Three questions on symmetric group algebras

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slides: http://www.cip.ifi.lmu.de/~grinberg/algebra/nfe2024.pdf

The symmetric group algebra

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- As a $\mathbf{k}$-module, $\mathcal{A}$ is just the n-th graded component of the Malvenuto–Reutenauer Hopf algebra FQSym, but its multiplication is the inner (not the standard) multiplication.

- Some of the nicest elements of $\mathcal{A}$ are the **Young–Jucys–Murphy elements**

$$\mathbf{m}_k := t_{1,k} + t_{2,k} + \cdots + t_{k-1,k} \qquad \text{for all } 1 \leq k \leq n,$$

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- Note that $\mathbf{m}_1 = 0$.

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- **Theorem (Murphy, ca. 1980?).** If $\mathbf{k}$ is a field of characteristic 0, then GZ_n (as a $\mathbf{k}$-vector space) has dimension equal to

$$\begin{aligned}&(\#\text{ of involutions in } S_n)\\ &= \sum_{\lambda \vdash n} (\#\text{ of standard Young tableaux of shape } \lambda)\end{aligned}$$

  (OEIS sequence A000085).

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- For $\mathbf{k} = \mathbb{Q}$, it has a basis $(\mathbf{e}_{T,T})_{\lambda \vdash n;\ T \text{ is a standard tableau of shape } \lambda}$ coming from the seminormal basis of $\mathbf{k}[S_n]$.

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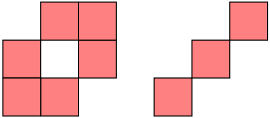
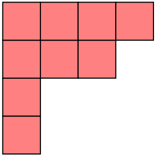
- Questions 1 and 1' true for $n \leq 6$.

Specht modules: a quick introduction

- Let D be a diagram with n cells. For instance, for $n = 9$, we can have

$D =$  or $D =$  .

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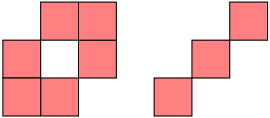
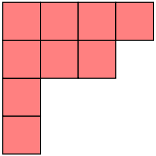
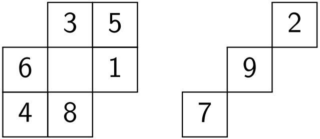
$D =$  or $D =$  .

- Let T be any filling of D with the numbers $1, 2, \ldots, n$. (Not necessarily standard!) For example, if D is the first diagram above, we can have

$T =$ 

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Are Specht modules pure?

- The **Specht module** $\mathcal{S}^D$ is the left ideal of $\mathcal{A}$ generated by

$$\left(\sum_{\substack{w \in S_n \text{ preserves} \\ \text{the columns of } T}} (-1)^w \, w \right) \left(\sum_{\substack{w \in S_n \text{ preserves} \\ \text{the rows of } T}} w \right).$$

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- Alternatively we can define $\mathcal{S}^D$ as a span of polytabloids or of determinants or in several other ways.

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- If D is a (skew) Young diagram, $\mathcal{S}^D$ has many famous properties and relates to Schur functions. I am interested here in the general case.

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- **Question 2:** Is $\mathcal{S}^D$ a direct addend of $\mathcal{A}$ as a $\mathbf{k}$-module?
- Proving this for $\mathbf{k} = \mathbb{Z}$ would suffice.
- **Question 2':** Let $\mathbf{k} = \mathbb{Z}/p$ for a prime p. Is $\dim_{\mathbf{k}} \mathcal{S}^D$ independent on p ?
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- Well-known positive answers when D is a skew Young diagram (Garnir's standard basis theorem).
- The answers are still positive when D is row-convex (Reiner/Shimozono 1993).
- Same questions exist for Schur and Weyl modules (over GL_n), but not sure if still equivalent.

Another unexpected commutativity

- For any permutation $w \in S_n$, define

$$\operatorname{exc} w := (\# \text{ of } i \in [n] \text{ such that } w(i) > i) \qquad \text{and}$$
$$\operatorname{anxc} w := (\# \text{ of } i \in [n] \text{ such that } w(i) < i).$$

- For any $a, b \in \mathbb{N}$, define

$$\mathbf{X}_{a,b} := \sum_{\substack{w \in S_n; \\ \operatorname{exc} w = a; \\ \operatorname{anxc} w = b}} w \in \mathcal{A} = \mathbf{k}[S_n].$$

- **Question 3:** Is it true that all these $\mathbf{X}_{a,b}$ commute (for fixed n and varying a, b) ? In other words, do we have $\mathbf{X}_{a,b}\mathbf{X}_{c,d} = \mathbf{X}_{c,d}\mathbf{X}_{a,b}$ for all $a, b, c, d \in \mathbb{N}$?
- Checked for all $n \leq 7$.
- This generalizes a limiting case of the Bethe subalgebra (Mukhin/Tarasov/Varchenko).