A remark on polyhedral cones from packed words and from finite topologies

Darij Grinberg

April 26, 2020 (unfinished!)

Contents

1. The main theorem 1

2. The proof 4

3. Application: an alternating sum identity 23

1. The main theorem

The purpose of this little note is to prove [2, Theorem 5.2] using the machinery of [1].

I shall use the notations of [1] (except that I write WQSym instead of WQSym).

Here is a brief overview of these notations:

• We fix a field $\mathbb{K}$.

• We let $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{N}_{>0} = \{1, 2, 3, \ldots\}$.

• For each $n \in \mathbb{N}$, we let $[n]$ denote the set $\{1, 2, \ldots, n\}$. In particular, $[0] = \emptyset$.

• A word means a $n$-tuple of positive integers for some $n \in \mathbb{N}$. In this case, the $n$ is called the length of the word. A word $w = (w_1, w_2, \ldots, w_n)$ is identified with the map $[n] \to \mathbb{N}_{>0}$, $i \mapsto w_i$.

• A word $w = (w_1, w_2, \ldots, w_n)$ is said to be packed if and only if $\{w_1, w_2, \ldots, w_n\} = [k]$ for some $k \in \mathbb{N}$. In this case, the $k$ is denoted by $\max w$. (Note that $k$ is the largest entry of $w$ if $w$ is nonempty.)
For example, the word \( (3, 1, 2, 1, 3) \) is packed (with \( \max (3, 1, 2, 1, 3) = 3 \)), and so is the empty word \( () \) (with \( \max () = 0 \)); but the word \( (3, 1, 3) \) is not packed.

- If \( w \) is any word, then the packing of \( w \) is the packed word \( \text{Pack} w \) obtained by replacing the smallest number that appears in \( w \) by 1 (as often as it appears), replacing the second-smallest number that appears in \( w \) by 2 (as often as it appears), and so on. More formally, it can be defined as follows: Write \( w \) as \( w = (w_1, w_2, \ldots, w_n) \). Let \( W = \{w_1, w_2, \ldots, w_n\} \) be the set of all entries of \( w \), and let \( m = |W| \). Let \( \phi \) be the unique increasing bijection from \( W \) to \([m]\). Then, \( \text{Pack} w \) is defined to be the word \( (\phi (w_1), \phi (w_2), \ldots, \phi (w_n)) \).

For example,

\[
\text{Pack} (4, 1, 7, 2, 4, 1) = (3, 1, 4, 2, 3, 1) \quad \text{and} \quad \text{Pack} (4, 2) = (2, 1) .
\]

Also, \( \text{Pack} w = w \) for any packed word \( w \).

- We let \( WQ\text{Sym} \) denote the free \( K \)-vector space with basis \( (w)_w \) is a packed word. We define a \( K \)-bilinear operation \( . \) (you’re reading right: our symbol for this operation is a period) on this vector space \( WQ\text{Sym} \) by setting

\[
f \cdot g = \sum_{h = (h_1, h_2, \ldots, h_n + m) \text{ is a packed word of length } n + m; \text{ Pack}(h_1, h_2, \ldots, h_n) = f \text{ and Pack}(h_{n+1}, h_{n+2}, \ldots, h_{n+m}) = g} h
\]

for any two packed words \( f \) and \( g \), where \( n \) and \( m \) are the lengths of \( f \) and \( g \). Equipping \( WQ\text{Sym} \) with this operation . as multiplication, we obtain a \( K \)-algebra with unity \( () \) (the empty word). When we refer to the \( K \)-algebra \( WQ\text{Sym} \) below, we shall always understand it to be equipped with this \( K \)-algebra structure.

For example, in \( WQ\text{Sym} \), we have

\[
(1, 1) . (2, 1) = (1, 1, 2, 1) + (2, 2, 2, 1) + (1, 1, 3, 2) + (2, 2, 3, 1) + (3, 3, 2, 1) .
\]

The \( K \)-algebra \( WQ\text{Sym} \) has various further structures – such as a Hopf algebra structure, and an embedding into the ring of noncommutative formal power series (see [2, §4.3.2], where \( WQ\text{Sym} \) is constructed via this embedding, and where the image of a packed word \( u \) under this embedding is denoted by \( M_u \)). We won’t need this extra structure.

Let me add a few more definitions\[1\]

---

\[1\]A set composition of a set \( X \) means a tuple \((X_1, X_2, \ldots, X_k)\) of disjoint nonempty subsets of \( X \) such that \( X_1 \cup X_2 \cup \cdots \cup X_k = X \).
Definition 1.1. Let $n \in \mathbb{N}$. Let $u$ be a packed word of length $n$. Let $r = \max u$. Define $B_i = u^{-1}(\{i\})$ for every $i \in [r]$. (Thus, $(B_1, B_2, \ldots, B_r)$ is a set composition of $[n]$; it is what is called the “set composition of $[n]$ encoded by $u$” in [2].) Now, we define a polyhedral cone $K_u$ in $\mathbb{R}^n$ by

$$K_u = \left\{ (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid \sum_{j=1}^{k} \sum_{i \in B_j} x_i \geq 0 \quad \text{for all } k = 1, 2, \ldots, r \right\}.$$ 

Definition 1.2. For any two sets $X$ and $Y$, let $\text{Map}(X, Y)$ denote the set of all maps from $X$ to $Y$. Define a $K$-vector space $M$ by $M = \bigoplus_{n \geq 0} \text{Map}(\mathbb{R}^n, K)$ (where each $\text{Map}(\mathbb{R}^n, K)$ becomes a $K$-vector space by pointwise addition and multiplication with scalars). We make $\mathfrak{M}$ into a $K$-algebra, whose multiplication is defined as follows: For any $n \in \mathbb{N}$, any $m \in \mathbb{N}$, any $f \in \text{Map}(\mathbb{R}^n, K)$ and $g \in \text{Map}(\mathbb{R}^m, K)$, we define $fg$ to be the element of $\text{Map}(\mathbb{R}^{n+m}, K)$ which sends every $(x_1, x_2, \ldots, x_{n+m}) \in \mathbb{R}^{n+m}$ to $f(x_1, x_2, \ldots, x_n)g(x_{n+1}, x_{n+2}, \ldots, x_{n+m})$.

Definition 1.3. For every $n \in \mathbb{N}$ and any subset $S$ of $\mathbb{R}^n$, we define a map $1_S \in \text{Map}(\mathbb{R}^n, K) \subseteq \mathfrak{M}$ as the indicator function of $S$ (that is, the map which sends every $s \in S$ to 1 and every $s \in \mathbb{R}^n \setminus S$ to 0).

Our goal is to show:

**Theorem 1.4.** The map

$$\alpha : \text{WQSym} \to \mathfrak{M},$$

$$u \mapsto (-1)^{\max u} 1_{K_u}$$

is a $K$-algebra homomorphism.

This is a stronger version of [2, Theorem 5.2] and a particular case of [2, Theorem 8.1].

---

2 Notice that [2, Theorem 5.2] talks not about our map $\alpha : \text{WQSym} \to \mathfrak{M}$, but rather about a map $P \to \text{WQSym}$ where $P$ is a certain subquotient of $\mathfrak{M}$ (namely, the subalgebra of $\mathfrak{M}$ generated by $1_{K_u}$, taken modulo functions with measure-zero support). These two maps are “in some sense” inverse (allowing us to derive [2, Theorem 5.2] from Theorem 1.4). I find Theorem 1.4 the more natural statement.

Notice that [2] denotes by $(M_u)_u$ is a packed word the basis of WQSym that we call $(u)_u$ is a packed word.

3 At least, I suspect so – I have not checked all the details. I also suspect that the whole [2, Theorem 8.1] can be obtained in a similar way as we prove Theorem 1.4 below.
2. The proof

We shall prove Theorem 1.4 using a detour via the algebra \( H_T \) defined in [1, Chapter 2]. We shall use the following notations from [1, Chapter 2]:

- If \( X \) is a set, then a **topology** on \( X \) is defined to be a family \( \mathcal{T} \) of subsets of \( X \) that satisfies the following three properties:
  - We have \( \emptyset \in \mathcal{T} \) and \( X \in \mathcal{T} \).
  - The union of any number of sets in \( \mathcal{T} \) is again a set in \( \mathcal{T} \).
  - The intersection of any finite number of sets in \( \mathcal{T} \) is again a set in \( \mathcal{T} \).

  We will only use this concept in the case when \( X \) is finite; in this case, the difference between “any number of sets in \( \mathcal{T} \)” and “any finite number of sets in \( \mathcal{T} \)” is immaterial (since \( \mathcal{T} \) itself must be finite), and therefore unions and intersections play symmetric roles in the notion of a topology on \( X \).

  - If \( \mathcal{T} \) is a topology on \( X \), then the sets belonging to \( \mathcal{T} \) are called the **open sets** of \( \mathcal{T} \). The complements of these open sets (inside \( X \)) are called the **closed sets** of \( \mathcal{T} \).

- If \( X \) is a set, then a **preorder** on \( X \) is defined to be a binary relation \( \preceq \) on \( X \) that is reflexive and transitive (but, unlike a partial order, doesn’t need to be antisymmetric). Both partial orders and equivalence relations are preorders.

- If \( X \) is a set, and if \( \preceq \) is a preorder on \( X \), then an **ideal** of \((X, \preceq)\) means a subset \( I \) of \( X \) that has the following property:
  - If \( i \in I \) and \( j \in X \) satisfy \( i \preceq j \), then \( j \in I \).

- If \( X \) is a finite set, then there is a canonical bijection between \{topologies on \( X \)\} and \{preorders on \( X \)\}. This bijection (sometimes called the **Alexandrov correspondence**) proceeds as follows:
  - If \( \preceq \) is a preorder on \( X \), then we can define a topology \( \mathcal{T}_\preceq \) on \( X \) by
    \[
    \mathcal{T}_\preceq = \{\text{ideals of } (X, \preceq)\}. 
    \]

    We shall denote this topology \( \mathcal{T}_\preceq \) as the **topology corresponding to \( \preceq \)**.
  - If \( \mathcal{T} \) is a topology on \( X \), then we can define two binary relations \( \leq_T, \geq_T \) and \( \sim_T \) on \( X \) by setting
    \[
    (a \leq_T b) \iff \text{(each } I \in \mathcal{T} \text{ satisfying } a \in I \text{ satisfies } b \in I); \\
    (a \geq_T b) \iff \text{(each } I \in \mathcal{T} \text{ satisfying } b \in I \text{ satisfies } a \in I); \\
    (a \sim_T b) \iff \text{(each } I \in \mathcal{T} \text{ satisfies the equivalence } (a \in I) \iff (b \in I));
    \]
Remark on polyhedral cones

April 26, 2020

(a <_T b) \iff (a \leq_T b \text{ but not } a \geq_T b) \iff (a \leq_T b \text{ but not } a \sim_T b);
(a >_T b) \iff (a \geq_T b \text{ but not } a \leq_T b) \iff (a \geq_T b \text{ but not } a \sim_T b).

The three binary relations \leq_T, \geq_T and \sim_T are preorders on X, and the relation \sim_T is an equivalence relation (whence the quotient set \, X/\sim_T is well-defined). The relations <_T and >_T are strict partial orders. We shall refer to the relation \leq_T as the preorder corresponding to T.

These assignments of a topology to a preorder and vice versa are mutually inverse: If \preccurlyeq is a preorder on X, then \leq_{\preccurlyeq} is precisely \preccurlyeq. Conversely, if T is a topology on X, then \leq_T is precisely T.

• For each \( n \in \mathbb{N} \), we let \( T_n \) denote the set of all topologies on the set \([n] = \{1, 2, \ldots, n\}\).

• We let \( T \) denote the set \( \bigsqcup_{n \in \mathbb{N}} T_n \).

• If \( f \) is a packed word of length \( n \in \mathbb{N} \), then we define a preorder \( \leq_f \) on
the set \([n]\) by setting

\[(a \leq_f b) \iff (f(a) \leq f(b)).\]

Furthermore, if \( f \) is a packed word of length \( n \in \mathbb{N} \), then we let \( T_f \) be the topology \( T_{\leq_f} \) corresponding to this preorder \( \leq_f \). The closed sets of this topology \( T_f \) are the sets \( f^{-1}(\{1, 2, \ldots, i\}) \) for \( i \in \{0, 1, \ldots, \max f\} \).

• If \( P \subseteq \mathbb{N} \) and \( n \in \mathbb{N} \), then \( P(+n) \) shall denote the set \( \{k+n \mid k \in P\} \). (In other words, \( P(+n) \) is the set \( P \) shifted right by \( n \) units on the number line.)

• If \( T \in T_n \) and \( S \in T_m \) are two topologies (on the sets \([n]\) and \([m]\), respectively) for some \( n \in \mathbb{N} \) and \( m \in \mathbb{N} \), then we define a topology \( T \cdot S \in T_{n+m} \) on the set \([n+m]\) by

\[T \cdot S = \{O \cup (P(+n)) \mid O \in T \text{ and } P \in S\} \].

Thus, we have defined a binary operation \( \cdot \) on \( T \). This binary operation \( \cdot \) is associative (by [1, Proposition 3]), and the topology \( \{\emptyset\} \in T_0 \) is its neutral element.

• We let \( H_T \) be the free \( K \)-vector space with basis \( T \). We equip \( H_T \) with a multiplication \( \cdot \) that linearly extends the operation \( \cdot \) on \( T \) (that is, the restriction of the multiplication \( H_T \) to the basis \( T \) should be the operation \( \cdot \) on \( T \)). Thus, \( H_T \) becomes a \( K \)-algebra with unity \( \{\emptyset\} \in T_0 \).
The $K$-algebra $H_T$ also has the structure of a Hopf algebra, but we shall not need it, so we don’t define it here.

We shall also use the following notation from [1, Chapter 4]:

- If $X$ is a set, and if $T$ is a topology on $X$, then we set
  $$P(T) = \bigsqcup_{p \in \mathbb{N}} \{\text{surjective maps } f : X \to [p] \text{ such that every } c \in X \text{ and } d \in X \text{ satisfying } c \leq_T d \text{ satisfy } f(x) \leq f(d)\}.$$  
  Thus, if $X = [n]$ for some $n \in \mathbb{N}$, then all elements of $P(T)$ are packed words of length $n$.

Next, we define a polyhedral cone for every $T \in T$:

**Definition 2.1.** Let $n \in \mathbb{N}$ and $T \in T_n$ (that is, let $T$ be a topology on the set $[n] = \{1, 2, \ldots, n\}$). Then, we define a polyhedral cone $K_T$ in $\mathbb{R}^n$ by

$$K_T = \left\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid \sum_{i \in C} x_i \geq 0 \text{ for all closed sets } C \text{ of } T\right\}.$$  

The following follows from the definitions:

**Remark 2.2.** Let $u$ be a packed word. Then, $K_u = K_{T_u}$, where $T_u$ is as defined in [1, §2.1].

Let us define a few more things:

**Definition 2.3.** Let $X$ be a finite totally ordered set, and let $T$ be a topology on $X$. We define $U(T)$ to be the set of all $f \in P(T)$ having the property that any two elements $i$ and $j$ of $X$ satisfying $i <_T j$ must satisfy $f(i) < f(j)$. Notice that $U(T) \subseteq P(T)$. (We can call the elements of $U(T)$ “strictly increasing packed words” for $T$.) (It can also be shown that $L(T) \subseteq U(T)$, where $L(T)$ is as defined in [1, Definition 15].)

**Definition 2.4.** We define a $K$-linear map $U : H_T \to WQSym$ by

$$U(T) = \sum_{f \in U(T)} f \quad \text{for every } T \in T.$$  

**Remark 2.5.** This map $U$ is easily seen to be the map $\Gamma_{(0,0,1)}$ in the notation of [1, Proposition 14]. Thus, $U$ is a surjective Hopf algebra homomorphism.
**Proposition 2.6.** The map

\[ \beta : H_T \to \mathfrak{M}, \]

\[ T \mapsto (-1)^{|[n]/\sim_T|} 1_{K_T} \]

is a $K$-algebra homomorphism from $H_T = (H_T, \cdot)$ to $\mathfrak{M}$.

**Proof of Proposition 2.6 (sketched).** The proof boils down to the observation that if $n \in \mathbb{N}$, $m \in \mathbb{N}$, $T \in T_n$ and $S \in T_m$, then

\[ K_{T,S} = \{(x_1, x_2, \ldots, x_{n+m}) \in \mathbb{R}^{n+m} \mid (x_1, x_2, \ldots, x_n) \in K_T \]

and \((x_{n+1}, x_{n+2}, \ldots, x_{n+m}) \in K_S\}.

Now, we claim:

**Theorem 2.7.** The diagram

\[
\begin{array}{ccc}
H_T & \xrightarrow{U} & \text{WQSym} \\
\downarrow{\beta} & & \downarrow{\alpha} \\
\mathfrak{M} & & \\
\end{array}
\]

commutes. That is, we have $\beta = \alpha \circ U$.

Before we prove this, we introduce some more notations.

**Definition 2.8.** We define a $K$-linear map $Z : H_T \to H_T$ by

\[ Z(T) = (-1)^{|[n]/\sim_T|} T \]

for every $n \in \mathbb{N}$ and $T \in T_n$.

It is easy to see that $Z$ is an involutive Hopf algebra isomorphism.

**Definition 2.9.** Let $X$ be a finite totally ordered set, and let $\mathcal{T}$ be a topology on $X$. Let $a$ and $b$ be two elements of $X$. We define three new topologies $\mathcal{T} \leftrightarrow (a \leq b)$, $\mathcal{T} \leftrightarrow (a \geq b)$ and $\mathcal{T} \leftrightarrow (a \sim b)$ on $X$ as follows:

\[
\begin{align*}
\mathcal{T} \leftrightarrow (a \leq b) &= \{O \in \mathcal{T} \mid (a \in O \implies b \in O)\}; \\
\mathcal{T} \leftrightarrow (a \geq b) &= \{O \in \mathcal{T} \mid (b \in O \implies a \in O)\}; \\
\mathcal{T} \leftrightarrow (a \sim b) &= \{O \in \mathcal{T} \mid (a \in O \iff b \in O)\}.
\end{align*}
\]

(It is easy to check that these are actually topologies. Of course, $\mathcal{T} \leftrightarrow (a \geq b) = \mathcal{T} \leftrightarrow (b \leq a)$.)

Here comes a collection of simple properties of these three new topologies:
Lemma 2.10. Let $X$ be a finite totally ordered set, and let $\mathcal{T}$ be a topology on $X$. Let $a$ and $b$ be two elements of $X$.

(a) We have
\[(\mathcal{T} \ni (a \leq b)) \cap (\mathcal{T} \ni (a \geq b)) = \mathcal{T} \ni (a \sim b) \quad \text{and} \quad (\mathcal{T} \ni (a \leq b)) \cup (\mathcal{T} \ni (a \geq b)) = \mathcal{T}.
\]

(b) We have
\[(T \ni (a \sim b)) = (T \ni (a \leq b)) \cup (T \ni (a \geq b)).
\]

(c) If $a \leq_T b$, then $T \ni (a \leq b) = T$ and $T \ni (a \sim b) = T \ni (a \geq b)$.

(d) If $b \leq_T a$, then $T \ni (a \geq b) = T$ and $T \ni (a \sim b) = T \ni (a \leq b)$.

(e) If $c$ and $d$ are two elements of $X$, then $c \leq_{\mathcal{T} \ni (a \leq b)} d$ holds if and only if
\[(c \leq_T d \text{ or } (c \leq_T a \text{ and } b \leq_T d)).
\]

(f) If $c$ and $d$ are two elements of $X$, then $c \leq_{\mathcal{T} \ni (a \geq b)} d$ holds if and only if
\[(c \leq_T d \text{ or } (c \leq_T b \text{ and } a \leq_T d)).
\]

(g) If $c$ and $d$ are two elements of $X$, then $c \leq_{\mathcal{T} \ni (a \sim b)} d$ holds if and only if
\[(c \leq_T d \text{ or } (c \leq_T a \text{ and } b \leq_T d) \text{ or } (c \leq_T b \text{ and } a \leq_T d)).
\]

(h) If $c$ and $d$ are two elements of $X$, then $c \leq_{\mathcal{T} \ni (a \sim b)} d$ holds if and only if
\[c \leq_{\mathcal{T} \ni (a \leq b)} d \text{ and } c \leq_{\mathcal{T} \ni (a \geq b)} d.
\]

(i) If $c$ and $d$ are two elements of $X$, then $c \tilde{\sim} T d$ holds if and only if
\[c \leq_{\mathcal{T} \ni (a \leq b)} d \text{ and } c \leq_{\mathcal{T} \ni (a \geq b)} d.
\]

(j) If $c$ and $d$ are two elements of $X$, then $c \sim_{\mathcal{T} \ni (a \leq b)} d$ holds if and only if
\[(c \sim_T d \text{ or } b \leq_T c \leq_T a \text{ and } b \leq_T d \leq_T a)).
\]

(k) If $c$ and $d$ are two elements of $X$, and if we have neither $a \leq_T b$ nor $b \leq_T a$, then $c \sim_{\mathcal{T} \ni (a \sim b)} d$ holds if and only if
\[(c \sim_T d \text{ or } c \sim_T a \text{ and } d \sim_T b) \text{ or } (c \sim_T b \text{ and } d \sim_T a)).
\]

(l) We have
\[\mathcal{P} (T \ni (a \leq b)) \cap \mathcal{P} (T \ni (a \geq b)) = \mathcal{P} (T \ni (a \sim b)) \quad \text{and} \quad \mathcal{P} (T \ni (a \leq b)) \cup \mathcal{P} (T \ni (a \geq b)) = \mathcal{P} (T)\n\]

(m) Assume that neither $a \leq_T b$ nor $b \leq_T a$. Then, the three sets $U \ni (T \ni (a \leq b))$, $U \ni (T \ni (a \geq b))$ and $U \ni (T \ni (a \sim b))$ are disjoint, and their union is $U \ni (T)$.

(n) Assume that neither $a \leq_T b$ nor $b \leq_T a$. Then,
\[|X/ \sim_{\mathcal{T} \ni (a \leq b)}| = |X/ \sim_{\mathcal{T} \ni (a \geq b)}| = |X/ \sim_T| \quad \text{and} \quad |X/ \sim_{\mathcal{T} \ni (a \sim b)}| = |X/ \sim_T| - 1.
\]
Remark on polyhedral cones April 26, 2020

Proof of Lemma 2.10 (sketched). Parts (a) and (b) are straightforward to check.

(c) Assume that \( a \leq_T b \). Then, every \( O \in \mathcal{T} \) satisfies \((a \in \mathcal{T} \implies b \in \mathcal{T})\). Hence, \( \mathcal{T} \not\supseteq (a \leq b) = \mathcal{T} \) by the definition of \( \mathcal{T} \not\supseteq (a \leq b) \). From Lemma 2.10 (b), we have \( \mathcal{T} \not\supseteq (a \sim b) = (\mathcal{T} \not\subseteq (a \leq b)) \not\supseteq (a \geq b) = \mathcal{T} \not\subseteq (a \geq b) \). Thus, \( \mathcal{T} = \mathcal{T} \).

Lemma 2.10 (c) is proven.

(d) The proof of part (d) is similar to that of (c).

(e) \iff: Assume that \((c \leq_T d \text{ or } (c \leq_T a \text{ and } b \leq_T d))\). We need to check that \( c \leq_{T-\supseteq(a \leq b)} d \) holds. In other words, we need to check that every \( O \in \mathcal{T} \not\supseteq (a \leq b) \) satisfying \( c \in O \) satisfies \( d \in O \). So let us fix an \( O \in \mathcal{T} \not\supseteq (a \leq b) \) satisfying \( c \in O \). We must prove that \( d \in O \).

We have \( O \in \mathcal{T} \not\supseteq (a \leq b) \subseteq \mathcal{T} \) (by the definition of \( \mathcal{T} \not\supseteq (a \leq b) \)). Thus, if \( c \leq_T d \), then \( d \in O \). Hence, for the rest of this proof, we WLOG assume that we don’t have \( c \leq_T d \). Thus, by assumption, we have \( c \leq_T a \) and \( b \leq_T d \). Therefore, \( a \in O \) (since \( c \in O \) and \( c \leq_T a \)). But \( O \in \mathcal{T} \not\supseteq (a \leq b) \), and therefore \((a \in O \implies b \in O)\) (by the definition of \( \mathcal{T} \not\supseteq (a \leq b) \)), so that \( b \in O \) (since \( a \in O \)), and thus \( d \in O \) (since \( b \leq_T d \)). This completes the proof of the \( \iff \) direction of Lemma 2.10 (e).

\( \implies: \) Assume that \( c \leq_{T-\supseteq(a \leq b)} d \) holds. We need to check that \((c \leq_T d \text{ or } (c \leq_T a \text{ and } b \leq_T d))\). We can WLOG assume that we don’t have \( c \leq_T d \). Then, we must prove that \((c \leq_T a \text{ and } b \leq_T d)\).

We don’t have \( c \leq_T d \). Hence, there exists a \( Q \in \mathcal{T} \) such that \( c \in Q \) but \( d \notin Q \). Consider this \( Q \). If we had \((a \in Q \implies b \in Q)\), then \( Q \) would belong to \( \mathcal{T} \not\supseteq (a \leq b) \), which would yield \( d \in Q \) (since \( c \leq_{T-\supseteq(a \leq b)} d \) and \( c \in Q \)), which would contradict \( d \notin Q \). Hence, we cannot have \((a \in Q \implies b \in Q)\). Thus, \( a \in Q \) and \( b \notin Q \).

Let \( O \in \mathcal{T} \) be such that \( c \in O \). We shall prove that \( a \in O \). Indeed, assume the contrary. Then, \( a \notin O \). Thus, \( a \notin Q \cap O \), so that \((a \in Q \cap O \implies b \in Q \cap O)\). Since \( Q \cap O \in \mathcal{T} \) (because \( Q \in \mathcal{T} \) and \( O \in \mathcal{T} \)), this yields \( Q \cap O \in \mathcal{T} \not\supseteq (a \leq b) \). Since we also have \( c \in Q \cap O \) (since \( c \in Q \) and \( c \in O \)), this yields \( d \in Q \cap O \) (since \( c \leq_{T-\supseteq(a \leq b)} d \)), so that \( d \in Q \cap O \subseteq Q \), which contradicts \( d \notin Q \). This contradiction proves that our assumption was wrong. Hence, \( a \in O \) is proven. Forget now that we fixed \( O \). Thus we have shown that \( a \in O \) for every \( O \in \mathcal{T} \) which satisfies \( c \in O \). In other words, \( c \leq_T a \).

Let \( O \in \mathcal{T} \) be such that \( b \in O \). We shall prove that \( d \in O \). Indeed, assume the contrary. Then, \( d \notin O \). Thus, \( d \notin Q \cup O \) (since \( d \notin Q \) and \( d \notin O \)). But \( b \in O \subseteq Q \cup O \), so that \((a \in Q \cup O \implies b \in Q \cup O)\). Since \( Q \cup O \in \mathcal{T} \) (because \( Q \in \mathcal{T} \) and \( O \in \mathcal{T} \)), this yields \( Q \cup O \in \mathcal{T} \not\supseteq (a \leq b) \). Since we also have \( c \in Q \cup O \) (since \( c \in Q \)), this yields \( d \in Q \cup O \) (since \( c \leq_{T-\supseteq(a \leq b)} d \)), which contradicts \( d \notin Q \cup O \). This contradiction proves that our assumption was wrong. Hence, \( d \in O \) is proven. Forget now that we fixed \( O \). Thus we have shown that \( d \in O \) for every \( O \in \mathcal{T} \) which satisfies \( b \in O \). In other words, \( b \leq_T d \).

We thus have shown that \((c \leq_T a \text{ and } b \leq_T d)\). This completes the proof of the \( \implies \) direction of Lemma 2.10 (e).
The proof of part (f) is analogous to that of (e).

Let $c$ and $d$ be two elements of $X$. Then, we have the following logical equivalence:

$$
\left(c \leq_{T \rightarrow P(a \sim b)} d \right)
\iff
\left(c \leq_{(T \rightarrow P(a \leq b)) \rightarrow P(a \geq b)} d \right)
\iff
\left(c \leq_{T \rightarrow P(a \leq b)} d \text{ or } \left(c \leq_{T \rightarrow P(a \leq b)} b \text{ and } a \leq_{(T \rightarrow P(a \leq b))} d \right) \right)
\text{(by Lemma 2.10 (f))}
\iff
\left((c \leq_T d \text{ or } (c \leq_T a \text{ and } b \leq_T d)) \text{ or } ((c \leq_T b \text{ or } (c \leq_T a \text{ and } b \leq_T b)) \text{ and } (a \leq_T d \text{ or } (a \leq_T a \text{ and } b \leq_T d))) \right)
\text{(by Lemma 2.10 (e), applied to each of the three inequalities)}
\iff
\left(c \leq_T d \text{ or } (c \leq_T a \text{ and } b \leq_T d) \text{ or } (c \leq_T b \text{ and } a \leq_T d) \right)
\text{(after simplifying using the transitivity and reflexivity of } \leq_T).

This proves Lemma 2.10 (g).

This is just a rewriting of Lemma 2.10 (g) using parts (e) and (f).

(i) $\implies$: This is clear.

$\iff$: Assume that $\left(c \leq_{T \rightarrow P(a \leq b)} d \text{ and } c \leq_{T \rightarrow P(a \geq b)} d \right)$. We need to show that $c \leq_T d$. Indeed, assume the contrary.

We have $c \leq_{T \rightarrow P(a \leq b)} d$. Thus, Lemma 2.10 (e) yields that we must have $(c \leq_T d \text{ or } (c \leq_T a \text{ and } b \leq_T d))$. Since we assumed that $c \leq_T d$ does not hold, this yields $(c \leq_T a \text{ and } b \leq_T d)$. Similarly, $(c \leq_T b \text{ and } a \leq_T d)$. Thus, $c \leq_T b \leq_T d$, which contradicts our assumption that not $c \leq_T d$. This contradiction completes the proof.

(j) We have $c \sim_{T \rightarrow P(a \leq b)} d$ if and only if $\left(c \leq_{T \rightarrow P(a \leq b)} d \text{ and } d \leq_{T \rightarrow P(a \leq b)} c \right)$. We can rewrite each of the two statements $c \leq_{T \rightarrow P(a \leq b)} d$ and $d \leq_{T \rightarrow P(a \leq b)} c$ using Lemma 2.10 (e), and then simplify the result; we end up with Lemma 2.10 (j).

(k) Let $c$ and $d$ be two elements of $X$. Assume that we have neither $a \leq_T b$ nor $b \leq_T a$. We have $c \sim_{T \rightarrow P(a \sim b)} d$ if and only if $\left(c \leq_{T \rightarrow P(a \sim b)} d \text{ and } d \leq_{T \rightarrow P(a \sim b)} c \right)$. We can rewrite each of the two statements $c \leq_{T \rightarrow P(a \sim b)} d$ and $d \leq_{T \rightarrow P(a \sim b)} c$ using Lemma 2.10 (g), and then simplify the result (a disjunction with 9 cases, of which many can be ruled out due to the assumption that neither $a \leq_T b$ nor $b \leq_T a$); we end up with Lemma 2.10 (k).

(l) Proof of $\mathcal{P} (T \rightarrow P (a \leq b)) \cap \mathcal{P} (T \rightarrow P (a \geq b)) = \mathcal{P} (T \rightarrow P (a \sim b))$: Whenever $f$ is a surjective map $X \rightarrow [p]$ for some $p \in \mathbb{N}$, we have the following
Remark on polyhedral cones April 26, 2020

logical equivalence:

\[(f \in \mathcal{P}(T \vdash (a \leq b)) \cap \mathcal{P}(T \vdash (a \geq b)))\]

\[\iff \left( \begin{aligned}
&\left( f \in \mathcal{P}(T \vdash (a \leq b)) \right) \\
&\land \left( f \in \mathcal{P}(T \vdash (b \leq a)) \right) \\
\end{aligned} \right)
\]

\[\iff \left( \begin{aligned}
&\left( \forall c \in X \text{ and } d \in X \text{ satisfying } c \leq_{T \vdash (a \leq b)} d \text{ satisfy } f(c) \leq f(d) \right) \\
&\land \left( \forall c \in X \text{ and } d \in X \text{ satisfying } c \leq_{T \vdash (b \leq a)} d \text{ satisfy } f(c) \leq f(d) \right) \\
\end{aligned} \right)
\]

\[\iff \left( \begin{aligned}
&\forall c \in X \text{ and } d \in X \text{ satisfying } c \leq_{T \vdash (a \leq b)} d \text{ or } c \leq_{T \vdash (a \geq b)} d \\
\end{aligned} \right)
\]

\[\iff \left( \begin{aligned}
&\forall c \in X \text{ and } d \in X \text{ satisfying } c \leq_{T \vdash (a \sim b)} d \\
\end{aligned} \right)
\]

\[\iff (f \in \mathcal{P}(T \vdash (a \sim b))).\]

Thus, \(\mathcal{P}(T \vdash (a \leq b)) \cap \mathcal{P}(T \vdash (a \geq b)) = \mathcal{P}(T \vdash (a \sim b))\) is proven.

It remains to prove \(\mathcal{P}(T \vdash (a \leq b)) \cup \mathcal{P}(T \vdash (a \geq b)) = \mathcal{P}(T)\). We shall achieve this by proving both inclusions separately:

Proof of \(\mathcal{P}(T) \subseteq \mathcal{P}(T \vdash (a \leq b)) \cup \mathcal{P}(T \vdash (a \geq b))\): Let \(f \in \mathcal{P}(T)\). We must prove that \(f \in \mathcal{P}(T \vdash (a \leq b)) \cup \mathcal{P}(T \vdash (a \geq b))\). We WLOG assume that \(f(a) \leq f(b)\). We shall now show that \(f \in \mathcal{P}(T \vdash (a \leq b))\).

This will yield that \(f \in \mathcal{P}(T \vdash (a \leq b)) \cup \mathcal{P}(T \vdash (a \geq b))\), and thus complete this proof of \(\mathcal{P}(T) \subseteq \mathcal{P}(T \vdash (a \leq b)) \cup \mathcal{P}(T \vdash (a \geq b))\).

Let \(c \in X\) and \(d \in X\) be such that \(c \leq_{T \vdash (a \leq b)} d\). In order to prove that \(f \in \mathcal{P}(T \vdash (a \leq b))\), we must now show that \(f(c) \leq f(d)\).

We have \(c \leq_{T \vdash (a \leq b)} d\). Due to Lemma 2.10 (e), this yields that \(c \leq_{T \vdash (a \leq b)} d\) (or \(c \leq_{T \vdash (a \geq b)} d\)). In the first of these cases, \(f(c) \leq f(d)\) follows immediately from \(f \in \mathcal{P}(T)\); thus, let us assume that we are in the second case. Thus, \(c \leq_{T \vdash (a \leq b)} d\) and \(c \leq_{T \vdash (a \geq b)} d\). From \(f \in \mathcal{P}(T)\), we thus obtain \(f(c) \leq f(a)\) and \(f(b) \leq f(d)\). Hence, \(f(c) \leq f(a) \leq f(b) \leq f(d)\), qed.

Proof of \(\mathcal{P}(T \vdash (a \leq b)) \cup \mathcal{P}(T \vdash (a \geq b)) \subseteq \mathcal{P}(T)\): We now need to show that \(\mathcal{P}(T \vdash (a \leq b)) \cup \mathcal{P}(T \vdash (a \geq b)) \subseteq \mathcal{P}(T)\). To do so, it is clearly enough to prove \(\mathcal{P}(T \vdash (a \leq b)) \subseteq \mathcal{P}(T)\) and \(\mathcal{P}(T \vdash (a \geq b)) \subseteq \mathcal{P}(T)\). We shall
only show the first of these two relations, as the second is analogous. Let \( f \in \mathcal{P}(\mathcal{T} \ni (a \leq b)) \). Then, every \( c \in X \) and \( d \in X \) satisfying \( c \leq_{T\ni(a \leq b)} d \) satisfy \( f(c) \leq f(d) \). Hence, every \( c \in X \) and \( d \in X \) satisfying \( c \leq_{T} d \) satisfy \( f(c) \leq f(d) \) (since every \( c \in X \) and \( d \in X \) satisfying \( c \leq_{T} d \) satisfy \( c \leq_{T\ni(a \leq b)} d \) (due to Lemma \ref{lem:2.10}(e))). In other words, \( f \in \mathcal{P}(\mathcal{T}). \) Since this is proven for every \( f \in \mathcal{P}(\mathcal{T} \ni (a \leq b)) \), we thus conclude that \( \mathcal{P}(\mathcal{T} \ni (a \leq b)) \subseteq \mathcal{P}(\mathcal{T}). \)

The proof of Lemma \ref{lem:2.10}(i) is thus complete.

**(m)** It is clearly enough to prove the three equalities

\[
\begin{align*}
\mathcal{U}(\mathcal{T} \ni (a \leq b)) &= \{ f \in \mathcal{U}(\mathcal{T}) \mid f(a) < f(b) \}; \quad (3) \\
\mathcal{U}(\mathcal{T} \ni (a \sim b)) &= \{ f \in \mathcal{U}(\mathcal{T}) \mid f(a) = f(b) \}; \quad (4) \\
\mathcal{U}(\mathcal{T} \ni (a \geq b)) &= \{ f \in \mathcal{U}(\mathcal{T}) \mid f(a) > f(b) \}. \quad (5)
\end{align*}
\]

We shall only check the first two of these three equalities (since the third one is analogous to the first).

Let us first check that \( a <_{T\ni(a \leq b)} b \). Indeed, it is clear from the definition of \( T \ni (a \leq b) \) that \( a \leq_{T\ni(a \leq b)} b \). Thus, in order to prove that \( a <_{T\ni(a \leq b)} b \), we must only show that we don’t have \( b \leq_{T\ni(a \leq b)} a \). To achieve this, we assume the contrary. Lemma \ref{lem:2.10}(e) (applied to \( c = b \) and \( d = a \)) thus yields that \( (b \leq_{T} a \text{ or } (b \leq_{T} a \text{ and } b \leq_{T} a)) \). In either of these cases, we must have \( b \leq_{T} a \), which contradicts the assumption that neither \( a \leq_{T} b \text{ nor } b \leq_{T} a \). So \( a <_{T\ni(a \leq b)} b \) is proven.

Next, we are going to prove (3) by showing its two inclusions separately:

Proof of \( \mathcal{U}(\mathcal{T} \ni (a \leq b)) \subseteq \{ f \in \mathcal{U}(\mathcal{T}) \mid f(a) < f(b) \} \): Let \( g \in \mathcal{U}(\mathcal{T} \ni (a \leq b)) \), and every two elements \( i \) and \( j \) of \( X \) satisfying \( i <_{T\ni(a \leq b)} j \) must satisfy \( g(i) < g(j) \). Applying the latter fact to \( i = a \) and \( j = b \), we obtain \( g(a) < g(b) \) (since \( a <_{T\ni(a \leq b)} b \)).

Moreover, \( g \in \mathcal{P}(\mathcal{T} \ni (a \leq b)) \subseteq \mathcal{P}(\mathcal{T} \ni (a \leq b)) \cup \mathcal{P}(\mathcal{T} \ni (a \geq b)) = \mathcal{P}(\mathcal{T}) \) (by Lemma \ref{lem:2.10}(i)).

Let now \( i \) and \( j \) be any two elements of \( X \) satisfying \( i <_{T} j \). We shall show that \( g(i) < g(j) \).

Indeed, \( i <_{T} j \), thus \( i \leq_{T} j \) and therefore \( i <_{T\ni(a \leq b)} j \) (due to Lemma \ref{lem:2.10}(e)). Assume (for the sake of contradiction) that \( j \leq_{T\ni(a \leq b)} i \). Then, \( i \sim_{T\ni(a \leq b)} j \), and thus (by Lemma \ref{lem:2.10}(j), applied to \( c = i \) and \( d = j \)) we have \( (i \sim_{T} j \text{ or } (b \leq_{T} i \leq_{T} a \text{ and } b \leq_{T} j \leq_{T} a)) \). But neither of these two cases can occur (since \( i <_{T} j \) precludes \( i \sim_{T} j \), and since \( b \leq_{T} i \leq_{T} a \) contradicts our assumption that not \( b \leq_{T} a \)). Hence, we have our contradiction. Thus, our assumption (that \( j \leq_{T\ni(a \leq b)} i \)) was false. We therefore have \( i \leq_{T\ni(a \leq b)} j \) but not \( j \leq_{T\ni(a \leq b)} i \). In other words, \( i <_{T\ni(a \leq b)} j \). Thus, \( g(i) < g(j) \) (since \( g \in \mathcal{U}(\mathcal{T} \ni (a \leq b)) \)).

Now, let us forget that we fixed \( i \) and \( j \). We thus have shown that any two elements \( i \) and \( j \) of \( X \) satisfying \( i <_{T} j \) satisfy \( g(i) < g(j) \). In other words, \( g \in \mathcal{U}(\mathcal{T}) \) (since we already know that \( g \in \mathcal{P}(\mathcal{T}) \)). Thus, \( g \) is an element of \( \mathcal{U}(\mathcal{T}) \) and satisfies \( g(a) < g(b) \). In other words, \( g \in \{ f \in \mathcal{U}(\mathcal{T}) \mid f(a) < f(b) \} \).
Since this is proven for every \( g \in U (T \mapsto (a \leq b)) \), we thus conclude that
\[ U (T \mapsto (a \leq b)) \subseteq \{ f \in U (T) \mid f (a) < f (b) \}. \]

**Proof of \( \{ f \in U (T) \mid f (a) < f (b) \} \subseteq U (T \mapsto (a \leq b)) \):** Let
\[ g \in \{ f \in U (T) \mid f (a) < f (b) \}. \]
Then, \( g \in U (T) \) and \( g (a) < g (b) \). From \( g \in U (T) \), we obtain \( g \in P (T) \).

Let now \( c \in X \) and \( d \in X \) be such that \( c \leq_T (a \leq b) d \). We now aim to show
that \( g (c) \leq g (d) \).

Indeed, from \( c \leq_T (a \leq b) d \), we obtain \((c \leq_T d \text{ or } (c \leq_T a \text{ and } b \leq_T d))\) (by Lemma 2.10 (e)). In the first of these two cases, we obtain \((c \leq_T a \text{ and } g \in P (T))\) immediately (since \( g \in P (T) \)), while in the second case we obtain
\[
g (c) \leq g (a) \quad \text{(since } c \leq_T a \text{ and } g \in P (T))
< g (b) \leq g (d) \quad \text{(since } b \leq_T d \text{ and } g \in P (T)).
\]

Thus, \( g (c) \leq g (d) \) is proven in either case.

Now, let us forget that we fixed \( c \) and \( d \). We thus have proven that \( g (c) \leq g (d) \) for any \( c \in X \) and \( d \in X \) satisfying \( c \leq_T (a \leq b) d \). In other words, \( g \in P (T \mapsto (a \leq b)) \).

Now, let \( c \in X \) and \( d \in X \) be such that \( c \leq_T (a \leq b) d \). We now aim to show
that \( g (c) < g (d) \).

Indeed, from \( c \leq_T (a \leq b) d \), we obtain \( c \leq_T (a \leq b) d \), and thus
\((c \leq_T d \text{ or } (c \leq_T a \text{ and } b \leq_T d))\) (by Lemma 2.10 (e)). In the second of these two cases, we have
\[
g (c) \leq g (a) \quad \text{(since } c \leq_T a \text{ and } g \in P (T))
< g (b) \leq g (d) \quad \text{(since } b \leq_T d \text{ and } g \in P (T)).
\]

Thus, \( g (c) < g (d) \) is proven in the second case. We thus WLOG assume that we are in the first case. That is, we have \( c \leq_T d \). If \( c \prec_T d \), then we can immediately conclude that \( g (c) < g (d) \) (since \( g \in U (T) \)). Hence, we WLOG assume that we don’t have \( c \prec_T d \). Thus, \( c \sim_T d \) (since \( c \leq_T d \)), so that \( d \leq_T c \). Hence, \((d \leq_T c \text{ or } (d \leq_T a \text{ and } b \leq_T c))\), so that Lemma 2.10 (e) (applied to \( d \) and \( c \) instead of \( c \) and \( d \)) yields \( d \leq_T (a \leq b) c \). But this contradicts \( c \prec_T (a \leq b) d \). Thus, we have obtained a contradiction, and our proof of \( g (c) < g (d) \) is complete.

Now, let us forget that we fixed \( c \) and \( d \). We thus have proven that \( g (c) < g (d) \) for any \( c \in X \) and \( d \in X \) satisfying \( c \prec_T (a \leq b) d \). In other words, \( g \in U (T \mapsto (a \leq b)) \) (since \( g \in P (T \mapsto (a \leq b)) \)). Since this is proven for every \( g \in \{ f \in U (T) \mid f (a) < f (b) \} \), we thus conclude that \( \{ f \in U (T) \mid f (a) < f (b) \} \subseteq U (T \mapsto (a \leq b)) \).

Combining \( U (T \mapsto (a \leq b)) \subseteq \{ f \in U (T) \mid f (a) < f (b) \} \) with \( \{ f \in U (T) \mid f (a) < f (b) \} \subseteq U (T \mapsto (a \leq b)) \), we obtain (3).

Let us next check that \( a \sim_T (a \sim b) b \). Indeed, it is clear from the definition of \( T \mapsto (a \sim b) \) that \( a \leq_T (a \sim b) b \) and that \( b \leq_T (a \sim b) a \). Combining these, we obtain \( a \sim_T (a \sim b) b \).
Next, we are going to prove (4) by showing its two inclusions separately:
Proof of \( \mathcal{U} (T \mapsto (a \sim b)) \subseteq \{ f \in \mathcal{U} (T) \mid f (a) = f (b) \} \): Let \( g \in \mathcal{U} (T \mapsto (a \sim b)) \).
Thus, \( g \in \mathcal{P} (T \mapsto (a \sim b)) \), and every two elements \( i \) and \( j \) of \( X \) satisfying \( i \prec_{T^+(a \sim b)} j \) must satisfy \( g (i) < g (j) \). We have \( a \sim_{T^+(a \sim b)} b \) and \( g \in \mathcal{P} (T \mapsto (a \sim b)) \); thus, \( g (a) = g (b) \).

Moreover,
\[
g \in \mathcal{P} \left( T \mapsto (a \sim b) \right) = \mathcal{P} \left( T \mapsto (a \leq b) \right) \cap \mathcal{P} \left( T \mapsto (a \geq b) \right)
\]
(by Lemma 2.10 (l))
\[
\subseteq \mathcal{P} \left( T \mapsto (a \leq b) \right) \subseteq \mathcal{P} \left( T \mapsto (a \leq b) \right) \cup \mathcal{P} \left( T \mapsto (a \geq b) \right) = \mathcal{P} (T)
\]
(by Lemma 2.10 (l)).

Now, let \( i \) and \( j \) be any two elements of \( X \) satisfying \( i \prec_{T} j \). We shall show that \( g (i) < g (j) \).

Indeed, \( i \prec_{T^+} j \), thus \( i \preceq_{T^+} j \) and therefore \( i \preceq_{T^+(a \sim b)} j \) (due to Lemma 2.10 (g)). Assume (for the sake of contradiction) that \( j \preceq_{T^+(a \sim b)} i \). Then, \( i \sim_{T^+(a \sim b)} j \), and thus (by Lemma 2.10 (k), applied to \( c = i \) and \( d = j \)) we have \( (i \sim_{T} j) \) or \( (i \sim_{T} a \text{ and } j \sim_{T} b) \) or \( (i \sim_{T} b \text{ and } j \sim_{T} a) \). But neither of these three cases can occur. Hence, we have our contradiction. Thus, our assumption (that \( j \preceq_{T^+(a \sim b)} i \)) was false. We therefore have \( i \prec_{T^+(a \sim b)} j \) but not \( j \preceq_{T^+(a \sim b)} i \). In other words, \( i \prec_{T^+(a \sim b)} j \). Thus, \( g (i) < g (j) \) (since \( g \in \mathcal{U} \left( T \mapsto (a \sim b) \right) \)).

Now, let us forget that we fixed \( i \) and \( j \). We thus have shown that any two elements \( i \) and \( j \) of \( X \) satisfying \( i \prec_{T} j \) satisfy \( g (i) < g (j) \). In other words, \( g \in \mathcal{U} (T) \) (since we already know that \( g \in \mathcal{P} (T) \)). Thus, \( g \) is an element of \( \mathcal{U} \left( T \mapsto (a \sim b) \right) \) and satisfies \( g (a) = g (b) \). In other words, \( g \in \{ f \in \mathcal{U} (T) \mid f (a) = f (b) \} \).

Since this is proven for every \( g \in \mathcal{U} \left( T \mapsto (a \sim b) \right) \), we thus conclude that \( \mathcal{U} \left( T \mapsto (a \sim b) \right) \subseteq \{ f \in \mathcal{U} (T) \mid f (a) = f (b) \} \).

Proof of \( \{ f \in \mathcal{U} (T) \mid f (a) = f (b) \} \subseteq \mathcal{U} \left( T \mapsto (a \sim b) \right) \): Let \( g \in \{ f \in \mathcal{U} (T) \mid f (a) = f (b) \} \). Then, \( g \in \mathcal{U} (T) \) and \( g (a) = g (b) \). From \( g \in \mathcal{U} (T) \), we obtain \( g \in \mathcal{P} (T) \).

Let now \( c \in X \) and \( d \in X \) be such that \( c \leq_{T^+(a \sim b)} d \). We now aim to show that \( g (c) \leq g (d) \).

Indeed, from \( c \leq_{T^+(a \sim b)} d \), we obtain
\[
(c \leq_{T} d \text{ or } (c \leq_{T} a \text{ and } b \leq_{T} d) \text{ or } (c \leq_{T} b \text{ and } a \leq_{T} d) \) (by Lemma 2.10 (g)).
\]
In the first of these three cases, we obtain \( g (c) \leq g (d) \) immediately (since \( g \in \mathcal{P} (T) \)). In the second case, we obtain
\[
g (c) \leq g (a) \text{ (since } c \leq_{T} a \text{ and } g \in \mathcal{P} (T) \) \]
\[
= g (b) \leq g (d) \text{ (since } b \leq_{T} d \text{ and } g \in \mathcal{P} (T) \).
\]

\[\text{Indeed, the first case (} i \sim_{T} j \) is precluded by the fact that \( i \prec_{T} j \). The second case (} i \sim_{T} a \text{ and } j \sim_{T} b \) cannot occur since it would lead to \( a \sim_{T} i \preceq_{T} j \sim_{T} b \), which would contradict the assumption that we have neither \( a \leq_{T} b \) nor \( b \leq_{T} a \). The third case (} i \sim_{T} b \text{ and } j \sim_{T} a \) cannot occur for a similar reason.\]
In the third case, we obtain
\[
  g(c) \leq g(b) \quad \text{(since } c \leq_T b \text{ and } g \in \mathcal{P}(T)\text{)}
\]
\[
  = g(a) \leq g(d) \quad \text{(since } a \leq_T d \text{ and } g \in \mathcal{P}(T)\text{)}.
\]

Thus, \( g(c) \leq g(d) \) is proven in either case.

Now, let us forget that we fixed \( c \) and \( d \). We thus have proven that \( g(c) \leq g(d) \) for any \( c \in X \) and \( d \in X \) satisfying \( c \leq_{T+\varphi(a\sim b)} d \). In other words, \( g \in \mathcal{P}(T \leftrightarrow (a \sim b)) \).

Now, let \( c \in X \) and \( d \in X \) be such that \( c <_{T+\varphi(a\sim b)} d \). We now aim to show that \( g(c) < g(d) \).

Indeed, from \( c <_{T+\varphi(a\sim b)} d \), we obtain \( c \leq_{T+\varphi(a\sim b)} d \), and thus
\[
  (c \leq_T d \text{ or } (c \leq_T a \text{ and } b \leq_T d) \text{ or } (c \leq_T b \text{ and } a \leq_T d))
\]
(by Lemma 2.10 (g)). We study these three cases separately:

- Assume that we are in the first case, i.e., we have \( c \leq_T d \). Then, \( c <_{T} d \) (since otherwise, we would have \( d \leq_T c \) and therefore \( d <_{T+\varphi(a\sim b)} c \) (by Lemma 2.10 (g)), which would contradict \( c <_{T+\varphi(a\sim b)} d \)). Hence, \( g(c) < g(d) \) (since \( g \in \mathcal{U}(T) \)).

- Assume that we are in the second case, i.e., we have \( (c \leq_T a \text{ and } b \leq_T d) \). Then,
\[
  g(c) \leq g(a) \quad \text{(since } c \leq_T a \text{ and } g \in \mathcal{P}(T)\text{)}
\]
\[
  = g(b) \leq g(d) \quad \text{(since } b \leq_T d \text{ and } g \in \mathcal{P}(T)\text{)}.
\]

If at least one of the strict inequalities \( c <_{T} a \) or \( b <_{T} d \) holds, then we can strengthen this to a strict inequality \( g(c) < g(d) \) (because \( g \in \mathcal{U}(T) \)), and thus be done. Hence, we WLOG assume that none of the inequalities \( c <_{T} a \) or \( b <_{T} d \) holds. Thus, \( c \sim_{T} a \) and \( b \sim_{T} d \). Hence, \( c \sim_{T+\varphi(a\sim b)} a \) and \( b \sim_{T+\varphi(a\sim b)} d \) (by Lemma 2.10 (k)), so that \( c \sim_{T+\varphi(a\sim b)} a \sim_{T+\varphi(a\sim b)} b \sim_{T+\varphi(a\sim b)} d \), which contradicts \( c <_{T+\varphi(a\sim b)} d \). Hence, we are done in the second case as well.

- The third case is similar to the second case.

Thus, our proof of \( g(c) < g(d) \) is complete in each case.

Now, let us forget that we fixed \( c \) and \( d \). We thus have proven that \( g(c) < g(d) \) for any \( c \in X \) and \( d \in X \) satisfying \( c <_{T+\varphi(a\sim b)} d \). In other words, \( g \in \mathcal{U}(T \leftrightarrow (a \sim b)) \) (since \( g \in \mathcal{P}(T \leftrightarrow (a \sim b)) \)). Since this is proven for every \( g \in \{f \in \mathcal{U}(T) \mid f(a) = f(b)\} \), we thus conclude that \( \{f \in \mathcal{U}(T) \mid f(a) = f(b)\} \subseteq \mathcal{U}(T \leftrightarrow (a \sim b)) \).

Combining \( \mathcal{U}(T \leftrightarrow (a \sim b)) \subseteq \{f \in \mathcal{U}(T) \mid f(a) = f(b)\} \) with \( \{f \in \mathcal{U}(T) \mid f(a) = f(b)\} \subseteq \mathcal{U}(T \leftrightarrow (a \sim b)) \), we obtain (4). Now, our proof of Lemma 2.10 (m) is complete.
(n) If \( c \) and \( d \) are two elements of \( X \), then \( c \sim_{T+\rho(a \leq b)} \) \( d \) holds if and only if
\[
(c \sim_T d \text{ or } (b \leq_T c \leq_T a \text{ and } b \leq_T d \leq_T a))
\]
(according to Lemma 2.10 (j)). Since \( (b \leq_T c \leq_T a \text{ and } b \leq_T d \leq_T a) \) cannot hold (because of our assumption that \( b \leq_T a \)), this simplifies as follows: If \( c \) and \( d \) are two elements of \( X \), then \( c \sim_{T+\rho(a \leq b)} \) \( d \) holds if and only if \( c \sim_T d \). Thus, the equivalence relation \( \sim_{T+\rho(a \leq b)} \) is identical to \( \sim_T \). Hence, \( |X/ \sim_{T+\rho(a \leq b)}| = |X/ \sim_T| \). Similarly, \( |X/ \sim_{T+\rho(a \geq b)}| = |X/ \sim_T| \). Thus,
\[
|X/ \sim_{T+\rho(a \leq b)}| = |X/ \sim_{T+\rho(a \geq b)}| = |X/ \sim_T| - 1.
\]
Lemma 2.10 (k) yields the following: If \( c \) and \( d \) are two elements of \( X \), then \( c \sim_{T+\rho(a \sim b)} \) \( d \) holds if and only if
\[
(c \sim_T d \text{ or } (c \sim_T a \text{ and } d \sim_T b) \text{ or } (c \sim_T b \text{ and } d \sim_T a)).
\]
In other words, two elements of \( X \) are equivalent under the equivalence relation \( \sim_{T+\rho(a \sim b)} \) if and only if either they are equivalent under \( \sim_T \), or one of them is in the \( \sim_T \)-class of \( a \) while the other is in the \( \sim_T \)-class of \( b \). Thus, when passing from the equivalence relation \( \sim_T \) to \( \sim_{T+\rho(a \sim b)} \), the equivalence classes of \( a \) and \( b \) get merged (and these two classes used to be separate for \( \sim_T \), because of our assumption that neither \( a \leq_T b \) nor \( b \leq_T a \)), while all other equivalence classes stay as they were. Thus, the total number of equivalence classes decreases by 1. In other words, \( |X/ \sim_{T+\rho(a \sim b)}| = |X/ \sim_T| - 1 \). This completes the proof of Lemma 2.10 (n). \( \square \)

**Lemma 2.11.** Let \( n \in \mathbb{N} \) and \( T \in T_n \). Let \( a \) and \( b \) be two elements of \( [n] \). Then,
\[
1k_T = 1k_{T+\rho(a \leq b)} + 1k_{T+\rho(a \geq b)} - 1k_{T+\rho(a \sim b)}.
\]

**Proof of Lemma 2.11.** It is clearly enough to prove that
\[
K_T = K_{T+\rho(a \leq b)} \cap K_{T+\rho(a \geq b)} 
\]
and
\[
K_{T+\rho(a \sim b)} = K_{T+\rho(a \leq b)} \cup K_{T+\rho(a \geq b)}.
\]
Before we start proving these statements, let us rewrite the definition of \( K_S \) for any topology \( S \) on \( [n] \). Namely, if \( O \) is a subset of \( [n] \), then we define a subset \( K_O \) of \( \mathbb{R}^n \) by
\[
K_O = \left\{ (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid \sum_{i \in [n] \setminus O} x_i \geq 0 \right\}.
\]

16
Remark on polyhedral cones

April 26, 2020

It is now clear that any topology $\mathcal{S}$ on $[n]$ satisfies

$$K_\mathcal{S} = \bigcap_{O \in \mathcal{S}} K_O.$$  \hfill (8)

(Indeed, this is just a restatement of the definition of $K_\mathcal{S}$, since the closed sets of $\mathcal{S}$ are the sets of the form $[n] \setminus O$ with $O$ being an open set of $\mathcal{S}$.)

**Proof of (6):** From (8), we obtain $K_T = \bigcap_{O \in T} K_O$ and $K_{T+\varrho(a \leq b)} = \bigcap_{O \in T+\varrho(a \leq b)} K_O$ and $K_{T+\varrho(a \geq b)} = \bigcap_{O \in T+\varrho(a \geq b)} K_O$. Thus,

$$K_{T+\varrho(a \leq b)} \cap K_{T+\varrho(a \geq b)} = \left( \bigcap_{O \in T+\varrho(a \leq b)} K_O \right) \cap \left( \bigcap_{O \in T+\varrho(a \geq b)} K_O \right) = \bigcap_{O \in (T+\varrho(a \leq b)) \cup (T+\varrho(a \geq b))} K_O = \bigcap_{O \in T} K_O \quad \text{(by (2))}$$

This proves (6).

**Proof of (7):** It is easy to see that $K_{T+\varrho(a \leq b)} \subseteq K_{T+\varrho(a \sim b)}$ and similarly $K_{T+\varrho(a \geq b)} \subseteq K_{T+\varrho(a \sim b)}$. Combining these two relations, we obtain $K_{T+\varrho(a \leq b)} \cup K_{T+\varrho(a \geq b)} \subseteq K_{T+\varrho(a \sim b)}$. Hence, in order to prove (7), it remains to show that $K_{T+\varrho(a \sim b)} \subseteq K_{T+\varrho(a \leq b)} \cup K_{T+\varrho(a \geq b)}$. So let us do this now.

Let $y \in K_{T+\varrho(a \sim b)}$. Our goal is to show that $y \in K_{T+\varrho(a \leq b)} \cup K_{T+\varrho(a \geq b)}$. In fact, assume the contrary. Then, $y \notin K_{T+\varrho(a \leq b)}$ and $y \notin K_{T+\varrho(a \geq b)}$.

We have $y \notin K_{T+\varrho(a \leq b)} = \bigcap_{O \in T+\varrho(a \leq b)} K_O$ (by (8)). Hence, there exists a $P \in T \varrho (a \leq b)$ such that $y \notin K_P$. Similarly, using $y \notin K_{T+\varrho(a \geq b)}$, we can see that there exists a $Q \in T \varrho (a \geq b)$ such that $y \notin K_Q$. Consider these $P$ and $Q$.

We have $P \in T \varrho (a \leq b) = \{O \in T \mid (a \in O \implies b \in O)\}$. Thus, $P \in T$

\[\text{Proof. Indeed, (1) yields } (T \varrho (a \leq b)) \cap (T \varrho (a \geq b)) = T \varrho (a \sim b), \text{ so that } T \varrho (a \sim b) \subseteq T \varrho (a \leq b). \text{ Now, from (8), we obtain } K_{T+\varrho(a \leq b)} = \bigcap_{O \in T+\varrho(a \leq b)} K_O \text{ and } K_{T+\varrho(a \sim b)} = \bigcap_{O \in T+\varrho(a \sim b)} K_O. \text{ Thus,} \]

$$K_{T+\varrho(a \leq b)} = \bigcap_{O \in T+\varrho(a \leq b)} K_O \subseteq \bigcap_{O \in T+\varrho(a \sim b)} K_O \quad \text{(since } T \varrho (a \sim b) \subseteq T \varrho (a \leq b))$$

$$= K_{T+\varrho(a \sim b)},$$

qed.
Remark on polyhedral cones

Remark on polyhedral cones April 26, 2020

and \((a \in P \implies b \in P)\). But we do not have \((b \in P \implies a \in P)\). Hence, 
\(a \notin P\) and \(b \in P\) (since \((a \in P \implies b \in P)\) but not \((b \in P \implies a \in P)\)).

We have thus shown that \(P \in \mathcal{T}, a \notin P\) and \(b \in P\). Similarly, we find that 
\(Q \in \mathcal{T}, b \notin Q\) and \(a \in Q\). Now, it is easy to see that \(P \cap Q \in \mathcal{T} \implies (a \sim b)\)
and \(P \cup Q \in \mathcal{T} \implies (a \sim b)\).

Let us write \(y \in \mathbb{R}^n\) in the form \(y = (y_1, y_2, \ldots, y_n)\). We have \((y_1, y_2, \ldots, y_n) = \ y \notin K_P \ \leadsto \ \sum_{i \in [n]} y_i \geq 0\).

We have
\[
(y_1, y_2, \ldots, y_n) = y \in K_{\mathcal{T} \implies (a \sim b)} = \bigcap_{O \in \mathcal{T} \implies (a \sim b)} K_O \quad \text{(by (8))}
\leq K_{P \cap Q} \quad \text{(since } P \cap Q \in \mathcal{T} \implies (a \sim b)\text{)}
= \left\{ (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid \sum_{i \in [n]} x_i \geq 0 \right\},
\]
so that \(\sum_{i \in [n] \setminus (P \cap Q)} y_i \geq 0\). The same argument can be applied to \(P \cup Q\) instead of \(P \cap Q\), and leads to \(\sum_{i \in [n] \setminus (P \cup Q)} y_i \geq 0\). But any two subsets \(A\) and \(B\) of \([n]\) satisfy \(\sum_{i \in A} y_i + \sum_{i \in B} y_i = \sum_{i \in A \cup B} y_i + \sum_{i \in A \cap B} y_i\).

\(6\)\textbf{Proof.} Assume the contrary. Then, \((b \in P \implies a \in P)\). Combining this with \((a \in P \implies b \in P)\), we obtain \((a \in P \iff b \in P)\). Hence, \(P \in \mathcal{T} \implies (a \sim b)\) (by the definition of \(\mathcal{T} \implies (a \sim b)\)). Now, \(y \in K_{\mathcal{T} \implies (a \sim b)} = \bigcap_{O \in \mathcal{T} \implies (a \sim b)} K_O \quad \text{(by (8))}
\)
\(\bigcap_{O \in \mathcal{T} \implies (a \sim b)} K_O \subseteq K_P \quad \text{(since } P \in \mathcal{T} \implies (a \sim b)\text{)}, so that \(y \in \bigcap_{O \in \mathcal{T} \implies (a \sim b)} K_O \subseteq K_P\), which contradicts \(y \notin K_P\). This contradiction proves that our assumption was wrong, qed.

\(7\)\textbf{Proof.} From \(P \in \mathcal{T}\) and \(Q \in \mathcal{T}\), we infer that \(P \cap Q \in \mathcal{T}\). Also, \(a \notin P \cap Q\) (since \(a \notin P\)), so that \((a \in P \cap Q \implies b \in P \cap Q)\). Hence, \(P \cap Q \in \mathcal{T} \implies (a \sim b)\) (by the definition of \(\mathcal{T} \implies (a \sim b)\)). Now, \(y \in K_{\mathcal{T} \implies (a \sim b)} = \bigcap_{O \in \mathcal{T} \implies (a \sim b)} K_O \quad \text{(by (8))}\)
\(\bigcap_{O \in \mathcal{T} \implies (a \sim b)} K_O \subseteq K_{P \cap Q} \quad \text{(since } P \cap Q \in \mathcal{T} \implies (a \sim b)\text{)}, so that \(y \in \bigcap_{O \in \mathcal{T} \implies (a \sim b)} K_O \subseteq K_{P \cap Q}\), which contradicts \(y \notin K_{P \cap Q}\). This contradiction proves that our assumption was wrong, qed.

\(8\)\textbf{Proof.} From \(P \in \mathcal{T}\) and \(Q \in \mathcal{T}\), we infer that \(P \cup Q \in \mathcal{T}\). Also, \(a \notin P \cup Q\) (since \(a \notin Q\)), so that \((a \in P \cup Q \implies b \in P \cup Q)\). Hence, \(P \cup Q \in \mathcal{T} \implies (a \sim b)\) (by the definition of \(\mathcal{T} \implies (a \sim b)\)). Now, \(y \in K_{\mathcal{T} \implies (a \sim b)} = \bigcap_{O \in \mathcal{T} \implies (a \sim b)} K_O \quad \text{(by (8))}\)
\(\bigcap_{O \in \mathcal{T} \implies (a \sim b)} K_O \subseteq K_{P \cup Q} \quad \text{(since } P \cup Q \in \mathcal{T} \implies (a \sim b)\text{)}, so that \(y \in \bigcap_{O \in \mathcal{T} \implies (a \sim b)} K_O \subseteq K_{P \cup Q}\), which contradicts \(y \notin K_{P \cup Q}\). This contradiction proves that our assumption was wrong, qed.
Applying this to $A = [n] \setminus P$ and $B = [n] \setminus Q$, we obtain

$$
\sum_{i \in [n] \setminus P} y_i + \sum_{i \in [n] \setminus Q} y_i = \sum_{i \in ([n] \setminus P) \cup ([n] \setminus Q)} y_i + \sum_{i \in ([n] \setminus P) \cap ([n] \setminus Q)} y_i
$$

(since $([n] \setminus P) \cup ([n] \setminus Q) = [n] \setminus (P \cap Q)$ and $([n] \setminus P) \cap ([n] \setminus Q) = [n] \setminus (P \cup Q)$).

Thus,

$$
\sum_{i \in [n] \setminus (P \cap Q)} y_i + \sum_{i \in [n] \setminus (P \cup Q)} y_i = \sum_{i \in [n] \setminus P} y_i + \sum_{i \in [n] \setminus Q} y_i < 0.
$$

This contradicts

$$
\sum_{i \in [n] \setminus (P \cap Q)} y_i + \sum_{i \in [n] \setminus (P \cup Q)} y_i \geq 0.
$$

This contradiction proves that our assumption was wrong. Hence, $y \in K_{T \leftrightarrow (a \leq b)} \cup K_{T \leftrightarrow (a > b)}$. Since we have proven this for every $y \in K_{T \leftrightarrow (a \leq b)}$, we thus conclude that $K_{T \leftrightarrow (a \leq b)} \subseteq K_{T \leftrightarrow (a \leq b)} \cup K_{T \leftrightarrow (a > b)}$. This finishes the proof of (7).

Now that both (6) and (7) are proven, Lemma 2.11 easily follows.

**Definition 2.12.** Let $V$ be a $\mathbb{K}$-vector space. A $\mathbb{K}$-linear map $f : H_T \to V$ is said to be $T$-additive if and only if every $n \in \mathbb{N}$, every $T \in T_n$ and every two distinct elements $a$ and $b$ of $[n]$ satisfy

$$
f(T) = f(T \leftrightarrow (a \leq b)) + f(T \leftrightarrow (a > b)) - f(T \leftrightarrow (a \sim b)). \quad (9)
$$

**Proposition 2.13.** Let $V$ be a $\mathbb{K}$-vector space. Let $f$ and $g$ be two $T$-additive $\mathbb{K}$-linear maps $H_T \to V$. Assume that $f(T_u) = g(T_u)$ for every packed word $u$. Then, $f = g$.

**Proof of Proposition 2.13** It is clearly enough to show that

$$
f(T) = g(T) \quad \text{for every } T \in T. \quad (10)
$$

For any topology $T$ on a finite set $X$, we let $h(T)$ denote the nonnegative integer $\sharp \{(x, y) \in X^2 \mid \text{neither } x \leq_T y \text{ nor } y \leq_T x\}$. We shall prove (10) by strong induction over $h(T)$. So we fix some $T \in T_n$, and we want to prove (10), assuming that every $S \in T$ satisfying $h(S) < h(T)$ satisfies

$$
f(S) = g(S). \quad (11)
$$

Let $n \in \mathbb{N}$ be such that $T \in T_n$. If there exist no two elements $a$ and $b$ of $[n]$ satisfying neither $a \leq_T b$ nor $b \leq_T a$, then we have $T = T_u$ for some
packed word $u$, and this $u$ satisfies $f(T_u) = g(T_u)$ (due to the assumption of the proposition); thus, (10) follows immediately (since $T = T_u$). Hence, we can WLOG assume that such two elements $a$ and $b$ exist. Consider these two elements. Of course, $a$ and $b$ are distinct.

If $S$ is any of the three posets $T \leftrightarrow (a \leq b)$, $T \leftrightarrow (a \geq b)$ and $T \leftrightarrow (a \sim b)$, then $h(S) < h(T)$. Hence, we can apply (11) to each of these three posets. We obtain

$$f(T \leftrightarrow (a \leq b)) = g(T \leftrightarrow (a \leq b));$$

$$f(T \leftrightarrow (a \geq b)) = g(T \leftrightarrow (a \geq b));$$

$$f(T \leftrightarrow (a \sim b)) = g(T \leftrightarrow (a \sim b)).$$

But since $f$ is $T$-additive, we have

$$f(T) = \frac{f(T \leftrightarrow (a \leq b))}{g(T \leftrightarrow (a \leq b))} + \frac{f(T \leftrightarrow (a \geq b))}{g(T \leftrightarrow (a \geq b))} - \frac{f(T \leftrightarrow (a \sim b))}{g(T \leftrightarrow (a \sim b))} = g(T \leftrightarrow (a \leq b)) + g(T \leftrightarrow (a \geq b)) - g(T \leftrightarrow (a \sim b)) = g(T)$$

(since $g$ is $T$-additive). Thus, (10) is proven, and the induction step is complete. \qed

Proof of Theorem 2.7 (sketched). We need to show that $\beta = \alpha \circ U$.

We notice that every topology $S$ on $[n]$ satisfies

$$f(S) = \begin{cases} f(T \leftrightarrow (a \leq b)) + f(T \leftrightarrow (a \geq b)) - f(T \leftrightarrow (a \sim b)) \\ = g(T \leftrightarrow (a \leq b)) + g(T \leftrightarrow (a \geq b)) - g(T \leftrightarrow (a \sim b)) \end{cases}$$

(by the definition of $Z$)

$$= (-1)^{|[n]/\sim S|} \beta(S)$$

(by the definition of $\beta$)

$$= 1_K_S$$

(12)

This is because $\{x,y \in X^2 \mid \text{neither } x \leq S y \text{ nor } y \leq S x\}$ is a proper subset of $\{x,y \in X^2 \mid \text{neither } x \leq T y \text{ nor } y \leq T x\}$. (Proper because $(a,b)$ or $(b,a)$ belongs to the latter but not to the former.)
and

\[(α ∘ U ∘ Z) (S) = α \left( U \left( Z (S) \right) \right) = (-1)^{|[n]/\sim_S|} \alpha \left( \sum_{f ∈ U(S)} f \right)\]

(by the definition of \(Z\)).

(13)

We shall now show that both maps \(β ∘ Z : H_T \to WQSym\) and \(α ∘ U ∘ Z : H_T \to WQSym\) are \(T\)-additive.

Proof that the map \(β ∘ Z\) is \(T\)-additive: Let \(n ∈ \mathbb{N}\). Let \(T ∈ T_n\). Let \(a\) and \(b\) be two distinct elements of \([n]\). In order to show that \(β ∘ Z\) is \(T\)-additive, we must prove that

\[(β ∘ Z) (T) = (β ∘ Z) (T \leftrightarrow (a ≤ b)) + (β ∘ Z) (T \leftrightarrow (a ≥ b)) − (β ∘ Z) (T \leftrightarrow (a ∼ b)) .\]

(14)

This rewrites as follows:

\[1_{K_T} = 1_{K_{T \leftrightarrow (a ≤ b)}} + 1_{K_{T \leftrightarrow (a ≥ b)}} - 1_{K_{T \leftrightarrow (a ∼ b)}} \]

(because of (12)). But this is precisely the claim of Lemma 2.11. Hence, (14) is proven. We thus have shown that the map \(β ∘ Z\) is \(T\)-additive.

Proof that the map \(α ∘ U ∘ Z\) is \(T\)-additive: Let \(n ∈ \mathbb{N}\). Let \(T ∈ T_n\). Let \(a\) and \(b\) be two distinct elements of \([n]\). In order to show that \(α ∘ U ∘ Z\) is \(T\)-additive, we must prove that

\[(α ∘ U ∘ Z) (T) = (α ∘ U ∘ Z) (T \leftrightarrow (a ≤ b)) + (α ∘ U ∘ Z) (T \leftrightarrow (a ≥ b)) - (α ∘ U ∘ Z) (T \leftrightarrow (a ∼ b)) .\]

(15)

This is rather obvious if \(a ≤_T b\) \[10\]. Hence, for the rest of this proof, we

\[\text{Assume that } a ≤_T b. \text{ Then, Lemma 2.10 (a) yields } T \leftrightarrow (a ≤ b) = T \text{ and } T \leftrightarrow (a ∼ b) = T \leftrightarrow (a ≥ b). \text{ Hence, (15) rewrites as}\]

\[(α ∘ U ∘ Z) (T) = (α ∘ U ∘ Z) (T) + (α ∘ U ∘ Z) (T \leftrightarrow (a ≥ b)) - (α ∘ U ∘ Z) (T \leftrightarrow (a ≥ b)).\]

But this is obvious.

\[\text{Proof. Assume that } a ≤_T b. \text{ Then, Lemma 2.10 (a) yields } T \leftrightarrow (a ≤ b) = T \text{ and } T \leftrightarrow (a ∼ b) = T \leftrightarrow (a ≥ b). \text{ Hence, (15) rewrites as}\]

\[(α ∘ U ∘ Z) (T) = (α ∘ U ∘ Z) (T) + (α ∘ U ∘ Z) (T \leftrightarrow (a ≥ b)) - (α ∘ U ∘ Z) (T \leftrightarrow (a ≥ b)).\]

But this is obvious.
WLOG assume that we don’t have \( a \leq T b \). Similarly, we WLOG assume that we don’t have \( b \leq T a \). Now, using (13), we can rewrite the equality (15) as follows:

\[
(-1)^{|n|/\sim_T} \sum_{f \in U(T)} \alpha(f) \]

\[
= (-1)^{|n|/\sim_T} \sum_{f \in U(T \sim (a \leq b))} \alpha(f) + (-1)^{|n|/\sim_{T \sim (a \geq b)}} \sum_{f \in U(T \sim (a \geq b))} \alpha(f) \\
- (-1)^{|n|/\sim_{T \sim (a \sim b)}} \sum_{f \in U(T \sim (a \sim b))} \alpha(f).
\]

This can be rewritten further as

\[
(-1)^{|n|/\sim_T} \sum_{f \in U(T)} \alpha(f) \\
= (-1)^{|n|/\sim_T} \sum_{f \in U(T \sim (a \leq b))} \alpha(f) + (-1)^{|n|/\sim_{T \sim (a \geq b)}} \sum_{f \in U(T \sim (a \geq b))} \alpha(f) \\
- (-1)^{|n|/\sim_{T \sim (a \sim b)}} \sum_{f \in U(T \sim (a \sim b))} \alpha(f)
\]

(because Lemma 2.10 (m) (applied to \( X = \{n\} \)) yields

\[
|n|/\sim_{T \sim (a \leq b)} = |n|/\sim_{T \sim (a \geq b)} = |n|/\sim_T \quad \text{and}
\]

\[
|n|/\sim_{T \sim (a \sim b)} = |n|/\sim_T - 1. \quad \text{Upon cancelling } (-1)^{|n|/\sim_T}, \text{this simplifies to}
\]

\[
\sum_{f \in U(T)} \alpha(f) = \sum_{f \in U(T \sim (a \leq b))} \alpha(f) + \sum_{f \in U(T \sim (a \geq b))} \alpha(f) + \sum_{f \in U(T \sim (a \sim b))} \alpha(f).
\]

But this follows immediately from Lemma 2.10 (m) (applied to \( X = \{n\} \)). Thus, (15) is proven. We have thus shown that \( \alpha \circ U \circ Z \) is \( T \)-additive.

Now, it is easy to see that \( (\beta \circ Z)(T_u) = (\alpha \circ U \circ Z)(T_u) \) for every packed word \( u \).\footnote{Proof. Let \( u \) be a packed word. Applying (12) to \( S = T_u \), we obtain \( (\beta \circ Z)(T_u) = 1_{K_{T_u}} = 1_{K_u} \) (since Remark 2.2 yields \( K_{T_u} = K_u \)). But applying (13) to \( S = T_u \) leads to

\[
(\alpha \circ U \circ Z)(T_u) = (-1)^{|n|/\sim_{T_u}} \sum_{f \in U(T_u)} \alpha(f)
\]

\[
= (-1)^{\max_u} \sum_{f \in U(T_u)} \alpha(f) \quad \text{(since } |n|/\sim_{T_u} = \max u \text{)}
\]

\[
= (-1)^{\max_u} \sum_{f \in U(T_u)} \alpha(f) \quad \text{(since } u(T_u) = \{u\} \text{)}
\]

\[
= (-1)^{\max_u} \alpha(u) \quad \text{(by the definition of } a \text{)}
\]

\[
= (-1)^{\max_u} \frac{1}{K_u} = (-1)^{\max_u} \frac{1}{K_u} 1_{K_u} = 1_{K_u} \quad \text{(by the definition of } a \text{)}
\]

\[
= (\beta \circ Z)(T_u).
\]

qed.
Proof of Theorem 1.4. Theorem 2.7 yields $\beta = \alpha \circ U$. Since both $\beta$ and $U$ are $K$-algebra homomorphisms, and since $U$ is surjective, this easily yields that $\alpha$ is a $K$-algebra homomorphism. (Indeed, let $p \in \text{WQSym}$ and $q \in \text{WQSym}$. Then, thanks to the surjectivity of $U$, there exist $P \in \mathcal{H}_T$ and $Q \in \mathcal{H}_T$ satisfying $p = U(P)$ and $q = U(Q)$. Consider these $P$ and $Q$. Since $U$ is a $K$-algebra homomorphism, we have $U(P \cdot Q) = U(P)U(Q) = pq$. Now,)

$$\alpha \left( \begin{array}{c} \frac{p}{U(P)} \end{array} \right) \cdot \alpha \left( \begin{array}{c} \frac{q}{U(Q)} \end{array} \right) = \alpha \left( \begin{array}{c} \frac{U(P)}{(\alpha \circ U)(P)} \end{array} \right) \cdot \alpha \left( \begin{array}{c} \frac{U(Q)}{(\alpha \circ U)(Q)} \end{array} \right) = \beta \cdot \beta = \beta \left( \begin{array}{c} \frac{P \cdot Q}{(\alpha \circ U)(P \cdot Q)} \end{array} \right) = \beta \left( \begin{array}{c} \frac{P}{\alpha \circ U} \end{array} \right) \cdot \beta \left( \begin{array}{c} \frac{Q}{\alpha \circ U} \end{array} \right) = \beta \left( \begin{array}{c} \frac{P \cdot Q}{pq} \end{array} \right) = \left( \begin{array}{c} \frac{\alpha \circ U}{\alpha} \end{array} \right) \left( \begin{array}{c} \frac{U(P \cdot Q)}{pq} \end{array} \right) = \left( \begin{array}{c} \frac{\alpha}{\beta} \end{array} \right) \left( \begin{array}{c} \frac{U(P \cdot Q)}{pq} \end{array} \right) = \alpha \left( \begin{array}{c} \frac{pq}{\alpha(U(P \cdot Q))} \end{array} \right),$$

and this shows that $\alpha$ is a $K$-algebra homomorphism.) Theorem 1.4 is proven. \hfill \square

3. Application: an alternating sum identity

As an application of Theorem 1.4 we can prove the following fact, which is analogous to [3, Corollary 4.8]:

**Corollary 3.1.** Let $n \in \mathbb{N}$. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}^n$. Then,

$$\sum_{u \text{ is a packed word of length } n; \lambda \in K_u} (-1)^{\text{max } u} = \begin{cases} (-1)^n, & \text{if } \lambda_1, \lambda_2, \ldots, \lambda_n \geq 0; \\ 0, & \text{otherwise.} \end{cases}$$

This will rely on the following equality in WQSym:

**Proposition 3.2.** Let $\zeta$ be the packed word (1) of length 1. Then, in WQSym, we have

$$\zeta^n = \sum_{u \text{ is a packed word of length } n} u.$$
**Proof sketch.** Induction on \( n \) (details are left to the reader). \( \square \)

**Proof of Corollary 3.1.** Let \( \mathbb{R}_+ \) denote the set of all nonnegative reals. Let \( \zeta \in \text{WQSym} \) be the packed word \((1)\) of length 1.

Consider the map \( \alpha \) from Theorem 1.4. The definition of this map \( \alpha \) yields

\[
\alpha (\zeta) = (-1)^{\max \zeta} 1_{K_\zeta} = -1_{K_\zeta} = -1_{\mathbb{R}_+} \quad \text{(since \( \max \zeta = 1 \))}
\]

(since the definition of \( K_\zeta \) yields \( K_\zeta = \mathbb{R}_+ \)). Hence,

\[
(\alpha (\zeta))^n = (-1_{\mathbb{R}_+})^n = (-1)^n 1_{\mathbb{R}_+}^n = (-1)^n 1_{\mathbb{R}_+}. \quad \text{(this follows easily from the definition of multiplication on \( M \))}
\]

But Proposition 3.2 yields

\[
\zeta^n = \sum_{u \text{ is a packed word of length } n} u.
\]

Applying the map \( \alpha \) to both sides of this equality, we obtain

\[
\alpha (\zeta^n) = \alpha \left( \sum_{u \text{ is a packed word of length } n} u \right) = \sum_{u \text{ is a packed word of length } n} \alpha (u) = (-1)^{\max u} 1_{K_u} \quad \text{(by the definition of } \alpha)\]

Applying both sides of this equality to \( \lambda \), we obtain

\[
(\alpha (\zeta^n)) (\lambda) = \sum_{u \text{ is a packed word of length } n} (-1)^{\max u} 1_{K_u} (\lambda) = \begin{cases} 1, & \text{if } \lambda \in K_u; \\ 0, & \text{if } \lambda \not\in K_u \end{cases} \quad \text{(by the definition of } 1_{K_u})
\]

\[
= \sum_{u \text{ is a packed word of length } n} (-1)^{\max u} \begin{cases} 1, & \text{if } \lambda \in K_u; \\ 0, & \text{if } \lambda \not\in K_u \end{cases} \]

\[
= \sum_{u \text{ is a packed word of length } n; \lambda \in K_u} (-1)^{\max u}. \]

---

24
Hence,\[
\sum_{\text{u is a packed word of length } n; \lambda \in K_u} (-1)^{\max u} = \left(\frac{a(\xi^n)}{a(\xi)}\right) (\lambda) = \left(\frac{a(\xi)}{1}\right)^n (\lambda)
\]

(since \(a\) is a \(K\)-algebra homomorphism)\[
= (-1)^n \left\langle 1_{R_+^n} \right\rangle (\lambda) = (-1)^n \begin{cases} 1, & \text{if } \lambda \in R_+^n; \\ 0, & \text{otherwise} \end{cases}
\]

\[
= \begin{cases} (-1)^n, & \text{if } \lambda \in R_+^n; \\ 0, & \text{otherwise} \end{cases}
\]

\[
= \begin{cases} (-1)^n, & \text{if } \lambda_1, \lambda_2, \ldots, \lambda_n \geq 0; \\ 0, & \text{otherwise} \end{cases}
\]

(since the condition "\(\lambda \in R_+^n\)" is equivalent to "\(\lambda_1, \lambda_2, \ldots, \lambda_n \geq 0\)"). This proves Corollary 3.1. \(\square\)

From Corollary 3.1, we can in turn derive the precise statement of [3, Corollary 4.8]:

**Corollary 3.3.** Let \(n \in \mathbb{N}\). Let \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}^n\). Then,

\[
\sum_{\text{u is a packed word of length } n; \lambda \in K_u^o} (-1)^{\max u} = \left\{ \begin{array}{ll} (-1)^n, & \text{if } \lambda_1, \lambda_2, \ldots, \lambda_n > 0; \\ 0, & \text{otherwise} \end{array} \right. 
\]

Here, for any packed word \(u\) of length \(n\), we define the subset \(K_u^o\) of \(\mathbb{R}^n\) in the same way as we defined \(K_u\), but with the "\(\geq\)" sign replaced by "\(>\).

**Proof sketch.** Pick a small \(\varepsilon > 0\), and let \(\lambda' := (\lambda_1 - \varepsilon, \lambda_2 - \varepsilon, \ldots, \lambda_n - \varepsilon)\). If \(\varepsilon\) has been chosen small enough (say, \(0 < \varepsilon < \frac{1}{n} \min \left\{ \sum_{i \in I} \lambda_i \mid I \subseteq [n] \text{ satisfying } \sum_{i \in I} \lambda_i > 0 \right\} \)), then any packed word \(u\) of length \(n\) will satisfy \(\lambda \in K_u^o\) if and only if it satisfies \(\lambda' \in K_u\), and we will have \(\lambda_1, \lambda_2, \ldots, \lambda_n > 0\) if and only if \(\lambda_1 - \varepsilon, \lambda_2 - \varepsilon, \ldots, \lambda_n - \varepsilon \geq 0\). Hence, Corollary 3.3 follows from Corollary 3.1 (applied to \(\lambda'\) and \(\lambda_i - \varepsilon\) instead of \(\lambda\) and \(\lambda_i\)). \(\square\)
References


   [http://alco.centre-mersenne.org/item/ALCO_2019__2_5_863_0](http://alco.centre-mersenne.org/item/ALCO_2019__2_5_863_0).