Schubert polynomial expansions revisited

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Errata and addenda by Darij Grinberg

I will refer to the results appearing in the article "Schubert polynomial expansions revisited" by the numbers under which they appear in this article (specifically, in its published version (doi:10.1017/fms.2025.10068)).

8. Errata

1. **Page 3:** The word "immediately" one line above Theorem 1.1 appears to be somewhat of an overstatement...

2. **Page 6, §2.1:** It should be added that the value $\ell(w)$ of the length function ℓ at a given element $w \in M$ is called the *length* or the *degree* of w (the two words are being used rather synonymously here).

3. **Page 6, §2.1:** I think you want to require M to be **equipped with** a specific enumeration $(a_1, a_2, a_3, \ldots)$ of M_1 (that is, a countably infinite list $(a_1, a_2, a_3, \ldots)$ of distinct elements of M_1 that contains each of these elements exactly once). This way, when you refer to $\{a_i\}_{i \geq 1}$ in Definition 2.4, it is clear what the a_i are, and the condition $\max \operatorname{supp} c(w) = \max \operatorname{Last}(w)$ on page 9 makes sense (since $\operatorname{Last}(w)$ is a set of positive integers just like $\operatorname{supp} c(w)$).

4. **From page 6, §2.1 onwards:** The word "right-cancellative" is sometimes spelled "right-cancellable" (once on page 4 and once on page 14). I think the first spelling is more reasonable, but they are both fine, as long as the synonymity is mentioned.

5. **Page 6, §2.2:** It should be said that Pol_n denotes the polynomial ring $\mathbb{Z}[x_1, x_2, \ldots, x_n]$.

6. **Page 7, Example 2.7:** Am I confused or have you never properly defined the monoids Codes ? As I guess, it should be defined as the monoid of all finitely supported infinite sequences $(c_i)_{i \geq 1} = (c_1, c_2, c_3, \ldots)$ of nonnegative integers (where "finitely supported" means that all but finitely many entries are 0). Its multiplication is entrywise addition of sequences: i.e.,

   $$(c_i)_{i \geq 1} (d_i)_{i \geq 1} := (c_i + d_i)_{i \geq 1} \qquad \text{for any } (c_i)_{i \geq 1}, (d_i)_{i \geq 1} \in \mathsf{Codes}.$$

   As an abelian monoid, it is freely generated by the "standard basis vectors" $e_i := (0, 0, \ldots, 0, 1, 0, 0, 0, \ldots)$ (with the 1 in the i-th position) for all $i \geq 1$. By abuse of notation, we denote these basis vectors e_i simply by i in Example 2.7, so that Codes is the free abelian monoid with free generators $1, 2, 3, \ldots$.

7. **Page 7, Example 2.10:** "to to" $\to$ "to".

8. **Page 8, proof of Proposition 2.11:** I think "such that $w = vu$" should be "such that $w = uv$".

9. **Page 9:** "Let M be a graded right cancellable monoid" should be "Let M be a graded right-cancellable partial monoid generated by $M_1 = \{a_1, a_2, a_3, \ldots\}$". (The latter requirement is needed to ensure that the condition $\max \operatorname{supp} c(w) = \max \operatorname{Last}(w)$ makes any sense; we want $\max \operatorname{Last}(w)$ to be a positive integer after all.)

10. **Page 9:** In the displayed equation

$$\begin{aligned} &\#\{w \in M \mid \ell(w) = n \text{ and } \max \operatorname{Last}(w) = k\} \\ &\leq \#\{c \in \mathsf{Codes} \mid |c(w)| = n \text{ and } \max \operatorname{supp} c(w) = k\}, \end{aligned}$$

replace both "$c(w)$"s by "c"s. (There should not be any w on the right hand side.)

11. **Page 10, proof of Theorem 2.20:** In the equality sign

$$\sum_{\ell(w)=k} \sum_{i \in \operatorname{Last}(w)} Y_i (X_w(f) S_{w/i}) = \sum_{\ell(w)=k} X_w(f) S_w,$$

you are tacitly using the fact that $X_w(f)$ is a constant (since $\deg f = k = \ell(w)$) and thus can be moved out of the Y_i. This is perhaps worth saying...

12. **Page 10, proof of Theorem 2.20:** In "Writing $\mathsf{Codes}_{k,d} = \{c \in \mathsf{Codes} \mid \max \operatorname{supp} c \leq d \text{ and } |c(w)| = k\}$", replace "$c(w)$" by "$c$".

13. **Page 10, proof of Theorem 2.20:** "for any $\lambda \in \mathbb{Z}$" should be "for any nonzero $\lambda \in \mathbb{Z}$".

14. **Page 10, proof of Theorem 2.20:** The argument you use to show that $\{S_w \mid w \in M_{k,d}\}$ is a basis for $\operatorname{Pol}_d^{(k)}$ is unnecessarily complicated. There is a general fact (well-known for fields, easy for $\mathbb{Z}$, not hard in general[1]) that if V is a free module of rank $m \in \mathbb{N}$ over a nontrivial commutative ring $\mathbf{k}$, then any family of $\leq m$ vectors in V that spans V must be a basis of V and must furthermore contain exactly m vectors. Applying this to $\mathbf{k} = \mathbb{Z}$ and $V = \operatorname{Pol}_d^{(k)}$ and $m = \operatorname{rank}(\operatorname{Pol}_d) = |\mathsf{Codes}_{k,d}|$ and to the family $(S_w)_{w \in M_{k,d}} \in V^{M_{k,d}}$ (which spans Pol_d by (2.6), and consists of $|M_{k,d}| \leq |\mathsf{Codes}_{k,d}| = \operatorname{rank}(\operatorname{Pol}_d)$ vectors), we conclude that this family $(S_w)_{w \in M_{k,d}}$ must be a basis of $\operatorname{Pol}_d^{(k)}$ and must contain exactly $|\mathsf{Codes}_{k,d}|$

[1] For a proof, see, e.g., Proposition II.5.2 in: Henri Lombardi & Claude Quitté, *Commutative algebra: Constructive methods*, arXiv:1605.04832v4.

many vectors, i.e., that $|M_{k,d}| = |\mathsf{Codes}_{k,d}|$. This is what you wanted to show.

A major advantage of this argument is that it works over any nontrivial commutative ring $\mathbf{k}$, not just over $\mathbb{Z}$ or $\mathbb{Q}$.

15. **Page 11, §3:** "The Lehmer code is the bijective map $S_\infty \to \mathsf{Codes}$" $\to$ "The Lehmer code is the bijective map lcode : $S_\infty \to \mathsf{Codes}$" (since you then speak of lcode (w)).

16. **Page 12, §3.2:** "staircase" $\to$ "staircase Young diagram". (Otherwise, it is not clear how the n-tuple $(n, n-1, \ldots, 1)$ suddenly has "columns".)

17. **Page 13, proof of Theorem 3.5:** In the first sentence, you are using the fact that id $-\mathsf{R}_1$ is invertible on Pol^+ (so that the recursion (3.1) can be solved for $\mathfrak{S}_w$ uniquely once $\mathfrak{S}_1 = 1$ is known).

18. **Page 13, proof of Theorem 3.5:** You talk of "pipes i and $i+1$". It should be explained that "pipe k" means the pipe that starts (enters) in row k at the left boundary of the diagram.

19. **Page 13, proof of Theorem 3.5:** In the last display, the "D" in "D has no crosses in row i" should be typeset in math mode (D, not D).

    The same applies to "nodes(D)" in the proof of Theorem 4.3 (page 17).

20. **Page 14, definition of T_i:** It should be explained that "$\dfrac{\mathsf{R}_{i+1} - \mathsf{R}_i}{x_i}$" means $x_i^{-1} (\mathsf{R}_{i+1} - \mathsf{R}_i)$, not $(\mathsf{R}_{i+1} - \mathsf{R}_i)\, x_i^{-1}$ (since operators don't commute).

21. **Page 14, definition of indexed forests:** The definition of an indexed forest is somewhat ambiguous about whether the internal nodes are labelled or unlabelled. As I understand, they should be unlabelled (i.e., strictly speaking, each T_i is an unlabelled rooted planar binary tree; its leaves of course obtain a natural labelling by the left-to-right reading order).

22. **Page 14, definition of the map c:** Replace "$c_i = \{v \mid \rho_F(v) = i\}$" by "$c_i = \#\{v \mid \rho_F(v) = i\}$".

23. **Page 14, definition of the map c:** If I am not mistaken, the bijectivity of the map c can be proved in a simple way: As we know, the monoid For is generated by the generators $\underline{1}, \underline{2}, \underline{3}, \ldots$ which are subject to the Thompson relations $\underline{i} \cdot \underline{j} = \underline{j} \cdot \underline{i+1}$ for all $i > j$. The latter relations allow us to write each element $F \in \mathsf{For}$ in the form $\underline{i_1} \cdot \underline{i_2} \cdot \cdots \cdot \underline{i_k}$ for some positive integers $i_1 \leq i_2 \leq \cdots \leq i_k$ (by successively rewriting $\underline{i} \cdot \underline{j}$ as $\underline{j} \cdot \underline{i+1}$ when $i > j$; it is not hard to check that this algorithm will always terminate, because the sum of all numbers keep increasing but none of the numbers can outgrow the label of the last leaf of a nontrivial tree in our forest F).

But it is easy to see that any positive integers $i_1 \leq i_2 \leq \cdots \leq i_k$ satisfy $\mathsf{c}\left(\underline{i_1} \cdot \underline{i_2} \cdot \cdots \cdot \underline{i_k}\right) = e_{i_1} e_{i_2} \cdots e_{i_k}$ in the monoid Codes (that is, the p-th entry of the sequence $\mathsf{c}\left(\underline{i_1} \cdot \underline{i_2} \cdot \cdots \cdot \underline{i_k}\right)$ is the number of times p appears in the k-tuple $(i_1, i_2, \ldots, i_k)$). Thus, the map $\mathsf{c} : \mathsf{For} \to \mathsf{Codes}$ is bijective. This also shows that each element $F \in \mathsf{For}$ can be written in the form $\underline{i_1} \cdot \underline{i_2} \cdot \cdots \cdot \underline{i_k}$ for a **unique** weakly increasing tuple of positive integers $i_1 \leq i_2 \leq \cdots \leq i_k$. This, in turn, easily yields that the Thompson relations give a presentation of For (that is, For is the quotient of the free monoid by the Thompson relations).

24. **Page 14:** Replace "The set $\operatorname{Last}(w)$ is identified with" by "The set $\operatorname{Last}(F)$ is identified with" (you don't call anything w here).

25. **Page 15, definition of the trimmed forest:** After "deleting the terminal node v satisfying $\rho_F(v) = i$", add "and compressing the leaves i and $i+1$ into a single leaf (this causes all later leaves $j > i+1$ to be relabelled as $j-1$)".

26. **Page 16, §4.2:** "the relation $\mathsf{R}_1 + \sum_{i \geq 1} x_i \mathsf{T}_i = \operatorname{id}$ from Corollary 3.2" should be "the relation $\mathsf{R}_1 + \sum_{i \geq 1} x_i \mathsf{T}_i = \operatorname{id}$ from Lemma 3.1".

27. **Page 17, §4.2:** It would be nice to explain how exactly the graphs correspond to forests. I don't see how to generalize Figure 6.

28. **Page 17, §4.3:** In the definition of $\mathsf{ThMon}^{\underline{m}}$, replace "for $i > m$" by "for $i > j$".

29. **Page 18, §5.2, definition of R_i^∞:** Replace "$\mathsf{R}_i^\infty(f) = f(x_1, \ldots, x_{i-1}, x_i, 0, 0, \ldots)$" by "$\mathsf{R}_i^\infty(f) = f(x_1, \ldots, x_{i-2}, x_{i-1}, 0, 0, \ldots)$".

30. **Page 19, Definition 5.4:** "Explicitly, B_i vanishes outside of Pol_i" should be "Explicitly, B_i vanishes on all monomials outside of Pol_i".

31. **Page 19, Definition 5.4:** "where $p_j > 0$ or $j = 0$" should be "where $j < i$ and $p_j > 0$ or $j = 0$".

32. **Page 19, proof of Proposition 5.5:** The computation of $\left(\mathsf{R}_{r+1}^\infty - \mathsf{R}_r^\infty\right) f$ goes a bit too fast for me. Here are some explanations I would make:

   - The equality

$$\begin{aligned} & f\left(x_1, \ldots, x_{r-1}, 0^j, x_r, 0, \ldots\right) - f\left(x_1, \ldots, x_{r-1}, 0^{j+1}, x_r, 0, \ldots\right) \\ & = \mathsf{R}_r^j \mathsf{R}_{r+j+1}^\infty \left(x_{r+j} - x_{r+j+1}\right) \partial_{r+j} f \end{aligned}$$

follows from observing that

$$\begin{aligned}&\left(x_{r+j} - x_{r+j+1}\right) \partial_{r+j} f \\ &= f - \left(f \text{ with the variables } x_{r+j} \text{ and } x_{r+j+1} \text{ swapped}\right)\end{aligned}$$

and

$$\mathsf{R}_r^j \mathsf{R}_{r+j+1}^\infty g = g\left(x_1, x_2, \ldots, x_{r-1}, 0^j, x_r, 0^\infty\right) \qquad \text{for each } g \in \mathrm{Pol}$$

(where 0^∞ means an infinite sequence $0, 0, 0, \ldots$).

- The equality

  $$\mathsf{R}_r^j \mathsf{R}_{r+j+1}^\infty \left(x_{r+j} - x_{r+j+1}\right) \partial_{r+j} f = x_r \mathsf{R}_r^j \mathsf{R}_{r+j+1}^\infty \partial_{r+j} f$$

  follows from the (easily proved) identities $\mathsf{R}_{r+j+1}^\infty x_{r+j+1} = 0$ and $\mathsf{R}_{r+j+1}^\infty x_{r+j} = x_{r+j} \mathsf{R}_{r+j+1}^\infty$ and $\mathsf{R}_r^j x_{r+j} = x_r \mathsf{R}_r^j$.

- The equality

  $$x_r \mathsf{R}_r^j \mathsf{R}_{r+j+1}^\infty \partial_{r+j} f = \left(x_r \mathsf{R}_r^j \mathsf{R}_{r+j+1}^\infty\right) \mathsf{D}_{r+j} f$$

  follows from the fact that the operator $\mathsf{R}_{r+j+1}^\infty$ (like any R_i^∞) is idempotent.

33. **Page 20, proof of Proposition 5.6:** In the definition of $A_{(i_1,\ldots,i_k)}$, it should be pointed out that i_0 is understood as 0 (this becomes necessary when all entries of $(i_1, \ldots, i_k)$ are equal to p).

34. **Page 20, proof of Proposition 5.6:** Replace "we have $a_{k-\ell} \geq a_{k-\ell+1} \geq \cdots \geq a_k \geq p$" by "we have $a_{\ell+1} \geq a_{\ell+2} \geq \cdots \geq a_k \geq p$".

35. **Page 21, proof of Proposition 5.7:** "We can then use Theorem 2.20" $\rightarrow$ "We can then use Proposition 2.16".

36. **Page 21, proof of Lemma 5.10:** Why does it suffice to show that "$\mathsf{D}_i \mathsf{R}_j \mathfrak{F}_a = 0$ for all i, except at most one for which $\mathsf{D}_i \mathsf{R}_j \mathfrak{F}_a = \mathfrak{F}_b$ for some $b \in \mathsf{Winc}$"? Shouldn't it also be shown that i is $\geq$ to the last entry of b, so that Theorem 5.3 can be applied?

    (Pedantic remark: The "Winc" here should be a "WInc".)

37. **Page 22, Slide Kostka matrix:** In the definition of $E_i(a)$, what does "$\in \{0, 1\} = 0$" mean?