

84th Putnam contest 2023 problem B6 with solution

Darij Grinberg

detailed version

December 2, 2023

0.1. Problem B6

The following is Problem B6 on the 84th Putnam contest 2023 in the form I proposed it (which is slightly easier than what made it on the contest, because the answer is provided).

Problem 1. Let m be a positive integer. Let n be either $2m$ or $2m - 1$.

Let A be the $n \times n$ -matrix $(a_{i,j})_{1 \leq i \leq n, 1 \leq j \leq n'}$ where $a_{i,j}$ is the number of all pairs (x, y) of nonnegative integers satisfying $xi + yj = n$.

Prove that $\det A = (-1)^{m-1} \cdot 2m$.

(For instance, for $m = 3$ and $n = 5$, we have $A = \begin{pmatrix} 6 & 3 & 2 & 2 & 2 \\ 3 & 0 & 1 & 0 & 1 \\ 2 & 1 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 & 1 \\ 2 & 1 & 1 & 1 & 2 \end{pmatrix}$.)

Remark 0.1. This is likely connected to gcd-matrices and Möbius inversion.

Solution to Problem 1. By assumption, n is either $2m$ or $2m - 1$. Hence, m is either $n/2$ or $(n + 1)/2$. Thus, it easily follows that $0 \leq m \leq n$ and $m - 1 < n/2$.

We shall use the *Iverson bracket notation*: If \mathcal{A} is a logical statement, then $[\mathcal{A}]$ shall mean $\begin{cases} 1, & \text{if } \mathcal{A} \text{ is true;} \\ 0, & \text{if } \mathcal{A} \text{ is false.} \end{cases}$

We set $[n] := \{1, 2, \dots, n\}$ and $[0, n] := \{0, 1, \dots, n\}$.

If M is any matrix, then we will use the notation $M_{i,j}$ for the entry of M in the i -th row and the j -th column. Thus, in particular, the entries of our matrix A are $A_{i,j} = a_{i,j}$ for all $i, j \in [n]$.

We shall consider $n \times (n+1)$ -matrices whose rows are indexed by $1, 2, \dots, n$ and whose columns are indexed by $0, 1, \dots, n$. Likewise, we shall consider $(n+1) \times n$ -matrices whose rows are indexed by $0, 1, \dots, n$ and whose columns are indexed by $1, 2, \dots, n$.

If M is an $n \times (n+1)$ -matrix, and $j \in [0, n]$, then $M^{\sim j}$ shall denote the $n \times n$ -matrix obtained from M by removing the j -th column. Likewise, if N is an $(n+1) \times n$ -matrix, and $i \in [0, n]$, then $N_{\sim i}$ shall denote the $n \times n$ -matrix obtained from N by removing the i -th column. The Cauchy–Binet formula (or, more precisely, a particular case thereof) says that if M is an $n \times (n+1)$ -matrix and N is an $(n+1) \times n$ -matrix, then

$$\det(MN) = \sum_{k=0}^n \det(M^{\sim k}) \cdot \det(N_{\sim k}). \quad (1)$$

Now, let M be the $n \times (n+1)$ -matrix $([i | j])_{i \in [n], j \in [0, n]}$ (so that its (i, j) -entry is 1 if $i | j$ and 0 otherwise). Let N be the $(n+1) \times n$ -matrix $([j | n-i])_{i \in [0, n], j \in [n]}$. For example, for $n = 8$, we have

$$M = \begin{pmatrix} & j=0 & j=1 & j=2 & j=3 & j=4 & j=5 & j=6 & j=7 & j=8 \\ i=1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ i=2 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ i=3 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ i=4 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ i=5 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ i=6 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ i=7 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ i=8 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$N = \begin{pmatrix} & j=1 & j=2 & j=3 & j=4 & j=5 & j=6 & j=7 & j=8 \\ i=0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ i=1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ i=2 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ i=3 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ i=4 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ i=5 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ i=6 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ i=7 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ i=8 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \dots$$

Now, it is easy to see that

$$A = MN. \quad (2)$$

(Indeed, for each $i, j \in [n]$, we have

$$\begin{aligned}
 (MN)_{i,j} &= \sum_{k=0}^n M_{i,k} N_{k,j} && \text{(by the definition of matrix multiplication)} \\
 &= \sum_{k=0}^n \underbrace{[i \mid k] [j \mid n-k]}_{=[i \mid k \text{ and } j \mid n-k]} && \text{(by the definitions of } M \text{ and } N) \\
 &= \sum_{k=0}^n [i \mid k \text{ and } j \mid n-k] \\
 &= (\text{number of } k \in [0, n] \text{ satisfying } i \mid k \text{ and } j \mid n-k) \\
 &= (\text{number of pairs } (x, y) \text{ of nonnegative integers satisfying } xi + yj = n) \\
 &\quad \left(\begin{array}{c} \text{by a simple bijection, explained in} \\ \text{the footnote following this computation} \end{array} \right) \\
 &= a_{i,j} && \text{(by the definition of } a_{i,j}) \\
 &= A_{i,j}.
 \end{aligned}$$

¹ Thus, $MN = A$, so that $A = MN$.)

¹Let me justify the equality

$$\begin{aligned}
 &(\text{number of } k \in [0, n] \text{ satisfying } i \mid k \text{ and } j \mid n-k) \\
 &= (\text{number of pairs } (x, y) \text{ of nonnegative integers satisfying } xi + yj = n)
 \end{aligned}$$

that was used above. Clearly, it suffices to find a bijection

$$\begin{aligned}
 &\text{from the set } \mathfrak{A} := \{k \in [0, n] \text{ satisfying } i \mid k \text{ and } j \mid n-k\} \\
 &\text{to the set } \mathfrak{B} := \{\text{pairs } (x, y) \text{ of nonnegative integers satisfying } xi + yj = n\}.
 \end{aligned}$$

We can construct such a bijection as follows:

- For each $k \in \mathfrak{A}$, we have $(k/i, (n-k)/j) \in \mathfrak{B}$ (since $k \in \mathfrak{A}$ entails $k \in [0, n]$ and $i \mid k$ and $j \mid n-k$ and therefore $k/i \in \mathbb{N}$ and $(n-k)/j \in \mathbb{N}$, and of course we have $(k/i)i + ((n-k)/j)j = k + (n-k) = n$). Thus, we can define a map

$$\begin{aligned}
 \phi : \mathfrak{A} &\rightarrow \mathfrak{B}, \\
 k &\mapsto (k/i, (n-k)/j).
 \end{aligned}$$

- For each $(x, y) \in \mathfrak{B}$, we have $xi \in \mathfrak{A}$ (in fact, $(x, y) \in \mathfrak{B}$ entails $x, y \in \mathbb{N}$ and $xi + yj = n$; now we clearly have $i \mid xi$ and $j \mid yj = \underbrace{xi + yj}_{=n} - xi = n - xi$ and finally $xi \in [0, n]$ (since $xi \geq 0$ and $xi \leq xi + yj = n$); but these three facts combined yield $xi \in \mathfrak{A}$). Thus, we can define a map

$$\begin{aligned}
 \psi : \mathfrak{B} &\rightarrow \mathfrak{A}, \\
 (x, y) &\mapsto xi.
 \end{aligned}$$

- The maps ϕ and ψ are easily seen to be mutually inverse (indeed, $\psi \circ \phi = \text{id}$ is obvious; and $\phi \circ \psi = \text{id}$ can be proved as follows: if $(x, y) \in \mathfrak{B}$, then $xi + yj =$

Now, from (2), we obtain

$$\det A = \det(MN) = \sum_{k=0}^n \det(M^{\sim k}) \cdot \det(N_{\sim k}) \quad (3)$$

(by (1)). It remains to manage the right hand side. We will need several claims:

Claim 1: Let $k \in [0, n]$ satisfy $k > \frac{n}{2}$. Then,

$$\det(M^{\sim k}) = (-1)^{k+1}.$$

Proof of Claim 1. We have $k > \frac{n}{2}$, so that $2k > n$. Hence, the only multiples of k in the set $[0, n]$ are 0 and k . In other words, we have $[k | j] = 0$ for all $j \in [0, n]$ other than 0 and k , whereas $[k | j] = 1$ holds for $j = 0$ and for $j = k$.

Thus, the k -th row of the matrix M has only two entries distinct from 0: namely, a 1 in the 0-st column, and a 1 in the k -th column. Therefore, the k -th row of the matrix $M^{\sim k}$ is $(1, 0, 0, \dots, 0)$ (since the 1 in the k -th column disappears when we remove this column). Expanding $\det(M^{\sim k})$ along this row, we thus obtain $\det(M^{\sim k}) = (-1)^{k+1} \cdot \det Q$, where Q is the result of removing the 0-th column and the k -th row from $M^{\sim k}$. But it is easy to see that the matrix Q is upper-triangular with its diagonal entries all being equal to 1 (since $[i | j] = 0$ if $i > j$, and since $[i | j] = 1$ if $i = j$). Therefore, its determinant is $\det Q = 1$. Thus, $\det(M^{\sim k}) = (-1)^{k+1} \cdot \underbrace{\det Q}_{=1} = (-1)^{k+1}$. This proves Claim

1. □

Claim 2: Assume that $n = 2m$. Then,

$$\det(M^{\sim(n/2)}) = 0.$$

Proof of Claim 2. From $n = 2m$, we see that n is even. Thus, $n/2 \in [n]$ (since $n/2 = m > 0$), so that the matrix M has an $(n/2)$ -th row. Moreover, this $(n/2)$ -th row is not the n -th row (since $n = 2m > 0$ and thus $n/2 < n$).

Now, it is easy to see that the $(n/2)$ -th row and the n -th row of M agree in all their entries except for the ones in the $(n/2)$ -th column (since any $j \in [0, n]$ satisfies $[n/2 | j] = \begin{cases} 1, & \text{if } j \in \{0, n/2, n\}; \\ 0, & \text{otherwise} \end{cases}$ and $[n | j] = \begin{cases} 1, & \text{if } j \in \{0, n\}; \\ 0, & \text{otherwise,} \end{cases}$ and

$$n \text{ and thus } n - xi = yj, \text{ so that } \phi(\psi((x, y))) = \phi(xi) = \left(xi/i, \underbrace{(n - xi)/j}_{=yj} \right) =$$

$(xi/i, yj/j) = (x, y)$, so that $\phi \circ \psi = \text{id}$ is proved). Thus, $\phi : \mathfrak{A} \rightarrow \mathfrak{B}$ is a bijection. This is exactly the bijection we need.

these two values disagree only for $j = n/2$). Since the $(n/2)$ -th column is removed in $M^{\sim(n/2)}$, this entails that the $(n/2)$ -th row and the n -th row of the matrix $M^{\sim(n/2)}$ completely agree. Hence, the matrix $M^{\sim(n/2)}$ has two equal rows, and therefore its determinant is 0. In other words, $\det\left(M^{\sim(n/2)}\right) = 0$. This proves Claim 2. \square

Claim 3: Let $k \in [0, n]$. Then, $\det(N_{\sim k}) = (-1)^{n(n-1)/2} \cdot \det\left(M^{\sim(n-k)}\right)$.

Proof of Claim 3. From $M = ([i \mid j])_{i \in [n], j \in [0, n]}$, we see that the transpose M^T of the matrix M is given by $M^T = ([j \mid i])_{i \in [0, n], j \in [n]}$. Comparing this with $N = ([j \mid n - i])_{i \in [0, n], j \in [n]}$, we observe that the matrix N can be obtained from the matrix M^T by reversing the order of the rows, i.e., by permuting the rows using the permutation

$$\begin{aligned} [0, n] &\rightarrow [0, n], \\ i &\mapsto n - i. \end{aligned}$$

Hence, the matrix $N_{\sim k}$ is obtained from the matrix $(M^T)_{\sim(n-k)}$ in the exact same way (i.e., by reversing the order of the rows). Therefore,

$$\det(N_{\sim k}) = (-1)^\sigma \cdot \det\left(\left(M^T\right)_{\sim(n-k)}\right),$$

where σ is a permutation of an n -element set that reverses the order of its elements. Basic combinatorics shows that the sign $(-1)^\sigma$ of this latter permutation equals $(-1)^\sigma = (-1)^{n(n-1)/2}$ (for example, because this permutations has $n(n-1)/2$ inversions). Thus, we obtain

$$\begin{aligned} \det(N_{\sim k}) &= \underbrace{(-1)^\sigma}_{=(-1)^{n(n-1)/2}} \cdot \det\left(\underbrace{\left(M^T\right)_{\sim(n-k)}}_{=(M^{\sim(n-k)})^T}\right) \\ &= (-1)^{n(n-1)/2} \cdot \underbrace{\det\left(\left(M^{\sim(n-k)}\right)^T\right)}_{=\det(M^{\sim(n-k)})} = (-1)^{n(n-1)/2} \cdot \det\left(M^{\sim(n-k)}\right). \end{aligned}$$

(since $\det(U^T) = \det U$
for any matrix U)

This proves Claim 3. \square

Claim 4: We have

$$\sum_{k=0}^{m-1} (-1)^k \det\left(M^{\sim k}\right) = m.$$

Proof of Claim 4. Let M' be the $(n+1) \times (n+1)$ -matrix obtained from M by attaching an extra row $(1, 1, \dots, 1)$ at the very top of the matrix. Then, the top-most two rows of the matrix M' are equal (since the top row of M already is $(1, 1, \dots, 1)$), so that $\det(M') = 0$. On the other hand, expanding the determinant of M' along the topmost row, we obtain

$$\begin{aligned} \det(M') &= \sum_{k=0}^n (-1)^k 1 \cdot \det(M^{\sim k}) = \sum_{k=0}^n (-1)^k \det(M^{\sim k}) \\ &= \sum_{k=0}^{m-1} (-1)^k \det(M^{\sim k}) + \sum_{k=m}^n (-1)^k \det(M^{\sim k}) \end{aligned} \quad (4)$$

(since $0 \leq m \leq n$). However, it is not hard to see that

$$\sum_{k=m}^n (-1)^k \det(M^{\sim k}) = -m, \quad (5)$$

for example by distinguishing between two cases:

- If $n = 2m$, then $m = n/2$ and thus

$$\begin{aligned} &\sum_{k=m}^n (-1)^k \det(M^{\sim k}) \\ &= \sum_{k=n/2}^n (-1)^k \det(M^{\sim k}) \\ &= \underbrace{(-1)^{n/2} \det(M^{\sim(n/2)})}_{=0 \text{ (by Claim 2)}} + \sum_{k=n/2+1}^n (-1)^k \underbrace{\det(M^{\sim k})}_{=(-1)^{k+1} \text{ (by Claim 1)}} \\ &= \underbrace{(-1)^{n/2} 0}_{=0} + \sum_{k=n/2+1}^n \underbrace{(-1)^k (-1)^{k+1}}_{=-1} = \sum_{k=n/2+1}^n (-1) = (n - n/2)(-1) \\ &= -n/2 = -m \quad (\text{since } m = n/2). \end{aligned}$$

- If $n \neq 2m$, then $n = 2m - 1$ (since n is either $2m$ or $2m - 1$) and thus
-

$m = (n + 1) / 2$ and therefore

$$\begin{aligned}
& \sum_{k=m}^n (-1)^k \det(M^{\sim k}) \\
&= \sum_{k=(n+1)/2}^n (-1)^k \underbrace{\det(M^{\sim k})}_{\substack{=(-1)^{k+1} \\ \text{(by Claim 1,} \\ \text{since } k \geq (n+1)/2 > n/2)}} \\
&= \sum_{k=(n+1)/2}^n \underbrace{(-1)^k (-1)^{k+1}}_{=-1} = (n - (n + 1) / 2 + 1) (-1) \\
&= - (n + 1) / 2 = -m \quad (\text{since } m = (n + 1) / 2).
\end{aligned}$$

In either case, we have proved (5). Now, (4) becomes

$$\begin{aligned}
\det(M') &= \sum_{k=0}^{m-1} (-1)^k \det(M^{\sim k}) + \underbrace{\sum_{k=m}^n (-1)^k \det(M^{\sim k})}_{\substack{=-m \\ \text{(by (5))}}} \\
&= \sum_{k=0}^{m-1} (-1)^k \det(M^{\sim k}) - m.
\end{aligned}$$

Hence,

$$\sum_{k=0}^{m-1} (-1)^k \det(M^{\sim k}) - m = \det(M') = 0,$$

so that

$$\sum_{k=0}^{m-1} (-1)^k \det(M^{\sim k}) = m.$$

This proves Claim 4. □

Now, (3) becomes

$$\begin{aligned}
\det A &= \sum_{k=0}^n \det(M^{\sim k}) \cdot \det(N_{\sim k}) \\
&= \sum_{k=0}^{m-1} \det(M^{\sim k}) \cdot \det(N_{\sim k}) + \sum_{k=m}^n \det(M^{\sim k}) \cdot \det(N_{\sim k}) \quad (6)
\end{aligned}$$

(since $0 \leq m \leq n$). However, it is easy to see that

$$\sum_{k=m}^n \det(M^{\sim k}) \cdot \det(N_{\sim k}) = \sum_{k=n-m+1}^n \det(M^{\sim k}) \cdot \det(N_{\sim k}), \quad (7)$$

for instance by distinguishing between two cases:

- If $n = 2m$, then $m = n/2$ and thus

$$\begin{aligned}
& \sum_{k=m}^n \det(M^{\sim k}) \cdot \det(N_{\sim k}) \\
&= \sum_{k=n/2}^n \det(M^{\sim k}) \cdot \det(N_{\sim k}) \\
&= \underbrace{\det(M^{\sim(n/2)}) \cdot \det(N_{\sim(n/2)})}_{=0 \text{ (by Claim 2)}} + \sum_{k=n/2+1}^n \det(M^{\sim k}) \cdot \det(N_{\sim k}) \\
&= \sum_{k=n/2+1}^n \det(M^{\sim k}) \cdot \det(N_{\sim k}) = \sum_{k=n-m+1}^n \det(M^{\sim k}) \cdot \det(N_{\sim k})
\end{aligned}$$

(since $n/2 = n - m$ (because $n - \underbrace{m}_{=n/2} = n - n/2 = n/2$)), so that (7)

holds.

- If $n \neq 2m$, then $n = 2m - 1$ (since n is either $2m$ or $2m - 1$) and thus $m = n - m + 1$, so that (7) holds tautologically (since the summation bounds on both sides are equal).
-

In either case, we have proved (7). Now, (6) becomes

$$\begin{aligned}
\det A &= \sum_{k=0}^{m-1} \det(M^{\sim k}) \cdot \det(N_{\sim k}) + \underbrace{\sum_{k=m}^n \det(M^{\sim k}) \cdot \det(N_{\sim k})}_{= \sum_{k=n-m+1}^n \det(M^{\sim k}) \cdot \det(N_{\sim k})} \\
&= \sum_{k=0}^{m-1} \det(M^{\sim k}) \cdot \underbrace{\det(N_{\sim k})}_{=(-1)^{n(n-1)/2} \cdot \det(M^{\sim(n-k)})} \\
&\quad \text{(by Claim 3)} \\
&\quad + \sum_{k=n-m+1}^n \det(M^{\sim k}) \cdot \underbrace{\det(N_{\sim k})}_{=(-1)^{n(n-1)/2} \cdot \det(M^{\sim(n-k)})} \\
&\quad \text{(by Claim 3)} \\
&= (-1)^{n(n-1)/2} \sum_{k=0}^{m-1} \det(M^{\sim k}) \cdot \det(M^{\sim(n-k)}) \\
&\quad + (-1)^{n(n-1)/2} \sum_{k=n-m+1}^n \det(M^{\sim k}) \cdot \det(M^{\sim(n-k)}) \\
&= (-1)^{n(n-1)/2} \sum_{k=0}^{m-1} \det(M^{\sim k}) \cdot \det(M^{\sim(n-k)}) \\
&\quad + (-1)^{n(n-1)/2} \sum_{k=0}^{m-1} \det(M^{\sim(n-k)}) \cdot \det(M^{\sim k}) \\
&\quad \text{(here, we have substituted } n-k \text{ for } k \text{ in the second sum)} \\
&= (-1)^{n(n-1)/2} \sum_{k=0}^{m-1} \det(M^{\sim k}) \cdot \det(M^{\sim(n-k)}) \\
&\quad + (-1)^{n(n-1)/2} \sum_{k=0}^{m-1} \det(M^{\sim k}) \cdot \det(M^{\sim(n-k)}) \\
&= 2 \cdot (-1)^{n(n-1)/2} \sum_{k=0}^{m-1} \det(M^{\sim k}) \cdot \underbrace{\det(M^{\sim(n-k)})}_{=(-1)^{(n-k)+1}} \\
&\quad \text{(by Claim 1, applied to } n-k \text{ instead of } k \\
&\quad \text{(since } k \leq m-1 < n/2 \text{ and thus } n-k > n-n/2 = n/2)) \\
&= 2 \cdot (-1)^{n(n-1)/2} \sum_{k=0}^{m-1} \det(M^{\sim k}) \cdot \underbrace{(-1)^{(n-k)+1}}_{=(-1)^{n+1}(-1)^k}
\end{aligned}$$

$$\begin{aligned}
&= 2 \cdot \underbrace{(-1)^{n(n-1)/2} (-1)^{n+1}}_{\substack{=(-1)^{n(n-1)/2+(n+1)} \\ =(-1)^{m-1} \\ \text{(by easy modulo-2} \\ \text{computations)}}} \underbrace{\sum_{k=0}^{m-1} (-1)^k \det(M^{\sim k})}_{\substack{\equiv m \\ \text{(by Claim 4)}}} \\
&= 2 \cdot (-1)^{m-1} \cdot m = (-1)^{m-1} \cdot 2m.
\end{aligned}$$

This solves the problem. □
