is

$$\bigsqcup_{j\in[k]} G_j = G_1 \sqcup G_2 \sqcup \cdots \sqcup G_k = G_1 \cup G_2 \cup \cdots \cup G_k = F.$$

Thus, $F = \bigsqcup_{j\in[k]} G_j$.

Altogether, we have now shown that $F = \bigsqcup_{j\in[k]} G_j$, and that each G_j is a linear subset of $C_j \cap A$. Hence, F can be written as $F = \bigsqcup_{j\in[k]} F_j$, where each F_j is a linear

[49] *Proof.* Let $j \in [k]$. Then, $G_j = \underbrace{F}_{\subseteq \mathbf{A}_\sigma \cap A \subseteq A} \cap C_j \subseteq A \cap C_j = C_j \cap A$. In other words, G_j is a subset of $C_j \cap A$. Furthermore, $G_j = F \cap C_j \subseteq F$, so that G_j is a subset of F. However, F is a linear subset of $V \times V$ (since F is linear, and $F \subseteq \mathbf{A}_\sigma \subseteq V \times V$). Thus, Proposition 2.7 shows that any subset of F is linear as well. Therefore, G_j is linear (since G_j is a subset of F). Hence, G_j is a linear subset of $C_j \cap A$ (since G_j is a subset of $C_j \cap A$), qed.

[50] *Proof.* Let i and j be two distinct elements of $[k]$. We shall show that $G_i \cap G_j = \varnothing$.

Indeed, i and j are distinct, and thus we have $C_i \cap C_j = \varnothing$ (since the k sets $C_1, C_2, \ldots, C_k$ are disjoint). However, the definition of G_j yields

$$G_j = F \cap C_j \subseteq C_j.$$

The same argument (applied to i instead of j) yields $G_i \subseteq C_i$. Hence,

$$\underbrace{G_i}_{\subseteq C_i} \cap \underbrace{G_j}_{\subseteq C_j} \subseteq C_i \cap C_j = \varnothing.$$

Therefore, $G_i \cap G_j = \varnothing$.

Forget that we fixed i and j. We thus have shown that $G_i \cap G_j = \varnothing$ whenever i and j are two distinct elements of $[k]$. In other words, the sets $G_1, G_2, \ldots, G_k$ are disjoint.

subset of $C_j \cap A$ (namely, for $F_j = G_j$). This proves the "$\Longrightarrow$" direction of Lemma 2.48 **(a)**.

(b) Assume that F is written as $F = \bigsqcup_{j\in[k]} F_j$, where each F_j is a linear subset of $C_j \cap A$. We must show that $F_j = F \cap C_j$ for each $j \in [k]$.

Indeed, we have $F = \bigsqcup_{j\in[k]} F_j = \bigsqcup_{i\in[k]} F_i$.

Now, let $j \in [k]$. Then, F_j is a subset of F (since $F = \bigsqcup_{i\in[k]} F_i$) and also a subset of $C_j \cap A$ (since we required that each F_j be a linear subset of $C_j \cap A$). In other words, F_j is a subset of both F and $C_j \cap A$. Thus, F_j is a subset of the intersection $F \cap \left(C_j \cap A\right)$ as well. In other words,

$$F_j \subseteq F \cap \underbrace{\left(C_j \cap A\right)}_{\subseteq C_j} \subseteq F \cap C_j.$$

Let us now show that $F \cap C_j \subseteq F_j$.

Indeed, let $\alpha \in F \cap C_j$. Then, $\alpha \in F \cap C_j \subseteq F = \bigsqcup_{i\in[k]} F_i$. Hence, $\alpha \in F_i$ for some $i \in [k]$. Consider this i. Recall that F_j is a subset of $C_j \cap A$; thus, $F_j \subseteq C_j \cap A \subseteq C_j$. The same argument (applied to i instead of j) yields $F_i \subseteq C_i$. Hence, $\alpha \in F_i \subseteq C_i$. However, $\alpha \in F \cap C_j \subseteq C_j$. Thus, the element α belongs to both sets C_i and C_j. Therefore, the sets C_i and C_j have at least one element in common. In other words, the sets C_i and C_j are not disjoint. However, we know that the sets $C_1, C_2, \ldots, C_k$ are disjoint. The only way to reconcile the previous two sentences is when $i = j$.

Thus, we obtain $i = j$. Hence, $\alpha \in F_i = F_j$ (since $i = j$).

Forget that we fixed α. We thus have shown that $\alpha \in F_j$ for each $\alpha \in F \cap C_j$. In other words, $F \cap C_j \subseteq F_j$. Combining this with $F_j \subseteq F \cap C_j$, we conclude that $F_j = F \cap C_j$. This completes the proof of Lemma 2.48 **(b)**. □

We can now prove Proposition 2.42:

Proof of Proposition 2.42. Let $\gamma_1, \gamma_2, \ldots, \gamma_k$ be the cycles of σ, listed with no repetition (as in Lemma 2.48). Thus, these cycles $\gamma_1, \gamma_2, \ldots, \gamma_k$ are distinct, and we have $\operatorname{Cycs}\sigma = \{\gamma_1, \gamma_2, \ldots, \gamma_k\}$ (since $\operatorname{Cycs}\sigma$ is defined to be the set of all cycles of σ).

For each $i \in [k]$, let $C_i := \operatorname{CArcs}(\gamma_i)$. Then, the sets $C_1, C_2, \ldots, C_k$ are disjoint[51].

Now, we observe the following: If F_j is a linear subset of $C_j \cap A$ for each $j \in [k]$, then the disjoint union $\bigsqcup_{j\in[k]} F_j$ is well-defined[52], and is a linear subset of $\mathbf{A}_\sigma \cap A$ (by

[51]Indeed, this was shown during our above proof of Lemma 2.48.

[52]*Proof.* Let F_j be a linear subset of $C_j \cap A$ for each $j \in [k]$. Then, in particular, we have $F_j \subseteq C_j \cap A \subseteq C_j$ for each $j \in [k]$. In other words, F_j is a subset of C_j for each $j \in [k]$. In other words, the sets $F_1, F_2, \ldots, F_k$ are subsets of $C_1, C_2, \ldots, C_k$, respectively.

However, the sets $C_1, C_2, \ldots, C_k$ are disjoint. Thus, their subsets $F_1, F_2, \ldots, F_k$ are disjoint as well (since the sets $F_1, F_2, \ldots, F_k$ are subsets of $C_1, C_2, \ldots, C_k$, respectively). In other words, the sets F_j for different $j \in [k]$ are disjoint. Thus, the disjoint union $\bigsqcup_{j\in[k]} F_j$ is well-defined.

the "$\Longleftarrow$" direction of Lemma 2.48 **(a)**, applied to $F = \bigsqcup\limits_{j\in[k]} F_j$). Hence, the map

$$\begin{aligned}&\text{from } \left\{\text{families } \left(F_j\right)_{j\in[k]}, \text{ where each } F_j \text{ is a linear subset of } C_j \cap A\right\}\\ &\text{to } \left\{\text{linear subsets of } \mathbf{A}_\sigma \cap A\right\}\\ &\text{that sends each family } \left(F_j\right)_{j\in[k]} \text{ to } \bigsqcup_{j\in[k]} F_j\end{aligned}$$

is well-defined. Moreover, this map is injective (since Lemma 2.48 **(b)** shows that the sets F_j are uniquely determined by their union $\bigsqcup\limits_{j\in[k]} F_j$) and surjective (by the "$\Longrightarrow$" direction of Lemma 2.48 **(a)**). Thus, it is bijective. Hence, we can substitute $\bigsqcup\limits_{j\in[k]} F_j$ for F in the sum $\sum\limits_{\substack{F\subseteq \mathbf{A}_\sigma\cap A \\ \text{is linear}}} (-1)^{|F|}$. We thus obtain

$$\begin{aligned}&\sum_{\substack{F\subseteq \mathbf{A}_\sigma\cap A \\ \text{is linear}}} (-1)^{|F|} \\ &= \sum_{\substack{\left(F_j\right)_{j\in[k]} \text{ is a family} \\ \text{of linear subsets } F_j\subseteq C_j\cap A}} (-1)^{\left|\bigsqcup_{j\in[k]} F_j\right|}.\end{aligned} \tag{47}$$

However, if $\left(F_j\right)_{j\in[k]}$ is a family of linear subsets $F_j \subseteq C_j \cap A$, then

$$\left|\bigsqcup_{j\in[k]} F_j\right| = \sum_{j\in[k]} \left|F_j\right| \qquad \text{(by the sum rule)}$$

and thus

$$(-1)^{\left|\bigsqcup_{j\in[k]} F_j\right|} = (-1)^{\sum_{j\in[k]} \left|F_j\right|} = \prod_{j\in[k]} (-1)^{\left|F_j\right|}. \tag{48}$$

Hence, (47) becomes

$$\begin{aligned}
&\sum_{\substack{F\subseteq\mathbf{A}_\sigma\cap A\\ \text{is linear}}}(-1)^{|F|}\\
&=\sum_{\substack{(F_j)_{j\in[k]}\text{ is a family}\\ \text{of linear subsets }F_j\subseteq C_j\cap A}}\underbrace{(-1)^{\left|\bigsqcup_{j\in[k]}F_j\right|}}_{\substack{=\prod_{j\in[k]}(-1)^{|F_j|}\\ \text{(by (48))}}}\\
&=\sum_{\substack{(F_j)_{j\in[k]}\text{ is a family}\\ \text{of linear subsets }F_j\subseteq C_j\cap A}}\prod_{j\in[k]}(-1)^{|F_j|}\\
&=\prod_{j\in[k]}\underbrace{\sum_{\substack{F_j\subseteq C_j\cap A\\ \text{is linear}}}}_{\substack{=\sum_{\substack{F_j\subseteq(\operatorname{CArcs}(\gamma_j))\cap A\\ \text{is linear}}}\\ \text{(since }C_j=\operatorname{CArcs}(\gamma_j)\\ \text{(by the definition of }C_j))}}(-1)^{|F_j|}\qquad\text{(by the product rule)}\\
&=\prod_{j\in[k]}\sum_{\substack{F_j\subseteq(\operatorname{CArcs}(\gamma_j))\cap A\\ \text{is linear}}}(-1)^{|F_j|}\\
&=\prod_{j\in[k]}\sum_{\substack{F\subseteq(\operatorname{CArcs}(\gamma_j))\cap A\\ \text{is linear}}}(-1)^{|F|}
\end{aligned}\tag{49}$$

(here, we have renamed the summation index F_j as F).

On the other hand, recall that the cycles $\gamma_1,\gamma_2,\ldots,\gamma_k$ are distinct, and that we have $\operatorname{Cycs}\sigma=\{\gamma_1,\gamma_2,\ldots,\gamma_k\}$. Hence, $\gamma_1,\gamma_2,\ldots,\gamma_k$ are the elements of the set $\operatorname{Cycs}\sigma$, listed with no repetition. In other words, the map

$$\begin{aligned}
[k]&\to\operatorname{Cycs}\sigma,\\
j&\mapsto\gamma_j
\end{aligned}$$

is a bijection. Hence, we can substitute γ_j for γ in the product $\prod_{\gamma\in\operatorname{Cycs}\sigma}\sum_{\substack{F\subseteq(\operatorname{CArcs}\gamma)\cap A\\ \text{is linear}}}(-1)^{|F|}$. We thus obtain

$$\prod_{\gamma\in\operatorname{Cycs}\sigma}\sum_{\substack{F\subseteq(\operatorname{CArcs}\gamma)\cap A\\ \text{is linear}}}(-1)^{|F|}=\prod_{j\in[k]}\sum_{\substack{F\subseteq(\operatorname{CArcs}(\gamma_j))\cap A\\ \text{is linear}}}(-1)^{|F|}.$$

Comparing this with (49), we obtain

$$\sum_{\substack{F\subseteq \mathbf{A}_\sigma\cap A \\ \text{is linear}}} (-1)^{|F|} = \prod_{\gamma\in\operatorname{Cycs}\sigma} \sum_{\substack{F\subseteq (\operatorname{CArcs}\gamma)\cap A \\ \text{is linear}}} (-1)^{|F|}. \qquad (50)$$

Next, the following three claims follow easily from Lemma 2.47:

Claim 1: Let $\gamma \in \operatorname{Cycs}\sigma$. Assume that γ is a D-cycle. Then,

$$\sum_{\substack{F\subseteq (\operatorname{CArcs}\gamma)\cap A \\ \text{is linear}}} (-1)^{|F|} = (-1)^{\ell(\gamma)-1}.$$

Claim 2: Let $\gamma \in \operatorname{Cycs}\sigma$. Assume that γ is a $\overline{D}$-cycle. Then,

$$\sum_{\substack{F\subseteq (\operatorname{CArcs}\gamma)\cap A \\ \text{is linear}}} (-1)^{|F|} = 1.$$

Claim 3: Let $\gamma \in \operatorname{Cycs}\sigma$. Assume that γ is neither a D-cycle nor a $\overline{D}$-cycle. Then,

$$\sum_{\substack{F\subseteq (\operatorname{CArcs}\gamma)\cap A \\ \text{is linear}}} (-1)^{|F|} = 0.$$

Proof of Claim 1. We have $\gamma \in \operatorname{Cycs}\sigma$. In other words, γ is a cycle of σ (since $\operatorname{Cycs}\sigma$ is the set of all cycles of σ). Hence, Lemma 2.47 **(a)** (applied to $C = \operatorname{CArcs}\gamma$) yields $\sum_{\substack{F\subseteq (\operatorname{CArcs}\gamma)\cap A \\ \text{is linear}}} (-1)^{|F|} = (-1)^{\ell(\gamma)-1}$. This proves Claim 1. □

Proof of Claim 2. We have $\gamma \in \operatorname{Cycs}\sigma$. In other words, γ is a cycle of σ (since $\operatorname{Cycs}\sigma$ is the set of all cycles of σ). Hence, Lemma 2.47 **(b)** (applied to $C = \operatorname{CArcs}\gamma$) yields $\sum_{\substack{F\subseteq (\operatorname{CArcs}\gamma)\cap A \\ \text{is linear}}} (-1)^{|F|} = 1$. This proves Claim 2. □

Proof of Claim 3. We have $\gamma \in \operatorname{Cycs}\sigma$. In other words, γ is a cycle of σ (since $\operatorname{Cycs}\sigma$ is the set of all cycles of σ). Hence, Lemma 2.47 **(c)** (applied to $C = \operatorname{CArcs}\gamma$) yields $\sum_{\substack{F\subseteq (\operatorname{CArcs}\gamma)\cap A \\ \text{is linear}}} (-1)^{|F|} = 0$. This proves Claim 3. □

Now, we shall prove the following two claims:

Claim 4: If $\sigma \in \mathfrak{S}_V\left(D, \overline{D}\right)$, then $\sum_{\substack{F\subseteq \mathbf{A}_\sigma\cap A \\ \text{is linear}}} (-1)^{|F|} = (-1)^{\varphi(\sigma)}$.

Claim 5: If $\sigma \notin \mathfrak{S}_V\left(D, \overline{D}\right)$, then $\sum\limits_{\substack{F \subseteq \mathbf{A}_\sigma \cap A \\ \text{is linear}}} (-1)^{|F|} = 0$.

Proof of Claim 4. Assume that $\sigma \in \mathfrak{S}_V\left(D, \overline{D}\right)$. Then, each cycle of σ is a D-cycle or a $\overline{D}$-cycle (by the definition of $\mathfrak{S}_V\left(D, \overline{D}\right)$). In other words, if $\gamma \in \operatorname{Cycs}\sigma$ is not a D-cycle, then γ is a $\overline{D}$-cycle[53] and thus satisfies

$$\sum_{\substack{F \subseteq (\operatorname{CArcs}\gamma) \cap A \\ \text{is linear}}} (-1)^{|F|} = 1 \tag{51}$$

(by Claim 2).

Now, (50) becomes

$$\begin{aligned}
&\sum_{\substack{F \subseteq \mathbf{A}_\sigma \cap A \\ \text{is linear}}} (-1)^{|F|} \\
&= \prod_{\gamma \in \operatorname{Cycs}\sigma} \quad \sum_{\substack{F \subseteq (\operatorname{CArcs}\gamma) \cap A \\ \text{is linear}}} (-1)^{|F|} \\
&= \left(\prod_{\substack{\gamma \in \operatorname{Cycs}\sigma; \\ \gamma \text{ is a } D\text{-cycle}}} \underbrace{\sum_{\substack{F \subseteq (\operatorname{CArcs}\gamma) \cap A \\ \text{is linear}}} (-1)^{|F|}}_{\substack{=(-1)^{\ell(\gamma)-1} \\ \text{(by Claim 1)}}} \right) \cdot \left(\prod_{\substack{\gamma \in \operatorname{Cycs}\sigma; \\ \gamma \text{ is not a } D\text{-cycle}}} \underbrace{\sum_{\substack{F \subseteq (\operatorname{CArcs}\gamma) \cap A \\ \text{is linear}}} (-1)^{|F|}}_{\substack{=1 \\ \text{(by (51))}}} \right) \\
&\qquad \left(\begin{array}{c} \text{here, we have split our product in two: one} \\ \text{that contains all } \gamma\text{'s that are } D\text{-cycles, and} \\ \text{one that contain all other } \gamma\text{'s} \end{array} \right) \\
&= \left(\prod_{\substack{\gamma \in \operatorname{Cycs}\sigma; \\ \gamma \text{ is a } D\text{-cycle}}} (-1)^{\ell(\gamma)-1} \right) \cdot \underbrace{\left(\prod_{\substack{\gamma \in \operatorname{Cycs}\sigma; \\ \gamma \text{ is not a } D\text{-cycle}}} 1 \right)}_{=1} \\
&= \prod_{\substack{\gamma \in \operatorname{Cycs}\sigma; \\ \gamma \text{ is a } D\text{-cycle}}} (-1)^{\ell(\gamma)-1}.
\end{aligned}$$

[53] *Proof.* Let $\gamma \in \operatorname{Cycs}\sigma$ be not a D-cycle. We must prove that γ is a $\overline{D}$-cycle.

We have $\gamma \in \operatorname{Cycs}\sigma$. In other words, γ is a cycle of σ (since $\operatorname{Cycs}\sigma$ is the set of all cycles of σ). Hence, γ is a D-cycle or a $\overline{D}$-cycle (since each cycle of σ is a D-cycle or a $\overline{D}$-cycle). Since γ is not a D-cycle, we thus conclude that γ is a $\overline{D}$-cycle. Qed.

Comparing this with

$$(-1)^{\varphi(\sigma)} = (-1)^{\sum\limits_{\substack{\gamma\in\operatorname{Cycs}\sigma;\\ \gamma\text{ is a }D\text{-cycle}}}(\ell(\gamma)-1)} \qquad \left(\begin{array}{c}\text{since } \varphi(\sigma) = \sum\limits_{\substack{\gamma\in\operatorname{Cycs}\sigma;\\ \gamma\text{ is a }D\text{-cycle}}}(\ell(\gamma)-1)\\ \text{(by the definition of } \varphi(\sigma)\text{)}\end{array}\right)$$

$$= \prod_{\substack{\gamma\in\operatorname{Cycs}\sigma;\\ \gamma\text{ is a }D\text{-cycle}}} (-1)^{\ell(\gamma)-1},$$

we obtain $\sum\limits_{\substack{F\subseteq\mathbf{A}_\sigma\cap A\\ \text{is linear}}} (-1)^{|F|} = (-1)^{\varphi(\sigma)}$. Thus, Claim 4 is proven. □

Proof of Claim 5. Assume that $\sigma \notin \mathfrak{S}_V\left(D, \overline{D}\right)$. Then, not each cycle of σ is a D-cycle or a $\overline{D}$-cycle (by the definition of $\mathfrak{S}_V\left(D, \overline{D}\right)$). In other words, there exists some cycle of σ that is neither a D-cycle nor a $\overline{D}$-cycle. Let δ be such a cycle. Then, δ is a cycle of σ. In other words, $\delta \in \operatorname{Cycs}\sigma$ (since $\operatorname{Cycs}\sigma$ is the set of all cycles of σ). We know that δ is neither a D-cycle nor a $\overline{D}$-cycle (by its definition). Thus, Claim 3 (applied to $\gamma = \delta$) yields

$$\sum_{\substack{F\subseteq(\operatorname{CArcs}\delta)\cap A\\ \text{is linear}}} (-1)^{|F|} = 0. \tag{52}$$

However, $\delta \in \operatorname{Cycs}\sigma$. Thus, the sum $\sum\limits_{\substack{F\subseteq(\operatorname{CArcs}\delta)\cap A\\ \text{is linear}}} (-1)^{|F|}$ is one of the factors of the product $\prod\limits_{\gamma\in\operatorname{Cycs}\sigma} \sum\limits_{\substack{F\subseteq(\operatorname{CArcs}\gamma)\cap A\\ \text{is linear}}} (-1)^{|F|}$ (namely, the factor for $\gamma = \delta$). Since the former sum is 0 (by (52)), we can rewrite this as follows: The number 0 is one of the factors of the product $\prod\limits_{\gamma\in\operatorname{Cycs}\sigma} \sum\limits_{\substack{F\subseteq(\operatorname{CArcs}\gamma)\cap A\\ \text{is linear}}} (-1)^{|F|}$. In other words, the latter product has a factor equal to 0. Therefore, this product must be 0 (because if a product has a factor equal to 0, then this product must be 0). In other words,

$$\prod_{\gamma\in\operatorname{Cycs}\sigma} \sum_{\substack{F\subseteq(\operatorname{CArcs}\gamma)\cap A\\ \text{is linear}}} (-1)^{|F|} = 0.$$

Now, (50) becomes

$$\sum_{\substack{F\subseteq\mathbf{A}_\sigma\cap A\\ \text{is linear}}} (-1)^{|F|} = \prod_{\gamma\in\operatorname{Cycs}\sigma} \sum_{\substack{F\subseteq(\operatorname{CArcs}\gamma)\cap A\\ \text{is linear}}} (-1)^{|F|} = 0.$$

This proves Claim 5. □

Combining Claim 4 with Claim 5, we obtain

$$\sum_{\substack{F \subseteq \mathbf{A}_\sigma \cap A \\ \text{is linear}}} (-1)^{|F|} = \begin{cases} (-1)^{\varphi(\sigma)}, & \text{if } \sigma \in \mathfrak{S}_V\left(D, \overline{D}\right); \\ 0, & \text{else.} \end{cases}$$

This proves Proposition 2.42. □

2.10. A trivial lemma

We need one more trivial "data conversion" lemma:

> **Lemma 2.49.** Let V be a finite set. Let $w = (w_1, w_2, \ldots, w_n)$ be a V-listing. Then, the map
>
> $$\{\text{maps } f : V \to \mathbb{P}\} \to \mathbb{P}^n, \\ f \mapsto (f(w_1), f(w_2), \ldots, f(w_n))$$
>
> is well-defined and is a bijection.

Proof. Recall that $(w_1, w_2, \ldots, w_n)$ is a V-listing, i.e., a list of elements of V that contains each element of V exactly once (by the definition of a V-listing). Hence, in particular, $(w_1, w_2, \ldots, w_n)$ is a list of elements of V. In other words, $w_i \in V$ for each $i \in [n]$.

Now, if $f : V \to \mathbb{P}$ is a map, then each $i \in [n]$ satisfies $f(w_i) \in \mathbb{P}$ (since $w_i \in V$ by the preceding sentence), and thus the n-tuple $(f(w_1), f(w_2), \ldots, f(w_n))$ belongs to $\mathbb{P}^n$. Thus, the map

$$\{\text{maps } f : V \to \mathbb{P}\} \to \mathbb{P}^n, \\ f \mapsto (f(w_1), f(w_2), \ldots, f(w_n))$$

is well-defined. Let us denote this map by K. It remains to prove that this map K is a bijection.

Let us first show that K is injective. Indeed, let f and g be two maps from V to $\mathbb{P}$ that satisfy $K(f) = K(g)$. We shall show that $f = g$.

The definition of K yields

$$K(f) = (f(w_1), f(w_2), \ldots, f(w_n)) \tag{53}$$

$$\text{and} \qquad K(g) = (g(w_1), g(w_2), \ldots, g(w_n)). \tag{54}$$

We assumed that $K(f) = K(g)$. In view of (53) and (54), we can rewrite this as

$$(f(w_1), f(w_2), \ldots, f(w_n)) = (g(w_1), g(w_2), \ldots, g(w_n)).$$

In other words,

$$f(w_i) = g(w_i) \qquad \text{for each } i \in [n]. \tag{55}$$

Now, let $v \in V$ be arbitrary. Then, the list $(w_1, w_2, \ldots, w_n)$ contains v exactly once (since this list $(w_1, w_2, \ldots, w_n)$ contains each element of V exactly once). In other words, there is exactly one $i \in [n]$ such that $w_i = v$. Consider this i. From (55), we obtain $f(w_i) = g(w_i)$. In view of $w_i = v$, we can rewrite this as $f(v) = g(v)$.

Forget that we fixed v. We thus have shown that $f(v) = g(v)$ for each $v \in V$. In other words, $f = g$.

Forget that we fixed f and g. We have thus shown that if f and g are two maps from V to $\mathbb{P}$ that satisfy $K(f) = K(g)$, then $f = g$. In other words, the map K is injective.

Now, let us prove that K is surjective. Indeed, let $a \in \mathbb{P}^n$ be arbitrary. We shall construct a map $f : V \to \mathbb{P}$ such that $K(f) = a$.

According to Convention 1.5, we can write the n-tuple $a \in \mathbb{P}^n$ as $a = (a_1, a_2, \ldots, a_n)$.

Recall that $(w_1, w_2, \ldots, w_n)$ is a V-listing, i.e., a list of elements of V that contains each element of V exactly once (by the definition of a V-listing).

Now, we define a map $f : V \to \mathbb{P}$ as follows:

Let $v \in V$. Then, the list $(w_1, w_2, \ldots, w_n)$ contains v exactly once (since this list $(w_1, w_2, \ldots, w_n)$ contains each element of V exactly once). In other words, there is exactly one $i \in [n]$ such that $w_i = v$. Consider this i. Define $f(v)$ to be a_i. This is an element of $\mathbb{P}$ (since it is an entry of the n-tuple $a \in \mathbb{P}^n$).

Thus, we have defined an element $f(v)$ of $\mathbb{P}$ for each $v \in V$. In other words, we have defined a map $f : V \to \mathbb{P}$. Its definition has the following consequence: If $v \in V$ is arbitrary, and if $i \in [n]$ is an element satisfying $w_i = v$, then

$$f(v) = a_i. \tag{56}$$

Now, we shall show that $K(f) = a$.

Indeed, the definition of K yields $K(f) = (f(w_1), f(w_2), \ldots, f(w_n))$. However, each $i \in [n]$ satisfies $w_i = w_i$ (obviously) and thus $f(w_i) = a_i$ (by (56), applied to $v = w_i$). In other words, we have

$$(f(w_1), f(w_2), \ldots, f(w_n)) = (a_1, a_2, \ldots, a_n).$$

In view of $K(f) = (f(w_1), f(w_2), \ldots, f(w_n))$ and $a = (a_1, a_2, \ldots, a_n)$, we can rewrite this as $K(f) = a$. Thus, the map K takes a as a value (namely, at the input f).

Forget that we fixed a. We thus have shown that if $a \in \mathbb{P}^n$ is arbitrary, then the map K takes a as a value. In other words, the map K is surjective.

Now we know that the map K is both injective and surjective. In other words, K is bijective. In other words, K is a bijection.

So we have shown that the map K is well-defined and is a bijection. In other words, the map

$$\begin{aligned} \{\text{maps } f : V \to \mathbb{P}\} &\to \mathbb{P}^n, \\ f &\mapsto (f(w_1), f(w_2), \ldots, f(w_n)) \end{aligned}$$

is well-defined and is a bijection (since this map is what we called K). This completes the proof of Lemma 2.49. $\square$

2.11. The proof of Theorem 1.31

We are now ready to prove Theorem 1.31:

Proof of Theorem 1.31. Let $n = |V|$. Thus, the digraph $D = (V, A)$ has n vertices. Moreover, each V-listing w has n entries (since $|V| = n$), thus satisfies $w = (w_1, w_2, \ldots, w_n)$.

We will use a definition that we made back in Lemma 2.40: If $f : V \to \mathbb{P}$ is a map, and if $v = (v_1, v_2, \ldots, v_n)$ is a V-listing, then this V-listing v will be called (f, D)*-friendly* if it has the properties that $f(v_1) \leq f(v_2) \leq \cdots \leq f(v_n)$ and that

$$f\left(v_p\right) < f\left(v_{p+1}\right) \text{ for each } p \in [n-1] \text{ satisfying } \left(v_p, v_{p+1}\right) \in A.$$

The definition of U_D yields

$$U_D = \sum_{w \text{ is a } V\text{-listing}} L_{\operatorname{Des}(w,D),\ n}.$$

We shall now try to understand the addends in this sum better.

We fix a V-listing w. Then, w has n entries (since $|V| = n$), and thus satisfies $w = (w_1, w_2, \ldots, w_n)$. Moreover, the list $(w_1, w_2, \ldots, w_n) = w$ is a V-listing, i.e., consists of all elements of V and contains each of these elements exactly once. In other words, $(w_1, w_2, \ldots, w_n)$ is a list of all elements of V, with no repetitions. Hence, if we are given an element c_v of $\mathbb{Z}[[x_1, x_2, x_3, \ldots]]$ for each $v \in V$, then

$$\prod_{v \in V} c_v = c_{w_1} c_{w_2} \cdots c_{w_n}. \tag{57}$$

Thus, if $f : V \to \mathbb{P}$ is any map, then

$$\prod_{v \in V} x_{f(v)} = x_{f(w_1)} x_{f(w_2)} \cdots x_{f(w_n)} \tag{58}$$

(by (57), applied to $c_v = x_{f(v)}$).

However, the definition of $L_{\operatorname{Des}(w,D),\ n}$ yields

$$\begin{aligned} L_{\operatorname{Des}(w,D),\ n} &= \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_n; \\ i_p < i_{p+1} \text{ for each } p \in \operatorname{Des}(w,D)}} x_{i_1} x_{i_2} \cdots x_{i_n} \\ &= \sum_{\substack{(i_1, i_2, \ldots, i_n) \in \mathbb{P}^n; \\ i_1 \leq i_2 \leq \cdots \leq i_n; \\ i_p < i_{p+1} \text{ for each } p \in \operatorname{Des}(w,D)}} x_{i_1} x_{i_2} \cdots x_{i_n} \end{aligned} \tag{59}$$

(here, we have added the "$(i_1, i_2, \ldots, i_n) \in \mathbb{P}^n$" condition under the summation sign, since this condition is tacitly implied when we sum over $i_1 \leq i_2 \leq \cdots \leq i_n$).

We recall that $\operatorname{Des}(w, D)$ is defined as the set of all D-descents of w, but these D-descents are defined as the elements $i \in [n-1]$ satisfying $(w_i, w_{i+1}) \in A$. Hence, $\operatorname{Des}(w, D)$ is the set of all elements $i \in [n-1]$ satisfying $(w_i, w_{i+1}) \in A$. Thus, an element of $\operatorname{Des}(w, D)$ is the same thing as an element $i \in [n-1]$ satisfying $(w_i, w_{i+1}) \in A$. Renaming the variable i as p in this sentence, we obtain the following: An element of $\operatorname{Des}(w, D)$ is the same thing as an element $p \in [n-1]$ satisfying $(w_p, w_{p+1}) \in A$.

Lemma 2.49 yields that the map

$$\begin{aligned} \{\text{maps } f : V \to \mathbb{P}\} &\to \mathbb{P}^n, \\ f &\mapsto (f(w_1), f(w_2), \ldots, f(w_n)) \end{aligned}$$

is well-defined and is a bijection. Hence, we can substitute $(f(w_1), f(w_2), \ldots, f(w_n))$ for $(i_1, i_2, \ldots, i_n)$ in the sum on the right hand side of (59). We thus obtain

$$\begin{aligned} &\sum_{\substack{(i_1,i_2,\ldots,i_n)\in\mathbb{P}^n;\\ i_1\le i_2\le\cdots\le i_n;\\ i_p<i_{p+1} \text{ for each } p\in\operatorname{Des}(w,D)}} x_{i_1}x_{i_2}\cdots x_{i_n} \\ &= \sum_{\substack{f:V\to\mathbb{P} \text{ is a map};\\ f(w_1)\le f(w_2)\le\cdots\le f(w_n);\\ f(w_p)<f(w_{p+1}) \text{ for each } p\in\operatorname{Des}(w,D)}} \underbrace{x_{f(w_1)}x_{f(w_2)}\cdots x_{f(w_n)}}_{\substack{=\prod_{v\in V} x_{f(v)}\\ \text{(by (58))}}} \\ &= \sum_{\substack{f:V\to\mathbb{P} \text{ is a map};\\ f(w_1)\le f(w_2)\le\cdots\le f(w_n);\\ f(w_p)<f(w_{p+1}) \text{ for each } p\in\operatorname{Des}(w,D)}} \prod_{v\in V} x_{f(v)} \\ &= \sum_{\substack{f:V\to\mathbb{P} \text{ is a map};\\ f(w_1)\le f(w_2)\le\cdots\le f(w_n);\\ f(w_p)<f(w_{p+1}) \text{ for each } p\in[n-1]\\ \text{satisfying } (w_p,w_{p+1})\in A}} \prod_{v\in V} x_{f(v)} \end{aligned}$$

(here, we have replaced the condition "$p \in \operatorname{Des}(w, D)$" under the summation sign by the equivalent condition "$p \in [n-1]$ satisfying $(w_p, w_{p+1}) \in A$", because an element of $\operatorname{Des}(w, D)$ is the same thing as an element $p \in [n-1]$ satisfying $(w_p, w_{p+1}) \in A$). Thus, (59) becomes

$$\begin{aligned} L_{\operatorname{Des}(w,D),\, n} &= \sum_{\substack{(i_1,i_2,\ldots,i_n)\in\mathbb{P}^n;\\ i_1\le i_2\le\cdots\le i_n;\\ i_p<i_{p+1} \text{ for each } p\in\operatorname{Des}(w,D)}} x_{i_1}x_{i_2}\cdots x_{i_n} \\ &= \sum_{\substack{f:V\to\mathbb{P} \text{ is a map};\\ f(w_1)\le f(w_2)\le\cdots\le f(w_n);\\ f(w_p)<f(w_{p+1}) \text{ for each } p\in[n-1]\\ \text{satisfying } (w_p,w_{p+1})\in A}} \prod_{v\in V} x_{f(v)}. \end{aligned} \tag{60}$$

The sum on the right hand side of (60) ranges over all maps $f : V \to \mathbb{P}$ that satisfy the condition

$$\begin{aligned} &\text{"}f(w_1) \le f(w_2) \le \cdots \le f(w_n)\text{"} \\ \wedge\ &\text{"}f(w_p) < f(w_{p+1}) \text{ for each } p \in [n-1] \text{ satisfying } (w_p, w_{p+1}) \in A\text{".} \end{aligned}$$

However, this condition is equivalent to the condition "the V-listing w is (f, D)-friendly" (because this is how the notion of "(f, D)-friendly" was defined). Therefore, we can replace the former condition by the latter condition under the summation sign on the right hand side of (60). Thus, we can rewrite (60) as follows:

$$L_{\operatorname{Des}(w,D),\ n} = \sum_{\substack{f:V\to\mathbb{P} \text{ is a map;} \\ \text{the } V\text{-listing } w \text{ is } (f,D)\text{-friendly}}} \prod_{v\in V} x_{f(v)}. \tag{61}$$

Now, forget that we fixed w. We thus have proved (61) for each V-listing w.

Now,

$$
\begin{aligned}
U_D &= \sum_{w \text{ is a } V\text{-listing}} L_{\operatorname{Des}(w,D),\ n} \\
&= \underbrace{\sum_{w \text{ is a } V\text{-listing}} \ \sum_{\substack{f:V\to\mathbb{P} \text{ is a map;} \\ \text{the } V\text{-listing } w \text{ is } (f,D)\text{-friendly}}}}_{= \sum\limits_{f:V\to\mathbb{P} \text{ is a map}} \ \sum\limits_{\substack{w \text{ is a } V\text{-listing;} \\ \text{the } V\text{-listing } w \text{ is } (f,D)\text{-friendly}}}} \prod_{v\in V} x_{f(v)} \qquad \text{(by (61))} \\
&= \underbrace{\sum_{f:V\to\mathbb{P} \text{ is a map}}}_{= \sum\limits_{f:V\to\mathbb{P}}} \ \underbrace{\sum_{\substack{w \text{ is a } V\text{-listing;} \\ \text{the } V\text{-listing } w \text{ is } (f,D)\text{-friendly}}}}_{= \sum\limits_{w \text{ is an } (f,D)\text{-friendly } V\text{-listing}}} \prod_{v\in V} x_{f(v)} \\
&= \sum_{f:V\to\mathbb{P}} \ \underbrace{\sum_{w \text{ is an } (f,D)\text{-friendly } V\text{-listing}} \ \prod_{v\in V} x_{f(v)}}_{\substack{= (\#\text{ of } (f,D)\text{-friendly } V\text{-listings}) \cdot \prod\limits_{v\in V} x_{f(v)} \\ \text{(since all addends of this sum have the same value } \prod\limits_{v\in V} x_{f(v)}\text{)}}} \\
&= \sum_{f:V\to\mathbb{P}} \underbrace{(\#\text{ of } (f,D)\text{-friendly } V\text{-listings})}_{\substack{= \sum\limits_{\substack{\sigma\in\mathfrak{S}_V; \\ f\circ\sigma=f}} \ \sum\limits_{\substack{F\subseteq\mathbf{A}_\sigma\cap A \\ \text{is linear}}} (-1)^{|F|} \\ \text{(by Lemma 2.40)}}} \cdot \prod_{v\in V} x_{f(v)} \\
&= \underbrace{\sum_{f:V\to\mathbb{P}} \ \sum_{\substack{\sigma\in\mathfrak{S}_V; \\ f\circ\sigma=f}} \ \sum_{\substack{F\subseteq\mathbf{A}_\sigma\cap A \\ \text{is linear}}}}_{= \sum\limits_{\sigma\in\mathfrak{S}_V} \ \sum\limits_{\substack{F\subseteq\mathbf{A}_\sigma\cap A \\ \text{is linear}}} \ \sum\limits_{\substack{f:V\to\mathbb{P}; \\ f\circ\sigma=f}}} (-1)^{|F|} \cdot \prod_{v\in V} x_{f(v)} \\
&= \sum_{\sigma\in\mathfrak{S}_V} \ \sum_{\substack{F\subseteq\mathbf{A}_\sigma\cap A \\ \text{is linear}}} \ \sum_{\substack{f:V\to\mathbb{P}; \\ f\circ\sigma=f}} (-1)^{|F|} \cdot \prod_{v\in V} x_{f(v)} \\
&= \sum_{\sigma\in\mathfrak{S}_V} \ \underbrace{\sum_{\substack{F\subseteq\mathbf{A}_\sigma\cap A \\ \text{is linear}}} (-1)^{|F|}}_{\substack{= \begin{cases} (-1)^{\varphi(\sigma)}, & \text{if } \sigma \in \mathfrak{S}_V\left(D,\overline{D}\right); \\ 0, & \text{else} \end{cases} \\ \text{(by Proposition 2.42)}}} \ \underbrace{\sum_{\substack{f:V\to\mathbb{P}; \\ f\circ\sigma=f}} \ \prod_{v\in V} x_{f(v)}}_{\substack{= p_{\operatorname{type}\sigma} \\ \text{(by Lemma 2.41)}}}
\end{aligned}
$$

$$= \sum_{\sigma \in \mathfrak{S}_V} \begin{cases} (-1)^{\varphi(\sigma)}, & \text{if } \sigma \in \mathfrak{S}_V\left(D, \overline{D}\right); \\ 0, & \text{else} \end{cases} p_{\operatorname{type}\sigma}$$

$$= \underbrace{\sum_{\substack{\sigma \in \mathfrak{S}_V; \\ \sigma \in \mathfrak{S}_V\left(D,\overline{D}\right)}}}_{\substack{= \sum_{\sigma \in \mathfrak{S}_V\left(D,\overline{D}\right)} \\ \text{(since } \mathfrak{S}_V\left(D,\overline{D}\right) \\ \text{is a subset of } \mathfrak{S}_V\text{)}}} \underbrace{\begin{cases} (-1)^{\varphi(\sigma)}, & \text{if } \sigma \in \mathfrak{S}_V\left(D, \overline{D}\right); \\ 0, & \text{else} \end{cases}}_{\substack{=(-1)^{\varphi(\sigma)} \\ \text{(since we have } \sigma \in \mathfrak{S}_V\left(D,\overline{D}\right)\text{)}}} p_{\operatorname{type}\sigma}$$

$$+ \sum_{\substack{\sigma \in \mathfrak{S}_V; \\ \text{we don't have } \sigma \in \mathfrak{S}_V\left(D,\overline{D}\right)}} \underbrace{\begin{cases} (-1)^{\varphi(\sigma)}, & \text{if } \sigma \in \mathfrak{S}_V\left(D, \overline{D}\right); \\ 0, & \text{else} \end{cases}}_{\substack{=0 \\ \text{(since we don't have } \sigma \in \mathfrak{S}_V\left(D,\overline{D}\right)\text{)}}} p_{\operatorname{type}\sigma}$$

$$\left(\text{since each } \sigma \in \mathfrak{S}_V \text{ either satisfies } \sigma \in \mathfrak{S}_V\left(D, \overline{D}\right) \text{ or doesn't}\right)$$

$$= \sum_{\sigma \in \mathfrak{S}_V\left(D,\overline{D}\right)} (-1)^{\varphi(\sigma)} p_{\operatorname{type}\sigma} + \underbrace{\sum_{\substack{\sigma \in \mathfrak{S}_V; \\ \text{we don't have } \sigma \in \mathfrak{S}_V\left(D,\overline{D}\right)}} 0 p_{\operatorname{type}\sigma}}_{=0}$$

$$= \sum_{\sigma \in \mathfrak{S}_V\left(D,\overline{D}\right)} (-1)^{\varphi(\sigma)} p_{\operatorname{type}\sigma}.$$

□

3. Proof of Theorem 1.39

Theorem 1.39 can be derived from Theorem 1.31 by combining some addends that have the same $p_{\operatorname{type}\sigma}$ factor. Depending on the respective $(-1)^{\varphi(\sigma)}$ factors, these addends either cancel each other out or combine to form a multiple of $p_{\operatorname{type}\sigma}$.

Proof of Theorem 1.39. We have assumed that D is a tournament. Hence, for any two distinct vertices u and v of D, we have the logical equivalences

$$((u,v) \text{ is an arc of } D) \iff \left((v,u) \text{ is an arc of } \overline{D}\right)$$

and

$$\left((u,v) \text{ is an arc of } \overline{D}\right) \iff ((v,u) \text{ is an arc of } D).$$

Therefore, the reversal[54] of a nontrivial D-cycle is always a nontrivial $\overline{D}$-cycle, and vice versa.

[54]See Definition 1.23 for the meanings of "reversal" and "nontrivial".

We define a map $\Psi : \mathfrak{S}_V\left(D, \overline{D}\right) \to \mathfrak{S}_V(D)$ as follows: If $\sigma \in \mathfrak{S}_V(D)$, then we let $\Psi(\sigma)$ be the permutation obtained from σ by reversing each cycle of σ that is a nontrivial $\overline{D}$-cycle (i.e., replacing this cycle of σ by its reversal, i.e., replacing σ by σ^{-1} on all entries of this cycle)[55]. This map Ψ is well-defined (i.e., we really have $\Psi(\sigma) \in \mathfrak{S}_V(D)$ for each $\sigma \in \mathfrak{S}_V\left(D, \overline{D}\right)$), because as we just said, the reversal of a nontrivial $\overline{D}$-cycle is always a nontrivial D-cycle. Moreover, the map Ψ preserves the cycle type of a permutation – i.e., we have

$$\operatorname{type}(\Psi(\sigma)) = \operatorname{type}\sigma \tag{62}$$

for each $\sigma \in \mathfrak{S}_V\left(D, \overline{D}\right)$.

Now, Theorem 1.31 yields

$$\begin{aligned}
U_D &= \sum_{\sigma \in \mathfrak{S}_V\left(D,\overline{D}\right)} (-1)^{\varphi(\sigma)} \underbrace{p_{\operatorname{type}\sigma}}_{\substack{=p_{\operatorname{type}(\Psi(\sigma))} \\ \text{(by (62))}}} = \sum_{\sigma \in \mathfrak{S}_V\left(D,\overline{D}\right)} (-1)^{\varphi(\sigma)} p_{\operatorname{type}(\Psi(\sigma))} \\
&= \sum_{\tau \in \mathfrak{S}_V(D)} \sum_{\substack{\sigma \in \mathfrak{S}_V\left(D,\overline{D}\right); \\ \Psi(\sigma)=\tau}} (-1)^{\varphi(\sigma)} p_{\operatorname{type}\tau} \qquad \left(\begin{array}{c}\text{here, we have split up the sum} \\ \text{according to the value of } \Psi(\sigma)\end{array}\right) \\
&= \sum_{\tau \in \mathfrak{S}_V(D)} \left(\sum_{\substack{\sigma \in \mathfrak{S}_V\left(D,\overline{D}\right); \\ \Psi(\sigma)=\tau}} (-1)^{\varphi(\sigma)} \right) p_{\operatorname{type}\tau}.
\end{aligned} \tag{63}$$

Now, we claim that each $\tau \in \mathfrak{S}_V(D)$ satisfies

$$\sum_{\substack{\sigma \in \mathfrak{S}_V\left(D,\overline{D}\right); \\ \Psi(\sigma)=\tau}} (-1)^{\varphi(\sigma)} = \begin{cases} 2^{\psi(\tau)}, & \text{if all cycles of } \tau \text{ have odd length;} \\ 0, & \text{otherwise.} \end{cases} \tag{64}$$

[*Proof of (64):* Let $\tau \in \mathfrak{S}_V(D)$. Then, τ has exactly $\psi(\tau)$ many nontrivial cycles (by the definition of $\psi(\tau)$), and all of these nontrivial cycles are D-cycles (by the

[55] Here is what this means in rigorous terms: We let $\Psi(\sigma)$ be the permutation of V defined by setting

$$(\Psi(\sigma))(z) = \begin{cases} \sigma^{-1}(z), & \text{if } z \text{ is an entry of a cycle of } \sigma \text{ that is a nontrivial } \overline{D}\text{-cycle;} \\ \sigma(z), & \text{otherwise} \end{cases}$$
$$\text{for each } z \in V.$$

The cycles of this permutation $\Psi(\sigma)$ are precisely

- the reversals of those cycles of σ that are nontrivial $\overline{D}$-cycles, and
- the remaining cycles of σ.

definition of $\mathfrak{S}_V(D)$). The permutations $\sigma \in \mathfrak{S}_V\left(D, \overline{D}\right)$ that satisfy $\Psi(\sigma) = \tau$ can be obtained by choosing some of these nontrivial cycles and reversing them, which turns them into $\overline{D}$-cycles. This can be done in $2^{\psi(\tau)}$ many ways, since each of the $\psi(\tau)$ many nontrivial cycles can be either reversed or not. If all cycles of τ have odd length, then all $2^{\psi(\tau)}$ permutations σ obtained in this way will satisfy $(-1)^{\varphi(\sigma)} = 1$ (because $\varphi(\sigma) = \sum\limits_{\substack{\gamma \in \operatorname{Cycs}\sigma; \\ \gamma \text{ is a } D\text{-cycle}}} \left(\underbrace{\ell(\gamma)}_{\text{odd}} - 1\right)$ will always be even); therefore, the sum $\sum\limits_{\substack{\sigma \in \mathfrak{S}_V\left(D, \overline{D}\right); \\ \Psi(\sigma) = \tau}} (-1)^{\varphi(\sigma)}$ will be a sum of $2^{\psi(\tau)}$ many 1s and therefore simplify to $2^{\psi(\tau)}$. On the other hand, if not all cycles of τ have odd length, then there is at least one cycle δ of τ that has even length, and of course this cycle δ will be nontrivial (since a trivial cycle has odd length); thus, among the permutations $\sigma \in \mathfrak{S}_V\left(D, \overline{D}\right)$ that satisfy $\Psi(\sigma) = \tau$, there will be as many that have δ reversed as ones that have δ not reversed, and the parities of $\varphi(\sigma)$ for the former will be opposite from the parities of $\varphi(\sigma)$ for the latter; thus, the sum $\sum\limits_{\substack{\sigma \in \mathfrak{S}_V\left(D, \overline{D}\right); \\ \Psi(\sigma) = \tau}} (-1)^{\varphi(\sigma)}$ will have equally many 1s and -1s among its addends, and therefore will simplify to 0. In either case, we obtain (64).]

Now, (63) becomes

$$\begin{aligned}
U_D &= \sum_{\tau \in \mathfrak{S}_V(D)} \underbrace{\left(\sum_{\substack{\sigma \in \mathfrak{S}_V\left(D, \overline{D}\right); \\ \Psi(\sigma) = \tau}} (-1)^{\varphi(\sigma)}\right)}_{\substack{= \begin{cases} 2^{\psi(\tau)}, & \text{if all cycles of } \tau \text{ have odd length;} \\ 0, & \text{otherwise} \end{cases} \\ \text{(by (64))}}} p_{\operatorname{type}\tau} \\
&= \sum_{\tau \in \mathfrak{S}_V(D)} \begin{cases} 2^{\psi(\tau)}, & \text{if all cycles of } \tau \text{ have odd length;} \\ 0, & \text{otherwise} \end{cases} p_{\operatorname{type}\tau} \\
&= \sum_{\substack{\tau \in \mathfrak{S}_V(D); \\ \text{all cycles of } \tau \text{ have odd length}}} 2^{\psi(\tau)} p_{\operatorname{type}\tau} = \sum_{\substack{\sigma \in \mathfrak{S}_V(D); \\ \text{all cycles of } \sigma \text{ have odd length}}} 2^{\psi(\sigma)} p_{\operatorname{type}\sigma}.
\end{aligned}$$

This proves Theorem 1.39. □

4. Proving the corollaries

Let us now quickly go through the proofs of the corollaries we stated after Theorem 1.31 and after Theorem 1.39:

Proof of Corollary 1.35. We let $\mathbb{N}[p_1, p_2, p_3, \ldots]$ denote the set of all polynomials in $p_1, p_2, p_3, \ldots$ with coefficients in $\mathbb{N}$.

For each integer partition λ, we have

$$p_\lambda \in \mathbb{N}[p_1, p_2, p_3, \ldots] \tag{65}$$

(by the definition of p_λ).

Theorem 1.31 yields

$$\begin{aligned} U_D &= \sum_{\sigma \in \mathfrak{S}_V(D, \overline{D})} (-1)^{\varphi(\sigma)} \underbrace{p_{\operatorname{type}\sigma}}_{\substack{\in \mathbb{N}[p_1, p_2, p_3, \ldots] \\ \text{(by (65))}}} \\ &\in \sum_{\sigma \in \mathfrak{S}_V(D, \overline{D})} (-1)^{\varphi(\sigma)} \mathbb{N}[p_1, p_2, p_3, \ldots] \subseteq \mathbb{Z}[p_1, p_2, p_3, \ldots]. \end{aligned}$$

This proves Corollary 1.35. $\square$

Proof of Corollary 1.36. Let $2\mathbb{Z}$ denote the set of all even integers.

Let $\sigma \in \mathfrak{S}_V(D, \overline{D})$. The definition of $\varphi(\sigma)$ in Theorem 1.31 yields

$$\varphi(\sigma) = \sum_{\substack{\gamma \in \operatorname{Cycs}\sigma; \\ \gamma \text{ is a } D\text{-cycle}}} \underbrace{(\ell(\gamma) - 1)}_{\substack{\in 2\mathbb{Z} \\ \text{(since } \ell(\gamma) \text{ is odd} \\ \text{(because every } D\text{-cycle} \\ \text{has odd length))}}} \in 2\mathbb{Z},$$

so that

$$(-1)^{\varphi(\sigma)} = 1. \tag{66}$$

Theorem 1.31 now yields

$$\begin{aligned} U_D &= \sum_{\sigma \in \mathfrak{S}_V(D, \overline{D})} \underbrace{(-1)^{\varphi(\sigma)}}_{\substack{=1 \\ \text{(by (66))}}} p_{\operatorname{type}\sigma} \\ &= \sum_{\sigma \in \mathfrak{S}_V(D, \overline{D})} \underbrace{p_{\operatorname{type}\sigma}}_{\substack{\in \mathbb{N}[p_1, p_2, p_3, \ldots] \\ \text{(by (65))}}} \\ &\in \sum_{\sigma \in \mathfrak{S}_V(D, \overline{D})} \mathbb{N}[p_1, p_2, p_3, \ldots] \subseteq \mathbb{N}[p_1, p_2, p_3, \ldots]. \end{aligned}$$

This proves Corollary 1.36. $\square$

Proof of Corollary 1.40. For each $\sigma \in \mathfrak{S}_V$, let $\psi(\sigma)$ denote the number of nontrivial cycles of σ.

Let $\sigma \in \mathfrak{S}_V(D)$ be a permutation whose all cycles have odd length. We shall show that $2^{\psi(\sigma)} p_{\operatorname{type}\sigma} \in \mathbb{N}[p_1, 2p_3, 2p_5, 2p_7, \ldots]$.

Indeed, let $k_1, k_2, \ldots, k_s$ be the lengths of all cycles of σ, listed in decreasing order. Then, the numbers $k_1, k_2, \ldots, k_s$ are odd (since all cycles of σ have odd length). Moreover, the definition of $\operatorname{type}\sigma$ yields $\operatorname{type}\sigma = (k_1, k_2, \ldots, k_s)$. Furthermore,

$$\begin{aligned}
\psi(\sigma) &= (\text{\# of nontrivial cycles of } \sigma) \\
&= (\text{\# of cycles of } \sigma \text{ that have length } > 1) \\
&= (\text{\# of } i \in [s] \text{ such that } k_i > 1) \\
&\qquad\qquad (\text{since the lengths of all cycles of } \sigma \text{ are } k_1, k_2, \ldots, k_s) \\
&= \sum_{i=1}^{s} [k_i > 1]
\end{aligned}$$

(here, we are using the Iverson bracket notation), so that

$$2^{\psi(\sigma)} = 2^{\sum_{i=1}^{s} [k_i > 1]} = \prod_{i=1}^{s} 2^{[k_i > 1]}. \tag{67}$$

Now, recall that $\operatorname{type}\sigma = (k_1, k_2, \ldots, k_s)$. Hence, the definition of $p_{\operatorname{type}\sigma}$ yields

$$p_{\operatorname{type}\sigma} = p_{k_1} p_{k_2} \cdots p_{k_s} = \prod_{i=1}^{s} p_{k_i}. \tag{68}$$

Multiplying the equalities (67) and (68), we obtain

$$\begin{aligned}
2^{\psi(\sigma)} p_{\operatorname{type}\sigma} &= \left(\prod_{i=1}^{s} 2^{[k_i > 1]} \right) \left(\prod_{i=1}^{s} p_{k_i} \right) = \prod_{i=1}^{s} \underbrace{\left(2^{[k_i > 1]} p_{k_i} \right)}_{\substack{\in \{p_1, 2p_3, 2p_5, 2p_7, \ldots\} \\ (\text{since } k_i \text{ is odd} \\ (\text{because } k_1, k_2, \ldots, k_s \text{ are odd}))}} \\
&= (\text{a product of } s \text{ elements of the set } \{p_1, 2p_3, 2p_5, 2p_7, \ldots\}) \\
&\in \mathbb{N}[p_1, 2p_3, 2p_5, 2p_7, \ldots].
\end{aligned} \tag{69}$$

Forget that we fixed σ. We thus have proved (69) for each permutation $\sigma \in \mathfrak{S}_V(D)$ whose all cycles have odd length. Now, Theorem 1.39 yields

$$U_D = \sum_{\substack{\sigma \in \mathfrak{S}_V(D); \\ \text{all cycles of } \sigma \text{ have odd length}}} \underbrace{2^{\psi(\sigma)} p_{\operatorname{type}\sigma}}_{\substack{\in \mathbb{N}[p_1, 2p_3, 2p_5, 2p_7, \ldots] \\ (\text{by (69)})}} \in \mathbb{N}[p_1, 2p_3, 2p_5, 2p_7, \ldots].$$

This proves Corollary 1.40. $\square$

5. Proof of Theorem 1.41

The proof of Theorem 1.41 is a slightly more complicated variant of our above proof of Theorem 1.39.

Proof of Theorem 1.41. **(b)** First, we attempt to gain a better understanding of risky cycles.

We start by noticing that the reversal of a risky rotation-equivalence class is again risky.

We have assumed that there exist no two distinct vertices u and v of D such that both pairs (u,v) and (v,u) belong to A. In other words, if (u,v) is an arc of D with $u \neq v$, then (v,u) is not an arc of D, and thus (v,u) must be an arc of $\overline{D}$.

Hence, if v is any D-cycle of length ≥ 2, then the reversal of v must be a $\overline{D}$-cycle, and thus cannot be a D-cycle. Therefore, in particular, if v is a risky rotation-equivalence class of tuples of elements of V, then either v or the reversal of v is a D-cycle (by the definition of "risky"), but not both at the same time.

Consequently, if v is a risky rotation-equivalence class of tuples of elements of V, then v and the reversal of v cannot be identical, i.e., we must have

$$v \neq \operatorname{rev} v. \tag{70}$$

We define a subset $\mathfrak{S}_V^\circ\left(D,\overline{D}\right)$ of $\mathfrak{S}_V\left(D,\overline{D}\right)$ by

$$\mathfrak{S}_V^\circ\left(D,\overline{D}\right) := \left\{\sigma \in \mathfrak{S}_V\left(D,\overline{D}\right) \ \mid \ \text{each risky cycle of } \sigma \text{ is a } D\text{-cycle}\right\}.$$

We define a map $\Gamma : \mathfrak{S}_V\left(D,\overline{D}\right) \to \mathfrak{S}_V^\circ\left(D,\overline{D}\right)$ as follows: If $\sigma \in \mathfrak{S}_V\left(D,\overline{D}\right)$, then we let $\Gamma(\sigma)$ be the permutation obtained from σ by reversing each risky cycle of σ that is not a D-cycle (i.e., replacing this cycle of σ by its reversal, i.e., replacing σ by σ^{-1} on all entries of this cycle). This map Γ is well-defined (i.e., we really have $\Gamma(\sigma) \in \mathfrak{S}_V^\circ\left(D,\overline{D}\right)$ for each $\sigma \in \mathfrak{S}_V\left(D,\overline{D}\right)$), because if a risky tuple is not a D-cycle, then its reversal is a D-cycle (by the definition of "risky"). Moreover, the map Γ preserves the cycle type of a permutation – i.e., we have

$$\operatorname{type}(\Gamma(\sigma)) = \operatorname{type}\sigma \tag{71}$$

for each $\sigma \in \mathfrak{S}_V\left(D,\overline{D}\right)$.

Now, Theorem 1.31 yields

$$\begin{aligned}
U_D &= \sum_{\sigma \in \mathfrak{S}_V\left(D,\overline{D}\right)} (-1)^{\varphi(\sigma)} \underbrace{p_{\operatorname{type}\sigma}}_{\substack{=p_{\operatorname{type}(\Gamma(\sigma))} \\ \text{(by (71))}}} = \sum_{\sigma \in \mathfrak{S}_V\left(D,\overline{D}\right)} (-1)^{\varphi(\sigma)} p_{\operatorname{type}(\Gamma(\sigma))} \\
&= \sum_{\tau \in \mathfrak{S}_V^\circ\left(D,\overline{D}\right)} \sum_{\substack{\sigma \in \mathfrak{S}_V\left(D,\overline{D}\right); \\ \Gamma(\sigma)=\tau}} (-1)^{\varphi(\sigma)} p_{\operatorname{type}\tau} \qquad \left(\begin{array}{c}\text{here, we have split up the sum} \\ \text{according to the value of } \Gamma(\sigma)\end{array}\right) \\
&= \sum_{\tau \in \mathfrak{S}_V^\circ\left(D,\overline{D}\right)} \left(\sum_{\substack{\sigma \in \mathfrak{S}_V\left(D,\overline{D}\right); \\ \Gamma(\sigma)=\tau}} (-1)^{\varphi(\sigma)} \right) p_{\operatorname{type}\tau}.
\end{aligned} \tag{72}$$

Now, we claim that each $\tau \in \mathfrak{S}_V^\circ \left(D, \overline{D}\right)$ satisfies

$$\sum_{\substack{\sigma \in \mathfrak{S}_V\left(D, \overline{D}\right);\\ \Gamma(\sigma)=\tau}} (-1)^{\varphi(\sigma)} = \begin{cases} (-1)^{\varphi(\tau)}, & \text{if no cycle of } \tau \text{ is risky;} \\ 0, & \text{otherwise.} \end{cases} \tag{73}$$

[*Proof of (73):* Let $\tau \in \mathfrak{S}_V^\circ \left(D, \overline{D}\right)$. Let $c_1, c_2, \ldots, c_k$ be the risky cycles of τ. All of these k risky cycles $c_1, c_2, \ldots, c_k$ are D-cycles (since $\tau \in \mathfrak{S}_V^\circ \left(D, \overline{D}\right)$). The permutations $\sigma \in \mathfrak{S}_V \left(D, \overline{D}\right)$ that satisfy $\Gamma(\sigma) = \tau$ can be obtained by choosing some of these k risky cycles $c_1, c_2, \ldots, c_k$ of τ and reversing them, which turns them into $\overline{D}$-cycles (because if v is any D-cycle of length ≥ 2, then the reversal of v must be a $\overline{D}$-cycle). This can be done in 2^k many ways, since each of the k risky cycles $c_1, c_2, \ldots, c_k$ can be either reversed or not[56]. The sum $\sum\limits_{\substack{\sigma \in \mathfrak{S}_V\left(D, \overline{D}\right);\\ \Gamma(\sigma)=\tau}} (-1)^{\varphi(\sigma)}$ thus has 2^k many addends, and each of these addends corresponds to one way to decide which of the k risky cycles $c_1, c_2, \ldots, c_k$ to reverse and which not to reverse. If $k = 0$, then this sum therefore simplifies to $(-1)^{\varphi(\tau)}$. If, on the other hand, we have $k \neq 0$, then this sum equals 0 [57]. Combining the results from both of these cases,

[56] Fineprint: All of these k risky cycles are distinct from their reversals (by (70)). Thus, each of the 2^k possible choices of risky cycles to reverse leads to a different permutation $\sigma \in \mathfrak{S}_V \left(D, \overline{D}\right)$.

[57] *Proof.* Assume that $k \neq 0$. Thus, $k \geq 1$, so that the risky cycle c_1 exists. If $\sigma \in \mathfrak{S}_V \left(D, \overline{D}\right)$ is such that $\Gamma(\sigma) = \tau$, then either the cycle c_1 or its reversal (but not both) is a cycle of σ. Thus,

$$\sum_{\substack{\sigma \in \mathfrak{S}_V\left(D, \overline{D}\right);\\ \Gamma(\sigma)=\tau}} (-1)^{\varphi(\sigma)} = \sum_{\substack{\sigma \in \mathfrak{S}_V\left(D, \overline{D}\right);\\ \Gamma(\sigma)=\tau;\\ c_1 \text{ is a cycle of } \sigma}} (-1)^{\varphi(\sigma)} + \sum_{\substack{\sigma \in \mathfrak{S}_V\left(D, \overline{D}\right);\\ \Gamma(\sigma)=\tau;\\ c_1 \text{ is not a cycle of } \sigma}} (-1)^{\varphi(\sigma)}. \tag{74}$$

The two sums on the right hand side of this equality have the same number of addends, and there is in fact a bijection between the addends of the former and those of the latter (given by replacing the cycle c_1 by its reversal or vice versa). Moreover, this bijection toggles the parity of the number $\varphi(\sigma)$ (that is, it changes this number from odd to even or vice versa), since $\varphi(\sigma)$ is defined to be the sum $\sum\limits_{\substack{\gamma \in \operatorname{Cycs} \sigma;\\ \gamma \text{ is a } D\text{-cycle}}} (\ell(\gamma) - 1)$ (which contains the odd addend $\ell(c_1) - 1$ when c_1 is a cycle of σ, but does not contain this addend when c_1 is not a cycle of σ). Hence, this bijection flips the sign $(-1)^{\varphi(\sigma)}$. Therefore, the addends in the first sum on the right hand side of (74) cancel those in the second. Therefore, the two sums add up to 0. The equality (74) thus simplifies to $\sum\limits_{\substack{\sigma \in \mathfrak{S}_V\left(D, \overline{D}\right);\\ \Gamma(\sigma)=\tau}} (-1)^{\varphi(\sigma)} = 0$, qed.

we obtain

$$\sum_{\substack{\sigma\in\mathfrak{S}_V\left(D,\overline{D}\right);\\ \Gamma(\sigma)=\tau}} (-1)^{\varphi(\sigma)} = \begin{cases} (-1)^{\varphi(\tau)}, & \text{if } k=0;\\ 0, & \text{otherwise}\end{cases}$$

$$= \begin{cases} (-1)^{\varphi(\tau)}, & \text{if no cycle of } \tau \text{ is risky};\\ 0, & \text{otherwise.}\end{cases}$$

(since k is the number of risky cycles of τ). This proves (73).]

Now, (72) becomes

$$U_D = \sum_{\tau\in\mathfrak{S}_V^{\circ}\left(D,\overline{D}\right)} \underbrace{\left(\sum_{\substack{\sigma\in\mathfrak{S}_V\left(D,\overline{D}\right);\\ \Gamma(\sigma)=\tau}} (-1)^{\varphi(\sigma)}\right)}_{\substack{=\begin{cases} (-1)^{\varphi(\tau)}, & \text{if no cycle of } \tau \text{ is risky};\\ 0, & \text{otherwise}\end{cases}\\ \text{(by (73))}}} p_{\operatorname{type}\tau}$$

$$= \sum_{\tau\in\mathfrak{S}_V^{\circ}\left(D,\overline{D}\right)} \begin{cases} (-1)^{\varphi(\tau)}, & \text{if no cycle of } \tau \text{ is risky};\\ 0, & \text{otherwise}\end{cases} p_{\operatorname{type}\tau}$$

$$= \sum_{\substack{\tau\in\mathfrak{S}_V^{\circ}\left(D,\overline{D}\right);\\ \text{no cycle of } \tau \text{ is risky}}} \underbrace{(-1)^{\varphi(\tau)}}_{\substack{=1\\ \text{(since no cycle of } \tau \text{ is risky,}\\ \text{and thus it is easy to see}\\ \text{that } \varphi(\tau) \text{ is even)}}} p_{\operatorname{type}\tau}$$

$$= \sum_{\substack{\tau\in\mathfrak{S}_V^{\circ}\left(D,\overline{D}\right);\\ \text{no cycle of } \tau \text{ is risky}}} p_{\operatorname{type}\tau} = \sum_{\substack{\tau\in\mathfrak{S}_V\left(D,\overline{D}\right);\\ \text{no cycle of } \tau \text{ is risky}}} p_{\operatorname{type}\tau}$$

(since the permutations $\tau \in \mathfrak{S}_V^{\circ}\left(D,\overline{D}\right)$ that have no risky cycles are precisely the permutations $\tau \in \mathfrak{S}_V\left(D,\overline{D}\right)$ that have no risky cycles[58]). Renaming the summation index τ as σ on the right hand side, we obtain

$$U_D = \sum_{\substack{\sigma\in\mathfrak{S}_V\left(D,\overline{D}\right);\\ \text{no cycle of } \sigma \text{ is risky}}} p_{\operatorname{type}\sigma}.$$

This proves Theorem 1.41 **(b)**.

(a) This follows trivially from part **(b)**, since $p_\lambda \in \mathbb{N}\left[p_1, p_2, p_3, \ldots\right]$ for each partition λ. □

[58]This follows trivially from the definition of $\mathfrak{S}_V^{\circ}\left(D,\overline{D}\right)$.

6. Recovering Redei's and Berge's theorems

We shall now derive two well-known theorems from Theorem 1.31 and Theorem 1.39.

We recall Convention 2.1 and Definition 2.22. The two theorems we shall derive are the following:

Theorem 6.1 (Rédei's Theorem)**.** Let D be a tournament. Then, the # of hamps of D is odd. Here, we agree to consider the empty list $()$ as a hamp of the empty tournament with 0 vertices.

Theorem 6.2 (Berge's Theorem)**.** Let D be a digraph. Then,

$$\left(\text{\# of hamps of } \overline{D}\right) \equiv (\text{\# of hamps of } D) \bmod 2.$$

Theorem 6.1 originates in Laszlo Rédei's 1933 paper [Redei33] (see [Moon13, proof of Theorem 14] for an English translation of his proof). Theorem 6.2 was found by Claude Berge (see [Berge76, §10.1, Theorem 1], [Berge91, §10.1, Theorem 1], [Tomesc85, solution to problem 7.8], [Lovasz07, Exercise 5.19] or [Grinbe17, Theorem 1.3.6] for his proof, and [Lass02, Corollaire 5.1] for another). Berge used Theorem 6.2 to give a new and simpler proof of Theorem 6.1 (see [Berge91, §10.2, Theorem 6] or [Lovasz07, Exercise 5.20] or [Grinbe17, Theorem 1.6.1]).

We can now give new proofs for both theorems. This will rely on the symmetric function U_D and also on a few simple tools:

We define $\zeta : \operatorname{QSym} \to \mathbb{Z}$ to be the evaluation homomorphism that sends each quasisymmetric function $f \in \operatorname{QSym}$ to its evaluation $f(1, 0, 0, 0, \ldots)$ (obtained by setting x_1 to be 1 and setting all other variables $x_2, x_3, x_4, \ldots$ to be 0). Note that ζ is a $\mathbb{Z}$-algebra homomorphism.[59] We shall show two simple lemmas:

Lemma 6.3. Let $n \in \mathbb{N}$. Let I be a subset of $[n-1]$. Then, $\zeta(L_{I,\,n}) = [I = \varnothing]$ (where we are using the Iverson bracket notation).

Proof. The definition of $L_{I,\,n}$ yields

$$L_{I,\,n} = \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_n; \\ i_p < i_{p+1} \text{ for each } p \in I}} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

When we apply ζ to the sum on the right hand side (i.e., substitute 1 for x_1 and substitute 0 for $x_2, x_3, x_4, \ldots$), any addend that contains at least one of the variables $x_2, x_3, x_4, \ldots$ becomes 0, whereas any addend that only contains copies

[59] We don't really need QSym here. We could just as well define ζ on the ring of bounded-degree power series (that is, of all power series $f \in \mathbb{Z}[[x_1, x_2, x_3, \ldots]]$ for which there exists an $N \in \mathbb{N}$ such that no monomial of degree $> N$ appears in f). However, we cannot define ζ on the whole ring $\mathbb{Z}[[x_1, x_2, x_3, \ldots]]$, since ζ would have to send $1 + x_1 + x_1^2 + x_1^3 + \cdots$ to $1 + 1 + 1^2 + 1^3 + \cdots$.

of x_1 becomes 1. Hence, $\zeta\left(L_{I,\,n}\right)$ is the number of addends that only contain copies of x_1. But this number is 1 if $I = \varnothing$ (namely, in this case, the addend for $(i_1, i_2, \ldots, i_n) = (1, 1, \ldots, 1)$ fits the bill), and is 0 if $I \neq \varnothing$ (because in this case, the condition "$i_p < i_{p+1}$ for each $p \in I$" forces at least one of the n numbers $i_1, i_2, \ldots, i_n$ in each addend $x_{i_1} x_{i_2} \cdots x_{i_n}$ to be larger than 1, and therefore each addend contains at least one of $x_2, x_3, x_4, \ldots$). Thus, altogether, this number is $[I = \varnothing]$. This proves Lemma 6.3. □

Lemma 6.4. Let λ be any partition. Then,

$$\zeta\left(p_\lambda\right) = 1.$$

Proof. Write the partition λ in the form $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, where the k entries $\lambda_1, \lambda_2, \ldots, \lambda_k$ are positive. Then, the definition of p_λ yields $p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_k}$. Hence,

$$\begin{aligned} \zeta\left(p_\lambda\right) &= \zeta\left(p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_k}\right) = \zeta\left(p_{\lambda_1}\right) \zeta\left(p_{\lambda_2}\right) \cdots \zeta\left(p_{\lambda_k}\right) \\ &\qquad\qquad (\text{since } \zeta \text{ is a } \mathbb{Z}\text{-algebra homomorphism}) \\ &= \prod_{i=1}^{k} \zeta\left(p_{\lambda_i}\right). \end{aligned} \tag{75}$$

However, for each positive integer n, we have $p_n = x_1^n + x_2^n + x_3^n + \cdots$ (by the definition of p_n) and

$$\begin{aligned} \zeta\left(p_n\right) &= p_n\left(1, 0, 0, 0, \ldots\right) \qquad (\text{by the definition of } \zeta) \\ &= \underbrace{1^n}_{=1} + \underbrace{0^n + 0^n + 0^n + \cdots}_{\substack{=0 \\ (\text{since } n \text{ is positive})}} \qquad (\text{since } p_n = x_1^n + x_2^n + x_3^n + \cdots) \\ &= 1. \end{aligned} \tag{76}$$

Hence, (75) becomes

$$\zeta\left(p_\lambda\right) = \prod_{i=1}^{k} \underbrace{\zeta\left(p_{\lambda_i}\right)}_{\substack{=1 \\ (\text{by (76)}, \\ \text{since } \lambda_i \text{ is positive})}} = \prod_{i=1}^{k} 1 = 1.$$

This proves Lemma 6.4. □

Lemma 6.5. Let D be a digraph. Then,

$$\zeta\left(U_D\right) = \left(\# \text{ of hamps of } \overline{D}\right).$$

Proof. Write D as $D = (V, A)$, and set $n := |V|$. Then, $\overline{D} = (V, (V \times V) \setminus A)$. Hence, a hamp of $\overline{D}$ is the same as a V-listing w such that each $i \in [n-1]$ satisfies $(w_i, w_{i+1}) \in (V \times V) \setminus A$. In other words, a hamp of $\overline{D}$ is the same as a V-listing w such that no $i \in [n-1]$ satisfies $(w_i, w_{i+1}) \in A$. In other words, a hamp of $\overline{D}$ is the same as a V-listing w that satisfies $\operatorname{Des}(w, D) = \varnothing$ (because $\operatorname{Des}(w, D)$ is defined to be the set of all $i \in [n-1]$ satisfying $(w_i, w_{i+1}) \in A$). Therefore,

$$\begin{aligned}
&\left(\# \text{ of hamps of } \overline{D}\right) \\
&= \left(\# \text{ of } V\text{-listings } w \text{ that satisfy } \operatorname{Des}(w, D) = \varnothing\right).
\end{aligned} \tag{77}$$

The definition of U_D yields

$$U_D = \sum_{w \text{ is a } V\text{-listing}} L_{\operatorname{Des}(w,D),\ n}.$$

Hence,

$$\begin{aligned}
\zeta(U_D) &= \zeta\left(\sum_{w \text{ is a } V\text{-listing}} L_{\operatorname{Des}(w,D),\ n}\right) \\
&= \sum_{w \text{ is a } V\text{-listing}} \underbrace{\zeta\left(L_{\operatorname{Des}(w,D),\ n}\right)}_{\substack{=[\operatorname{Des}(w,D)=\varnothing] \\ \text{(by Lemma 6.3)}}} \qquad \text{(since the map } \zeta \text{ is } \mathbb{Z}\text{-linear)} \\
&= \sum_{w \text{ is a } V\text{-listing}} [\operatorname{Des}(w, D) = \varnothing] \\
&= \left(\# \text{ of } V\text{-listings } w \text{ that satisfy } \operatorname{Des}(w, D) = \varnothing\right) \\
&= \left(\# \text{ of hamps of } \overline{D}\right) \qquad (\text{by } (77)).
\end{aligned}$$

This proves Lemma 6.5. □

We can now state a formula for the # of hamps of $\overline{D}$ for any digraph D:

Theorem 6.6. Let $D = (V, A)$ be a digraph. Then:

(a) Set

$$\varphi(\sigma) := \sum_{\substack{\gamma \in \operatorname{Cycs} \sigma; \\ \gamma \text{ is a } D\text{-cycle}}} (\ell(\gamma) - 1) \qquad \text{for each } \sigma \in \mathfrak{S}_V.$$

Then,

$$\left(\# \text{ of hamps of } \overline{D}\right) = \sum_{\sigma \in \mathfrak{S}_V\left(D, \overline{D}\right)} (-1)^{\varphi(\sigma)}.$$

(b) We have $\left(\# \text{ of hamps of } \overline{D}\right) \equiv \left|\mathfrak{S}_V\left(D, \overline{D}\right)\right| \bmod 2$.

Proof. **(a)** Theorem 1.31 yields

$$U_D = \sum_{\sigma \in \mathfrak{S}_V\left(D, \overline{D}\right)} (-1)^{\varphi(\sigma)}\, p_{\operatorname{type} \sigma}.$$

Hence,

$$\begin{aligned}
\zeta\left(U_D\right) &= \zeta\left(\sum_{\sigma\in\mathfrak{S}_V\left(D,\overline{D}\right)} (-1)^{\varphi(\sigma)}\, p_{\operatorname{type}\sigma}\right) \\
&= \sum_{\sigma\in\mathfrak{S}_V\left(D,\overline{D}\right)} (-1)^{\varphi(\sigma)} \underbrace{\zeta\left(p_{\operatorname{type}\sigma}\right)}_{\substack{=1 \\ \text{(by Lemma 6.4,} \\ \text{applied to } \lambda=\operatorname{type}\sigma)}} \qquad \left(\text{since } \zeta \text{ is } \mathbb{Z}\text{-linear}\right) \\
&= \sum_{\sigma\in\mathfrak{S}_V\left(D,\overline{D}\right)} (-1)^{\varphi(\sigma)}.
\end{aligned}$$

However, Lemma 6.5 yields

$$\zeta\left(U_D\right) = \left(\#\text{ of hamps of } \overline{D}\right).$$

Comparing these two equalities, we find

$$\left(\#\text{ of hamps of } \overline{D}\right) = \sum_{\sigma\in\mathfrak{S}_V\left(D,\overline{D}\right)} (-1)^{\varphi(\sigma)}.$$

This proves Theorem 6.6 **(a)**.

(b) Theorem 6.6 **(a)** yields

$$\left(\#\text{ of hamps of } \overline{D}\right) = \sum_{\sigma\in\mathfrak{S}_V\left(D,\overline{D}\right)} \underbrace{(-1)^{\varphi(\sigma)}}_{\substack{\equiv 1 \bmod 2 \\ \text{(since } (-1)^k \equiv 1 \bmod 2 \\ \text{for any } k\in\mathbb{Z})}} \equiv \sum_{\sigma\in\mathfrak{S}_V\left(D,\overline{D}\right)} 1 = \left|\mathfrak{S}_V\left(D,\overline{D}\right)\right| \bmod 2.$$

This proves Theorem 6.6 **(b)**. □

We are now ready to prove Rédei's and Berge's theorems:

Proof of Theorem 6.2. We have $\mathfrak{S}_V\left(\overline{D},D\right) = \mathfrak{S}_V\left(D,\overline{D}\right)$ (since the digraphs D and $\overline{D}$ play symmetric roles in the definition of $\mathfrak{S}_V\left(D,\overline{D}\right)$). However, it is also easy to see (using the definition of the complement of a digraph) that $\overline{\overline{D}} = D$.

Theorem 6.6 **(b)** yields

$$\left(\#\text{ of hamps of } \overline{D}\right) \equiv \left|\mathfrak{S}_V\left(D,\overline{D}\right)\right| \bmod 2. \tag{78}$$

However, Theorem 6.6 **(b)** (applied to $\overline{D}$ instead of D) yields

$$\left(\#\text{ of hamps of } \overline{\overline{D}}\right) \equiv \left|\mathfrak{S}_V\left(\overline{D},\overline{\overline{D}}\right)\right| \bmod 2.$$

We can rewrite this as

$$\left(\#\text{ of hamps of } D\right) \equiv \left|\mathfrak{S}_V\left(\overline{D},D\right)\right| \bmod 2$$

(since $\overline{\overline{D}} = D$). Hence,

$$\begin{aligned}
(\text{\# of hamps of } D) &\equiv \left|\mathfrak{S}_V\left(\overline{D}, D\right)\right| = \left|\mathfrak{S}_V\left(D, \overline{D}\right)\right| && \left(\text{since } \mathfrak{S}_V\left(\overline{D}, D\right) = \mathfrak{S}_V\left(D, \overline{D}\right)\right) \\
&\equiv \left(\text{\# of hamps of } \overline{D}\right) \bmod 2 && (\text{by (78)}).
\end{aligned}$$

This proves Theorem 6.2. □

Proof of Theorem 6.1. Write the tournament D as $D = (V, A)$. Set $n := |V|$.

For each $\sigma \in \mathfrak{S}_V$, let $\psi(\sigma)$ denote the number of nontrivial cycles of σ. Then, Theorem 1.39 yields

$$U_D = \sum_{\substack{\sigma \in \mathfrak{S}_V(D); \\ \text{all cycles of } \sigma \text{ have odd length}}} 2^{\psi(\sigma)} p_{\operatorname{type}\sigma}.$$

Hence,

$$\begin{aligned}
\zeta(U_D) &= \zeta\left(\sum_{\substack{\sigma \in \mathfrak{S}_V(D); \\ \text{all cycles of } \sigma \text{ have odd length}}} 2^{\psi(\sigma)} p_{\operatorname{type}\sigma}\right) \\
&= \sum_{\substack{\sigma \in \mathfrak{S}_V(D); \\ \text{all cycles of } \sigma \text{ have odd length}}} 2^{\psi(\sigma)} \underbrace{\zeta\left(p_{\operatorname{type}\sigma}\right)}_{\substack{=1 \\ \text{(by Lemma 6.4,} \\ \text{applied to } \lambda = \operatorname{type}\sigma)}} && (\text{since } \zeta \text{ is } \mathbb{Z}\text{-linear}) \\
&= \sum_{\substack{\sigma \in \mathfrak{S}_V(D); \\ \text{all cycles of } \sigma \text{ have odd length}}} 2^{\psi(\sigma)} \qquad (79) \\
&= \underbrace{2^{\psi(\mathrm{id}_V)}}_{\substack{=1 \\ (\text{since } \psi(\mathrm{id}_V)=0)}} + \sum_{\substack{\sigma \in \mathfrak{S}_V(D); \\ \text{all cycles of } \sigma \text{ have odd length;} \\ \sigma \neq \mathrm{id}_V}} \underbrace{2^{\psi(\sigma)}}_{\substack{\equiv 0 \bmod 2 \\ (\text{since } \psi(\sigma) \geq 1 \\ (\text{because } \sigma \neq \mathrm{id}_V \text{ shows} \\ \text{that } \sigma \text{ has at least} \\ \text{one nontrivial cycle}))}} \\
&\qquad \left(\begin{array}{c}\text{here, we have split off the addend for } \sigma = \mathrm{id}_V \\ \text{from the sum (since } \mathrm{id}_V \in \mathfrak{S}_V(D) \text{, and since} \\ \text{all cycles of } \mathrm{id}_V \text{ have odd length)}\end{array}\right) \\
&\equiv 1 + \underbrace{\sum_{\substack{\sigma \in \mathfrak{S}_V(D); \\ \text{all cycles of } \sigma \text{ have odd length;} \\ \sigma \neq \mathrm{id}_V}} 0}_{=0} = 1 \bmod 2.
\end{aligned}$$

In view of

$$\begin{aligned}
\zeta(U_D) &= \left(\text{\# of hamps of } \overline{D}\right) && (\text{by Lemma 6.5}) \\
&\equiv (\text{\# of hamps of } D) \bmod 2 && (\text{by Theorem 6.2}),
\end{aligned}$$

we can rewrite this as

$$(\text{\# of hamps of } D) \equiv 1 \bmod 2.$$

In other words, the # of hamps of D is odd. This proves Theorem 6.1. $\square$

7. A modulo-4 improvement of Redei's theorem

We can extend Redei's theorem (Theorem 6.1) to a somewhat stronger result:

Theorem 7.1. Let D be a tournament. Then,

$$(\text{\# of hamps of } D) \equiv 1 + 2\,(\text{\# of nontrivial odd } D\text{-cycles}) \bmod 4.$$

Here:

- We agree to consider the empty list $()$ as a hamp of the empty tournament with 0 vertices (even though it is not a path).
- We say that a D-cycle is *odd* if its length is odd.
- We say that a D-cycle is *nontrivial* if its length is > 1. (This was already said in Definition 1.23 **(e)**.)

To prove this, we shall need a simple lemma:

Lemma 7.2. Let $D = (V, A)$ be a digraph. For each $\sigma \in \mathfrak{S}_V$, let $\psi(\sigma)$ denote the number of nontrivial cycles of σ. Let $\mathfrak{S}_V^{\text{odd}}(D)$ denote the set of all permutations $\sigma \in \mathfrak{S}_V(D)$ such that all cycles of σ have odd length. Then,

$$\begin{aligned}&\left(\text{\# of permutations } \sigma \in \mathfrak{S}_V^{\text{odd}}(D) \text{ satisfying } \psi(\sigma) = 1\right)\\ &= (\text{\# of nontrivial odd } D\text{-cycles}).\end{aligned}$$

(We are here using the same notations as in Theorem 7.1.)

Proof. If $\gamma = (a_1, a_2, \ldots, a_k)_\sim$ is any D-cycle (or, more generally, any cycle of the digraph $(V,\ V \times V)$), then $\operatorname{perm}\gamma$ shall denote the permutation of V that sends the elements $a_1, a_2, \ldots, a_{k-1}, a_k$ to $a_2, a_3, \ldots, a_k, a_1$ (respectively) while leaving all other elements of V unchanged. (This permutation $\operatorname{perm}\gamma$ is what is usually called "the cycle $(a_1, a_2, \ldots, a_k)$" in group theory.)

If γ is any nontrivial D-cycle, then the permutation $\operatorname{perm}\gamma$ belongs to $\mathfrak{S}_V(D)$ (since its only nontrivial cycle is γ, which is a D-cycle) and satisfies $\psi(\operatorname{perm}\gamma) = 1$ (by the definition of $\psi(\operatorname{perm}\gamma)$). Moreover, if γ is a nontrivial **odd** D-cycle, then this permutation $\operatorname{perm}\gamma$ furthermore has the property that all its cycles have odd length (since its only nontrivial cycle γ is odd, whereas its trivial cycles have length

1, which is also odd), i.e., belongs to $\mathfrak{S}_V^{\operatorname{odd}}(D)$ (since we know that it belongs to $\mathfrak{S}_V(D)$). Thus, we obtain a map

$$\begin{aligned}&\text{from } \{\text{nontrivial odd } D\text{-cycles}\}\\&\text{to } \left\{\text{permutations } \sigma \in \mathfrak{S}_V^{\operatorname{odd}}(D) \text{ satisfying } \psi(\sigma) = 1\right\}\end{aligned}$$

which sends each nontrivial odd D-cycle γ to the permutation $\operatorname{perm}\gamma$. This map is furthermore injective (because any distinct nontrivial D-cycles γ and δ will always give rise to different permutations $\operatorname{perm}\gamma$ and $\operatorname{perm}\delta$) and surjective[60]. Thus, this map is bijective. Hence, the bijection principle yields

$$\begin{aligned}&(\#\text{ of nontrivial odd } D\text{-cycles})\\&= \left(\#\text{ of permutations } \sigma \in \mathfrak{S}_V^{\operatorname{odd}}(D) \text{ satisfying } \psi(\sigma) = 1\right).\end{aligned}$$

This proves Lemma 7.2. □

We can now prove Theorem 7.1:

Proof of Theorem 7.1. We use the same notations as in Section 6. Write the tournament D as $D = (V, A)$.

For each $\sigma \in \mathfrak{S}_V$, let $\psi(\sigma)$ denote the number of nontrivial cycles of σ. Let $\mathfrak{S}_V^{\operatorname{odd}}(D)$ denote the set of all permutations $\sigma \in \mathfrak{S}_V(D)$ such that all cycles of σ have odd length. Note that the identity permutation id_V belongs to $\mathfrak{S}_V^{\operatorname{odd}}(D)$, since all its cycles are trivial.

[60]*Proof.* If $\sigma \in \mathfrak{S}_V(D)$ is a permutation satisfying $\psi(\sigma) = 1$, then $\sigma = \operatorname{perm}\gamma$ where γ is the unique nontrivial cycle of σ. Moreover, this cycle γ is a D-cycle (since $\sigma \in \mathfrak{S}_V(D)$). If we furthermore assume that $\sigma \in \mathfrak{S}_V^{\operatorname{odd}}(D)$, then this cycle γ has odd length (since $\sigma \in \mathfrak{S}_V^{\operatorname{odd}}(D)$ entails that all cycles of σ have odd length), i.e., is odd.

Then, from (79), we have

$$\zeta(U_D) = \sum_{\substack{\sigma\in\mathfrak{S}_V(D);\\ \text{all cycles of } \sigma \text{ have odd length}}} 2^{\psi(\sigma)} = \sum_{\sigma\in\mathfrak{S}_V^{\text{odd}}(D)} 2^{\psi(\sigma)}$$

$$\left(\begin{array}{c}\text{since the permutations } \sigma\in\mathfrak{S}_V(D) \text{ such that all cycles}\\ \text{of } \sigma \text{ have odd length are precisely the elements of } \mathfrak{S}_V^{\text{odd}}(D)\end{array}\right)$$

$$\equiv \sum_{\substack{\sigma\in\mathfrak{S}_V^{\text{odd}}(D);\\ \psi(\sigma)=0}} \underbrace{2^{\psi(\sigma)}}_{\substack{=1\\ (\text{since } \psi(\sigma)=0)}} + \sum_{\substack{\sigma\in\mathfrak{S}_V^{\text{odd}}(D);\\ \psi(\sigma)=1}} \underbrace{2^{\psi(\sigma)}}_{\substack{=2\\ (\text{since } \psi(\sigma)=1)}} + \sum_{\substack{\sigma\in\mathfrak{S}_V^{\text{odd}}(D);\\ \psi(\sigma)\geq 2}} \underbrace{2^{\psi(\sigma)}}_{\substack{\equiv 0 \bmod 4\\ (\text{since } \psi(\sigma)\geq 2)}}$$

$$\left(\begin{array}{c}\text{here, we have split our sum according to}\\ \text{whether } \psi(\sigma) \text{ is 0 or 1 or } \geq 2\end{array}\right)$$

$$\equiv \sum_{\substack{\sigma\in\mathfrak{S}_V^{\text{odd}}(D);\\ \psi(\sigma)=0}} 1 + \sum_{\substack{\sigma\in\mathfrak{S}_V^{\text{odd}}(D);\\ \psi(\sigma)=1}} 2 + \underbrace{\sum_{\substack{\sigma\in\mathfrak{S}_V^{\text{odd}}(D);\\ \psi(\sigma)\geq 2}} 0}_{=0}$$

$$= \underbrace{\sum_{\substack{\sigma\in\mathfrak{S}_V^{\text{odd}}(D);\\ \psi(\sigma)=0}} 1}_{=\left(\#\text{ of permutations } \sigma\in\mathfrak{S}_V^{\text{odd}}(D) \text{ satisfying } \psi(\sigma)=0\right)\cdot 1} + \underbrace{\sum_{\substack{\sigma\in\mathfrak{S}_V^{\text{odd}}(D);\\ \psi(\sigma)=1}} 2}_{=\left(\#\text{ of permutations } \sigma\in\mathfrak{S}_V^{\text{odd}}(D) \text{ satisfying } \psi(\sigma)=1\right)\cdot 2}$$

$$= \underbrace{\left(\#\text{ of permutations } \sigma\in\mathfrak{S}_V^{\text{odd}}(D) \text{ satisfying } \psi(\sigma)=0\right)}_{\substack{=1\\ (\text{since the only permutation } \sigma\in\mathfrak{S}_V^{\text{odd}}(D)\\ \text{satisfying } \psi(\sigma)=0 \text{ is the identity permutation})}} \cdot 1$$

$$+ \underbrace{\left(\#\text{ of permutations } \sigma\in\mathfrak{S}_V^{\text{odd}}(D) \text{ satisfying } \psi(\sigma)=1\right)}_{\substack{=(\#\text{ of nontrivial odd } D\text{-cycles})\\ (\text{by Lemma 7.2})}} \cdot 2$$

$$= 1\cdot 1 + (\#\text{ of nontrivial odd } D\text{-cycles})\cdot 2$$
$$= 1 + 2\,(\#\text{ of nontrivial odd } D\text{-cycles}) \bmod 4.$$

Comparing this with

$$\zeta(U_D) = \left(\#\text{ of hamps of } \overline{D}\right) \qquad (\text{by Lemma 6.5}),$$

we obtain

$$\left(\#\text{ of hamps of } \overline{D}\right) \equiv 1 + 2\,(\#\text{ of nontrivial odd } D\text{-cycles}) \bmod 4. \tag{80}$$

However, recall that D is a tournament. Hence, the tournament axiom shows that a pair (u, v) of two distinct elements of V is an arc of D if and only if the pair (v, u) is not. In other words, a pair (u, v) of two distinct elements of V is an arc of D if and only if the pair (v, u) is an arc of $\overline{D}$. Thus, if $v = (v_1, v_2, \ldots, v_k)$ is a hamp of D, then its reversal $\operatorname{rev} v = (v_k, v_{k-1}, \ldots, v_1)$ is a hamp of $\overline{D}$. Hence, we obtain a map

$$\begin{aligned} \{\text{hamps of } D\} &\to \{\text{hamps of } \overline{D}\}, \\ v &\mapsto \operatorname{rev} v. \end{aligned}$$

This map is furthermore easily seen to be injective and surjective. Hence, it is bijective. Thus, the bijection principle yields

$$\begin{aligned} (\text{\# of hamps of } D) &= (\text{\# of hamps of } \overline{D}) \\ &\equiv 1 + 2\,(\text{\# of nontrivial odd } D\text{-cycles}) \bmod 4 \end{aligned}$$

(by (80)). This proves Theorem 7.1. $\square$

8. The antipode and the omega involution

Next, we will discuss how the Redei-Berge symmetric functions U_D interplay with two well-known involutions on the ring Λ: the omega involution ω and the antipode map S.

We shall not recall the standard definitions of these involutions ω and S (see, e.g., [GriRei20, §2.4]); however, we shall briefly state the few properties that will be used in what follows. Both the *omega involution* ω and the *antipode* S of Λ are endomorphisms of the $\mathbb{Z}$-algebra Λ; they satisfy the equalities

$$S(p_n) = -p_n \tag{81}$$

and

$$\omega(p_n) = (-1)^{n-1} p_n \tag{82}$$

for every positive integer n (see [GriRei20, Proposition 2.4.1 (i)] and [GriRei20, Proposition 2.4.3 (c)]). Moreover, if $f \in \Lambda$ is a homogeneous power series of degree n, then

$$S(f) = (-1)^n \omega(f) \tag{83}$$

(this is [GriRei20, Proposition 2.4.3 (e)]). We now claim the following theorem:

Theorem 8.1. Let $D = (V, A)$ be a digraph. Then,

$$\omega(U_D) = U_{\overline{D}}. \tag{84}$$

Furthermore, if $n := |V|$, then

$$S(U_D) = (-1)^n U_{\overline{D}}. \tag{85}$$

Proof. The definition of $\overline{D}$ yields that $\overline{\overline{D}} = D$. Hence, the definition of $\mathfrak{S}_V\left(D, \overline{D}\right)$ yields that $\mathfrak{S}_V\left(\overline{D}, D\right) = \mathfrak{S}_V\left(D, \overline{D}\right)$.

For each $\sigma \in \mathfrak{S}_V$, we set

$$\varphi(\sigma) := \sum_{\substack{\gamma \in \operatorname{Cycs}\sigma; \\ \gamma \text{ is a } D\text{-cycle}}} (\ell(\gamma) - 1) \qquad \text{and} \qquad \overline{\varphi}(\sigma) := \sum_{\substack{\gamma \in \operatorname{Cycs}\sigma; \\ \gamma \text{ is a } \overline{D}\text{-cycle}}} (\ell(\gamma) - 1).$$

Now, it is easy to see that

$$\omega\left((-1)^{\varphi(\sigma)} p_{\operatorname{type}\sigma}\right) = (-1)^{\overline{\varphi}(\sigma)} p_{\operatorname{type}\sigma} \tag{86}$$

for each $\sigma \in \mathfrak{S}_V\left(D, \overline{D}\right)$.

[*Proof of (86):* Let $\sigma \in \mathfrak{S}_V\left(D, \overline{D}\right)$. Let $k_1, k_2, \ldots, k_s$ be the lengths of all cycles of σ, listed in decreasing order. Then, the definition of $\operatorname{type}\sigma$ yields $\operatorname{type}\sigma = (k_1, k_2, \ldots, k_s)$. Hence,

$$p_{\operatorname{type}\sigma} = p_{(k_1, k_2, \ldots, k_s)} = p_{k_1} p_{k_2} \cdots p_{k_s} = \prod_{\gamma \in \operatorname{Cycs}\sigma} p_{\ell(\gamma)} \tag{87}$$

(since $k_1, k_2, \ldots, k_s$ are the lengths of all cycles of σ). Hence,

$$\begin{aligned}
\omega\left((-1)^{\varphi(\sigma)} p_{\operatorname{type}\sigma}\right) &= \omega\left((-1)^{\varphi(\sigma)} \prod_{\gamma \in \operatorname{Cycs}\sigma} p_{\ell(\gamma)}\right) \\
&= (-1)^{\varphi(\sigma)} \prod_{\gamma \in \operatorname{Cycs}\sigma} \underbrace{\omega\left(p_{\ell(\gamma)}\right)}_{\substack{= (-1)^{\ell(\gamma)-1} p_{\ell(\gamma)} \\ \text{(by (82))}}} \qquad \left(\begin{array}{c}\text{since } \omega \text{ is a } \mathbb{Z}\text{-algebra} \\ \text{homomorphism}\end{array}\right) \\
&= (-1)^{\varphi(\sigma)} \prod_{\gamma \in \operatorname{Cycs}\sigma} \left((-1)^{\ell(\gamma)-1} p_{\ell(\gamma)}\right) \\
&= (-1)^{\varphi(\sigma)} \underbrace{\left(\prod_{\gamma \in \operatorname{Cycs}\sigma} (-1)^{\ell(\gamma)-1}\right)}_{= (-1)^{\sum_{\gamma \in \operatorname{Cycs}\sigma} (\ell(\gamma)-1)}} \underbrace{\prod_{\gamma \in \operatorname{Cycs}\sigma} p_{\ell(\gamma)}}_{\substack{= p_{\operatorname{type}\sigma} \\ \text{(by (87))}}} \\
&= (-1)^{\varphi(\sigma)} (-1)^{\sum_{\gamma \in \operatorname{Cycs}\sigma} (\ell(\gamma)-1)} p_{\operatorname{type}\sigma}.
\end{aligned} \tag{88}$$

However, each $\gamma \in \operatorname{Cycs}\sigma$ is either a D-cycle or a $\overline{D}$-cycle (since $\sigma \in \mathfrak{S}_V\left(D, \overline{D}\right)$), but cannot be both at the same time (since D and $\overline{D}$ have no arcs in common). Thus,

$$\begin{aligned}
\sum_{\gamma \in \operatorname{Cycs}\sigma} (\ell(\gamma) - 1) &= \underbrace{\sum_{\substack{\gamma \in \operatorname{Cycs}\sigma; \\ \gamma \text{ is a } D\text{-cycle}}} (\ell(\gamma) - 1)}_{\substack{= \varphi(\sigma) \\ \text{(by the definition of } \varphi(\sigma))}} + \underbrace{\sum_{\substack{\gamma \in \operatorname{Cycs}\sigma; \\ \gamma \text{ is a } \overline{D}\text{-cycle}}} (\ell(\gamma) - 1)}_{\substack{= \overline{\varphi}(\sigma) \\ \text{(by the definition of } \overline{\varphi}(\sigma))}} \\
&= \varphi(\sigma) + \overline{\varphi}(\sigma).
\end{aligned}$$

Thus, (88) rewrites as

$$\omega\left((-1)^{\varphi(\sigma)}\,p_{\operatorname{type}\sigma}\right) = \underbrace{(-1)^{\varphi(\sigma)}\,(-1)^{\varphi(\sigma)+\overline{\varphi}(\sigma)}}_{=(-1)^{\overline{\varphi}(\sigma)}}\,p_{\operatorname{type}\sigma} = (-1)^{\overline{\varphi}(\sigma)}\,p_{\operatorname{type}\sigma}.$$

This proves (86).]

Now, Theorem 1.31 yields

$$U_D = \sum_{\sigma\in\mathfrak{S}_V\left(D,\overline{D}\right)} (-1)^{\varphi(\sigma)}\,p_{\operatorname{type}\sigma}. \tag{89}$$

Also, Theorem 1.31 (applied to $\overline{D}$, $(V\times V)\setminus A$ and $\overline{\varphi}$ instead of D, A and φ) yields

$$\begin{aligned}
U_{\overline{D}} &= \sum_{\sigma\in\mathfrak{S}_V\left(\overline{D},D\right)} (-1)^{\overline{\varphi}(\sigma)}\,p_{\operatorname{type}\sigma} \\
&= \sum_{\sigma\in\mathfrak{S}_V\left(D,\overline{D}\right)} \underbrace{(-1)^{\overline{\varphi}(\sigma)}\,p_{\operatorname{type}\sigma}}_{\substack{=\omega\left((-1)^{\varphi(\sigma)}p_{\operatorname{type}\sigma}\right)\\ \text{(by (86))}}} \qquad \left(\text{since } \mathfrak{S}_V\left(\overline{D},D\right) = \mathfrak{S}_V\left(D,\overline{D}\right)\right) \\
&= \sum_{\sigma\in\mathfrak{S}_V\left(D,\overline{D}\right)} \omega\left((-1)^{\varphi(\sigma)}\,p_{\operatorname{type}\sigma}\right) \\
&= \omega\left(\underbrace{\sum_{\sigma\in\mathfrak{S}_V\left(D,\overline{D}\right)} (-1)^{\varphi(\sigma)}\,p_{\operatorname{type}\sigma}}_{\substack{=U_D\\ \text{(by (89))}}}\right) \qquad (\text{since } \omega \text{ is } \mathbb{Z}\text{-linear}) \\
&= \omega\left(U_D\right).
\end{aligned}$$

This proves (84).

Now, let $n := |V|$. Then, the definition of U_D easily yields that U_D is homogeneous of degree n. Hence, (83) (applied to $f = U_D$) yields

$$S\left(U_D\right) = (-1)^n\,\omega\left(U_D\right) = (-1)^n\,U_{\overline{D}} \qquad (\text{by (84)}).$$

Thus, (85) is proved. This completes the proof of Theorem 8.1. □

Theorem 8.1 can also be proved directly from the definition of U_D, using the formula for the antipode of a fundamental quasisymmetric function ([GriRei20, (5.2.7)]). Indeed, three different proofs of Theorem 8.1 (specifically, of (84)) are found in [Chow96] (where (84) appears as [Chow96, Corollary 2]), one of which is doing just this. A fourth proof can be found in [Wisema07, (6)].

We can use Theorem 8.1 to give a new proof of Berge's theorem (Theorem 6.2). For this purpose, we recall the $\mathbb{Z}$-algebra homomorphism ζ introduced in Section 6. We need another simple property of this ζ:

Lemma 8.2. Let $f \in \mathbb{Z}[p_1, p_2, p_3, \ldots]$. Then, $\zeta(\omega(f)) \equiv \zeta(f) \operatorname{mod} 2$.

Proof. Let $\pi : \mathbb{Z} \to \mathbb{Z}/2$ be the projection map that sends each integer to its congruence class modulo 2. This π is a $\mathbb{Z}$-algebra homomorphism.

For each positive integer n, we have

$$\begin{aligned} \zeta(\omega(p_n)) &= \zeta\left((-1)^{n-1} p_n\right) && \text{(by (82))} \\ &= \underbrace{(-1)^{n-1}}_{\equiv 1 \operatorname{mod} 2} \zeta(p_n) && \text{(since } \zeta \text{ is } \mathbb{Z}\text{-linear)} \\ &\equiv \zeta(p_n) \operatorname{mod} 2 \end{aligned}$$

and thus

$$\pi(\zeta(\omega(p_n))) = \pi(\zeta(p_n))$$

(since two integers a and b satisfy $a \equiv b \operatorname{mod} 2$ if and only if $\pi(a) = \pi(b)$). In other words, for each positive integer n, we have

$$(\pi \circ \zeta \circ \omega)(p_n) = (\pi \circ \zeta)(p_n).$$

In other words, the two maps $\pi \circ \zeta \circ \omega$ and $\pi \circ \zeta$ agree on each of the generators $p_1, p_2, p_3, \ldots$ of the $\mathbb{Z}$-algebra $\mathbb{Z}[p_1, p_2, p_3, \ldots]$. Since these two maps are $\mathbb{Z}$-algebra homomorphisms (because π, ζ and ω are $\mathbb{Z}$-algebra homomorphisms), this shows that these two maps agree on the entire $\mathbb{Z}$-algebra $\mathbb{Z}[p_1, p_2, p_3, \ldots]$. Hence, $(\pi \circ \zeta \circ \omega)(f) = (\pi \circ \zeta)(f)$. In other words, $\pi(\zeta(\omega(f))) = \pi(\zeta(f))$. In other words, $\zeta(\omega(f)) \equiv \zeta(f) \operatorname{mod} 2$ (since two integers a and b satisfy $a \equiv b \operatorname{mod} 2$ if and only if $\zeta(a) = \zeta(b)$). This proves Lemma 8.2. $\square$

Second proof of Theorem 6.2. From (84), we obtain $\omega(U_D) = U_{\overline{D}}$.

Corollary 1.35 yields $U_D \in \mathbb{Z}[p_1, p_2, p_3, \ldots]$. Hence, Lemma 8.2 (applied to $f = U_D$) yields that

$$\zeta(\omega(U_D)) \equiv \zeta(U_D) \operatorname{mod} 2.$$

In view of

$$\zeta(U_D) = \left(\#\text{ of hamps of } \overline{D}\right) \qquad \text{(by Lemma 6.5)}$$

and

$$\begin{aligned} \zeta\left(\underbrace{\omega(U_D)}_{=U_{\overline{D}}}\right) &= \zeta\left(U_{\overline{D}}\right) = \left(\#\text{ of hamps of } \overline{\overline{D}}\right) && \left(\begin{array}{c}\text{by Lemma 6.5,} \\ \text{applied to } \overline{D} \text{ instead of } D\end{array}\right) \\ &= (\#\text{ of hamps of } D) && \left(\text{since } \overline{\overline{D}} = D\right), \end{aligned}$$

we can rewrite this as

$$(\#\text{ of hamps of } D) \equiv \left(\#\text{ of hamps of } \overline{D}\right) \operatorname{mod} 2.$$

This proves Theorem 6.2 again. $\square$

9. A multiparameter deformation

Let us now briefly discuss a multiparameter deformation of the Redei-Berge symmetric functions U_D, which replaces the digraph D by an arbitrary matrix.

We fix a commutative ring $\mathbf{k}$, which we shall now be using instead of $\mathbb{Z}$ as a base ring for our power series.

We fix an $n \in \mathbb{N}$, and a set V with n elements.

For any $a \in V \times V$, we fix an element $t_a \in \mathbf{k}$. (Thus, the family $\left(t_{(i,j)}\right)_{i,j \in V}$ of these elements can be viewed as a $V \times V$-matrix.)

For any $a \in V \times V$, we set $s_a := t_a + 1 \in \mathbf{k}$.

The following definition is inspired by a comment from Mike Zabrocki:

Definition 9.1. We define the *deformed Redei–Berge symmetric function* $\widetilde{U}_t$ to be the formal power series

$$\widetilde{U}_t = \sum_{\substack{w=(w_1,w_2,\ldots,w_n) \\ \text{is a } V\text{-listing}}} \sum_{i_1 \leq i_2 \leq \cdots \leq i_n} \left(\prod_{\substack{k \in [n-1]; \\ i_k = i_{k+1}}} s_{(w_k, w_{k+1})} \right) x_{i_1} x_{i_2} \cdots x_{i_n}$$
$$\in \mathbf{k}\left[\left[x_1, x_2, x_3, \ldots\right]\right].$$

For example, if $n = 2$ and $V = \{1, 2\}$, then

$$\begin{aligned}
\widetilde{U}_t &= \sum_{i_1 < i_2} x_{i_1} x_{i_2} + \sum_{i_1 = i_2} t_{(1,2)} x_{i_1} x_{i_2} + \sum_{i_1 < i_2} x_{i_1} x_{i_2} + \sum_{i_1 = i_2} t_{(2,1)} x_{i_1} x_{i_2} \\
&= \sum_{i<j} x_i x_j + \sum_i t_{(1,2)} x_i^2 + \sum_{i<j} x_i x_j + \sum_i t_{(2,1)} x_i^2 \\
&= p_1^2 + \left(s_{(1,2)} + s_{(2,1)} - 1\right) p_2 \\
&= p_1^2 + \left(t_{(1,2)} + t_{(2,1)} + 1\right) p_2
\end{aligned}$$

For a more complicated example, if $n = 3$ and $V = \{1, 2, 3\}$, then a longer computation shows that

$$\begin{aligned}
\widetilde{U}_t &= p_1^3 + \left(s_{(1,2)} + s_{(2,1)} + s_{(1,3)} + s_{(3,1)} + s_{(2,3)} + s_{(3,2)} - 3\right) p_2 p_1 \\
&\qquad + \Big(s_{(1,2)} s_{(2,3)} + s_{(2,3)} s_{(3,1)} + s_{(3,1)} s_{(1,2)} \\
&\qquad\qquad + s_{(1,3)} s_{(3,2)} + s_{(3,2)} s_{(2,1)} + s_{(2,1)} t_{(1,3)} \\
&\qquad\qquad - s_{(1,2)} - s_{(2,1)} - s_{(1,3)} - s_{(3,1)} - s_{(2,3)} - s_{(3,2)} + 2\Big) p_3 \\
&= p_1^3 + \left(t_{(1,2)} + t_{(2,1)} + t_{(1,3)} + t_{(3,1)} + t_{(2,3)} + t_{(3,2)} + 3\right) p_2 p_1 \\
&\qquad + \Big(t_{(1,2)} t_{(2,3)} + t_{(2,3)} t_{(3,1)} + t_{(3,1)} t_{(1,2)} \\
&\qquad\qquad + t_{(1,3)} t_{(3,2)} + t_{(3,2)} t_{(2,1)} + t_{(2,1)} t_{(1,3)} \\
&\qquad\qquad + t_{(1,2)} + t_{(2,1)} + t_{(1,3)} + t_{(3,1)} + t_{(2,3)} + t_{(3,2)} + 2\Big) p_3 .
\end{aligned}$$

Why are we calling $\widetilde{U}_t$ a deformation of U_D ?

Example 9.2. Let $D = (V, A)$ be a digraph. Set $\mathbf{k} = \mathbb{Z}$, and let

$$t_a := \begin{cases} -1, & \text{if } a \in A; \\ 0, & \text{if } a \notin A \end{cases} \qquad \text{for each } a \in V \times V.$$

Then, $\widetilde{U}_t = U_D$, as can be seen by comparing the definitions.

All the above results leading up to Theorem 1.31 can be extended to this deformation, culminating in the following deformation of Theorem 1.31:

Theorem 9.3. We have

$$\widetilde{U}_t = \sum_{\sigma \in \mathfrak{S}_V} \left(\prod_{\gamma \text{ is a cycle of } \sigma} \left(\prod_{a \in \operatorname{CArcs} \gamma} s_a - \prod_{a \in \operatorname{CArcs} \gamma} t_a \right) \right) p_{\operatorname{type} \sigma}.$$

Alternatively, Theorem 9.3 can be deduced from Theorem 1.31 via the "multilinearity trick": View each t_a as an indeterminate, and argue that both sides in Theorem 9.3 are polynomials in degree ≤ 1 in these indeterminates (over the base ring $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$). Thus, in order to prove their equality, it suffices to prove that they are equal when each t_a is specialized to either 0 or -1. But this is precisely the claim of Theorem 1.31. (Thus, Theorem 9.3 is not essentially more general than Theorem 1.31.)

Theorem 9.3 shows that the $\widetilde{U}_t$ are p-integral symmetric functions (taking the t_a as "integers"). There do not seem to be any good opportunities for generalizing any of Theorem 1.39 and Theorem 1.41, however.

References

[Berge76] Claude Berge, *Graphs and Hypergraphs*, North-Holland Mathematical Library **6**, 2nd edition, North-Holland 1976.

[Berge91] Claude Berge, *Graphs*, North-Holland Mathematical Library **6.1**, 3rd edition, North-Holland 1991.

[Chow96] Timothy Y. Chow, *The Path-Cycle Symmetric Function of a Digraph*, Advances in Mathematics **118** (1996), pp. 71–98. `http://timothychow.net/pathcycle.pdf`

[EC2sup22] Richard P. Stanley, *Supplementary problems for Chapter 7 of "Enumerative Combinatorics"*, `https://math.mit.edu/~rstan/ec/`

[FoaZei96] Dominique Foata, Doron Zeilberger, *Graphical major indices*, J. Comput. Appl. Math. **68** (1996), no. 1-2, pp. 79–101. `https://doi.org/10.1016/0377-0427(95)00254-5`

[Grinbe17] Darij Grinberg, *UMN, Spring 2017, Math 5707: Lecture 7 (Hamiltonian paths in digraphs)*, 14 May 2022.
`https://www.cip.ifi.lmu.de/~grinberg/t/17s/5707lec7.pdf`

[Grinbe21] Darij Grinberg, *Notes on mathematical problem solving*, 10 February 2021.
`http://www.cip.ifi.lmu.de/~grinberg/t/20f/mps.pdf`

[GriRei20] Darij Grinberg, Victor Reiner, *Hopf algebras in Combinatorics*, version of 27 July 2020, `arXiv:1409.8356v7`.

[GriSta23] Darij Grinberg, Richard P. Stanley, *The Redei–Berge symmetric functions*, standard version of the present paper. Available as main version of this arXiv preprint, or at `http://www.cip.ifi.lmu.de/~grinberg/algebra/redeiberge.pdf`

[Lass02] Bodo Lass, *Variations sur le thème E+E=XY*, Advances in Applied Mathematics **29** (2002), Issue 2, pp. 215–242. `https://doi.org/10.1016/S0196-8858(02)00010-6`

[Lovasz07] László Lovász, *Combinatorial Problems and Exercises*, 2nd edition, AMS 2007.

[MO232751] bof and Gordon Royle, *MathOverflow question #232751 ("The number of Hamiltonian paths in a tournament")*.
`https://mathoverflow.net/questions/232751/the-number-of-hamiltonian-paths-in-a-tournament`

[Moon13] John W. Moon, *Topics on Tournaments*, Project Gutenberg EBook, 5 June 2013.
`https://www.gutenberg.org/ebooks/42833`

[Redei33] László Rédei, *Ein kombinatorischer Satz*, Acta Litteraria Szeged **7** (1934), pp. 39–43.

[Stanle01] Richard P. Stanley, *Enumerative Combinatorics, volume 2*, First edition, Cambridge University Press 2001.
See `http://math.mit.edu/~rstan/ec/` for errata.

[Tomesc85] Ioan Tomescu, *Problems in Combinatorics and Graph Theory*, Wiley 1985.

[Wisema07] Gus Wiseman, *Enumeration of paths and cycles and e-coefficients of incomparability graphs*, arXiv:0709.0430v1. `https://arxiv.org/abs/0709.0430v1`