Polynomials over Symmetric Polynomials

[expository note based on the literature and emails of Darij Grinberg and Victor Reiner]

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Abstract

Abstract. Let $\mathbf{k}$ be a commutative ring, and let the symmetric group $\mathfrak{S}_n$ act on $P = \mathbf{k}[x_1, x_2, \ldots, x_n]$ by permuting the variables. We prove four classical results. First, the coinvariant algebra (the quotient of P by the ideal generated by the symmetric polynomials with constant term 0) is a free $\mathbf{k}$-module of rank $n!$, with the residue classes of the Artin monomials as a basis. Second, P is a free module of rank $n!$ over the ring $P^{\mathfrak{S}_n}$ of symmetric polynomials, again with the Artin monomials as a basis. Third, if $n!$ is invertible in $\mathbf{k}$, the coinvariant algebra is the regular $\mathbf{k}[\mathfrak{S}_n]$-module. Fourth, under the same hypothesis, P is a free left $P^{\mathfrak{S}_n}[\mathfrak{S}_n]$-module of rank 1. The first result follows from an elementary normal-form lemma for monic polynomials with pairwise relatively prime leading monomials. The second is proved by lifting the Artin basis. For the third, we use orbit harmonics with a strongly discrete point orbit, and the fourth follows by equivariantly lifting a regular basis of the coinvariant algebra.

Manifest. What follows is a series of folklore arguments that are not easily found in the literature, at least not all in one place. None of these is new; I have learned some of them from Vic Reiner and Brendon Rhoades. The text below was produced by GPT-5.6 based on a telegraphic outline. I have then edited it. – DG[1]

1 Statement and notation

Let n be a positive integer. Set $[n] = \{1, 2, \ldots, n\}$, and let $\mathfrak{S}_n$ denote the n-th *symmetric group*, that is, the group of all bijections $[n] \to [n]$. Throughout, $\mathbf{k}$ is a commutative ring with identity and $1_{\mathbf{k}} \neq 0$. (The requirement $1_{\mathbf{k}} \neq 0$ is unnecessary in essence; it just serves to make certain statements hold literally, e.g., ensuring that each free $\mathbf{k}$-module has a unique rank.[2]) For specific claims, we will impose further requirements on $\mathbf{k}$.

Consider the polynomial ring

$$P = \mathbf{k}[x_1, x_2, \ldots, x_n].$$

[1] This work is in the public domain.

[2] Of course, the case when $1_{\mathbf{k}} = 0$ is trivial, since $\mathbf{k}$ is a one-element set in this case.

We give P its usual grading by total degree:[3]

$$P = \bigoplus_{d \geq 0} P_d,$$

where P_d is the free $\mathbf{k}$-module of homogeneous polynomials of total degree d. The group $\mathfrak{S}_n$ acts on P (and on each P_d) by permuting the variables:

$$\sigma(x_i) = x_{\sigma(i)} \qquad \text{for all } \sigma \in \mathfrak{S}_n \text{ and } i \in [n].$$

This is an action by $\mathbf{k}$-algebra automorphisms.

Let

$$\Lambda = P^{\mathfrak{S}_n} = \{f \in P \mid \sigma(f) = f \text{ for all } \sigma \in \mathfrak{S}_n\}$$

be the subring of symmetric polynomials. For each $i \in [n]$, let

$$e_i = e_i(x_1, x_2, \ldots, x_n) = \sum_{1 \leq j_1 < j_2 < \cdots < j_i \leq n} x_{j_1} x_{j_2} \cdots x_{j_i}$$

denote the i-th elementary symmetric polynomial. The fundamental theorem of symmetric polynomials gives

$$\Lambda = \mathbf{k}[e_1, e_2, \ldots, e_n], \tag{1}$$

and says that $e_1, e_2, \ldots, e_n$ are algebraically independent over $\mathbf{k}$. Equivalently, the substitution map

$$\mathbf{k}[t_1, t_2, \ldots, t_n] \longrightarrow \Lambda, \qquad t_i \longmapsto e_i$$

(a $\mathbf{k}$-algebra homomorphism from the polynomial ring $\mathbf{k}[t_1, t_2, \ldots, t_n]$), is an isomorphism. The ring Λ is a graded subring of P; write

$$\Lambda_+ = \bigoplus_{d > 0} \Lambda_d$$

for its ideal of elements with constant term 0. This ideal Λ_+ is generated by the elementary symmetric polynomials $e_1, e_2, \ldots, e_n$:

Proposition 1. *We have*

$$\Lambda_+ = e_1\Lambda + e_2\Lambda + \cdots + e_n\Lambda. \tag{2}$$

Proof. The inclusion "$\supseteq$" is clear, since each e_i belongs to Λ_i and thus to Λ_+.

Conversely, let $a \in \Lambda_+$. By (1), we can write $a = F(e_1, e_2, \ldots, e_n)$ for some $F \in T := \mathbf{k}[t_1, t_2, \ldots, t_n]$. Evaluating at $x_1 = x_2 = \cdots = x_n = 0$ gives

$$\begin{aligned} F(0, 0, \ldots, 0) &= a(0, 0, \ldots, 0) \qquad (\text{since } e_i(0, 0, \ldots, 0) = 0 \text{ for all } i > 0) \\ &= 0 \qquad (\text{since } a \text{ has constant term } 0)\,. \end{aligned}$$

Hence $F \in t_1T + t_2T + \cdots + t_nT$. Substituting $t_i = e_i$ yields $a \in e_1\Lambda + e_2\Lambda + \cdots + e_n\Lambda$ (since $F(e_1, e_2, \ldots, e_n) = a$). This proves the "$\subseteq$" inclusion. $\square$

[3]All gradings are $\mathbb{N}$-gradings in this note. If V is a graded abelian group, then V_d shall mean the d-th graded component of V.

Extending the ideal $\Lambda_+ = e_1\Lambda + e_2\Lambda + \cdots + e_n\Lambda$ from Λ to P gives the ideal

$$\begin{aligned} J := \Lambda_+ P &= \left(e_1\Lambda + e_2\Lambda + \cdots + e_n\Lambda\right) P \qquad \text{(by (2))} \\ &= e_1 P + e_2 P + \cdots + e_n P \end{aligned} \tag{3}$$

of P. The *coinvariant algebra* (of $\mathfrak{S}_n$ acting on P) is defined to be the quotient $\mathbf{k}$-algebra

$$C := P/J.$$

For any $f \in \Lambda$, $g \in P$, and $\sigma \in \mathfrak{S}_n$, we have

$$\begin{aligned} \sigma(fg) &= \sigma(f)\sigma(g) \qquad \text{(since } \mathfrak{S}_n \text{ acts by algebra automorphisms)} \\ &= f\sigma(g) \qquad \text{(since } f \in \Lambda \text{ entails } \sigma(f) = f\text{)}. \end{aligned} \tag{4}$$

Thus the action of $\mathfrak{S}_n$ on P is Λ-linear, and P is naturally a left module over the group algebra $\Lambda[\mathfrak{S}_n]$.

The action of $\mathfrak{S}_n$ on P also preserves the ideal J, since J is generated by symmetric polynomials. Hence the quotient algebra $C = P/J$ inherits an action of $\mathfrak{S}_n$ and is a left module over $\mathbf{k}[\mathfrak{S}_n]$.

We will prove the following folklore results mentioned, among other places, in [11, Section 1.5]:

Theorem 2.

(a) *The $\mathbf{k}$-module C is free of rank $n!$. More explicitly, call the $n!$ monomials*

$$x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} \qquad \textit{with} \qquad 0 \leq a_i < i \quad \textit{for every } i \in [n]$$

the Artin monomials. *Their residue classes in C form a $\mathbf{k}$-basis of C.*

(b) *The Λ-module P is free of rank $n!$; the Artin monomials form a Λ-basis of P.*

(c) *Assume that $n!$ is invertible in $\mathbf{k}$. Then* [4]

$$C \cong \mathbf{k}[\mathfrak{S}_n] \qquad \textit{as a } \mathbf{k}[\mathfrak{S}_n]\textit{-module.}$$

In other words, the coinvariant algebra $C = P/J$ is isomorphic to the regular $\mathbf{k}[\mathfrak{S}_n]$-module.

(d) *Assume that $n!$ is invertible in $\mathbf{k}$. Then*

$$P \cong \Lambda[\mathfrak{S}_n] \qquad \textit{as a } \Lambda[\mathfrak{S}_n]\textit{-module.}$$

In other words, P is isomorphic to the regular $\Lambda[\mathfrak{S}_n]$-module. Equivalently, there exists an element $f \in P$ such that the map

$$\Phi : \Lambda[\mathfrak{S}_n] \longrightarrow P, \qquad \sum_{\sigma \in \mathfrak{S}_n} a_\sigma \sigma \longmapsto \sum_{\sigma \in \mathfrak{S}_n} a_\sigma \sigma(f) \tag{5}$$

is an isomorphism of $\Lambda[\mathfrak{S}_n]$-modules.

[4] Here and in the following, "A-module" for a ring A shall always mean "left A-module". Thus, a $\mathbf{k}[\mathfrak{S}_n]$-module is the same thing as a representation of $\mathfrak{S}_n$ over $\mathbf{k}$.

We first prove an elementary normal-form lemma and use it to compute the coinvariant algebra, proving **(a)**. A lifting argument then proves **(b)**. Next, an orbit-harmonics argument proves **(c)**. Finally, combining the lifting argument with **(c)** proves **(d)**. The invertibility hypothesis in **(c)** and **(d)** cannot simply be omitted; we give a modular counterexample at the end.

2 A coprime-leading-monomial lemma

In this section, we shall prove a general property of certain ideals in polynomial rings. This property is part of the theory of Gröbner bases (see Remark 4), but we will give a self-contained proof avoiding the general theory, as the property is the only thing we will need.

Let R be a commutative ring with identity. Consider the polynomial ring

$$A := R[z_1, z_2, \ldots, z_N],$$

and fix a monomial order $\prec$ on A. That is, $\prec$ is a total well-order on the set of all monomials in A, and

$$u \prec v \quad \Longrightarrow \quad uw \prec vw$$

for all monomials u, v, w. For a nonzero polynomial $f \in A$, its *leading monomial* $\mathrm{LM}(f)$ is the largest monomial that occurs in f with a nonzero coefficient. We call f *monic* if the coefficient of $\mathrm{LM}(f)$ is 1. We say that two monomials u and v are *relatively prime* if the only monomial that divides them both is 1; equivalently, this means that they have no variable in common. (For instance, the monomials $z_1 z_3^2$ and $z_2 z_5$ are relatively prime, but the monomials $z_1 z_3^2$ and $z_3 z_5$ are not.)

Lemma 3 (Coprime-leading-monomial lemma)**.** *Let $g_1, g_2, \ldots, g_k \in A$ be monic polynomials whose leading monomials*

$$m_i = \mathrm{LM}(g_i) \qquad (\text{for } i \in [k])$$

are pairwise relatively prime. A monomial q will be called reduced *if it is not divisible by any of $m_1, m_2, \ldots, m_k$ (that is, if $m_i \nmid q$ for each $i \in [k]$). Define an ideal I of A by*

$$I = g_1 A + g_2 A + \cdots + g_k A.$$

Then the residue classes of the reduced monomials form an R-module basis of A/I.

Proof. For each $i \in [k]$, write

$$g_i = m_i - r_i, \tag{6}$$

where every monomial occurring in r_i is strictly smaller than m_i. (We can write g_i in this form, since g_i is monic with leading monomial m_i.) Let A_{red} be the free R-submodule of A spanned by the reduced monomials.

We shall construct an R-linear map

$$\rho : A \longrightarrow A_{\mathrm{red}}$$

by recursively defining the values $\rho(u)$ for all monomials u; the recursion shall proceed along the well-founded order relation $\prec$. If a monomial u is reduced, set $\rho(u) = u$. If u is not reduced, choose an $i \in [k]$ such that $m_i \mid u$ and set

$$\rho(u) = \rho\left(\frac{u}{m_i} r_i\right). \tag{7}$$

On the right-hand side, ρ is applied termwise and R-linearly. This is legitimate because every monomial occurring in $(u/m_i)r_i$ is strictly smaller than u: indeed, every monomial of r_i is smaller than m_i, and a monomial order is compatible with multiplication.

We must check that (7) is independent of the choice of i. Suppose that both m_i and m_j divide u, where $i \neq j$. We must show that

$$\rho\left(\frac{u}{m_i} r_i\right) = \rho\left(\frac{u}{m_j} r_j\right). \tag{8}$$

Since the monomials m_i and m_j are relatively prime and both divide u, their product must divide u as well; thus

$$u = q m_i m_j$$

for some monomial q. By the recursive induction hypothesis, the definition of ρ is already independent of all choices on monomials strictly smaller than u. Every monomial t occurring in r_i satisfies $t \prec m_i$ and thus (since $\prec$ is a monomial order)

$$q t m_j \prec q m_i m_j = u.$$

When evaluating $\rho(q t m_j)$, we may therefore use the divisor m_j in (7), and obtain

$$\rho(q t m_j) = \rho\left(\frac{q t m_j}{m_j} r_j\right) = \rho(q t r_j).$$

Multiplying by the coefficient of t and summing over all monomials of r_i gives

$$\rho(q r_i m_j) = \rho(q r_i r_j).$$

The same argument (with the roles of i and j switched) yields

$$\rho(q r_j m_i) = \rho(q r_j r_i).$$

Since A is commutative, this is the same as

$$\rho(q m_i r_j) = \rho(q r_i r_j).$$

Hence, from $u = q m_i m_j$, we obtain

$$\rho\left(\frac{u}{m_i} r_i\right) = \rho(q r_i m_j) = \rho(q r_i r_j) = \rho(q m_i r_j) = \rho\left(\frac{u}{m_j} r_j\right),$$

which proves (8). Thus we have shown that the R-linear map $\rho : A \to A_{\mathrm{red}}$ is well-defined.

We next record two properties of ρ. First, for every monomial q and every $i \in [k]$, the definition allows us to reduce the monomial qm_i using m_i (that is, apply (7) to $u = qm_i$); hence

$$\rho(qm_i) = \rho(qr_i). \tag{9}$$

By linearity of ρ, this entails $\rho(q(m_i - r_i)) = 0$. In view of (6), this rewrites as

$$\rho(qg_i) = 0. \tag{10}$$

By R-linearity, (10) shows that

$$\rho(I) = 0.$$

Second, every $f \in A$ satisfies

$$f - \rho(f) \in I. \tag{11}$$

Indeed, by R-linearity, it is enough to prove this when f is a monomial; i.e., it is enough to show that $u - \rho(u) \in I$ for any monomial u. We do so by induction on u. If u is reduced, the claim is clear. Otherwise, choose i with $m_i \mid u$ and put $q = u/m_i$. Then $u = qm_i$ and therefore $\rho(u) = \rho(qm_i) = \rho(qr_i)$ by (9); hence,

$$\begin{aligned} u - \rho(u) &= qm_i - \rho(qr_i) \\ &= q(m_i - r_i) + \big(qr_i - \rho(qr_i)\big) \\ &= qg_i + \big(qr_i - \rho(qr_i)\big) \qquad \text{(by (6))}. \end{aligned}$$

The first addend belongs to I. Every monomial occurring in qr_i is strictly smaller than $qm_i = u$. Applying the induction hypothesis to these monomials and then using R-linearity gives $qr_i - \rho(qr_i) \in I$. Thus the second addend belongs to I as well. Thus, $u - \rho(u) \in I$. This proves (11).

Equation (11) shows that

$$A = I + A_{\mathrm{red}}.$$

This sum is direct. Indeed, if $h \in I \cap A_{\mathrm{red}}$, then $\rho(h) = 0$ because $\rho(I) = 0$, whereas $\rho(h) = h$ because ρ fixes every reduced monomial. Thus $h = 0$. Consequently,

$$A = I \oplus A_{\mathrm{red}}$$

as R-modules. Therefore, the canonical projection $A \to A/I$, restricted to A_{red}, becomes an R-module isomorphism. Since the reduced monomials form a basis of A_{red}, this shows that their residue classes form a basis of A/I. □

Remark 4. *In Gröbner-basis language, Lemma 3 says that $g_1, g_2, \ldots, g_k$ form a monic Gröbner basis and then applies the standard-monomial basis theorem. Over a field, this follows from Buchberger's first criterion; see, for example, [4, Chapter 1, especially the conclusion after Lemma 1.1.38]. The proof above avoids S-polynomials and Buchberger's criterion altogether. It is the particularly simple commutative case of a diamond-lemma argument in which the only competing reductions are*

$$qm_im_j \longrightarrow qr_im_j \longrightarrow qr_ir_j \quad \text{and} \quad qm_im_j \longrightarrow qm_ir_j \longrightarrow qr_ir_j.$$

For a more detailed treatment of monic reductions and Gröbner bases over arbitrary commutative rings, see [9, Section 3.2]. Compare also Bergman's diamond lemma [1, Theorem 1.2], which makes similar statements about noncommutative polynomial rings.

3 The Artin basis

We prove Theorem 2 **(a)** using Lemma 3. Our proof follows the ideas of [16, proof of Theorem 1.2.7]. Other proofs can be found in [12, (DIFF.1.3)], [3, Chapter IV, §6, no. 1, Theorem 1, part (c)], [6, Theorem, part (c)] and (over $\mathbb{Z}$ instead of $\mathbf{k}$) in [5, Proposition 3.4] and [13, (5.1)].

We shall use the shorthand notation x^a for the monomial $x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$, where $a = (a_1, a_2, \ldots, a_n) \in \mathbb{N}^n$.

For $r \in \mathbb{N}$ and a finite list of variables $y_1, y_2, \ldots, y_m$, let $h_r(y_1, y_2, \ldots, y_m)$ denote the complete homogeneous symmetric polynomial of degree r in $y_1, y_2, \ldots, y_m$; this is defined as the sum of all monomials $y_1^{a_1} y_2^{a_2} \cdots y_m^{a_m}$ with $a_1 + a_2 + \cdots + a_m = r$. (In particular, $h_0 = 1$.) For $1 \leq i \leq n$, set

$$g_i = h_i(x_i, x_{i+1}, \ldots, x_n).$$

Lemma 5 (A generating-function identity)**.** *For each $i \in [n]$, we have the identity*

$$\left(\sum_{j=0}^{n} (-1)^j e_j t^j \right) \left(\sum_{r \geq 0} h_r(x_i, x_{i+1}, \ldots, x_n) t^r \right) = \prod_{j=1}^{i-1} (1 - x_j t) \tag{12}$$

in the formal power-series ring $P[[t]]$, where $e_0 = 1$.

Proof. The elementary symmetric polynomials satisfy Viète's identity

$$\sum_{j=0}^{n} (-1)^j e_j t^j = \prod_{j=1}^{n} (1 - x_j t). \tag{13}$$

Indeed, when the product on the right-hand side is expanded, choosing the term $-x_j t$ from exactly r of its factors produces $(-1)^r e_r t^r$. On the other hand, every power series $1 - x_j t$ in $P[[t]]$ has constant term 1 and is therefore a unit. We have

$$\sum_{r \geq 0} h_r(x_i, x_{i+1}, \ldots, x_n) t^r = \prod_{j=i}^{n} \frac{1}{1 - x_j t}. \tag{14}$$

To see this, expand each factor on the right-hand side as

$$\frac{1}{1 - x_j t} = (1 - x_j t)^{-1} = \sum_{a \geq 0} x_j^a t^a$$

by the geometric-series formula. Thus, the whole product rewrites as follows:

$$\prod_{j=i}^{n} \frac{1}{1 - x_j t} = \prod_{j=i}^{n} \sum_{a \geq 0} x_j^a t^a = \sum_{(a_i, a_{i+1}, \ldots, a_n) \in \mathbb{N}^{n-i+1}} x_i^{a_i} x_{i+1}^{a_{i+1}} \cdots x_n^{a_n} t^{a_i + a_{i+1} + \cdots + a_n}.$$

The coefficient of t^r in this is the sum of all monomials $x_i^{a_i} x_{i+1}^{a_{i+1}} \cdots x_n^{a_n}$ with $a_i + a_{i+1} + \cdots + a_n = r$. This is precisely $h_r(x_i, x_{i+1}, \ldots, x_n)$. Thus, (14) is proved.

Multiplying (13) and (14), we obtain

$$\left(\sum_{j=0}^{n}(-1)^j e_j t^j\right)\left(\sum_{r\geq 0} h_r(x_i, x_{i+1}, \ldots, x_n)t^r\right) = \prod_{j=1}^{n}(1-x_j t)\prod_{j=i}^{n}\frac{1}{1-x_j t} = \prod_{j=1}^{i-1}(1-x_j t),$$

since the factors $1-x_i t, 1-x_{i+1}t, \ldots, 1-x_n t$ cancel. Thus (12) is proved. □

Lemma 6 (Asymmetric generating set of J)**.** *We have the equality*

$$g_1 P + g_2 P + \cdots + g_n P = J \tag{15}$$

of ideals of P.

Proof. For each $i \in [n]$, the right-hand side of (12) has degree at most $i-1$ in t. Comparing coefficients of t^i in (12) therefore gives

$$\sum_{j=0}^{i}(-1)^j e_j h_{i-j}(x_i, x_{i+1}, \ldots, x_n) = 0. \tag{16}$$

The $j = 0$ addend on the left-hand side of this equality is $(-1)^0 e_0 h_i(x_i, x_{i+1}, \ldots, x_n) = h_i(x_i, x_{i+1}, \ldots, x_n) = g_i$; thus, we can rewrite (16) as

$$g_i + \sum_{j=1}^{i}(-1)^j e_j h_{i-j}(x_i, x_{i+1}, \ldots, x_n) = 0,$$

or, equivalently,

$$\begin{aligned} g_i &= -\sum_{j=1}^{i}(-1)^j e_j h_{i-j}(x_i, x_{i+1}, \ldots, x_n) \\ &= -\sum_{j=1}^{i-1}(-1)^j e_j h_{i-j}(x_i, x_{i+1}, \ldots, x_n) - (-1)^i e_i \end{aligned} \tag{17}$$

(since $h_{i-i}(x_i, x_{i+1}, \ldots, x_n) = 1$). Thus every g_i belongs to J (since $e_1, e_2, \ldots, e_i \in J$). This proves the "$\subseteq$" inclusion of (15).

Conversely, (17) gives

$$(-1)^i e_i = -g_i - \sum_{j=1}^{i-1}(-1)^j e_j h_{i-j}(x_i, x_{i+1}, \ldots, x_n). \tag{18}$$

We now prove by strong induction on i that

$$e_i \in g_1 P + g_2 P + \cdots + g_i P.$$

Indeed, assume that this claim is known for all smaller indices. Every e_j occurring on the right-hand side of (18) has $j < i$, so the induction hypothesis places it in $g_1 P + g_2 P + \cdots + g_j P$. Hence the left-hand side $(-1)^i e_i$, and therefore also e_i, belongs to $g_1 P + g_2 P + \cdots + g_i P$. Thus we conclude that $e_i \in g_1 P + g_2 P + \cdots + g_n P$ for each $i \in [n]$. In view of (3), this entails $J \subseteq g_1 P + g_2 P + \cdots + g_n P$. This proves the "$\supseteq$" inclusion in (15). Hence the equality of ideals (15) is proved. □

Proposition 7 (Artin basis)**.** *The residue classes of the Artin monomials*

$$x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} \qquad \text{with} \qquad 0 \leq a_i < i \quad \text{for every } i \in [n] \tag{19}$$

form a $\mathbf{k}$*-basis of* $P/J = C$*.*

Proof. From Lemma 6, we know that $J = g_1 P + g_2 P + \cdots + g_n P$. To obtain a $\mathbf{k}$-module basis of $C = P/J$, we shall apply Lemma 3 with $R = \mathbf{k}$, $N = n$, $A = P$, $k = n$, and $I = J$, using the generators $g_1, g_2, \ldots, g_n$. We therefore need a monomial order $\succ$ on P for which the g_i are monic and have pairwise relatively prime leading monomials. There are several valid choices here. The simplest one is to let $\succ$ be the lexicographic order $>_{\text{lex}}$ with

$$x_1 > x_2 > \cdots > x_n.$$

Thus, for exponent vectors $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$, we have $x^a \succ x^b$ if, at the first index r for which $a_r \neq b_r$, we have $a_r > b_r$. Alternatively, we can let $\succ$ be the degree-lexicographic order $>_{\text{grlex}}$, in which two monomials of equal degree are compared as in the lexicographic order, whereas two monomials of distinct degrees are compared by their degrees (that is, the monomial of larger degree is declared to be larger). Both of these orders are monomial orders (this is particularly easy to check for $>_{\text{grlex}}$, since there are only finitely many monomials of a given degree), and both have the property that each of the polynomials g_i is monic with leading monomial $\operatorname{LM}(g_i) = x_i^i$ (since $g_i = h_i(x_i, x_{i+1}, \ldots, x_n) = x_i^i +$ smaller monomials). These leading monomials are pairwise relatively prime.

Hence Lemma 3 shows that the residue classes of the reduced monomials (i.e., of the monomials not divisible by any of $x_1^1, x_2^2, \ldots, x_n^n$) form a $\mathbf{k}$-basis of P/J. A monomial $x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ is reduced if and only if $0 \leq a_i < i$ for every $i \in [n]$. These are exactly the Artin monomials in (19). Thus their residue classes form a basis of P/J.

This proves Proposition 7 and therefore Theorem 2 **(a)**. □

If $V = \bigoplus_{d \geq 0} V_d$ is a graded $\mathbf{k}$-module such that every V_d is finite free, its *Hilbert series* is the formal power series

$$\operatorname{Hilb}(V; q) := \sum_{d \geq 0} (\operatorname{rank}_{\mathbf{k}} V_d) q^d \in \mathbb{Z}[[q]]. \tag{20}$$

We will use Hilbert series only for graded $\mathbf{k}$-modules with finite free homogeneous components. Note that P, C and Λ are such graded $\mathbf{k}$-modules; indeed, the basis of C constructed in Proposition 7 is a graded basis (i.e., each of its basis vectors is homogeneous).

Corollary 8. *We have*

$$\operatorname{Hilb}(C; q) = \prod_{i=1}^{n} (1 + q + \cdots + q^{i-1}) \qquad \text{and} \qquad \operatorname{rank}_{\mathbf{k}} C = n!.$$

Proof. By Proposition 7, the degree-d component of C has as a basis the residue classes of the Artin monomials $x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ satisfying $0 \leq a_i < i$ and $a_1 + a_2 + \cdots + a_n = d$. Hence

its rank $\operatorname{rank}_{\mathbf{k}} C_d$ is the number of exponent tuples $(a_1, a_2, \ldots, a_n)$ satisfying $0 \leq a_i < i$ for all i as well as $a_1 + a_2 + \cdots + a_n = d$. Therefore

$$\begin{aligned}
\operatorname{Hilb}(C; q) &= \sum_{d \geq 0} \sum_{\substack{(a_1, a_2, \ldots, a_n); \\ 0 \leq a_i < i \text{ for each } i; \\ a_1 + a_2 + \cdots + a_n = d}} q^d \\
&= \sum_{\substack{(a_1, a_2, \ldots, a_n); \\ 0 \leq a_i < i \text{ for each } i}} q^{a_1 + a_2 + \cdots + a_n} \\
&= \sum_{0 \leq a_1 < 1} \sum_{0 \leq a_2 < 2} \cdots \sum_{0 \leq a_n < n} q^{a_1 + a_2 + \cdots + a_n} \\
&= \prod_{i=1}^{n} \left(\sum_{a=0}^{i-1} q^a \right) \\
&= \prod_{i=1}^{n} (1 + q + \cdots + q^{i-1}).
\end{aligned}$$

Evaluating this polynomial at $q = 1$ counts all Artin monomials and gives $\operatorname{rank}_{\mathbf{k}} C = 1 \cdot 2 \cdot \cdots \cdot n = n!$. □

4 Lifting the coinvariant algebra

We next prove Theorem 2 **(b)**. Over a field, one can start with an arbitrary graded vector-space complement to J in P. Over a general coefficient ring such complements need not exist, so we formulate the lifting argument for a chosen graded set of representatives. The Artin monomials provide such representatives.

We shall use the following elementary fact.

Lemma 9. *Let R be a commutative ring. Any surjective R-linear map between two finite free R-modules of the same rank is an isomorphism.*

Proof. Let f be a surjective R-linear map between two free R-modules of the same rank. We must prove that f is an isomorphism.

After choosing bases, the map f is represented by a square matrix A. Since its target is free, the surjection f has an R-linear right inverse g, represented by a matrix B. Thus $AB = I$. Taking determinants gives $\det(A) \det(B) = 1$, so $\det(A)$ is a unit. The adjugate formula $A \operatorname{adj}(A) = \operatorname{adj}(A) A = \det(A) \cdot I$ then shows that A is invertible. Hence, the corresponding linear map f is an isomorphism. □

Proposition 10. *Let*

$$H = \bigoplus_{d \geq 0} H_d$$

be a graded $\mathbf{k}$*-submodule of P such that each H_d is finite free and the quotient map $P \to C = P/J$ restricts to a graded* $\mathbf{k}$*-module isomorphism*

$$H \xrightarrow{\sim} C. \tag{21}$$

Then:

(a) *Multiplication induces an isomorphism of graded Λ-modules*

$$m : \Lambda \otimes_{\mathbf{k}} H \xrightarrow{\sim} P, \qquad a \otimes h \longmapsto ah.$$

Here the tensor product has its usual grading:

$$(\Lambda \otimes_{\mathbf{k}} H)_d = \bigoplus_{r+s=d} \Lambda_r \otimes_{\mathbf{k}} H_s.$$

(b) *If H is $\mathfrak{S}_n$-stable, then m is also $\mathfrak{S}_n$-equivariant, where $\mathfrak{S}_n$ acts on the second tensor factor.*

Proof. The map m is plainly graded and Λ-linear. We first prove that it is surjective, or equivalently that $P = \Lambda H$, by strong induction on degree. Let $p \in P_d$ be homogeneous. By (21), the quotient map $H \xrightarrow{\sim} C$ is an isomorphism; since it is graded, it thus restricts to an isomorphism $H_d \xrightarrow{\sim} C_d$. Thus, the residue class of p in C_d lies in the image of this restricted isomorphism; that is, there is an $h \in H_d$ such that $p - h \in J_d$. Since

$$J = e_1 P + e_2 P + \cdots + e_n P,$$

we can thus write

$$p - h = \sum_{i=1}^{n} e_i q_i,$$

where each $q_i \in P$. Moreover, by projecting both sides of this equality onto the d-th graded component of P, we obtain

$$p - h = \sum_{i=1}^{n} e_i \widetilde{q}_i, \tag{22}$$

where $\widetilde{q}_i$ is the $(d-i)$-th graded component of q_i (since each e_i is homogeneous of degree i). As usual, $\widetilde{q}_i = 0$ when $i > d$. By induction, each $\widetilde{q}_i$ lies in ΛH (since $\deg \widetilde{q}_i = d - i < d$). Since $e_i \in \Lambda$, the equality (22) thus yields $p - h \in \underbrace{\Lambda\Lambda}_{=\Lambda} H = \Lambda H$, and therefore $p \in \Lambda H$ as well (since $h \in H_d \subseteq H \subseteq \Lambda H$). So we have proved the surjectivity of m.

It remains to prove the injectivity of m. We do this degree by degree. By (1), the monomials

$$e_1^{a_1} e_2^{a_2} \cdots e_n^{a_n} \qquad (\text{with } a_1, a_2, \ldots, a_n \in \mathbb{N})$$

form a homogeneous $\mathbf{k}$-basis of Λ, and they are homogeneous of degree $1a_1 + 2a_2 + \cdots + na_n$. Hence

$$\begin{aligned} \operatorname{Hilb}(\Lambda; q) &= \sum_{d \geq 0} \sum_{\substack{a_1, a_2, \ldots, a_n \in \mathbb{N}; \\ 1a_1 + 2a_2 + \cdots + na_n = d}} q^d \\ &= \sum_{a_1, a_2, \ldots, a_n \in \mathbb{N}} q^{1a_1 + 2a_2 + \cdots + na_n} \\ &= \prod_{i=1}^{n} \sum_{a \in \mathbb{N}} q^{ia} = \prod_{i=1}^{n} \frac{1}{1 - q^i} \end{aligned} \tag{23}$$

(since $\sum_{a\in\mathbb{N}} q^{ia} = \dfrac{1}{1-q^i}$ for each $i > 0$). Likewise, the ordinary monomial basis of P gives

$$\operatorname{Hilb}(P;q) = \frac{1}{(1-q)^n}. \tag{24}$$

Furthermore, (21) and Corollary 8 give

$$\begin{aligned}\operatorname{Hilb}(H;q) = \operatorname{Hilb}(C;q) &= \prod_{i=1}^{n} \underbrace{\left(1+q+\cdots+q^{i-1}\right)}_{=\frac{1-q^i}{1-q}} \\ &= \frac{\prod_{i=1}^{n}(1-q^i)}{(1-q)^n}. \end{aligned} \tag{25}$$

If V and W are two graded $\mathbf{k}$-modules whose homogeneous components are finite free, then their Hilbert series multiply under tensor products, i.e., we have

$$\operatorname{Hilb}\left(V \otimes_{\mathbf{k}} W; q\right) = \operatorname{Hilb}\left(V;q\right) \cdot \operatorname{Hilb}\left(W;q\right),$$

since each $d \geq 0$ satisfies

$$\operatorname{rank}_{\mathbf{k}}(V \otimes_{\mathbf{k}} W)_d = \sum_{r+s=d} (\operatorname{rank}_{\mathbf{k}} V_r)(\operatorname{rank}_{\mathbf{k}} W_s).$$

Consequently,

$$\begin{aligned}\operatorname{Hilb}(\Lambda \otimes_{\mathbf{k}} H; q) &= \operatorname{Hilb}(\Lambda;q) \cdot \operatorname{Hilb}(H;q) \\ &= \left(\prod_{i=1}^{n} \frac{1}{1-q^i}\right) \cdot \frac{\prod_{i=1}^{n}(1-q^i)}{(1-q)^n} \qquad \text{(by (23) and (25))} \\ &= \frac{1}{(1-q)^n} = \operatorname{Hilb}(P;q) \qquad \text{(by (24))}. \end{aligned}$$

In other words, for each $d \geq 0$, we have $\operatorname{rank}_{\mathbf{k}}\left(\Lambda \otimes_{\mathbf{k}} H\right)_d = \operatorname{rank}_{\mathbf{k}} P_d$.

Thus, in every degree d, the map $m : \left(\Lambda \otimes_{\mathbf{k}} H\right)_d \to P_d$ is a surjection between finite free $\mathbf{k}$-modules of the same rank. Therefore, Lemma 9 shows that each homogeneous component of m is an isomorphism. Altogether, m is thus an isomorphism of graded $\mathbf{k}$-modules. This proves part **(a)**.

(b) If H is $\mathfrak{S}_n$-stable, then m is equivariant because of (4). □

Corollary 11. *The polynomial ring P is a free Λ-module of rank $n!$. More explicitly, the Artin monomials*

$$x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} \qquad \text{(with } 0 \leq a_i < i \text{ for every } i \in [n]\text{)}$$

form a Λ-basis of P.

Proof. Let H be the free $\mathbf{k}$-submodule of P spanned by the Artin monomials. This is a graded $\mathbf{k}$-submodule of P, since each Artin monomial is homogeneous. The Artin monomials form a $\mathbf{k}$-basis of H. Meanwhile, their residue classes in $P/J = C$ form a $\mathbf{k}$-basis of C, by Proposition 7.

Restricting the quotient map $P \to P/J = C$ to H, we obtain a graded $\mathbf{k}$-linear map $\phi : H \to C$. This map ϕ sends the Artin monomials in H to their residue classes in C. Since the former form a $\mathbf{k}$-basis of H while the latter form a $\mathbf{k}$-basis of C, we thus conclude that ϕ is a $\mathbf{k}$-module isomorphism. Hence, Proposition 10 **(a)** shows that multiplication induces an isomorphism of graded Λ-modules

$$m : \Lambda \otimes_{\mathbf{k}} H \xrightarrow{\sim} P, \qquad a \otimes h \longmapsto ah.$$

But the Artin monomials $x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ form a $\mathbf{k}$-basis of H. Hence, the tensors $1 \otimes_{\mathbf{k}} x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ form a Λ-basis of $\Lambda \otimes_{\mathbf{k}} H$ (since base change respects bases). Therefore, the images of these tensors under m form a Λ-basis of P (since m is a Λ-module isomorphism). But these images are again the Artin monomials $x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$. So we have shown that the Artin monomials form a Λ-basis of P. This proves Corollary 11 and, with it, Theorem 2 **(b)**. □

Remark 12 (Harmonic representatives)**.** *When* $\mathbf{k}$ *is a field of characteristic* 0*, one customary choice of* H *is the space of harmonic polynomials; compare Haiman's discussion of harmonics and the one-set case [11, Sections 1.3 and 1.5]. On* $\mathbb{Q}[x_1, x_2, \ldots, x_n]$*, consider the* apolar pairing*, i.e., the* $\mathbb{Q}$*-bilinear form given by*

$$\langle p, q \rangle = \big(p(\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_n})q\big)(0).$$

The monomials are orthogonal with respect to this pairing, and satisfy

$$\langle x^a, x^a \rangle = a_1! a_2! \cdots a_n!.$$

Thus, after extension to $\mathbb{R}$*, this pairing is positive definite, and* $J^\perp$ *is a graded* $\mathfrak{S}_n$*-stable complement to* J*. After base change to any characteristic-*0 *field, one may therefore take* $H = J^\perp$ *in Proposition 10. This realization is not needed for the ring-valued theorem.*

Another proof of Theorem 2 **(b)** can be found in [8, Proposition 3.3].

5 Orbit harmonics and the regular representation

We now prove Theorem 2 **(c)**. The construction is the orbit-harmonics, or point-orbit, method used by Oh and Rhoades [14, Section 1]; their Section 3.1 applies it to the coinvariant algebra of a reflection group. We first record the associated-graded facts in the form needed over a commutative coefficient ring.

Recall that P is graded. If

$$f = f_0 + f_1 + \cdots + f_d \qquad (\text{with } f_i \in P_i \text{ and } f_d \neq 0)$$

is a nonzero polynomial in P, then its *leading homogeneous part* is defined to be

$$\tau(f) := f_d.$$

For an arbitrary ideal $I \subseteq P$, define the graded ideal

$$\operatorname{in}_{\deg}(I) = \sum_{\substack{f \in I; \\ f \neq 0}} \tau(f) P. \tag{26}$$

Thus $\operatorname{in}_{\deg}(I)$ is generated by leading homogeneous parts (not by leading monomials!). Reiner and Rhoades call this ideal $\operatorname{gr} I$; compare [15, Proposition 2.1].

For any $d \in \mathbb{Z}$, define the $\mathbf{k}$-submodule

$$P_{\leq d} := \begin{cases} \bigoplus_{i=0}^{d} P_i, & \text{if } d \geq 0, \\ 0, & \text{if } d < 0. \end{cases}$$

Thus, $P_{\leq d}$ consists of the polynomials of degree at most d (when $d \geq 0$), and

$$P_{\leq d} P_{\leq e} \subseteq P_{\leq d+e} \qquad \text{for all } d, e \in \mathbb{Z}.$$

Hence P is a filtered $\mathbf{k}$-algebra, with filtration $0 = P_{\leq -1} \subseteq P_{\leq 0} \subseteq P_{\leq 1} \subseteq \cdots$.

Lemma 13 (The associated graded quotient)**.** *Let I be any ideal of P. Give P/I the filtration $0 = F_{-1} \subseteq F_0 \subseteq F_1 \subseteq \cdots$, where*

$$F_d := F_d(P/I) := \frac{P_{\leq d} + I}{I} \qquad \text{for each } d \in \mathbb{Z}.$$

Thus F_d is the image of $P_{\leq d}$ under the quotient map $P \to P/I$. This is a multiplicative filtration: $F_d F_e \subseteq F_{d+e}$.

The associated graded algebra *of P/I is the graded $\mathbf{k}$-algebra*

$$\operatorname{gr}_F(P/I) := \bigoplus_{d \geq 0} F_d / F_{d-1}.$$

Its multiplication is given explicitly by

$$(a + F_{d-1})(b + F_{e-1}) = ab + F_{d+e-1} \qquad (a \in F_d,\ b \in F_e).$$

There is a canonical isomorphism of graded $\mathbf{k}$-algebras

$$P/\operatorname{in}_{\deg}(I) \xrightarrow{\sim} \operatorname{gr}_F(P/I). \tag{27}$$

More explicitly, if $p \in P_d$ is homogeneous, then this isomorphism sends

$$p + \operatorname{in}_{\deg}(I) \quad \longmapsto \quad (p + I) + F_{d-1} \in F_d / F_{d-1}.$$

Equivalently, it identifies the degree-d component of the left-hand side with

$$\left(P/\operatorname{in}_{\deg}(I)\right)_d \cong \frac{P_{\leq d} + I}{P_{\leq d-1} + I}. \tag{28}$$

Proof. The inclusion $P_{\le d}P_{\le e} \subseteq P_{\le d+e}$ implies $F_dF_e \subseteq F_{d+e}$. It also shows that the displayed multiplication is independent of the chosen representatives. For example, if $a' \in F_{d-1}$, then $a'b \in F_{d+e-1}$, so replacing a by $a+a'$ does not change the product in F_{d+e}/F_{d+e-1}; the same argument applies to b. Thus $\mathrm{gr}_F(P/I)$ is indeed a graded algebra.

For each $d \ge 0$, define a $\mathbf{k}$-linear map

$$\varphi_d : P_d \longrightarrow F_d/F_{d-1}, \qquad p \longmapsto (p+I) + F_{d-1}.$$

Taking the direct sum over all d gives a graded $\mathbf{k}$-linear map

$$\varphi : P = \bigoplus_{d\ge 0} P_d \longrightarrow \mathrm{gr}_F(P/I).$$

This map is an algebra homomorphism. Indeed, if $p \in P_d$ and $q \in P_e$ are homogeneous, then

$$\begin{aligned}\varphi(p)\varphi(q) &= \big((p+I)+F_{d-1}\big)\big((q+I)+F_{e-1}\big)\\ &= (pq+I)+F_{d+e-1}\\ &= \varphi(pq).\end{aligned}$$

It also sends 1 to 1, so it is a homomorphism of graded $\mathbf{k}$-algebras.

The map φ is surjective. Indeed, let an element of F_d/F_{d-1} be represented by $q+I$, where $q \in P_{\le d}$. Write

$$q = q_0 + q_1 + \cdots + q_d \qquad (q_i \in P_i).$$

Since $q - q_d \in P_{\le d-1}$, the class of $q+I$ in F_d/F_{d-1} is $\varphi(q_d)$.

We next compute the kernel. Let $0 \ne f = f_0 + f_1 + \cdots + f_d \in I$, with $f_i \in P_i$ and $f_d \ne 0$. In P/I we have

$$f_d + I = -(f_0 + f_1 + \cdots + f_{d-1}) + I \in F_{d-1}.$$

Therefore $\varphi(\tau(f)) = \varphi(f_d) = 0$. Since $\ker\varphi$ is an ideal, it follows that

$$\mathrm{in}_{\mathrm{deg}}(I) \subseteq \ker\varphi.$$

For the reverse inclusion, first note that φ is graded. Thus, if $p = \sum_{d\ge 0} p_d \in \ker\varphi$, where $p_d \in P_d$, then the direct sum decomposition of the target gives $\varphi(p_d) = 0$ for every d. Hence it is enough to consider a homogeneous element $p \in P_d$ satisfying $\varphi(p) = 0$. If $p = 0$, then there is nothing to prove. Otherwise, $\varphi(p) = 0$ means that $p + I \in F_{d-1}$. By the definition of F_{d-1}, there exists $r \in P_{\le d-1}$ such that

$$p - r \in I.$$

The polynomial $p - r$ is nonzero, because p is a nonzero homogeneous polynomial of degree d, whereas r has degree at most $d-1$. Its leading homogeneous part is therefore p. Hence

$$p = \tau(p-r) \in \mathrm{in}_{\mathrm{deg}}(I).$$

This proves $\ker\varphi \subseteq \mathrm{in}_{\mathrm{deg}}(I)$, and thus

$$\ker\varphi = \mathrm{in}_{\mathrm{deg}}(I).$$

The first isomorphism theorem now yields (27). For each d, the standard quotient-of-a-quotient isomorphism gives

$$F_d/F_{d-1} = \frac{(P_{\le d}+I)/I}{(P_{\le d-1}+I)/I} \cong \frac{P_{\le d}+I}{P_{\le d-1}+I},$$

which proves the componentwise description (28). No choices were made in the construction of φ, which is why the isomorphism is canonical. □

Lemma 14 (Maschke splitting lemma)**.** *Let G be a finite group, and assume that $|G|$ is invertible in $\mathbf{k}$. Let*

$$0 \longrightarrow U \longrightarrow V \xrightarrow{\pi} W \longrightarrow 0 \tag{29}$$

be a short exact sequence of $\mathbf{k}[G]$-modules. If (29) splits as a sequence of $\mathbf{k}$-modules, then it also splits as a sequence of $\mathbf{k}[G]$-modules. In particular, this holds whenever W is projective as a $\mathbf{k}$-module.

Proof. Choose a $\mathbf{k}$-linear section $s : W \to V$ of π. Define

$$\widetilde{s}(w) = \frac{1}{|G|}\sum_{g\in G} g\, s(g^{-1}w) \qquad \text{for every } w \in W. \tag{30}$$

This is a $\mathbf{k}$-linear map. It is a section of π, since the G-equivariance of π gives

$$\pi\big(\widetilde{s}(w)\big) = \frac{1}{|G|}\sum_{g\in G} g\,\pi\big(s(g^{-1}w)\big) = \frac{1}{|G|}\sum_{g\in G} g(g^{-1}w) = w.$$

It is also G-equivariant. Indeed, for $h \in G$, substitute $g = hk$ in the sum to obtain

$$\widetilde{s}(hw) = \frac{1}{|G|}\sum_{k\in G} hk\, s(k^{-1}w) = h\widetilde{s}(w).$$

Thus $\widetilde{s}$ is a $\mathbf{k}[G]$-linear section of π. The final assertion follows because a surjection onto a projective $\mathbf{k}$-module has a $\mathbf{k}$-linear section. This is the usual averaging argument behind Maschke's theorem; compare [10, Theorem 4.4.14]. □

Lemma 15 (Equivariance and splitting)**.** *Let a group G act on P by degree-preserving $\mathbf{k}$-algebra automorphisms, and let $I \subseteq P$ be a G-stable ideal.*

- ***(a)*** *The ideal $\mathrm{in}_{\deg}(I)$ is G-stable, and the canonical isomorphism (27) is G-equivariant.*
- ***(b)*** *Assume, in addition, that G is finite, that $|G|$ is invertible in $\mathbf{k}$, that the filtration of P/I stabilizes, and that every $\mathbf{k}$-module*

  $$F_d(P/I)/F_{d-1}(P/I)$$

  is projective. Then there is a generally noncanonical isomorphism of ungraded $\mathbf{k}[G]$-modules

  $$P/I \cong \mathrm{gr}_F(P/I). \tag{31}$$

  In particular, the projectivity hypothesis holds if the associated graded algebra has a homogeneous $\mathbf{k}$-basis.

Proof. **(a)** For $g \in G$ and $0 \neq f \in I$, degree preservation gives $\tau(gf) = g\tau(f)$. Hence $\mathrm{in}_{\deg}(I)$ is G-stable. The map φ in the proof of Lemma 13 is defined only from the grading, the quotient map, and the induced filtration, all of which are G-equivariant. Thus (27) is G-equivariant.

(b) Write $F_d = F_d(P/I)$. Since the filtration stabilizes, only finitely many of its successive quotients are nonzero. For every d, projectivity of F_d/F_{d-1} gives a $\mathbf{k}$-linear splitting of the short exact sequence

$$0 \longrightarrow F_{d-1} \longrightarrow F_d \longrightarrow F_d/F_{d-1} \longrightarrow 0.$$

Lemma 14 upgrades this to a $\mathbf{k}[G]$-linear splitting. Choose a G-stable complement $L_d \subseteq F_d$ such that

$$F_d = F_{d-1} \oplus L_d \qquad \text{and} \qquad L_d \cong F_d/F_{d-1}$$

as $\mathbf{k}[G]$-modules. If D is large enough that $F_D = P/I$, then induction on d gives

$$P/I = F_D = \bigoplus_{d=0}^{D} L_d \cong \bigoplus_{d=0}^{D} F_d/F_{d-1} = \mathrm{gr}_F(P/I),$$

which is (31). □

The symmetric group $\mathfrak{S}_n$ acts on $\mathbf{k}^n$ by

$$\sigma \cdot (b_1, b_2, \ldots, b_n) = (b_{\sigma^{-1}(1)}, b_{\sigma^{-1}(2)}, \ldots, b_{\sigma^{-1}(n)}).$$

This action is compatible with the action on P:

$$(\sigma f)(\sigma \cdot v) = f(v).$$

Definition 16. *A finite subset $Y \subseteq \mathbf{k}^n$ will be called* strongly discrete *if, for any two distinct points*

$$y = (y_1, y_2, \ldots, y_n) \quad \text{and} \quad z = (z_1, z_2, \ldots, z_n)$$

of Y, at least one coordinate difference $y_i - z_i$ is a unit of $\mathbf{k}$.

If Y is a finite set, let $\mathbf{k}^Y$ denote the $\mathbf{k}$-algebra of all functions $Y \to \mathbf{k}$, with pointwise addition and multiplication.

Lemma 17. *Let Y be a finite strongly discrete subset of $\mathbf{k}^n$, and set*

$$I(Y) = \{p \in P \mid p(y) = 0 \text{ for every } y \in Y\}.$$

Then:

(a) Evaluation induces a canonical $\mathbf{k}$-algebra isomorphism

$$\mathrm{ev}_Y : P/I(Y) \xrightarrow{\sim} \mathbf{k}^Y, \qquad p + I(Y) \longmapsto (y \longmapsto p(y)). \tag{32}$$

(b) If Y is $\mathfrak{S}_n$-stable, then (32) is $\mathfrak{S}_n$-equivariant, where

$$(\sigma\varphi)(y) = \varphi(\sigma^{-1} \cdot y).$$

Thus it is an isomorphism of $\mathfrak{S}_n$-algebras and, in particular, of $\mathbf{k}[\mathfrak{S}_n]$-modules.

Proof. For $y = (y_1, y_2, \ldots, y_n) \in Y$, set

$$\mathfrak{m}_y = (x_1 - y_1)P + (x_2 - y_2)P + \cdots + (x_n - y_n)P.$$

Evaluation at y is a surjective $\mathbf{k}$-algebra homomorphism $P \to \mathbf{k}$ with kernel $\mathfrak{m}_y$. Indeed, $p - p(y)$ lies in $\mathfrak{m}_y$ for every $p \in P$, as follows by replacing $x_1, x_2, \ldots, x_n$ successively by $y_1, y_2, \ldots, y_n$ and factoring each successive difference by $x_i - y_i$. Hence

$$P/\mathfrak{m}_y \cong \mathbf{k}.$$

Moreover,

$$I(Y) = \bigcap_{y \in Y} \mathfrak{m}_y.$$

The ideals $\mathfrak{m}_y$ are pairwise comaximal. Indeed, if $y \neq z$, strong discreteness gives an i for which $u := y_i - z_i$ is a unit, and

$$1 = u^{-1}(x_i - z_i) - u^{-1}(x_i - y_i) \in \mathfrak{m}_z + \mathfrak{m}_y.$$

The Chinese remainder theorem now gives

$$P/I(Y) \cong \prod_{y \in Y} P/\mathfrak{m}_y \cong \prod_{y \in Y} \mathbf{k} = \mathbf{k}^Y.$$

This is precisely the evaluation map in (32).

For equivariance, let $\sigma \in \mathfrak{S}_n$, $p \in P$, and $y \in Y$. Then

$$\big(\sigma \ \mathrm{ev}_Y(p + I(Y))\big)(y) = p(\sigma^{-1} \cdot y) = (\sigma p)(y) = \mathrm{ev}_Y(\sigma p + I(Y))(y).$$

This proves **(b)**. □

Proposition 18. *Assume that $n!$ is invertible in $\mathbf{k}$. Then the coinvariant algebra $C = P/J$ is isomorphic, as an ungraded $\mathbf{k}[\mathfrak{S}_n]$-module, to the regular module $\mathbf{k}[\mathfrak{S}_n]$.*

Proof. Put

$$v = (1_\mathbf{k}, 2 \cdot 1_\mathbf{k}, \ldots, n \cdot 1_\mathbf{k}) \in \mathbf{k}^n \qquad \text{and} \qquad X = \mathfrak{S}_n \cdot v.$$

If $r, s \in [n]$ are distinct, then $(r - s) \cdot 1_\mathbf{k}$ is a unit. Indeed, up to sign, the positive integer $|r - s|$ divides $n!$; since the image of $n!$ in $\mathbf{k}$ is a unit, so is the image of $|r - s|$. It follows that the coordinates of v are pairwise distinct, that X is an $\mathfrak{S}_n$-torsor, and that X is strongly discrete. Lemma 17 therefore gives

$$P/I(X) \cong \mathbf{k}^X \tag{33}$$

as $\mathfrak{S}_n$-algebras.

For $x \in X$, let $\delta_x \in \mathbf{k}^X$ be its indicator function. The functions $(\delta_x)_{x \in X}$ form a $\mathbf{k}$-basis, and $\sigma \delta_x = \delta_{\sigma \cdot x}$. Since X is a torsor,

$$\mathbf{k}[\mathfrak{S}_n] \longrightarrow \mathbf{k}^X, \qquad \sigma \longmapsto \delta_{\sigma \cdot v}, \tag{34}$$

is an isomorphism of $\mathbf{k}[\mathfrak{S}_n]$-modules.

Set

$$c_i = e_i(v) \in \mathbf{k} \qquad \text{and} \qquad K = (e_1 - c_1)P + (e_2 - c_2)P + \cdots + (e_n - c_n)P.$$

Every generator of K vanishes on X, so $K \subseteq I(X)$.

We next identify P/K explicitly. Evaluation at v restricts to a surjective homomorphism

$$\epsilon_v : \Lambda \longrightarrow \mathbf{k}, \qquad f \longmapsto f(v),$$

whose kernel is

$$(e_1 - c_1)\Lambda + (e_2 - c_2)\Lambda + \cdots + (e_n - c_n)\Lambda.$$

Indeed, under the isomorphism

$$\mathbf{k}[t_1, t_2, \ldots, t_n] \xrightarrow{\sim} \Lambda, \qquad t_i \longmapsto e_i,$$

this map is evaluation at $(c_1, c_2, \ldots, c_n)$, whose kernel is $(t_1 - c_1)\mathbf{k}[t_1, t_2, \ldots, t_n] + \cdots + (t_n - c_n)\mathbf{k}[t_1, t_2, \ldots, t_n]$. Corollary 11 gives the graded direct-sum decomposition

$$P = \bigoplus_{m \in \mathcal{A}} \Lambda m,$$

where $\mathcal{A}$ is the set of Artin monomials. Reducing this decomposition modulo $K = (\ker \epsilon_v)P$ shows that the residue classes of the Artin monomials form a $\mathbf{k}$-basis of P/K.

The inclusion $K \subseteq I(X)$ gives a surjection

$$P/K \twoheadrightarrow P/I(X) \cong \mathbf{k}^X.$$

The source has the $n!$ Artin monomials as a basis, while the target has the $n!$ indicator functions as a basis. Hence Lemma 9 shows that this surjection is an isomorphism. Thus

$$K = I(X). \tag{35}$$

We now determine the degree filtration on $P/I(X) = P/K$. For every $d \geq 0$,

$$F_d(P/K) = \bigoplus_{\substack{m \in \mathcal{A}; \\ \deg m \leq d}} \mathbf{k}\,\overline{m}, \tag{36}$$

where $\overline{m}$ denotes the residue class of m modulo K. The inclusion "$\supseteq$" is immediate. For the reverse inclusion, let $p \in P_{\leq d}$. Write it uniquely as

$$p = \sum_{m \in \mathcal{A}} \lambda_m m \qquad (\lambda_m \in \Lambda).$$

Since the decomposition $P = \bigoplus_m \Lambda m$ is graded, every homogeneous component of λ_m has degree at most $d - \deg m$. In particular, $\lambda_m = 0$ when $\deg m > d$. Modulo K, each λ_m becomes the scalar $\epsilon_v(\lambda_m)$, so the class of p lies in the right-hand side of (36). Directness follows from the fact that all Artin classes form a basis of P/K.

Consequently, $\mathrm{gr}_F(P/I(X))$ has the homogeneous $\mathbf{k}$-basis

$$m^* \qquad (m \in \mathcal{A}),$$

where m^* is the class of $\overline{m}$ in filtration degree $\deg m$.

Since the leading homogeneous part of $e_i - c_i$ is e_i, we have

$$J = e_1P + e_2P + \cdots + e_nP \subseteq \mathrm{in}_{\mathrm{deg}}(I(X)).$$

Thus Lemma 13 gives a canonical surjective graded homomorphism

$$C = P/J \twoheadrightarrow P/\,\mathrm{in}_{\mathrm{deg}}(I(X)) \xrightarrow{\sim} \mathrm{gr}_F(P/I(X)). \tag{37}$$

This map sends the residue class of each Artin monomial m to m^*. Proposition 7 says that the former classes are a basis of C, and the preceding paragraph says that the latter classes are a basis of the associated graded algebra. Hence (37) is an isomorphism. In particular,

$$J = \mathrm{in}_{\mathrm{deg}}(I(X)). \tag{38}$$

The filtration in (36) stabilizes, and each of its successive quotients is free, with basis given by the Artin monomials of the corresponding degree. Lemma 15 therefore gives an isomorphism of ungraded $\mathbf{k}[\mathfrak{S}_n]$-modules

$$P/I(X) \cong \mathrm{gr}_F(P/I(X)) \cong C.$$

Combining this with (33) and (34) yields

$$C \cong \mathbf{k}[\mathfrak{S}_n].$$

This proves Theorem 2 **(c)**. It also recovers the classical statement recorded by Haiman that the coinvariant algebra affords the regular representation [11, Section 1.5]. □

Remark 19. *The preceding proof uses the orbit-harmonics argument of Oh and Rhoades at two distinct points: the passage from the point quotient to its associated graded algebra, and the use of a free orbit whose function module is regular. The proof only concerns the coinvariant algebra as an* ungraded *$\mathfrak{S}_n$-module. Special choices involving roots of unity produce an additional cyclic symmetry that encodes the grading, but no such refinement is needed here.*

Remark 20. *The part of the construction preceding the equivariant splitting requires less than the invertibility of $n!$. Namely, suppose that $a_1, a_2, \ldots, a_n \in \mathbf{k}$ have pairwise invertible differences, and let*

$$v = (a_1, a_2, \ldots, a_n), \qquad X = \mathfrak{S}_n \cdot v.$$

Then X is a strongly discrete $\mathfrak{S}_n$-torsor, and the same argument gives the canonical graded isomorphism

$$C \cong \mathrm{gr}_F(P/I(X)).$$

The invertibility of $n!$ is used only to average the splittings that identify the filtered module $P/I(X)$ with its associated graded module equivariantly.

6 Proof of the equivariant normal-basis statement

Proof of Theorem 2 ***(d).*** Assume that $n!$ is invertible in $\mathbf{k}$. For each d, the quotient map

$$P_d \twoheadrightarrow C_d$$

has a $\mathbf{k}$-linear section, because C_d is free by Proposition 7. Applying Lemma 14 to the short exact sequence

$$0 \longrightarrow J_d \longrightarrow P_d \longrightarrow C_d \longrightarrow 0$$

gives an $\mathfrak{S}_n$-equivariant section. Let H_d be its image and set

$$H = \bigoplus_{d \geq 0} H_d.$$

Then H is a graded $\mathfrak{S}_n$-stable $\mathbf{k}$-submodule of P, and the quotient map restricts to an isomorphism

$$H \xrightarrow{\sim} C$$

of graded $\mathbf{k}[\mathfrak{S}_n]$-modules.

By Proposition 18, choose a $\mathbf{k}[\mathfrak{S}_n]$-module isomorphism

$$\theta : \mathbf{k}[\mathfrak{S}_n] \xrightarrow{\sim} H,$$

and put $f = \theta(1)$. For every $\sigma \in \mathfrak{S}_n$,

$$\theta(\sigma) = \theta(\sigma \cdot 1) = \sigma \cdot \theta(1) = \sigma(f).$$

Thus the family

$$\{\sigma(f) \mid \sigma \in \mathfrak{S}_n\}$$

is a $\mathbf{k}$-basis of H. Proposition 10 shows that it is a Λ-basis of P.

Under the canonical identification

$$\Lambda[\mathfrak{S}_n] \cong \Lambda \otimes_{\mathbf{k}} \mathbf{k}[\mathfrak{S}_n],$$

the map Φ from (5) is the composite

$$\Lambda \otimes_{\mathbf{k}} \mathbf{k}[\mathfrak{S}_n] \xrightarrow{\mathrm{id}_\Lambda \otimes \theta} \Lambda \otimes_{\mathbf{k}} H \xrightarrow{m} P.$$

Both arrows are $\Lambda[\mathfrak{S}_n]$-linear isomorphisms. Hence so is Φ. □

Remark 21 (The isomorphism is not graded)**.** *The element f cannot be chosen homogeneous when $n \geq 2$. Indeed, if* $\deg f = d$*, then all $\sigma(f)$ have degree d, and the resulting isomorphism would imply*

$$\mathrm{Hilb}(P; q) = q^d n!\, \mathrm{Hilb}(\Lambda; q),$$

which is false. What is graded is the decomposition

$$P \cong \Lambda \otimes_{\mathbf{k}} H,$$

where H has the nontrivial coinvariant grading. Only after forgetting this grading is H the regular module.

Example 22. *Let $n = 2$ and assume that 2 is invertible in $\mathbf{k}$. Then*

$$\Lambda = \mathbf{k}[x_1 + x_2, x_1x_2] \qquad \text{and} \qquad H = \operatorname{span}_{\mathbf{k}}\{1, x_1 - x_2\}.$$

For the transposition $s = (1\ 2)$, the element

$$f = 1 + (x_1 - x_2)$$

has orbit

$$f = 1 + (x_1 - x_2), \qquad s(f) = 1 - (x_1 - x_2),$$

which is a $\mathbf{k}$-basis of H and therefore a Λ-basis of P.

7 A modular counterexample

The invertibility hypothesis in Theorem 2 **(c)** and **(d)** cannot simply be omitted. Take $n = 2$ and $\mathbf{k} = \mathbb{F}_2$. Writing $s = (1\ 2)$, we have

$$C = \frac{\mathbf{k}[x_1, x_2]}{(x_1 + x_2)P + x_1x_2P} \cong \frac{\mathbf{k}[t]}{t^2\mathbf{k}[t]}.$$

The relation $x_1 + x_2 = 0$ becomes $x_1 = x_2$, so s acts trivially on all of C. On the other hand, s does not act trivially on the regular module $\mathbf{k}[\mathfrak{S}_2]$: it interchanges the basis vectors 1 and s. Hence, C is not isomorphic to $\mathbf{k}[\mathfrak{S}_2]$ as $\mathbf{k}[\mathfrak{S}_2]$-modules, and Theorem 2 **(c)** fails.

Theorem 2 **(d)** fails as well. Indeed, tensoring a hypothetical $\Lambda[\mathfrak{S}_2]$-module isomorphism $P \cong \Lambda[\mathfrak{S}_2]$ over Λ with $\Lambda/\Lambda_+ \cong \mathbf{k}$ — or, equivalently, reducing modulo $\Lambda_+P = e_1P + e_2P$ — would produce the impossible isomorphism $C \cong \mathbf{k}[\mathfrak{S}_2]$.

Notes on the origins of the proofs

The two field-valued ingredients underlying Theorem 2 are stated in Haiman's treatment of the ordinary coinvariant algebra [11, Section 1.5]: the coinvariant algebra affords the regular representation, and a homogeneous space of representatives freely generates the polynomial ring over the invariant subring. The proofs above formulate both statements over a commutative coefficient ring, under the hypotheses stated in Theorem 2.

The degeneration from a finite point locus to the quotient by leading homogeneous parts follows the orbit-harmonics framework of Oh and Rhoades [14, Section 1]; their Section 3.1 explains the reflection-group coinvariant case via a regular orbit. Our orbit

$$(1_{\mathbf{k}}, 2 \cdot 1_{\mathbf{k}}, \ldots, n \cdot 1_{\mathbf{k}})$$

is chosen so that distinct orbit points are strongly discrete. The additional cyclic action used in graded refinements of orbit harmonics is not needed here. The associated-graded construction is also recorded by Reiner and Rhoades [15, Propositions 2.1 and 2.7].

Our proof of Theorem 2 **(a)** imitates [16, proof of Theorem 1.2.7], but replaces the additional variables $y_1, y_2, \ldots, y_n$ by $0, 0, \ldots, 0$.

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