Monomial identities in the Weyl algebra

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detailed version
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Dedicated to a Hip Trendy Rascal on his 80th birthday

Abstract. Motivated by a question and some enumerative conjectures of Richard Stanley, we explore the equivalence classes of words in the Weyl algebra, $\mathbf{k}\langle D, U \mid DU - UD = 1 \rangle$. We show that each class is generated by the swapping of adjacent *balanced subwords*, i.e., those which have the same number of D's as U's, and give several other characterizations, as well as a linear-time algorithm for equivalence checking.

Armed with this, we deduce several enumerative results about such equivalence classes and their sizes. We extend these results to the class of c-Dyck words, where every prefix has at least c times as many U's as D's. We also connect these results to previous work on bond percolation and rook theory, and generalize them to some other algebras.

Keywords: Weyl algebra, words, lattice paths, rook placements, Ferrers boards, Dyck words, monoid kernel, bond percolation, PBW bases, down-up algebra, noncommutative algebra, rings, combinatorics, finite fields.

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Contents

1. Introduction

In 2023 Richard Stanley proposed the following problem (private communication). Let $\mathcal{M}$ denote the monoid freely generated by the two non-commuting variables D and U. Consider the action of $\mathcal{M}$ on the polynomial ring $\mathbb{Q}[x]$ in which D acts as differentiation ($\frac{d}{dx}$) and U acts as multiplication by x. This action is not free, as it is known to satisfy the relation

$$DU - UD = 1, \tag{1}$$

which is the defining relation of the *Weyl algebra* (also known as the *Heisenberg–Weyl algebra* [BHDPS08] due to its obvious quantum-physical significance).

Consider two words in $\mathcal{M}$ to be *equivalent* if they act equally on $\mathbb{Q}[x]$ (that is, become equal in the Weyl algebra). It can be shown that equivalent words have the same number of U's and the same number of D's. Thus we can ask: *How many distinct equivalence classes are there* for words with k many D's and $n-k$ many U's? We call this number $a(n,k)$. For example, among the six words with two D's and two U's, there is one equivalence: $DUUD = UDDU$, so that $a(4,2) = 5$.

We prove explicit formulas for $a(n,k)$ and for $\sum_k a(n,k)$ originally conjectured by Stanley (Section 6). Along the way, we study the equivalence from several directions and give several equivalent descriptions of it. Call a word in $\mathcal{M}$ *balanced* if it has the same number of U's as D's. We show (in Theorem 2.1, another conjecture of Stanley) that two words v and w are equivalent if and only if one can be obtained from another by a series of *balanced commutations*, i.e., by a sequence of swaps of adjacent balanced factors. In the example above, the transposition of DU with UD gives the equivalence. This and several further criteria serve as the linchpin for the enumerative results.

In much earlier work, Stanley identified a particular class of posets, including Young's lattice of integer partitions, which he called *differential posets* [Stanle88]. Standard and semi-standard Young tableaux are in bijection with certain chains in Young's lattice, $\mathbb{Y}$, which can be studied using the standard Down and Up operators in $\mathbb{Y}$. For example, applying U^n to the empty shape $\varnothing$ gives a formal sum of all partitions λ of n, each weighted by the number of standard tableaux of that shape f^λ. Applying D^nU^n to $\varnothing$ yields $\left(\sum_{\lambda\vdash n}(f^\lambda)^2\right)\varnothing$, and a simple inductive argument shows that $D^nU^n\varnothing = n!\varnothing$, recovering the basic enumerative identity shown by the Robinson–Schensted correspondence. The key insight is that these operators satisfy the fundamental relation of the Weyl algebra, Equation (1), allowing counting problems to be expressed in terms of these operators. Many enumerative identities, expressed as generating functions in these operators, can be proved by solving certain elementary partial differential equations. This study motivated the current problem, though it is natural enough on its own.

The outline of the paper is as follows. In Section 2 we formally define the monoid $\mathcal{M}$, the above monoid, the Weyl algebra $\mathcal{W}$, and many further related combinatorial objects. Then we state our main result (Theorem 2.1), which gives several different criteria for words in the monoid $\mathcal{M}$ to be equivalent (i.e., to represent the same operator in $\mathcal{W}$). [1] A second main result (Theorem 2.3) says that each balanced word u is equivalent to its "reverse toggle-image" (i.e., to the word obtained from

[1]We note that one of our criteria in Theorem 2.1 gives rise to an efficient (linear time) algorithm for the word problem in the monoid generated by D and U in $\mathcal{W}$, in contrast to the naive "expand and compare coefficients" approach (which requires quadratic time at best). See Remark 2.2 for details.

u by reversing the order of the letters and also replacing each D by U and vice versa). The proofs of these two results occupy the next three sections.

Section 3 provides some basic formulas for products of D's and U's in the Weyl algebra. In Section 4 we define a normal form for our words and start working our way towards the proof of our main result, which we finish in Section 5.

Enumerative results – including the formulas for $a(n,k)$ and for $\sum_k a(n,k)$ – are then obtained in Section 6. The formula for $\sum_k a(n,k)$ (Corollary 6.7) is surprisingly intricate, despite involving nothing more complicated than the Fibonacci sequence. Then we turn our attention to *c-Dyck words*, where every prefix has at least c times as many U's as D's, finding analogous results for this situation, and exploring some interesting special cases. Finally, we give an explicit formula for the *size* of the Weyl-equivalence class of a word w.

An intriguing digression is pursued in Section 7. Indeed, a search for the sequence of the numbers $\sum_k a(n,k)$ in the OEIS reveals a sequence [OEIS, A006727] originating in statistical physics (bond percolation on the directed square lattice). This sequence, however, agrees with ours only for $n \leq 11$, and in fact contains negative terms later on. We briefly introduce the physical context and explain this seeming coincidence.

Section 8 continues the study of equivalence of words and relates it to the part of combinatorics known as rook theory. The equivalence of two words u and v is revealed to be a stronger version of the *rook equivalence* of two Ferrers boards B_u and B_v induced by these words. This leads to two further "main theorems" (Theorem 8.2 and 8.3) that provide further equivalent conditions for two words to be equivalent. Their proofs piggyback on work by Navon, Haglund, Cotardo, Gruica and Ravagnani. We note that our equivalent criteria in Theorem 2.1 can thus also be seen as equivalent criteria for rook-equivalence of Ferrers boards, although some care is needed to ensure that the correspondence really is one-to-one (see Remark 8.4).

In Section 9, we take a closer look at our balanced commutations, and show that a subset of these commutations actually suffices to connect any two equivalent words. It is not hard to see that we can generate Weyl equivalence by transpositions only of *irreducible* balanced words, i.e., those which themselves cannot be factored into two or more balanced words. Even better, we show that we only need to swap irreducible balanced words starting with a U with ones starting with a D (Theorem 9.1).

In the final Section 10, we discuss other algebras that allow for the same or similar questions to be asked instead of the Weyl algebra. We generalize our results to multivariate Weyl algebras, and give a partial result for the down-up algebras of Benkart and Roby [BenRob98]. The case of Weyl algebras in positive characteristic appears to be more intricate, and we offer several open questions for exploration.

Remark. A more detailed version of this work, with some proofs expanded, is available as an ancillary file to this preprint on the arXiv.

2. Definitions and main results

In this section, we introduce the main notions and notations involved in our main results, which will be stated at the end of the section.

2.1. The monoid $\mathcal{M}$ and the Weyl algebra $\mathcal{W}$

Let $\mathcal{M}$ be the free (noncommutative) monoid generated by two symbols D and U. Its elements are the words with letters D and U, such as $DUUDDUDD$.

Let $\mathbf{k}$ be a field of characteristic 0, and let $\mathcal{W}$ be the Weyl algebra over $\mathbf{k}$ with generators D and U and relation $DU - UD = 1$. This algebra $\mathcal{W}$ acts on the univariate polynomial ring $\mathbf{k}[x]$ in a standard way: D acts as the derivative operator $\frac{\partial}{\partial x}$, whereas U acts as multiplication by x. It is known that this action is faithful, and $\mathcal{W}$ has a basis[2] $\left(D^i U^j\right)_{i,j\in\mathbb{N}}$ as well as a basis $\left(U^j D^i\right)_{i,j\in\mathbb{N}}$. See [ManSch16], [vanOys13] and several exercises in [Lorenz18] for much there is to know about $\mathcal{W}$ and then some. (The Weyl algebra $\mathcal{W}$ is often denoted by $A_1(\mathbf{k})$ or $\mathbb{A}_1(\mathbf{k})$.)

Let $\phi : \mathcal{M} \to \mathcal{W}$ be the canonical monoid morphism[3] from $\mathcal{M}$ to the monoid $(\mathcal{W}, \cdot, 1)$ that sends D to D and U to U. Thus, ϕ sends any product of D's and U's to the same product of D's and U's, but now computed in $\mathcal{W}$ instead of $\mathcal{M}$. This morphism ϕ is **not** injective, since (for example) $DUUD = UDDU$ in $\mathcal{W}$ (but not in $\mathcal{M}$). Thus, one may naturally wonder what pairs of words $u, v \in \mathcal{M}$ have equal images under ϕ. In the parlance of monoid theory, this is asking about the *kernel* of the monoid morphism ϕ – that is, the equivalence relation "$\phi(u) = \phi(v)$" on $\mathcal{M}$.

We will give several descriptions of this equivalence relation in terms of different objects, and subsequently study its enumerative properties (such as the number of equivalence classes of a given word length). We now define some of these objects.

2.2. Words

- The word "word" will always mean an element of $\mathcal{M}$ (that is, a word built of D's and U's), unless we say otherwise.
- A word $v \in \mathcal{M}$ is said to be a *factor* of a word $w \in \mathcal{M}$ if there exist words $u, u' \in \mathcal{M}$ (possibly empty) such that $w = uvu'$.
- A word $v \in \mathcal{M}$ is said to be a *prefix* of a word $w \in \mathcal{M}$ if there exists a word $u' \in \mathcal{M}$ (possibly empty) such that $w = vu'$.
- A word $v \in \mathcal{M}$ is said to be a *suffix* of a word $w \in \mathcal{M}$ if there exists a word $u \in \mathcal{M}$ (possibly empty) such that $w = uv$.

[2] Here and in the following, $\mathbb{N}$ denotes the set $\{0, 1, 2, \ldots\}$.
[3] "Morphism" means "homomorphism" throughout this work.

For example, the word DUD is a prefix of $DUDUDD$ and is a suffix of $UDDUD$. Furthermore, the word DUD is a factor of $UDUDDD$, but neither a prefix nor a suffix. The word DUD is not a factor of $DUUD$ (since a factor must appear contiguously). Note that each word w is a factor, a prefix and a suffix of itself.

2.3. Diagonal paths

A notion closely related to words are *diagonal paths*, which we will now introduce along with their various features:

- The *diagonal lattice* means the digraph (i.e., directed graph) with vertex set $\mathbb{Z}^2$ and arcs $(i,j) \to (i+1, j+1)$ (called *NE-arcs*) and $(i,j) \to (i+1, j-1)$ (called *SE-arcs*). Given two vertices u and v of the diagonal lattice, we write "$u \nearrow v$" for "$u \to v$ is an NE-arc", and we write "$u \searrow v$" for "$u \to v$ is an SE-arc".

  We imagine the diagonal lattice as being drawn in the Cartesian plane, but its arcs are not parallel to the axes but rather parallel to the two diagonals ($x = y$ and $x = -y$). Thus, NE-arcs and SE-arcs look like the arrows $\nearrow$ and $\searrow$, respectively (whence our notations for them).

- A *diagonal path* means a walk on the diagonal lattice. Since the diagonal lattice is acyclic (i.e., has no directed cycles), any such walk is a path.

- If $\mathbf{p} = (p_0, p_1, \ldots, p_k)$ is a diagonal path, then
  - the *vertices* of $\mathbf{p}$ are $p_0, p_1, \ldots, p_k$;
  - the *NE-steps* of $\mathbf{p}$ are the vertices p_i of $\mathbf{p}$ for which $i < k$ and $p_i \nearrow p_{i+1}$;
  - the *SE-steps* of $\mathbf{p}$ are the vertices p_i of $\mathbf{p}$ for which $i < k$ and $p_i \searrow p_{i+1}$.

  For example, if $\mathbf{p} = (p_0, p_1, p_2, p_3)$ is a diagonal path with $p_0 \nearrow p_1 \searrow p_2 \searrow p_3$, then its vertices are p_0, p_1, p_2, p_3; its only NE-step is p_0; and its SE-steps are p_1 and p_2.

- The *height* $\operatorname{ht}(i,j)$ of a vertex (i,j) of $\mathbb{Z}^2$ is its y-coordinate j.

- If $\mathbf{p} = (p_0, p_1, \ldots, p_k)$ is a diagonal path, then the *initial height* of $\mathbf{p}$ is the height $\operatorname{ht}(p_0)$ of its initial vertex, whereas the *final height* of $\mathbf{p}$ is the height $\operatorname{ht}(p_k)$ of its final vertex. We say that the path $\mathbf{p}$ *starts at height* $\operatorname{ht}(p_0)$ and *ends at height* $\operatorname{ht}(p_k)$.

- If $\mathbf{p} = (p_0, p_1, \ldots, p_k)$ is any diagonal path, then we associate three Laurent polynomials (in the indeterminate z) to $\mathbf{p}$:
  - the *height polynomial* $H(\mathbf{p}, z) = \sum\limits_{i=0}^{k} z^{\operatorname{ht}(p_i)}$;
  - the *NE-height polynomial* $H_{\mathrm{NE}}(\mathbf{p}, z) = \sum\limits_{p_i \text{ is an NE-step of } \mathbf{p}} z^{\operatorname{ht}(p_i)}$;

  – and the *SE-height polynomial* $H_{\text{SE}}(\mathbf{p}, z) = \sum_{p_i \text{ is an SE-step of } \mathbf{p}} z^{\text{ht}(p_i)}$.

- The *reading word* $\text{w}(\mathbf{p})$ of a diagonal path $\mathbf{p} = (p_0, p_1, \ldots, p_k)$ is defined to be the word $w_0 w_1 \cdots w_{k-1} \in \mathcal{M}$, where

$$w_i = \begin{cases} U, & \text{if } p_i \nearrow p_{i+1}; \\ D, & \text{if } p_i \searrow p_{i+1}. \end{cases}$$

  For instance, if $\mathbf{p} = (p_0, p_1, p_2, p_3, p_4)$ with $p_0 \nearrow p_1 \searrow p_2 \searrow p_3 \searrow p_4$, then $\text{w}(\mathbf{p}) = UDDD$.

For example, if $\mathbf{p}$ is the diagonal path shown in Figure 1, then the heights of all vertices of $\mathbf{p}$ are shown on Figure 2; in particular, its initial height is 0, its final height is -1, its height polynomial is $H(\mathbf{p}, z) = z^{-1} + 3z^0 + 3z^1 + z^2$, its NE-height polynomial is $H_{\text{NE}}(\mathbf{p}, z) = 2z^0 + z^1$, its SE-height polynomial is $H_{\text{SE}}(\mathbf{p}, z) = z^0 + 2z^1 + z^2$, and its reading word is $\text{w}(\mathbf{p}) = UDUUDDD$.

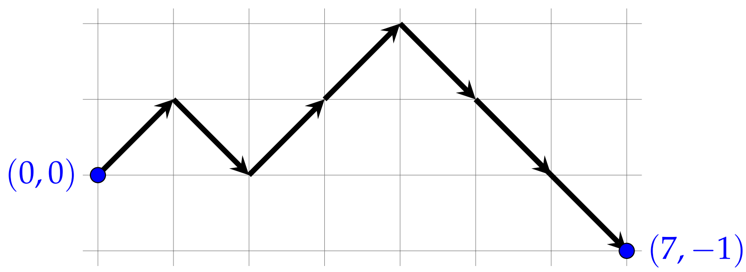


Figure 1: A diagonal path.

We note that if $\mathbf{p}$ is any diagonal path, then

$$\begin{aligned} &(\text{final height of } \mathbf{p}) - (\text{initial height of } \mathbf{p}) \\ &= (\# \text{ of } U\text{'s in } \text{w}(\mathbf{p})) - (\# \text{ of } D\text{'s in } \text{w}(\mathbf{p})). \end{aligned} \tag{2}$$

This is because, as we walk the path $\mathbf{p}$ from its beginning to its end, our height increases by 1 with each NE-step (which corresponds to a U in $\text{w}(\mathbf{p})$) and decreases by 1 with each SE-step (which corresponds to a D in $\text{w}(\mathbf{p})$).

Note that a diagonal path $\mathbf{p}$ is uniquely determined by its initial vertex p_0 and its reading word $\text{w}(\mathbf{p})$. (Knowing p_0 and $\text{w}(\mathbf{p})$, we can reconstruct $\mathbf{p}$ by starting at p_0 and walking in the directions provided by $\text{w}(\mathbf{p})$: namely, we make an NE-step for each U in w and an SE-step for each D in w.) In particular, for any word $w \in \mathcal{M}$, there is a unique diagonal path $\mathbf{p}$ that starts at $(0, 0)$ and has reading word $\text{w}(\mathbf{p}) = w$. We will call this path $\mathbf{p}$ the *standard path* of w. For example, the path shown in Figure 1 is the standard path of $UDUUDDD$.

Given a word $w \in \mathcal{M}$, we define the Laurent polynomials

$$H(w, z) = H(\mathbf{p}, z) \quad \text{and} \quad H_{\text{NE}}(w, z) = H_{\text{NE}}(\mathbf{p}, z) \quad \text{and} \quad H_{\text{SE}}(w, z) = H_{\text{SE}}(\mathbf{p}, z),$$

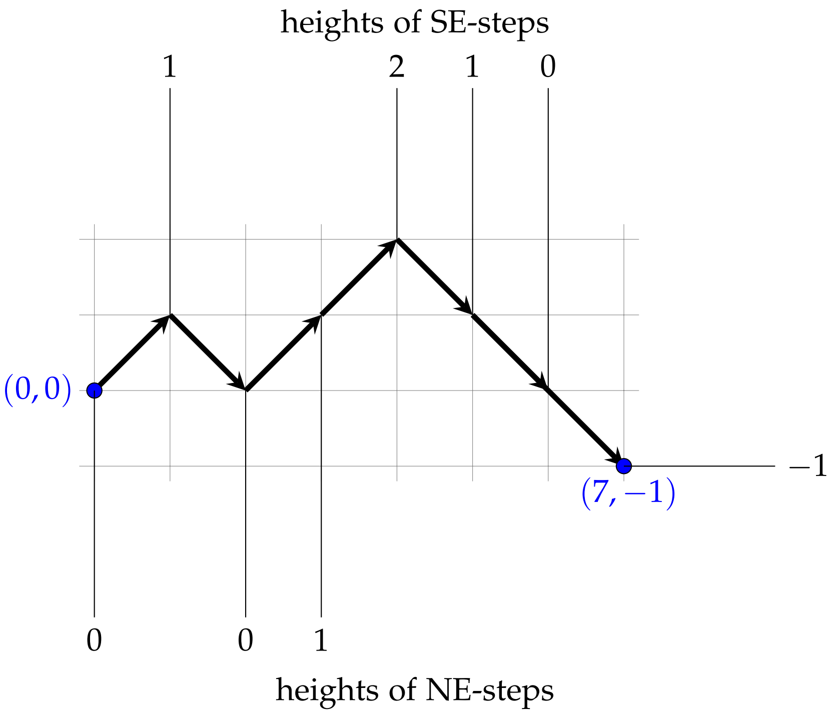


Figure 2: The heights of the vertices of **p**. The heights of the NE-steps are written at the bottom, while those of the SE-steps are written at the top. The last vertex of **p** counts neither as an SE-step nor as an NE-step.

where $\mathbf{p}$ is the standard path of w. We call $H(w,z)$ the *height polynomial* of w. Furthermore, we define the *final height* of our word w to be the final height of its standard path $\mathbf{p}$. Since its initial height is 0 (because it starts at $(0,0)$), and since it satisfies $\mathrm{w}(\mathbf{p}) = w$, we obtain from (2) the equality

$$\begin{aligned} &(\text{final height of } w) \\ &= (\text{\# of } U\text{'s in } w) - (\text{\# of } D\text{'s in } w). \end{aligned} \tag{3}$$

We note that the heights of the SE-steps of a diagonal path have come up in the study of Weyl algebras before (see the last paragraph of [BlaFla11, Section 7.1]).

2.4. The ω maps

We furthermore define two useful maps, both of which we call ω:

- Let $\omega : \mathcal{M} \to \mathcal{M}$ be the monoid anti-morphism[4] that sends U and D to D and U. Thus, acting on a word $w \in \mathcal{M}$, it reverses the word and toggles[5] every letter.

  For example, $\omega(UDD) = UUD$.

- Let $\omega : \mathcal{W} \to \mathcal{W}$ be the $\mathbf{k}$-algebra anti-morphism[6] that sends U and D to D and U. (This is well-defined, since the operation of swapping U with D transforms the defining relation $DU - UD = 1$ of $\mathcal{W}$ into the relation $UD - DU = 1$, which holds in the opposite algebra of $\mathcal{W}$.)

  For example, $\omega(UDD) = UUD$ (but now in $\mathcal{W}$).

Both maps ω are involutions, i.e., satisfy $\omega \circ \omega = \mathrm{id}$. Moreover, the two ω's commute with ϕ: That is, we have $\omega \circ \phi = \phi \circ \omega$. In other words, the diagram

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\phi} & \mathcal{W} \\ {\scriptstyle\omega}\downarrow & & \downarrow{\scriptstyle\omega} \\ \mathcal{M} & \xrightarrow[\phi]{} & \mathcal{W} \end{array} \tag{4}$$

commutes. This justifies us calling the two ω's by the same letter.

[4] A *monoid anti-morphism* is a map $f : M \to N$ between two monoids such that $f(1_M) = 1_N$ and $f(ab) = f(b) f(a)$ for all $a, b \in M$. In other words, it is a monoid morphism from M to the opposite monoid of N.

[5] To *toggle* a letter means to replace it by a D if it is a U, and to replace it by a U if it is a D.

[6] A $\mathbf{k}$-*algebra anti-morphism* is a map $f : A \to B$ between two $\mathbf{k}$-algebras such that f is a morphism of additive groups and a monoid anti-morphism of multiplicative monoids. In other words, it is a $\mathbf{k}$-algebra morphism from A to the opposite algebra of B.

2.5. Balanced words, commutations and flips

Finally, we define the concept of a *balanced word* and two equivalence relations on words:

- A word $w \in \mathcal{M}$ is said to be *balanced* if it has the same number of D's and U's. For example, $DUUDDU$ is balanced, whereas $DUUUD$ is not.

- Given two words $v, w \in \mathcal{M}$, we say that v is obtained from w by a *balanced commutation* if and only if we can write v and w as $v = pxyq$ and $w = pyxq$, where $p, q \in \mathcal{M}$ are two words and where $x, y \in \mathcal{M}$ are two balanced words. Roughly speaking, this means that v can be obtained from w by swapping two balanced factors that abut each other in w.

  For instance, from $DUDDUUDUUD$ we can obtain $DDUUDUDUUD$ by a balanced commutation (swapping the prefix DU with the infix $DDUU$, both of which are balanced). By a further balanced commutation, we can turn $DDUUDUDUUD$ into $DDUUDUUDDU$ (swapping the infix DU with the suffix UD, both of which are balanced).

  We define an equivalence relation $\overset{\text{bal}}{\sim}$ on $\mathcal{M}$ by stipulating that two words $w, v \in \mathcal{M}$ satisfy $w \overset{\text{bal}}{\sim} v$ if and only if v can be obtained from w by a sequence (possibly empty) of balanced commutations. Thus, our above examples show that $DUDDUUDUUD \overset{\text{bal}}{\sim} DDUUDUUDDU$.

- Given two words $v, w \in \mathcal{M}$, we say that v is obtained from w by a *balanced flip* if and only if we can write v and w as $v = pxq$ and $w = p\omega(x)q$, where $p, q \in \mathcal{M}$ are two words and where $x \in \mathcal{M}$ is a balanced word. Roughly speaking, this means that v can be obtained from w by picking a balanced factor and applying the involution ω to it.

  For instance, from $DUDDUU$ we can obtain $DDUUDU$ by a balanced flip (applying ω to the balanced factor $UDDU$).

  We define an equivalence relation $\overset{\text{flip}}{\sim}$ on $\mathcal{M}$ by stipulating that two words $w, v \in \mathcal{M}$ satisfy $w \overset{\text{flip}}{\sim} v$ if and only if v can be obtained from w by a sequence (possibly empty) of balanced flips. Thus, our above example shows that $DUDDUU \overset{\text{flip}}{\sim} DDUUDU$.

Another example for a balanced transformation and a balanced flip can be seen on Figure 3.

2.6. Main result: Equivalent descriptions of Weyl equivalence

Everything is now in place to state our main result, which gives several (necessary and sufficient) criteria for when two words $u, v \in \mathcal{M}$ have the same image under ϕ.

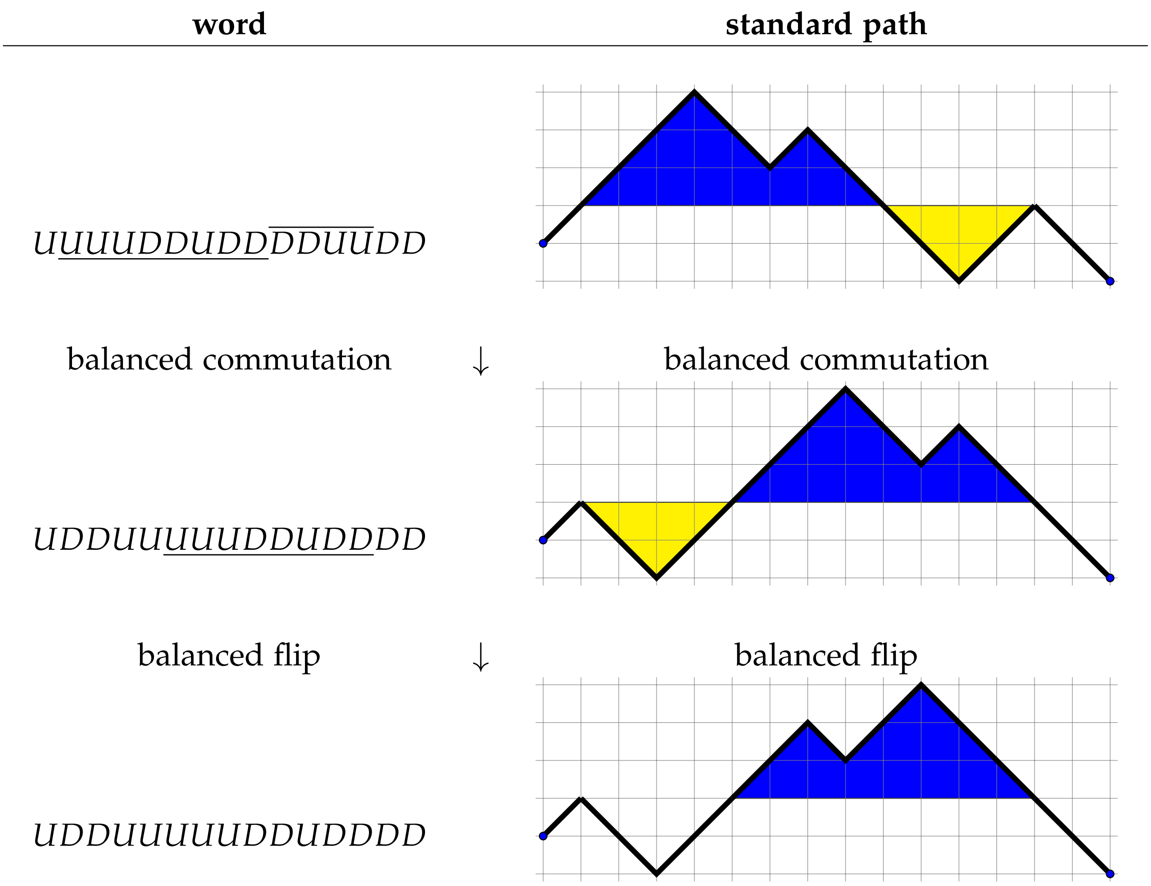


Figure 3: A word undergoing a balanced commutation followed by a balanced flip (left); the corresponding standard paths (right). The factors that are swapped or transformed are marked by underlines and overlines.

Theorem 2.1. Let u and v be two words in $\mathcal{M}$. Then, the following seven statements are equivalent:

- $\mathcal{S}_1$: We have $\phi(u) = \phi(v)$.
- $\mathcal{S}_2$: The elements $\phi(u)$ and $\phi(v)$ act equally on the polynomial ring $\mathbf{k}[x]$. (That is, we have $(\phi(u))(p) = (\phi(v))(p)$ for each polynomial $p \in \mathbf{k}[x]$. Note that the action of $\mathcal{W}$ on $\mathbf{k}[x]$ was defined in Subsection 2.1.)
- $\mathcal{S}_3$: The words u and v have the same final height and satisfy $H_{\text{NE}}(u,z) = H_{\text{NE}}(v,z)$.
- $\mathcal{S}_3'$: The words u and v have the same final height and satisfy $H_{\text{SE}}(u,z) = H_{\text{SE}}(v,z)$.
- $\mathcal{S}_4$: The words u and v have the same final height and satisfy $H(u,z) = H(v,z)$.
- $\mathcal{S}_5$: We have $u \overset{\text{bal}}{\sim} v$.
- $\mathcal{S}_6$: We have $u \overset{\text{flip}}{\sim} v$.

In particular, this shows that the two relations $\overset{\text{bal}}{\sim}$ and $\overset{\text{flip}}{\sim}$ are the same.

Remark 2.2. Criterion $\mathcal{S}_3$ (or $\mathcal{S}_3'$) can also be turned into an efficient algorithm to decide whether or not $\phi(u) = \phi(v)$, which requires linear time and space in the length of u and v:

We use an array A with both positive and negative indices. Start at $i = 0$ and read the word u. When a value of i is reached for the first time, $A[i]$ is initialized as 0. In particular, $A[0]$ is initialized as 0 right at the beginning. Each time the letter U is read, increase $A[i]$ by 1 and increase i by 1; else decrease i by 1 (and leave the values of the array unchanged). The final value of i is the final height of u, whereas the final entries of the array are the coefficients of $H_{\text{NE}}(u,z)$. Once u has been read completely, do the same with v (starting at $i = 0$ again), but decrease $A[i]$ whenever the letter U occurs. Condition $\mathcal{S}_3$ is clearly only satisfied if the final heights of u and v are the same and the array at the end consists entirely of zeros. If a value of i is reached in the second part that was not reached in the first part, or if an entry of the array becomes negative during the second stage of the algorithm, one can stop immediately: $\phi(u) \neq \phi(v)$ in this case. The length of the array is thus bounded by the length of u. Even more, since the sum of the entries of the array is bounded by the length of word u, a linear total number of bits is sufficient for all entries of the array.

In Section 8, we will add some more equivalent statements to the list in Theorem 2.1, albeit under the additional assumption that u and v have the same number of U's and the same number of D's.

The following result is a curious consequence of Theorem 2.1 (although we will prove it first and then use it in the proof of Theorem 2.1).

Theorem 2.3. Let $u \in \mathcal{M}$ be a balanced word. Then, $\phi(u) = \phi(\omega(u))$ and $u \overset{\text{bal}}{\sim} \omega(u)$.

3. Basic formulas for the Weyl algebra action

The proof of our main result requires significant build-up and preparation. We begin with a closer look at the Weyl algebra $\mathcal{W}$ and its action on the polynomial ring $\mathbf{k}[x]$.

Lemma 3.1. The action of the Weyl algebra $\mathcal{W}$ on the polynomial ring $\mathbf{k}[x]$ is faithful: That is, if two elements $a, b \in \mathcal{W}$ satisfy $a(p) = b(p)$ for all $p \in \mathbf{k}[x]$, then $a = b$.

Proof. This is a folklore result, and can be easily derived from facts in the literature. For instance, [Milici17, Theorem 5.11] (applied to $n = 1$) shows that the $\mathbf{k}$-algebra $D(1)$ of differential operators on the polynomial ring $\mathbf{k}[x]$ is isomorphic to the Weyl algebra $\mathbf{k}\langle D, U \mid DU - UD = 1\rangle = \mathcal{W}$. The actual proof of [Milici17, Theorem 5.11] shows that the $\mathbf{k}$-algebra morphism $\mathcal{W} \to D(1)$ that sends D and U to $\dfrac{\partial}{\partial x}$ and the multiplication by x is injective. But this morphism is precisely the action of $\mathcal{W}$ on $\mathbf{k}[x]$ (except that its target has been restricted to $D(1)$). Thus, its injectivity means that the action is faithful. This proves Lemma 3.1. $\square$

Remark 3.2. The Weyl algebra $\mathcal{W}$ acts not only on the polynomial ring $\mathbf{k}[x]$, but also on the rings of Laurent polynomials $\mathbf{k}\left[x, x^{-1}\right]$, of formal power series $\mathbf{k}[[x]]$, of formal Laurent series $\mathbf{k}((x))$, and (if $\mathbf{k} = \mathbb{R}$ or $\mathbf{k} = \mathbb{C}$) of infinitely differentiable functions $\mathcal{C}^{\infty}(\mathbf{k})$. In each case, the action is faithful (since the polynomial ring $\mathbf{k}[x]$ embeds into each of these rings), and thus all our results still apply.

Even more generally: Any nontrivial representation of $\mathcal{W}$ is faithful. Indeed, recall that we assumed $\operatorname{char} \mathbf{k} = 0$. Thus, the Weyl algebra $\mathcal{W}$ is simple (see, e.g., [Lam01, Corollary 3.17]), and thus its only proper ideal is 0. But the annihilator of a nontrivial representation V of $\mathcal{W}$ (that is, the set of all $w \in \mathcal{W}$ satisfying $wV = 0$) is a proper ideal of $\mathcal{W}$, and thus must be 0. Hence, V must be faithful. This gives an alternative proof of Lemma 3.1.

Each word $u \in \mathcal{M}$ is mapped by ϕ to an element of the Weyl algebra $\mathcal{W}$, which in turn acts on the polynomial ring $\mathbf{k}[x]$. Our next goal is to give an explicit formula for this action in terms of a diagonal path $\mathbf{p}$ that has u as its reading word. For the sake of simplicity, we extend the action of $\mathcal{W}$ from the polynomial ring $\mathbf{k}[x]$ to the

Laurent polynomial ring $\mathbf{k}\left[x, x^{-1}\right]$ (so that we don't have to worry about possible negative exponents on a power of x).

Proposition 3.3. Let $\mathbf{p} = (p_0, p_1, \ldots, p_k)$ be a diagonal path. Let $h_i := \operatorname{ht}(p_i)$ for each $i \in \{0, 1, \ldots, k\}$. Then, for each $s \in \mathbb{Z}$, we have

$$(\phi(\mathrm{w}(\mathbf{p})))(x^s) = \left(\prod_{p_i \text{ is an SE-step of } \mathbf{p}} (s + h_k - h_{i+1}) \right) \cdot x^{s+h_k-h_0}. \tag{5}$$

Example 3.4. If the last steps of $\mathbf{p}$ are $\cdots p_{k-4} \searrow p_{k-3} \nearrow p_{k-2} \nearrow p_{k-1} \searrow p_k$, then $\mathrm{w}(\mathbf{p}) = \cdots DUUD$ and thus any $s \in \mathbb{N}$ satisfies

$$\begin{aligned}(\phi(\mathrm{w}(\mathbf{p})))(x^s) &= \cdots DUUDx^s = \cdots \frac{\partial}{\partial x} xx \underbrace{\frac{\partial}{\partial x} x^s}_{= sx^{s-1}} = s \cdots \frac{\partial}{\partial x} x \underbrace{xx^{s-1}}_{= x^s} \\ &= s \cdots \frac{\partial}{\partial x} \underbrace{xx^s}_{= x^{s+1}} = s \cdots \underbrace{\frac{\partial}{\partial x} x^{s+1}}_{= (s+1)x^s} = s(s+1)\cdots x^s.\end{aligned}$$

The two factors s and $s+1$ that we have found correspond precisely to the two SE-steps p_{k-1} and p_{k-4} of $\mathbf{p}$. Of course, further SE-steps in $\mathbf{p}$ will contribute more such factors.

The above example illustrates where Proposition 3.3 comes from: In general, we can compute $(\phi(\mathrm{w}(\mathbf{p})))(x^s)$ in the same way, decomposing $\phi(\mathrm{w}(\mathbf{p}))$ into a product of D's and U's (which correspond, respectively, to SE-steps and NE-steps of $\mathrm{w}(\mathbf{p})$), and letting each of these D's and U's act on the monomial x^s sequentially (starting with the last one). The letter U acts as multiplication by x and thus sends x^k to x^{k+1}, whereas the letter D acts as $\dfrac{\partial}{\partial x}$ and thus sends x^k to kx^{k-1}. Thus, in total, the exponent on our monomial x^s is incremented once for each U (that is, for each NE-step of $\mathrm{w}(\mathbf{p})$) and decremented once for each D (that is, for each SE-step of $\mathrm{w}(\mathbf{p})$), so that it becomes $s + h_k - h_0$ at the end (an easy consequence of (2)). The factors accumulating in front of the monomial are precisely the k's coming from the D's, and thus are precisely the $s + h_k - h_{i+1}$ corresponding to the SE-steps p_i of $\mathbf{p}$, since it is these SE-steps that turn into letters D in $\phi(\mathrm{w}(\mathbf{p}))$ (and the current degree of the monomial at the time when D is applied is exactly $s + h_k - h_{i+1}$). The final result is precisely the right hand side of (5).

This argument can be translated into the following rigorous proof of Proposition 3.3:

Proof of Proposition 3.3. We induct on k.

The *base case* ($k = 0$) is obvious (here, $h_k = h_0$, whereas $\mathrm{w}(\mathbf{p})$ is the empty word, and thus $\phi(\mathrm{w}(\mathbf{p}))$ is the unity of $\mathcal{W}$, and the product over the SE-steps of $\mathbf{p}$ is empty).

For the *induction step*, we fix a positive integer k, and assume that the proposition is already proved for $k-1$ instead of k. Let $s \in \mathbb{Z}$ be arbitrary. Our goal is to prove (5).

The definition of the action of $\mathcal{W}$ on $\mathbf{k}\left[x, x^{-1}\right]$ yields

$$Ux^s = xx^s = x^{s+1} \qquad \text{and} \qquad Dx^s = \frac{\partial}{\partial x} x^s = sx^{s-1}.$$

By removing the last step from our path $\mathbf{p} = (p_0, p_1, \ldots, p_k)$, we obtain the shorter diagonal path $\mathbf{q} = (p_0, p_1, \ldots, p_{k-1})$. We are in one of the following two cases:

Case 1: We have $p_{k-1} \nearrow p_k$.

Case 2: We have $p_{k-1} \searrow p_k$.

Let us first consider Case 1. In this case, we have $p_{k-1} \nearrow p_k$. Thus, the diagonal path $\mathbf{p}$ is $\mathbf{q}$ followed by the NE-step $p_{k-1} \nearrow p_k$. Hence, the definition of a reading word yields $w(\mathbf{p}) = w(\mathbf{q})\, U$. Therefore,

$$\begin{aligned} \phi(\mathrm{w}(\mathbf{p})) &= \phi(\mathrm{w}(\mathbf{q})\, U) = \phi(\mathrm{w}(\mathbf{q}))\, \underbrace{\phi(U)}_{=U} \qquad (\text{since } \phi \text{ is a monoid morphism}) \\ &= \phi(\mathrm{w}(\mathbf{q}))\, U. \end{aligned} \tag{6}$$

Therefore,

$$\begin{aligned} (\phi(\mathrm{w}(\mathbf{p})))(x^s) = (\phi(\mathrm{w}(\mathbf{q}))\, U)(x^s) &= (\phi(\mathrm{w}(\mathbf{q})))\, \underbrace{(Ux^s)}_{=x^{s+1}} \\ &= (\phi(\mathrm{w}(\mathbf{q})))\left(x^{s+1}\right). \end{aligned} \tag{7}$$

Recall that the path $\mathbf{p}$ is $\mathbf{q}$ followed by the NE-step $p_{k-1} \nearrow p_k$. Hence, the SE-steps of $\mathbf{p}$ are exactly the SE-steps of $\mathbf{q}$.

Since $p_{k-1} \nearrow p_k$, we have $h_{k-1} = h_k - 1$ (since $h_{k-1} = \operatorname{ht}(p_{k-1})$ and $h_k = \operatorname{ht}(p_k)$). Thus, $1 + h_{k-1} = h_k$.

However, $\mathbf{q}$ is a diagonal path of length $k-1$. Hence, the induction hypothesis allows us to apply Proposition 3.3 to $k-1$, $\mathbf{q}$ and $s+1$ instead of k, $\mathbf{p}$ and s. We thus obtain

$$\begin{aligned} &(\phi(\mathrm{w}(\mathbf{q})))\left(x^{s+1}\right) \\ &= \left(\prod_{p_i \text{ is an SE-step of } \mathbf{q}} (s + 1 + h_{k-1} - h_{i+1})\right) \cdot x^{s+1+h_{k-1}-h_0} \\ &= \left(\prod_{p_i \text{ is an SE-step of } \mathbf{q}} (s + h_k - h_{i+1})\right) \cdot x^{s+h_k-h_0} \qquad (\text{since } 1 + h_{k-1} = h_k) \\ &= \left(\prod_{p_i \text{ is an SE-step of } \mathbf{p}} (s + h_k - h_{i+1})\right) \cdot x^{s+h_k-h_0} \end{aligned}$$

(here, we have replaced **q** by **p** under the product sign, since the SE-steps of **p** are exactly the SE-steps of **q**). In view of (7), we can rewrite this as

$$(\phi(\mathrm{w}(\mathbf{p})))(x^s) = \left(\prod_{p_i \text{ is an SE-step of } \mathbf{p}} (s + h_k - h_{i+1})\right) \cdot x^{s+h_k-h_0}.$$

Thus, (5) is proved in Case 1.

Let us now consider Case 2. In this case, we have $p_{k-1} \searrow p_k$. Thus, the diagonal path **p** is **q** followed by the SE-step $p_{k-1} \searrow p_k$. Hence, just like we proved (6) in Case 1, we can now show that

$$\phi(\mathrm{w}(\mathbf{p})) = \phi(\mathrm{w}(\mathbf{q}))\,D.$$

Therefore,

$$\begin{aligned}(\phi(\mathrm{w}(\mathbf{p})))(x^s) &= (\phi(\mathrm{w}(\mathbf{q}))\,D)(x^s) = (\phi(\mathrm{w}(\mathbf{q})))\underbrace{(Dx^s)}_{=sx^{s-1}} \\ &= s \cdot (\phi(\mathrm{w}(\mathbf{q})))\left(x^{s-1}\right). \qquad (8)\end{aligned}$$

Recall that the path **p** is **q** followed by the SE-step $p_{k-1} \searrow p_k$. Hence, the SE-steps of **p** are exactly the SE-steps of **q** as well as the additional SE-step p_{k-1}. Therefore,

$$\begin{aligned}&\prod_{p_i \text{ is an SE-step of } \mathbf{p}} (s + h_k - h_{i+1}) \\ &= \underbrace{\left(s + h_k - h_{(k-1)+1}\right)}_{=s+h_k-h_k=s} \cdot \left(\prod_{p_i \text{ is an SE-step of } \mathbf{q}} (s + h_k - h_{i+1})\right) \\ &= s \cdot \left(\prod_{p_i \text{ is an SE-step of } \mathbf{q}} (s + h_k - h_{i+1})\right). \qquad (9)\end{aligned}$$

Since $p_{k-1} \searrow p_k$, we have $h_{k-1} = h_k + 1$ (since $h_{k-1} = \mathrm{ht}(p_{k-1})$ and $h_k = \mathrm{ht}(p_k)$). Thus, $-1 + h_{k-1} = h_k$.

However, **q** is a diagonal path of length $k-1$. Hence, the induction hypothesis allows us to apply Proposition 3.3 to $k-1$, **q** and $s-1$ instead of k, **p** and s. We thus obtain

$$\begin{aligned}&(\phi(\mathrm{w}(\mathbf{q})))\left(x^{s-1}\right) \\ &= \left(\prod_{p_i \text{ is an SE-step of } \mathbf{q}} (s - 1 + h_{k-1} - h_{i+1})\right) \cdot x^{s-1+h_{k-1}-h_0} \\ &= \left(\prod_{p_i \text{ is an SE-step of } \mathbf{q}} (s + h_k - h_{i+1})\right) \cdot x^{s+h_k-h_0} \qquad \left(\text{since } -1 + h_{k-1} = h_k\right).\end{aligned}$$

Thus, (8) can be rewritten as

$$(\phi(\mathrm{w}(\mathbf{p})))(x^s) = \underbrace{s \cdot \left(\prod_{p_i \text{ is an SE-step of } \mathbf{q}} (s + h_k - h_{i+1}) \right)}_{\substack{= \prod_{p_i \text{ is an SE-step of } \mathbf{p}} (s+h_k-h_{i+1}) \\ \text{(by (9))}}} \cdot x^{s+h_k-h_0}$$

$$= \left(\prod_{p_i \text{ is an SE-step of } \mathbf{p}} (s + h_k - h_{i+1}) \right) \cdot x^{s+h_k-h_0}.$$

Thus, (5) is proved in Case 2.

We have now proved (5) in both Cases 1 and 2. Thus, the induction step is complete. This completes the proof of Proposition 3.3. □

The following is a version of Proposition 3.3 in which the reading word $\mathrm{w}(\mathbf{p})$ additionally undergoes the "toggle-and-reverse" anti-automorphism ω:

Proposition 3.5. Let $\mathbf{p} = (p_0, p_1, \ldots, p_k)$ be a diagonal path. Let $h_i := \mathrm{ht}(p_i)$ for each $i \in \{0, 1, \ldots, k\}$. Then, for each $s \in \mathbb{Z}$, we have

$$(\omega(\phi(\mathrm{w}(\mathbf{p}))))(x^s) = \left(\prod_{p_i \text{ is an NE-step of } \mathbf{p}} (s + h_0 - h_i) \right) \cdot x^{s+h_0-h_k}.$$

Proof. This can be proved similarly to Proposition 3.3. Alternatively, this can be derived from Proposition 3.3 as follows:

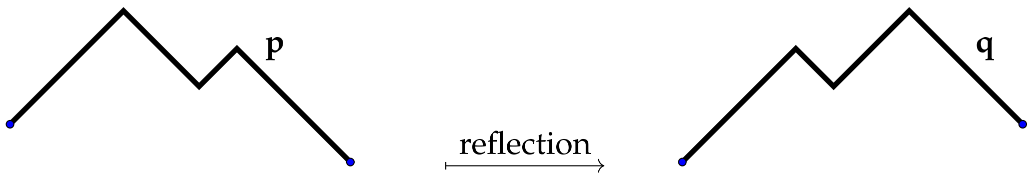


Figure 4: A path $\mathbf{p}$ (left) and its reflection $\mathbf{q}$ across a vertical axis (right).

Let $s \in \mathbb{Z}$. Let $\mathbf{q}$ be a diagonal path obtained by reflecting $\mathbf{p}$ across a vertical axis. Then, each NE-step of $\mathbf{p}$ becomes an SE-step in $\mathbf{q}$ and vice versa, and moreover, the order of all steps is reversed (see Figure 4). Hence, $\mathrm{w}(\mathbf{q}) = \omega(\mathrm{w}(\mathbf{p}))$ and thus

$$\phi(\mathrm{w}(\mathbf{q})) = \phi(\omega(\mathrm{w}(\mathbf{p}))) = \omega(\phi(\mathrm{w}(\mathbf{p}))) \qquad (\text{since } \omega \circ \phi = \phi \circ \omega).$$

Let us write the path $\mathbf{q}$ as $\mathbf{q} = (q_0, q_1, \ldots, q_k)$ (we can do this, since it clearly has the same length as $\mathbf{p} = (p_0, p_1, \ldots, p_k)$). Let $g_i := \mathrm{ht}(q_i)$ for each $i \in \{0, 1, \ldots, k\}$. Since $\mathbf{q}$ is the reflection of $\mathbf{p}$ across a vertical axis, we thus have

$$g_j = h_{k-j} \qquad \text{for each } j \in \{0, 1, \ldots, k\} \tag{10}$$

(since the height of a point does not change when we reflect it across a vertical axis). In particular, $g_0 = h_k$ and $g_k = h_0$. However, Proposition 3.3 (applied to $\mathbf{q}$, q_i and g_i instead of $\mathbf{p}$, p_i and h_i) yields

$$\begin{aligned} (\phi(\mathrm{w}(\mathbf{q})))(x^s) &= \left(\prod_{q_i \text{ is an SE-step of } \mathbf{q}} (s + g_k - g_{i+1}) \right) \cdot x^{s+g_k-g_0} \\ &= \left(\prod_{q_i \text{ is an SE-step of } \mathbf{q}} (s + h_0 - h_{k-i-1}) \right) \cdot x^{s+h_0-h_k} \qquad (11) \end{aligned}$$

(since (10) yields $g_{i+1} = h_{k-(i+1)} = h_{k-1-i}$ and $g_k = h_0$ and $g_0 = h_k$). But the SE-steps of $\mathbf{q}$ correspond to the NE-steps of $\mathbf{p}$; more specifically, a given vertex q_i of $\mathbf{q}$ is an SE-step of $\mathbf{q}$ if and only if p_{k-1-i} is an NE-step of $\mathbf{p}$. Hence,

$$\begin{aligned} &\prod_{q_i \text{ is an SE-step of } \mathbf{q}} (s + h_0 - h_{k-i-1}) \\ &= \prod_{p_{k-1-i} \text{ is an NE-step of } \mathbf{p}} (s + h_0 - h_{k-i-1}) \\ &= \prod_{p_i \text{ is an NE-step of } \mathbf{p}} (s + h_0 - h_i) \qquad \left(\begin{array}{c} \text{here, we have substituted } i \\ \text{for } k-i-1 \text{ in the product} \end{array} \right). \end{aligned}$$

Thus, we can rewrite (11) as

$$(\phi(\mathrm{w}(\mathbf{q})))(x^s) = \left(\prod_{p_i \text{ is an NE-step of } \mathbf{p}} (s + h_0 - h_i) \right) \cdot x^{s+h_0-h_k}.$$

In view of $\phi(\mathrm{w}(\mathbf{q})) = \omega(\phi(\mathrm{w}(\mathbf{p})))$, we can rewrite this further as

$$(\omega(\phi(\mathrm{w}(\mathbf{p}))))(x^s) = \left(\prod_{p_i \text{ is an NE-step of } \mathbf{p}} (s + h_0 - h_i) \right) \cdot x^{s+h_0-h_k}.$$

This proves Proposition 3.5. □

As a consequence of Proposition 3.5, we can easily obtain the following:

Proposition 3.6. Let $\mathbf{p} = (p_0, p_1, \ldots, p_k)$ and $\mathbf{q} = (q_0, q_1, \ldots, q_m)$ be two diagonal paths with the same initial height that satisfy $\phi(\mathrm{w}(\mathbf{p})) = \phi(\mathrm{w}(\mathbf{q}))$. Then, the final heights of $\mathbf{p}$ and $\mathbf{q}$ are equal, and we have

$$\begin{aligned} &\{\mathrm{ht}(p_i) \mid p_i \text{ is an NE-step of } \mathbf{p}\}_{\text{multiset}} \\ &= \{\mathrm{ht}(q_i) \mid q_i \text{ is an NE-step of } \mathbf{q}\}_{\text{multiset}}. \qquad (12) \end{aligned}$$

Proof. Let $h_i := \operatorname{ht}(p_i)$ for each $i \in \{0, 1, \ldots, k\}$. Let $g_i := \operatorname{ht}(q_i)$ for each $i \in \{0, 1, \ldots, m\}$. Note that $h_0 = g_0$ (since $\mathbf{p}$ and $\mathbf{q}$ have the same initial height).

Let $s \in \mathbb{N}$ be high enough to be larger than all numbers $h_i - h_0$ for all NE-steps p_i of $\mathbf{p}$ and also larger than all numbers $g_i - g_0$ for all NE-steps q_i of $\mathbf{q}$.

Then, Proposition 3.5 yields

$$(\omega(\phi(\mathrm{w}(\mathbf{p}))))(x^s) = \left(\prod_{p_i \text{ is an NE-step of } \mathbf{p}} (s + h_0 - h_i)\right) \cdot x^{s+h_0-h_k}$$

and similarly

$$(\omega(\phi(\mathrm{w}(\mathbf{q}))))(x^s) = \left(\prod_{q_i \text{ is an NE-step of } \mathbf{q}} (s + g_0 - g_i)\right) \cdot x^{s+g_0-g_m}.$$

But the left hand sides of these two equalities are equal, since $\phi(\mathrm{w}(\mathbf{p})) = \phi(\mathrm{w}(\mathbf{q}))$. Thus, the right hand sides are also equal. In other words, we have

$$\left(\prod_{p_i \text{ is an NE-step of } \mathbf{p}} (s + h_0 - h_i)\right) \cdot x^{s+h_0-h_k} = \left(\prod_{q_i \text{ is an NE-step of } \mathbf{q}} (s + g_0 - g_i)\right) \cdot x^{s+g_0-g_m}. \tag{13}$$

The products on both sides of this equality are nonzero (since s is larger than all numbers $h_i - h_0$ for all NE-steps p_i of $\mathbf{p}$ and also larger than all numbers $g_i - g_0$ for all NE-steps q_i of $\mathbf{q}$). Thus, (13) entails that the exponents $s + h_0 - h_k$ and $s + g_0 - g_m$ are equal, and therefore we conclude that $h_0 - h_k = g_0 - g_m$. Since $h_0 = g_0$, this entails $h_k = g_m$. In other words, the final heights of $\mathbf{p}$ and $\mathbf{q}$ are equal.

Since the two exponents $s + h_0 - h_k$ and $s + g_0 - g_m$ in (13) are equal, we obtain

$$\prod_{p_i \text{ is an NE-step of } \mathbf{p}} (s + h_0 - h_i) = \prod_{q_i \text{ is an NE-step of } \mathbf{q}} (s + g_0 - g_i)$$

by comparing coefficients in (13).

Forget that we fixed s. We just proved the equality

$$\prod_{p_i \text{ is an NE-step of } \mathbf{p}} (s + h_0 - h_i) = \prod_{q_i \text{ is an NE-step of } \mathbf{q}} (s + g_0 - g_i)$$

for each sufficiently high $s \in \mathbb{N}$. But this equality is a polynomial identity in s, and thus must hold formally (since it holds for each sufficiently high $s \in \mathbb{N}$). In other words, we have

$$\prod_{p_i \text{ is an NE-step of } \mathbf{p}} (x + h_0 - h_i) = \prod_{q_i \text{ is an NE-step of } \mathbf{q}} (x + g_0 - g_i)$$

in the polynomial ring $\mathbf{k}[x]$. In view of $h_0 = g_0$, we can rewrite this equality as

$$\prod_{p_i \text{ is an NE-step of } \mathbf{p}} (x + g_0 - h_i) = \prod_{q_i \text{ is an NE-step of } \mathbf{q}} (x + g_0 - g_i).$$

Substituting x for $x + g_0$ on both sides of this equality, we transform it into

$$\prod_{p_i \text{ is an NE-step of } \mathbf{p}} (x - h_i) = \prod_{q_i \text{ is an NE-step of } \mathbf{q}} (x - g_i).$$

Since $\mathbf{k}[x]$ is a unique factorization domain, this yields that

$$\{h_i \mid p_i \text{ is an NE-step of } \mathbf{p}\}_{\text{multiset}} = \{g_i \mid q_i \text{ is an NE-step of } \mathbf{q}\}_{\text{multiset}}.$$

In other words,

$$\{\operatorname{ht}(p_i) \mid p_i \text{ is an NE-step of } \mathbf{p}\}_{\text{multiset}} = \{\operatorname{ht}(q_i) \mid q_i \text{ is an NE-step of } \mathbf{q}\}_{\text{multiset}}$$

(since $h_i = \operatorname{ht}(p_i)$ and $g_i = \operatorname{ht}(q_i)$ for all respective i). This completes the proof of Proposition 3.6 (since we have already shown that the final heights of $\mathbf{p}$ and $\mathbf{q}$ are equal). □

Proposition 3.7. Let u and v be two words in $\mathcal{M}$ such that $\phi(u) = \phi(v)$. Then, u and v contain the same number of D's and the same number of U's.

Proof. Let $\mathbf{p} = (p_0, p_1, \ldots, p_k)$ be the standard path of u, and let $\mathbf{q} = (q_0, q_1, \ldots, q_m)$ be the standard path of v. The paths $\mathbf{p}$ and $\mathbf{q}$ both start at $(0, 0)$ (by the definition of a standard path), and thus have the same initial height (namely, 0). Moreover, their reading words are $\operatorname{w}(\mathbf{p}) = u$ and $\operatorname{w}(\mathbf{q}) = v$ (since $\mathbf{p}$ and $\mathbf{q}$ are the standard paths of u and v). Thus, from $\phi(u) = \phi(v)$, we obtain $\phi(\operatorname{w}(\mathbf{p})) = \phi(\operatorname{w}(\mathbf{q}))$. Hence, Proposition 3.6 yields that the final heights of $\mathbf{p}$ and $\mathbf{q}$ are equal, and that

$$\{\operatorname{ht}(p_i) \mid p_i \text{ is an NE-step of } \mathbf{p}\}_{\text{multiset}} = \{\operatorname{ht}(q_i) \mid q_i \text{ is an NE-step of } \mathbf{q}\}_{\text{multiset}}.$$

The latter equality yields (in particular) that the number of NE-steps of $\mathbf{p}$ equals the number of NE-steps of $\mathbf{q}$. Since the NE-steps of $\mathbf{p}$ are in bijection with the U's in $\operatorname{w}(\mathbf{p}) = u$, and similarly for $\mathbf{q}$, we can rewrite this as follows: The number of U's in u equals the number of U's in v. In other words, u and v contain the same number of U's.

It remains to see that the words u and v contain the same number of D's. But (2) shows that

$$(\text{final height of } \mathbf{p}) - (\text{initial height of } \mathbf{p}) = (\text{\# of } U\text{'s in } \operatorname{w}(\mathbf{p})) - (\text{\# of } D\text{'s in } \operatorname{w}(\mathbf{p}))$$

and

$$(\text{final height of } \mathbf{q}) - (\text{initial height of } \mathbf{q}) = (\text{\# of } U\text{'s in } \mathrm{w}(\mathbf{q})) - (\text{\# of } D\text{'s in } \mathrm{w}(\mathbf{q})).$$

The left hand sides of these two equalities are equal (since the paths $\mathbf{p}$ and $\mathbf{q}$ have the same initial height and the same final height). Thus, so are the right hand sides:

$$(\text{\# of } U\text{'s in } \mathrm{w}(\mathbf{p})) - (\text{\# of } D\text{'s in } \mathrm{w}(\mathbf{p})) = (\text{\# of } U\text{'s in } \mathrm{w}(\mathbf{q})) - (\text{\# of } D\text{'s in } \mathrm{w}(\mathbf{q})).$$

In other words,

$$(\text{\# of } U\text{'s in } u) - (\text{\# of } D\text{'s in } u) = (\text{\# of } U\text{'s in } v) - (\text{\# of } D\text{'s in } v)$$

(since $u = \mathrm{w}(\mathbf{p})$ and $v = \mathrm{w}(\mathbf{q})$). Since $(\text{\# of } U\text{'s in } u) = (\text{\# of } U\text{'s in } v)$ (because u and v contain the same number of U's), we thus conclude that $(\text{\# of } D\text{'s in } u) = (\text{\# of } D\text{'s in } v)$. In other words, u and v contain the same number of D's. Proposition 3.7 is thus fully proved. □

Lemma 3.8. Let $u \in \mathcal{M}$ be a balanced word. Let $s \in \mathbb{Z}$. Then, $(\phi(u))(x^s) = \lambda_{u,s} x^s$ for some $\lambda_{u,s} \in \mathbb{Z}$.

Proof. Let $\mathbf{p} = (p_0, p_1, \ldots, p_k)$ be the standard path of u. This path $\mathbf{p}$ starts at $(0,0)$ and thus has initial height 0. Thus, the final height of $\mathbf{p}$ is also 0 (by (2), since the word $\mathrm{w}(\mathbf{p}) = u$ is balanced).

Let $h_i := \mathrm{ht}(p_i)$ for each $i \in \{0, 1, \ldots, k\}$. Thus, $h_k = 0$ (since the final height of $\mathbf{p}$ is 0) and $h_0 = 0$ (since the initial height of $\mathbf{p}$ is 0). Now, Proposition 3.3 yields

$$(\phi(\mathrm{w}(\mathbf{p})))(x^s) = \left(\prod_{p_i \text{ is an SE-step of } \mathbf{p}} (s + h_k - h_{i+1}) \right) \cdot x^{s + h_k - h_0}.$$

Since $\mathrm{w}(\mathbf{p}) = u$ and $s + \underbrace{h_k}_{=0} - \underbrace{h_0}_{=0} = s$, we can rewrite this as

$$(\phi(u))(x^s) = \left(\prod_{p_i \text{ is an SE-step of } \mathbf{p}} (s + h_k - h_{i+1}) \right) \cdot x^s.$$

In other words, $(\phi(u))(x^s) = \lambda_{u,s} x^s$ for some $\lambda_{u,s} \in \mathbb{Z}$ (namely, for $\lambda_{u,s} = \prod_{p_i \text{ is an SE-step of } \mathbf{p}} (s + h_k - h_{i+1})$). This proves Lemma 3.8. □

Lemma 3.9. Let a and b be two balanced words in $\mathcal{M}$. Then, $\phi(a)\phi(b) = \phi(b)\phi(a)$.

First proof. Let $s \in \mathbb{N}$. Then, Lemma 3.8 (applied to $u = a$) shows that $(\phi(a))(x^s) = \lambda_{a,s} x^s$ for some $\lambda_{a,s} \in \mathbb{Z}$. Similarly, $(\phi(b))(x^s) = \lambda_{b,s} x^s$ for some $\lambda_{b,s} \in \mathbb{Z}$. Consider these $\lambda_{a,s}$ and $\lambda_{b,s}$. Now,

$$\begin{aligned}(\phi(a)\phi(b))(x^s) &= (\phi(a))\underbrace{(\phi(b))(x^s)}_{=\lambda_{b,s}x^s} = (\phi(a))(\lambda_{b,s} x^s) = \lambda_{b,s} \underbrace{(\phi(a))(x^s)}_{=\lambda_{a,s}x^s} \\ &= \lambda_{b,s} \lambda_{a,s} x^s.\end{aligned}$$

Similarly,

$$(\phi(b)\,\phi(a))\,(x^s) = \lambda_{a,s}\lambda_{b,s}x^s.$$

The right hand sides of these two equalities are equal (since $\lambda_{b,s}\lambda_{a,s} = \lambda_{a,s}\lambda_{b,s}$). Hence, so are their left hand sides. In other words, $(\phi(a)\,\phi(b))\,(x^s) = (\phi(b)\,\phi(a))\,(x^s)$.

Forget that we fixed s. We thus have shown that $(\phi(a)\,\phi(b))\,(x^s) = (\phi(b)\,\phi(a))\,(x^s)$ for each $s \in \mathbb{N}$. By linearity, this entails that $(\phi(a)\,\phi(b))\,(p) = (\phi(b)\,\phi(a))\,(p)$ for any polynomial $p \in \mathbf{k}[x]$. Since the action of $\mathcal{W}$ on $\mathbf{k}[x]$ is faithful, this shows that $\phi(a)\,\phi(b) = \phi(b)\,\phi(a)$ in $\mathcal{W}$. This proves Lemma 3.9. □

Second proof. The following alternative proof has been suggested to us by Jörgen Backelin. It shows that Lemma 3.9 is a disguised form of a classical result known already to Dixmier [Dixmie68].

We equip the Weyl algebra $\mathcal{W}$ with a $\mathbb{Z}$-grading by deciding that its generators U and D be homogeneous of degrees 1 and -1, respectively. The 0-th graded component $\mathcal{W}_0$ of $\mathcal{W}$ is then spanned by the images of the balanced words under ϕ. In particular, both $\phi(a)$ and $\phi(b)$ belong to $\mathcal{W}_0$ (since a and b are balanced words).

However, a result of Dixmier ([Dixmie68, last equation in §3.2]) says that the $\mathbf{k}$-algebra $\mathcal{W}_0$ is generated by the single element $\phi(DU)$. The proof of this result in [Dixmie68] is fairly easy: We abbreviate $\phi(D)$ and $\phi(U)$ as $\overline{D}$ and $\overline{U}$. First it is shown that $\mathcal{W}$ is spanned by the elements of the form $\overline{D}^i\overline{U}^j$ with $i, j \in \mathbb{N}$, which are homogeneous of respective degrees $j - i$. Therefore the 0-th graded component $\mathcal{W}_0$ is spanned by the elements of the form $\overline{D}^i\overline{U}^i$ with $i \in \mathbb{N}$. But each of the latter elements can be rewritten as

$$\overline{D}^i\overline{U}^i = \left(\overline{DU}\right)\left(\overline{DU}+1\right)\left(\overline{DU}+2\right)\cdots\left(\overline{DU}+i-1\right)$$

(as can be proved by induction on i), which is clearly a polynomial in $\overline{DU} = \phi(DU)$. Thus the algebra $\mathcal{W}_0$ is generated by the single element $\phi(DU)$.

Now, the algebra $\mathcal{W}_0$ is commutative (since we have just shown that it is generated by a single element). Therefore, any two of its elements commute. In particular, $\phi(a)$ and $\phi(b)$ commute (since both $\phi(a)$ and $\phi(b)$ belong to $\mathcal{W}_0$). In other words, $\phi(a)\,\phi(b) = \phi(b)\,\phi(a)$. This proves Lemma 3.9 again. □

Lemma 3.10. Let $u, v \in \mathcal{M}$ be two words such that $u \overset{\text{bal}}{\sim} v$. Then, $\phi(u) = \phi(v)$.

Proof. We have $u \overset{\text{bal}}{\sim} v$. Thus, by the definition of the relation $\overset{\text{bal}}{\sim}$, the word u can be transformed into v by a sequence of balanced commutations. Thus, we must show that the image $\phi(w)$ of a word $w \in \mathcal{M}$ is preserved whenever we apply a balanced commutation to w. In other words, we must show that if a word v is obtained from a word w by a balanced commutation, then $\phi(v) = \phi(w)$.

So let v and w be two words such that v is obtained from w by a balanced commutation. Thus, we can write v and w as $v = pxyq$ and $w = pyxq$, where $p, q \in \mathcal{M}$ are two words and where $x, y \in \mathcal{M}$ are two balanced words. Consider these p, q, x, y. Since x and y are balanced, we have $\phi(x)\phi(y) = \phi(y)\phi(x)$ by Lemma 3.9. However, from $v = pxyq$, we obtain

$$\phi(v) = \phi(pxyq) = \phi(p)\phi(x)\phi(y)\phi(q) \qquad \text{(since } \phi \text{ is a monoid morphism)}.$$

Similarly,

$$\phi(w) = \phi(p)\phi(y)\phi(x)\phi(q) \qquad \text{(since } w = pyxq).$$

The right hand sides of these two equalities are equal (since $\phi(x)\phi(y) = \phi(y)\phi(x)$). Thus, so are the left hand sides. In other words, $\phi(v) = \phi(w)$. This completes our proof of Lemma 3.10. □

4. Some words on words

Next, we take a closer look at some properties of the monoid $\mathcal{M}$ of words. We introduce some more terminology:

- A word $w \in \mathcal{M}$ is said to be *rising* if it has at least as many U's as it has D's.
- A word $w \in \mathcal{M}$ is said to be *falling* if it has at least as many D's as it has U's.

Thus, each word $w \in \mathcal{M}$ is rising or falling or both. Moreover, the balanced words $w \in \mathcal{M}$ are exactly the words $w \in \mathcal{M}$ that are both rising and falling.

- A *down-zig* means a word of the form UD^kU for some $k \geq 2$.
- A rising word $w \in \mathcal{M}$ is said to be *up-normal* if it contains no down-zig as a factor.

For example, the rising word $UUDUDUDD$ is up-normal, whereas the rising word $UUDDUU$ is not (since it has the down-zig $UDDU = UD^2U$ as a factor).

We could similarly define "up-zigs" and "down-normal words" (by toggling each letter in down-zigs and up-normal words, respectively), but we will have no need for them.

For what follows, we need a simple property of products in a monoid:

Lemma 4.1. Let a and b be two elements of a monoid M. Let $w \in M$ be any product of a's and b's ending with a b. Then, w can be written in the form $a^{r_1} b\, a^{r_2} b \cdots a^{r_h} b$ for some nonnegative integers h and $r_1, r_2, \ldots, r_h$. (Note that these integers are allowed to be 0.)

Proof. We assumed that w is a product of a's and b's ending with a b. Locate all the b factors in this product. Between any two consecutive b factors lies a (possibly empty) product of a's. Rewrite this product as a^r for some nonnegative integer r. Do the same for the product of a's that lies to the left of the first b. Thus, the total product becomes $a^{r_1} b\, a^{r_2} b \cdots a^{r_h} b$, where h is the number of all b factors and where the r_i are the exponents r obtained from the rewriting process. This proves Lemma 4.1. □

Proposition 4.2. Every up-normal word has the form

$$D^a \, (UD)^{r_1} U \, (UD)^{r_2} U \, \cdots \, (UD)^{r_h} U \, D^b$$

for some nonnegative integers a, b, h and $r_1, r_2, \ldots, r_h$ (that is, a power of D, followed by a product of several factors of the form $(UD)^r U$, followed by a further power of D, where all powers are allowed to be empty).

Proof. If the word consists entirely of D's, then this is trivial. Otherwise, we first remove the initial and the final run of D's (of length a and b respectively, both of which can also be 0) from our word. The remaining word is still up-normal, but starts and ends with a U.

This remaining word therefore has no two consecutive Ds, since any run of Ds longer than a single D would create a down-zig factor (when combined with the last U before the run and the first U after it). Thus, each D in this remaining word has to be preceded by a U (since the word starts with U). This allows us to decompose this word into a product of UD's and U's (for instance, by reading it from right to left, and pairing each D with the U that necessarily precedes it); this product ends with a U (since our word ends with a U). Thus, our word can be written as

$$(UD)^{r_1} U \, (UD)^{r_2} U \, \cdots \, (UD)^{r_h} U$$

for some nonnegative integers h and $r_1, r_2, \ldots, r_h$ (by Lemma 4.1, applied to $a = UD$ and $b = U$). This proves Proposition 4.2. $\square$

The converse of Proposition 4.2 also holds: Each rising word of the form shown in Proposition 4.2 is up-normal. The proof is nearly trivial, but we will not use this fact in the following.

Next, we observe a near-trivial symmetry of balanced commutations:

Proposition 4.3. Let u and v be two words in $\mathcal{M}$. Then, $u \overset{\text{bal}}{\sim} v$ if and only if $\omega(u) \overset{\text{bal}}{\sim} \omega(v)$.

Proof. The map ω transforms a word by reversing it and toggling each letter[7]. Clearly, both of these operations turn balanced factors of our word into balanced factors of the resulting word. Thus, if a word a is obtained from a word b by a balanced commutation, then $\omega(a)$ is obtained from $\omega(b)$ by a balanced commutation as well. The same must therefore hold for multiple balanced commutations applied in sequence. In other words, if $u \overset{\text{bal}}{\sim} v$, then $\omega(u) \overset{\text{bal}}{\sim} \omega(v)$. The converse holds for similar reasons (or can also be obtained by applying the preceding sentence to $\omega(u)$ and $\omega(v)$ instead of u and v, since $\omega \circ \omega = \text{id}$). Thus, Proposition 4.3 is proved. $\square$

[7]We recall: To *toggle* a letter means to replace it by the opposite letter (i.e., to replace a U by a D or a D by a U).

Our main goal in this section is to prove the following proposition:

Proposition 4.4. Let $w \in \mathcal{M}$ be a rising word. Then, there exists a unique up-normal word $t \in \mathcal{M}$ such that $t \overset{\text{bal}}{\sim} w$.

This up-normal word t will be called the *up-normal form* of w.

In order to prove Proposition 4.4, we need some lemmas. First, we show some simple identities that allow us to convert between the three height polynomials (height, NE-height and SE-height) of a diagonal path:

Lemma 4.5. Let $\mathbf{p}$ be any diagonal path starting at height a and ending at height b. We have

$$H(\mathbf{p}, z) = (1+z) H_{\mathrm{NE}}(\mathbf{p}, z) + \sum_{j \geq b} z^j - \sum_{j \geq a+1} z^j \tag{14}$$

and

$$H(\mathbf{p}, z) = (1+z^{-1}) H_{\mathrm{SE}}(\mathbf{p}, z) + \sum_{j \geq a} z^j - \sum_{j \geq b+1} z^j. \tag{15}$$

(The infinite sums are formal Laurent series, but only finitely many addends survive the cancellation.)

Proof. We only prove (14), since the proof of (15) is completely analogous. If the path $\mathbf{p}$ has length 0, then $a = b$ as well as $H(\mathbf{p}, z) = z^a = z^b$ and $H_{\mathrm{NE}}(\mathbf{p}, z) = H_{\mathrm{SE}}(\mathbf{p}, z) = 0$. The identity clearly holds in this case. Now proceed by induction, and let $\mathbf{p}'$ be the diagonal path obtained by removing the last vertex from $\mathbf{p}$. Then $H(\mathbf{p}, z) = H(\mathbf{p}', z) + z^b$, and the final height of $\mathbf{p}'$ is $b-1$ if the last step of $\mathbf{p}$ is an NE-step, and $b+1$ otherwise. In the former case, we have $H_{\mathrm{NE}}(\mathbf{p}, z) = H_{\mathrm{NE}}(\mathbf{p}', z) + z^{b-1}$, and the induction hypothesis gives us

$$H(\mathbf{p}', z) = (1+z) H_{\mathrm{NE}}(\mathbf{p}', z) + \sum_{j \geq b-1} z^j - \sum_{j \geq a+1} z^j,$$

from which the desired statement follows by adding z^b on both sides (since $z^{b-1} + z^b = (1+z) z^{b-1}$). In the latter case, we have $H_{\mathrm{NE}}(\mathbf{p}, z) = H_{\mathrm{NE}}(\mathbf{p}', z)$, and the induction hypothesis gives us

$$H(\mathbf{p}', z) = (1+z) H_{\mathrm{NE}}(\mathbf{p}', z) + \sum_{j \geq b+1} z^j - \sum_{j \geq a+1} z^j.$$

Again, we add z^b on both sides to obtain the desired identity. This completes the induction and thus the proof. $\square$

Remark 4.6. Let $\mathbf{p}$ and b be as in Lemma 4.5. By definition, we have

$$H(\mathbf{p}, z) = H_{\text{NE}}(\mathbf{p}, z) + H_{\text{SE}}(\mathbf{p}, z) + z^b,$$

since H_{NE} covers all NE-steps, H_{SE} covers all SE-steps, and z^b the final vertex. This can also be used to derive (15) from (14).

Recall that the height polynomial $H(w, z)$ of a word w was defined as the height polynomial $H(\mathbf{p}, z)$ of its standard path $\mathbf{p}$. Thus, Lemma 4.5 can be applied to words:

Lemma 4.7. Let $w \in \mathcal{M}$ be a word with final height b. Then,

$$H(w, z) = (1 + z) H_{\text{NE}}(w, z) + \sum_{j \geq b} z^j - \sum_{j \geq 1} z^j \tag{16}$$

and

$$H(w, z) = (1 + z^{-1}) H_{\text{SE}}(w, z) + \sum_{j \geq 0} z^j - \sum_{j \geq b+1} z^j. \tag{17}$$

Proof. Let $\mathbf{p}$ be the standard path of w. Then, the initial height of $\mathbf{p}$ is 0 (since $\mathbf{p}$ starts at $(0, 0)$), whereas the final height of $\mathbf{p}$ is the final height of w (by the definition of the latter), and we have

$$H(w, z) = H(\mathbf{p}, z) \quad \text{and} \quad H_{\text{NE}}(w, z) = H_{\text{NE}}(\mathbf{p}, z) \quad \text{and} \quad H_{\text{SE}}(w, z) = H_{\text{SE}}(\mathbf{p}, z)$$

(again by the definitions of the respective left hand sides). Thus, Lemma 4.7 follows from Lemma 4.5 (applied to $a = 0$). □

As we said, height polynomials of words are a particular case of height polynomials of diagonal paths. But the general case can easily be reduced to this particular case:

Lemma 4.8. Let $\mathbf{r}$ be a diagonal path. Let j be its initial height, and let $w = \mathrm{w}(\mathbf{r})$ be its reading word. Then, $H(\mathbf{r}, z) = z^j H(w, z)$.

Proof. Let $\mathbf{p} = (p_0, p_1, \ldots, p_k)$ be the standard path of w. Then, $H(w, z) = H(\mathbf{p}, z)$ (by the definition of $H(w, z)$).

However, the path $\mathbf{p}$ has reading word w (by the definition of a standard path). Thus, the path $\mathbf{r}$ has the same reading word as $\mathbf{p}$ (since both paths have reading word w). Thus, it is an image of $\mathbf{p}$ under the parallel translation by some vector (a, b). Consider this (a, b). Clearly, b must be the difference between the initial height of $\mathbf{r}$ and the initial height of $\mathbf{p}$. Since the initial height of $\mathbf{r}$ is j, while the initial height of $\mathbf{p}$ is 0 (since any standard path has initial height 0), we thus obtain $b = j - 0 = j$.

On the other hand, $\mathbf{r}$ is the image of the path $\mathbf{p} = (p_0, p_1, \ldots, p_k)$ under the translation by the vector (a, b). Hence, $\mathbf{r} = (r_0, r_1, \ldots, r_k)$, where each r_i is the image of the corresponding p_i under this translation. Thus, for each $i \in \{0, 1, \ldots, k\}$, we have

$$\operatorname{ht}(r_i) = \operatorname{ht}(p_i) + b. \tag{18}$$

Now, the definition of $H(\mathbf{r}, z)$ yields

$$\begin{aligned} H(\mathbf{r}, z) &= \sum_{i=0}^{k} z^{\operatorname{ht}(r_i)} = \sum_{i=0}^{k} \underbrace{z^{\operatorname{ht}(p_i)+b}}_{= z^b z^{\operatorname{ht}(p_i)}} \qquad \text{(by (18))} \\ &= \underbrace{z^b}_{\substack{= z^j \\ \text{(since } b = j)}} \underbrace{\sum_{i=0}^{k} z^{\operatorname{ht}(p_i)}}_{= H(\mathbf{p}, z) = H(w, z)} = z^j H(w, z). \end{aligned}$$

This proves Lemma 4.8. ◻

Lemma 4.9. Let u and v be two words. Let

$$k = (\text{\# of } U\text{'s in } u) - (\text{\# of } D\text{'s in } u).$$

Then,

$$H(uv, z) = H(u, z) + z^k (H(v, z) - 1); \tag{19}$$

$$H_{\mathrm{NE}}(uv, z) = H_{\mathrm{NE}}(u, z) + z^k H_{\mathrm{NE}}(v, z); \tag{20}$$

$$H_{\mathrm{SE}}(uv, z) = H_{\mathrm{SE}}(u, z) + z^k H_{\mathrm{SE}}(v, z). \tag{21}$$

Proof. Let $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ be the standard paths of the words u, v and uv, respectively. Then, the path $\mathbf{p}$ has initial height 0 (like any standard path) and reading word $\mathrm{w}(\mathbf{p}) = u$. Hence, the path $\mathbf{p}$ has final height k (by the definition of k, and by (2)). Let (ℓ, k) be the final vertex of $\mathbf{p}$.

The reading words of the paths $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ are u, v and uv (since $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ are the standard paths of u, v and uv). In particular, the reading word of the path $\mathbf{r}$ is uv, which is the concatenation of the reading words of $\mathbf{p}$ and $\mathbf{q}$. Thus, the path $\mathbf{r}$ is obtained by gluing together $\mathbf{p}$ with a copy of $\mathbf{q}$ that has been translated by the vector (ℓ, k) (since the final vertex of $\mathbf{p}$ is (ℓ, k)). Hence, the vertices of $\mathbf{r}$ are the vertices of $\mathbf{p}$ as well as the non-initial[8] vertices of $\mathbf{q}$ translated by (ℓ, k). Clearly, the translation increases the height of each of the latter vertices by k. Thus, the sum that defines $H(\mathbf{r}, z)$ can be split into two sub-sums, one of which collects all the vertices of $\mathbf{p}$ and thus amounts to $H(\mathbf{p}, z)$, while the other collects all the non-initial vertices of $\mathbf{q}$ (translated by (ℓ, k)) and thus amounts to $z^k (H(\mathbf{q}, z) - 1)$ (the

[8]"Non-initial" means that we exclude the initial vertex.

z^k factor comes from the translation, whereas the "-1" stems from removing the initial vertex). Thus, we obtain

$$H(\mathbf{r}, z) = H(\mathbf{p}, z) + z^k (H(\mathbf{q}, z) - 1). \quad (22)$$

However, $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ are the standard paths of the words u, v and uv. Hence, $H(\mathbf{p}, z) = H(u, z)$ and $H(\mathbf{q}, z) = H(v, z)$ and $H(\mathbf{r}, z) = H(uv, z)$. Thus, we can rewrite (22) as

$$H(uv, z) = H(u, z) + z^k (H(v, z) - 1).$$

This proves (19). The other two equalities can be proved in similar ways (but we no longer need to remove the initial vertex of $\mathbf{q}$, since the last vertex of a diagonal path counts neither as an NE-step nor as an SE-step). $\square$

Lemma 4.10. An up-normal word w is uniquely determined by its final height (i.e., the difference # of U's $-$ # of D's) and the height polynomial $H(w, z)$.

Proof. Let f be the final height of w. We shall recover w from f and $H(w, z)$.

From (16) (applied to $b = f$), we obtain

$$H(w, z) = (1 + z) H_{\text{NE}}(w, z) + \sum_{j \geq f} z^j - \sum_{j \geq 1} z^j.$$

All terms in this equality other than $H_{\text{NE}}(w, z)$ are determined by $H(w, z)$ and f. Thus, we can use this equality to determine $H_{\text{NE}}(w, z)$ from $H(w, z)$ and f (using polynomial division by $1 + z$).

Now, let $\mathbf{p}$ be the standard path of w. Thus, the path $\mathbf{p}$ starts at $(0, 0)$ and has reading word $\mathrm{w}(\mathbf{p}) = w$. Moreover, the definition of $H_{\text{NE}}(w, z)$ yields $H_{\text{NE}}(w, z) = H_{\text{NE}}(\mathbf{p}, z)$. Thus, $H_{\text{NE}}(\mathbf{p}, z)$ can be determined from $H(w, z)$ and f (since we can determine $H_{\text{NE}}(w, z)$ from these inputs).

By Proposition 4.2, we can write w in the form

$$w = D^a (UD)^{r_1} U (UD)^{r_2} U \cdots (UD)^{r_h} U D^b. \quad (23)$$

Each U here corresponds to an NE-step of the diagonal path $\mathbf{p}$ (since $w = \mathrm{w}(\mathbf{p})$). Thus, $\mathbf{p}$ has $r_1 + 1$ NE-steps of height $-a$ (corresponding to the $r_1 + 1$ many U's in the $(UD)^{r_1} U$ factor), followed by $r_2 + 1$ NE-steps of height $-a + 1$ (corresponding to the $r_2 + 1$ many U's in the $(UD)^{r_2} U$ factor), and so on. Altogether, we thus obtain

$$H_{\text{NE}}(\mathbf{p}, z) = \sum_{i=1}^{h} (r_i + 1) z^{-a+i-1}.$$

Hence, the Laurent polynomial $H_{\text{NE}}(\mathbf{p}, z)$ has the nonzero coefficients $r_1 + 1, r_2 + 1, \ldots, r_h + 1$ in front of the respective monomials $z^{-a}, z^{-a+1}, \ldots, z^{-a+h-1}$, and zero coefficients in front of all other monomials. In particular, the number $-a$ is the smallest exponent that appears with nonzero coefficient in the Laurent polynomial $H_{\text{NE}}(\mathbf{p}, z)$, whereas h is the total number of nonzero coefficients of this Laurent polynomial, and furthermore, the numbers $r_1+1, r_2+1, \ldots, r_h+1$ are the coefficients of the monomials $z^{-a}, z^{-a+1}, \ldots, z^{-a+h-1}$ in this Laurent polynomials.

Thus, the numbers a, h, and $r_1, r_2, \ldots, r_h$ can be determined from $H_{\mathrm{NE}}(\mathbf{p}, z)$. Since $H_{\mathrm{NE}}(\mathbf{p}, z)$ can be determined from $H(w, z)$ and f, we thus conclude that the numbers a, h, and $r_1, r_2, \ldots, r_h$ can be determined from $H(w, z)$ and f. Knowing these numbers, we can now determine b from f (since f is the final height of w, that is, the # of U's in w minus the # of D's in w). Knowing $a, b, h, r_1, r_2, \ldots, r_h$, we can now reconstruct w using (23). □

Remark 4.11. The statement of Lemma 4.10 would be false without the assumption that the final height is known. For example, the up-normal words $UDUU$ and $UUDD$ both have the height polynomial $2+2z+z^2$.

An analogue of Lemma 4.10 is true for down-normal words (with a similar proof).

Lemma 4.12. The height polynomial of a word is invariant under balanced commutations. In other words: If two words v and w satisfy $v \overset{\mathrm{bal}}{\sim} w$, then $H(v, z) = H(w, z)$.

Proof. It suffices to prove that $H(v, z) = H(w, z)$ whenever v can be obtained from w by a single balanced commutation. (The general case will then follow by induction.)

So let v be obtained from w by a single balanced commutation. Thus $v = pxyq$ and $w = pyxq$, where $p, q \in \mathcal{M}$ are two words and where $x, y \in \mathcal{M}$ are two balanced words. Consider these words p, q, x, y. Let a be the final height of p. Since x and y are balanced words, their final heights are 0, and thus the final heights of the four words px, py, pxy and pyx equal a as well. Thus, by repeated application of (19), we find

$$\begin{aligned} H(pxyq, z) &= H(pxy, z) + z^a (H(q, z) - 1) \qquad \text{(by (19))} \\ &= H(px, z) + z^a (H(y, z) - 1) + z^a (H(q, z) - 1) \qquad \text{(by (19))} \\ &= H(p, z) + z^a (H(x, z) - 1) + z^a (H(y, z) - 1) + z^a (H(q, z) - 1) \end{aligned}$$

(by (19)) and similarly

$$H(pyxq, z) = H(p, z) + z^a (H(y, z) - 1) + z^a (H(x, z) - 1) + z^a (H(q, z) - 1).$$

The right hand sides of these two equalities are visibly equal. Hence, so are their left hand sides: $H(pxyq, z) = H(pyxq, z)$. In other words, $H(v, z) = H(w, z)$ (since $v = pxyq$ and $w = pyxq$), and our proof is complete. □

The above lemmas will be used in showing the uniqueness part of Proposition 4.4. Let us now present some lemmas for the existence part.

Lemma 4.13. Let $w \in \mathcal{M}$ be a balanced word that starts with a U and ends with a U. Then, we can write w as a concatenation $w = pq$, where p is a balanced word starting with a U, and where q is a balanced word starting with a D.

Proof of Lemma 4.13. Write w as $w = w_1 w_2 \cdots w_\ell$, where $w_1, w_2, \ldots, w_\ell \in \{U, D\}$ are the letters of w. For each $k \in \{0, 1, \ldots, \ell\}$, define $w_{:k} := w_1 w_2 \cdots w_k$ to be the factor of w consisting of the first k letters of w, and define the number

$$h_k := (\text{\# of } U\text{'s in } w_{:k}) - (\text{\# of } D\text{'s in } w_{:k}).$$

(Note that if w is the reading word $\mathrm{w}(\mathbf{p})$ of a diagonal path $\mathbf{p} = (p_0, p_1, \ldots, p_\ell)$ that starts on the x-axis, then h_k is the height of p_k. Thus the notation h_k. Note also that $h_0 = 0$, since $w_{:0}$ is an empty word.)

It is clear that each $k \in \{1, 2, \ldots, \ell\}$ satisfies

$$h_k - h_{k-1} = \begin{cases} 1, & \text{if } w_k = U; \\ -1, & \text{if } w_k = D. \end{cases} \tag{24}$$

More generally, for any factor $w_i w_{i+1} \cdots w_j$ of w, we have

$$h_j - h_{i-1} = \left(\text{\# of } U\text{'s in } w_i w_{i+1} \cdots w_j\right) - \left(\text{\# of } D\text{'s in } w_i w_{i+1} \cdots w_j\right). \tag{25}$$

In particular, a factor $w_i w_{i+1} \cdots w_j$ of w is balanced if and only if $h_{i-1} = h_j$. Thus, $h_0 = h_\ell$ (since $w = w_1 w_2 \cdots w_\ell$ is balanced).

The word w ends with a U. In other words, $w_\ell = U$. Hence, (24) shows that $h_\ell - h_{\ell-1} = 1$, so that $h_\ell - 1 = h_{\ell-1}$.

Pick the largest $c \in \{0, 1, \ldots, \ell - 1\}$ such that $h_c \geq h_0$. (Such a c exists, since $h_c \geq h_0$ holds for $c = 0$.) Then, $c \neq \ell - 1$ (since $h_c \geq h_0 = h_\ell > h_\ell - 1 = h_{\ell-1}$ and thus $h_c \neq h_{\ell-1}$), so that $c \in \{0, 1, \ldots, \ell - 2\}$ and therefore $c + 1 \in \{0, 1, \ldots, \ell - 1\}$. Therefore, we cannot have $h_{c+1} \geq h_0$ (since this would contradict the maximality of c). In other words, we have $h_{c+1} < h_0$. Therefore, $h_{c+1} \leq h_0 - 1$ (since h_{c+1} and h_0 are integers). But (24) (applied to $k = c + 1$) shows that $h_{c+1} - h_c = \pm 1 \geq -1$, so that $h_{c+1} \geq h_c - 1$ and therefore $h_c - 1 \leq h_{c+1} \leq h_0 - 1$. Thus, $h_c \leq h_0$. Combined with $h_c \geq h_0$, this yields $h_c = h_0$. By (25), this shows that the factor $w_1 w_2 \cdots w_c$ of w is balanced. Moreover, combining $h_c = h_0$ with $h_0 = h_\ell$, we obtain $h_c = h_\ell$, and this shows that the factor $w_{c+1} w_{c+2} \cdots w_\ell$ of w is balanced (again by (25)).

If we had $w_{c+1} = U$, then we would have $h_{c+1} - h_c = 1$ (by (24)), whence we would obtain $h_{c+1} = h_c + 1 > h_c = h_0$, which would contradict the fact that we cannot have $h_{c+1} \geq h_0$. Thus, we cannot have $w_{c+1} = U$. Hence, $w_{c+1} = D$. In other words, the word $w_{c+1} w_{c+2} \cdots w_\ell$ starts with a D.

But the word w starts with a U. Thus, $w_1 = U \neq D = w_{c+1}$. Therefore, $1 \neq c + 1$, so that $c \neq 0$. The word $w_1 w_2 \cdots w_c$ is thus nonempty. Moreover, this word starts with a U (since $w_1 = U$).

Now, we have

$$w = w_1 w_2 \cdots w_\ell = \underbrace{(w_1 w_2 \cdots w_c)}_{\substack{\text{a balanced word} \\ \text{starting with a } U}} \underbrace{(w_{c+1} w_{c+2} \cdots w_\ell)}_{\substack{\text{a balanced word} \\ \text{starting with a } D}}$$

Hence, we can write w as a concatenation $w = pq$, where p is a balanced word starting with a U, and where q is a balanced word starting with a D (namely, $p = w_1 w_2 \cdots w_c$ and $q = w_{c+1} w_{c+2} \cdots w_\ell$). This proves Lemma 4.13. □

Lemma 4.14. Let $w \in \mathcal{M}$ be a rising word that is not up-normal. Then, we can write w in the form $w = upqv$, where u and v are two words, where p is a balanced word starting with a U, and where q is a balanced word starting with a D.

Proof of Lemma 4.14. Write w as $w = w_1 w_2 \cdots w_\ell$, where $w_1, w_2, \ldots, w_\ell \in \{U, D\}$ are the letters of w. For each $k \in \{0, 1, \ldots, \ell\}$, define $w_{:k} := w_1 w_2 \cdots w_k$ to be the factor of w consisting of the first k letters of w, and define the number

$$h_k := (\# \text{ of } U\text{'s in } w_{:k}) - (\# \text{ of } D\text{'s in } w_{:k}).$$

Clearly, the equalities (24) and (25) hold, just as in the proof of Lemma 4.13. In particular, from (25), we obtain $h_0 \leq h_\ell$, since w is rising.

Moreover, the word w is not up-normal, so that w contains a down-zig as a factor. Let $w_i w_{i+1} \cdots w_j$ be this factor. Thus, by the definition of a down-zig, we have $i < j - 2$ (since a down-zig must always have length ≥ 4) and $w_i = U$ and $w_{i+1} = w_{i+2} = \cdots = w_{j-1} = D$ and $w_j = U$. From $w_i = U$, we obtain $h_i - h_{i-1} = 1$ (by (24)). From $w_j = U$, we obtain $h_j - h_{j-1} = 1$ (by (24)). Moreover, the factor $w_i w_{i+1} \cdots w_j$ is a down-zig and thus contains at least as many D's as it contains U's; therefore, $h_j - h_{i-1} \leq 0$ (by (25)), so that $h_{i-1} \geq h_j$.

A *copair* means a pair (a, b) of elements of $\{0, 1, \ldots, \ell\}$ satisfying

$$a \leq i - 1 \text{ and } j \leq b \text{ and } h_a \leq h_b.$$

The *span* of a copair (a, b) will mean the difference $b - a$. Note that $(0, \ell)$ is a copair (since $h_0 \leq h_\ell$), so that there exists at least one copair.

Pick a copair (a, b) with minimum span. Thus, $a, b \in \{0, 1, \ldots, \ell\}$ and $a \leq i - 1$ and $j \leq b$ and $h_a \leq h_b$.

If we had $w_{a+1} = D$, then we would have $a + 1 \leq i - 1$ (since $w_{a+1} = D \neq U = w_i$ would entail $a + 1 \neq i$, so that $a \neq i - 1$, and therefore the inequality $a \leq i - 1$ could be improved to $a < i - 1$, so that $a \leq (i - 1) - 1$ and thus $a + 1 \leq i - 1$) and $h_{a+1} \leq h_b$ (since (24) would show that $h_{a+1} - h_a = -1$ (because of $w_{a+1} = D$), thus $h_{a+1} = h_a - 1 < h_a \leq h_b$). Therefore, $(a + 1, b)$ would again be a copair. This copair $(a + 1, b)$ would have a smaller span than (a, b) (since $b - (a + 1) < b - a$), which is impossible since (a, b) was chosen to have minimum span. Thus, we cannot have $w_{a+1} = D$. Hence, $w_{a+1} = U$. Therefore, (24) yields $h_{a+1} - h_a = 1$.

If we had $w_b = D$, then we would have $j \leq b - 1$ (since $w_b = D \neq U = w_j$ would entail $b \neq j$, so that the inequality $j \leq b$ could be improved to $j < b$, and this would entail $j \leq b - 1$) and $h_a \leq h_{b-1}$ (since (24) would show that $h_b - h_{b-1} = -1$ (because of $w_b = D$), thus $h_b = h_{b-1} - 1 < h_{b-1}$ and therefore $h_a \leq h_b < h_{b-1}$). Therefore, $(a, b - 1)$ would again be a copair. This copair $(a, b - 1)$ would have a smaller span than (a, b) (since $(b - 1) - a < b - a$), which is impossible since (a, b) was chosen to have minimum span. Thus, we cannot have $w_b = D$. Hence, $w_b = U$. Therefore, (24) yields $h_b - h_{b-1} = 1$.

Now, we show that $h_a = h_b$. To prove this, we assume the contrary. Thus, $h_a \neq h_b$, so that $h_a < h_b$ (since $h_a \leq h_b$). Therefore, $h_a \leq h_b - 1$ (since h_a and h_b are integers). If we had $a = i-1$ and $b = j$, then we could rewrite $h_a < h_b$ as $h_{i-1} < h_j$, which would contradict $h_{i-1} \geq h_j$. Thus, we cannot have $a = i-1$ and $b = j$. Hence, we are in one (or both) of the following two cases:

Case 1: We have $a \neq i-1$.

Case 2: We have $b \neq j$.

Let us first consider Case 1. In this case, we have $a \neq i-1$. Hence, $a < i-1$ (since $a \leq i-1$). Therefore, $a+1 \leq i-1$. Moreover, (24) yields $h_{a+1} - h_a \leq 1$, so that $h_{a+1} \leq h_a + 1 \leq h_b$ (since $h_a \leq h_b - 1$). Consequently, $(a+1, b)$ is a copair (since $a+1 \leq i-1$ and $j \leq b$). This copair $(a+1, b)$ has smaller span than (a, b), but this contradicts the fact that (a, b) was chosen to have minimum span. Thus, we have found a contradiction in Case 1.

Let us next consider Case 2. In this case, we have $b \neq j$. Hence, $j < b$ (since $j \leq b$). Therefore, $j \leq b-1$. Moreover, (24) yields $h_b - h_{b-1} \leq 1$, so that $h_b - 1 \leq h_{b-1}$. Now, $h_a \leq h_b - 1 \leq h_{b-1}$. Consequently, $(a, b-1)$ is a copair (since $a \leq i-1$ and $j \leq b-1$). This copair $(a, b-1)$ has smaller span than (a, b), but this contradicts the fact that (a, b) was chosen to have minimum span. Thus, we have found a contradiction in Case 2.

We have now found contradictions in both Cases 1 and 2. Hence, our assumption was false, and $h_a = h_b$ is proved.

Because of (25), this equality $h_a = h_b$ shows that the word $w_{a+1}w_{a+2}\cdots w_b$ is balanced. This balanced word is furthermore nonempty (since $a \leq i-1 < i < j-2 < j \leq b$) and starts with a U (since $w_{a+1} = U$) and ends with a U (since $w_b = U$). Thus, by Lemma 4.13 (applied to $w_{a+1}w_{a+2}\cdots w_b$ instead of w), we can write $w_{a+1}w_{a+2}\cdots w_b$ as a concatenation $w_{a+1}w_{a+2}\cdots w_b = pq$, where p is a balanced word starting with a U, and where q is a balanced word starting with a D. Consider these p and q.

Let us furthermore set $u := w_1 w_2 \cdots w_a$ and $v := w_{b+1}w_{b+2}\cdots w_\ell$. Then,

$$w = w_1 w_2 \cdots w_\ell = \underbrace{(w_1 w_2 \cdots w_a)}_{=u} \underbrace{(w_{a+1}w_{a+2}\cdots w_b)}_{=pq} \underbrace{(w_{b+1}w_{b+2}\cdots w_\ell)}_{=v} = upqv.$$

Hence, we have written w in the form $w = upqv$, where u and v are two words, where p is a balanced word starting with a U, and where q is a balanced word starting with a D. This proves Lemma 4.14. □

Proof of Proposition 4.4. We equip the set $\mathcal{M}$ with the lexicographic order, where $D < U$. If the word w is not yet up-normal, then by Lemma 4.14 we can write it as $w = upqv$, where p and q are balanced, p starts with U, and q starts with D. We then perform a balanced commutation to obtain the word $w' = uqpv$, which is lexicographically smaller than w (since the first letter of p, which was U, has been replaced by the first letter of q, which is D). We have $w' \overset{\text{bal}}{\sim} w$ by construction. This procedure can be iterated until we end up with an up-normal word t that satisfies $t \overset{\text{bal}}{\sim} w$. Indeed, the procedure cannot go on forever, since each balanced commutation makes our word lexicographically smaller while preserving its length. Moreover, the word remains rising throughout this procedure, since a balanced commutation does not change the total numbers of U's and D's in the word.

Thus, we have proved the existence of an up-normal word $t \in \mathcal{M}$ such that $t \overset{\text{bal}}{\sim} w$. It remains to prove its uniqueness.

The condition $t \overset{\text{bal}}{\sim} w$ ensures that the words t and w have the same final height (since balanced commutations do not change the numbers of U's and D's, and thus – by (3) – leave the final height unchanged as well) and the same height polynomial (since Lemma 4.12 shows that balanced commutations do not change the height polynomial). By Lemma 4.10, the up-normal word t is thus uniquely determined. □

5. Proofs of the main results

5.1. A lemma

We are getting close to the proofs of the main results (Theorems 2.1 and 2.3). First, we show a lemma that combines some results of the previous sections:

Lemma 5.1. Let $\mathbf{p}$ and $\mathbf{q}$ be two diagonal paths with the same initial height and the same final height. Assume that $H(\mathbf{p}, z) = H(\mathbf{q}, z)$. Then, $\mathrm{w}(\mathbf{p}) \overset{\text{bal}}{\sim} \mathrm{w}(\mathbf{q})$.

Proof. Set $u = \mathrm{w}(\mathbf{p})$ and $v = \mathrm{w}(\mathbf{q})$.

By assumption, the paths $\mathbf{p}$ and $\mathbf{q}$ have the same initial height and the same final height. Call these two heights i and f. We are in one of the following two cases:

Case 1: We have $f \geq i$.

Case 2: We have $f < i$.

Consider Case 1 first. In this case, $f \geq i$. Hence, the reading words $\mathrm{w}(\mathbf{p})$ and $\mathrm{w}(\mathbf{q})$ are rising (by (2)). In other words, the words u and v are rising (since $u = \mathrm{w}(\mathbf{p})$ and $v = \mathrm{w}(\mathbf{q})$). Thus, Proposition 4.4 shows that there exist unique up-normal words t_u and t_v such that $t_u \overset{\text{bal}}{\sim} u$ and $t_v \overset{\text{bal}}{\sim} v$. Consider these t_u and t_v. Lemma 4.12 shows that $H(t_u, z) = H(u, z)$ (since $t_u \overset{\text{bal}}{\sim} u$) and $H(t_v, z) = H(v, z)$ (since $t_v \overset{\text{bal}}{\sim} v$). Moreover, the relation $t_u \overset{\text{bal}}{\sim} u$ shows that the words t_u and u have the same # of U's (since balanced commutations do not change the # of U's). In other words, (# of U's in t_u) = (# of U's in u). Similarly, (# of D's in t_u) = (# of D's in u).

However, the path $\mathbf{p}$ has initial height i and reading word $\mathrm{w}(\mathbf{p}) = u$. Thus, Lemma 4.8 (applied to $\mathbf{r} = \mathbf{p}$ and $j = i$ and $w = u$) shows that $H(\mathbf{p}, z) = z^i H(u, z)$. Similarly, $H(\mathbf{q}, z) = z^i H(v, z)$. Hence, $z^i H(u, z) = H(\mathbf{p}, z) = H(\mathbf{q}, z) = z^i H(v, z)$.

Cancelling z^i, we obtain $H(u,z) = H(v,z)$. In view of $H(t_u,z) = H(u,z)$ and $H(t_v,z) = H(v,z)$, we can rewrite this as $H(t_u,z) = H(t_v,z)$. Furthermore, (3) yields

$$\begin{aligned}
(\text{final height of } t_u) &= \underbrace{(\text{\# of } U\text{'s in } t_u)}_{=(\text{\# of } U\text{'s in } u)} - \underbrace{(\text{\# of } D\text{'s in } t_u)}_{=(\text{\# of } D\text{'s in } u)} \\
&= (\text{\# of } U\text{'s in } u) - (\text{\# of } D\text{'s in } u) \\
&= (\text{\# of } U\text{'s in } \mathrm{w}(\mathbf{p})) - (\text{\# of } D\text{'s in } \mathrm{w}(\mathbf{p})) \qquad (\text{since } u = \mathrm{w}(\mathbf{p})) \\
&= \underbrace{(\text{final height of } \mathbf{p})}_{=f} - \underbrace{(\text{initial height of } \mathbf{p})}_{=i} \qquad (\text{by (2)}) \\
&= f - i
\end{aligned}$$

and similarly $(\text{final height of } t_v) = f - i$. Comparing these two equalities, we find $(\text{final height of } t_u) = (\text{final height of } t_v)$. In other words, the two words t_u and t_v have the same final height.

Now, we know that the two up-normal words t_u and t_v have the same final height and the same height polynomial (since $H(t_u,z) = H(t_v,z)$). Hence, Lemma 4.10 shows that they must be equal. That is, $t_u = t_v$. From $t_u \overset{\text{bal}}{\sim} u$ and $t_v \overset{\text{bal}}{\sim} v$, we thus obtain $u \overset{\text{bal}}{\sim} t_u = t_v \overset{\text{bal}}{\sim} v$. In other words, $\mathrm{w}(\mathbf{p}) \overset{\text{bal}}{\sim} \mathrm{w}(\mathbf{q})$ (since $u = \mathrm{w}(\mathbf{p})$ and $v = \mathrm{w}(\mathbf{q})$). Thus, Lemma 5.1 is proved in Case 1.

Let us now consider Case 2. In this case, $f < i$.

Let $\mathbf{p}'$ and $\mathbf{q}'$ be the reflections of the diagonal paths $\mathbf{p}$ and $\mathbf{q}$ across a vertical line. Then, the initial heights of $\mathbf{p}'$ and $\mathbf{q}'$ are the final heights of $\mathbf{p}$ and $\mathbf{q}$, which (as we know) are f. Likewise, the final heights of $\mathbf{p}'$ and $\mathbf{q}'$ are i. Obviously, from $f < i$, we obtain $i > f$, thus $i \geq f$.

Now, we claim that $\mathrm{w}(\mathbf{p}') = \omega(\mathrm{w}(\mathbf{p}))$. Indeed, the path $\mathbf{p}'$ is the reflection of $\mathbf{p}$ across a vertical line; thus, its steps are the toggle-images[9] of the steps of $\mathbf{p}$ read in the reverse order. But this means precisely that $\mathrm{w}(\mathbf{p}') = \omega(\mathrm{w}(\mathbf{p}))$ (since the anti-automorphism ω of $\mathcal{M}$ sends U to D and D to U and reverses the order of letters in a word). In view of $\mathrm{w}(\mathbf{p}) = u$, we can rewrite this as $\mathrm{w}(\mathbf{p}') = \omega(u)$. Similarly, $\mathrm{w}(\mathbf{q}') = \omega(v)$.

Furthermore, when we reflect a diagonal path across a vertical line, the heights of its vertices are preserved, and thus its height polynomial remains unchanged. Hence, $H(\mathbf{p}',z) = H(\mathbf{p},z)$ and $H(\mathbf{q}',z) = H(\mathbf{q},z)$. Thus, $H(\mathbf{p},z) = H(\mathbf{q},z)$ rewrites as $H(\mathbf{p}',z) = H(\mathbf{q}',z)$.

Now, we know that $\mathbf{p}'$ and $\mathbf{q}'$ are two diagonal paths with the same initial height f and the same final height i, and that $H(\mathbf{p}',z) = H(\mathbf{q}',z)$. Moreover, $i \geq f$. Hence, the claim of Lemma 5.1 in Case 1 (which we have already proved above) can be applied to $\mathbf{p}'$, $\mathbf{q}'$, i and f instead of $\mathbf{p}$, $\mathbf{q}$, f and i. As a result, we obtain $\mathrm{w}(\mathbf{p}') \overset{\text{bal}}{\sim} \mathrm{w}(\mathbf{q}')$. In other words, $\omega(u) \overset{\text{bal}}{\sim} \omega(v)$ (since $\mathrm{w}(\mathbf{p}') = \omega(u)$ and $\mathrm{w}(\mathbf{q}') =$

[9] The *toggle-image* of an edge of the diagonal lattice is defined as follows: The toggle-image of an NE-step is an SE-step; the toggle-image of an SE-step is an NE-step.

$\omega(v)$). By Proposition 4.3, this entails $u \overset{\text{bal}}{\sim} v$. In other words, $\mathrm{w}(\mathbf{p}) \overset{\text{bal}}{\sim} \mathrm{w}(\mathbf{q})$ (since $u = \mathrm{w}(\mathbf{p})$ and $v = \mathrm{w}(\mathbf{q})$). Thus, Lemma 5.1 is proved in Case 2.

Now, Lemma 5.1 is proved in both cases. □

5.2. Proof of Theorem 2.3

Now we can prove the second of our main results:

Proof of Theorem 2.3. Let $v = \omega(u)$. Then, the word v is balanced (since u is balanced).

Let $\mathbf{p}$ and $\mathbf{q}$ be the standard paths of u and v. Thus, $\mathbf{p}$ and $\mathbf{q}$ are diagonal paths starting at $(0,0)$ whose reading words are $\mathrm{w}(\mathbf{p}) = u$ and $\mathrm{w}(\mathbf{q}) = v$. The path $\mathbf{p}$ ends at the same height as it starts (since its reading word $\mathrm{w}(\mathbf{p}) = u$ is balanced), and thus ends at height 0 (since it starts at height 0). Similarly, the same holds for $\mathbf{q}$. Thus, the paths $\mathbf{p}$ and $\mathbf{q}$ both have final height 0. Of course, they also have initial height 0.

But recall that $v = \omega(u)$. Hence, the k-th letter of v from the left is the toggle-image[10] of the k-th letter of u from the right. Therefore, the k-th step of $\mathbf{q}$ from the left is the toggle-image of the k-th step of $\mathbf{p}$ from the right (since $\mathrm{w}(\mathbf{p}) = u$ and $\mathrm{w}(\mathbf{q}) = v$). Hence, the path $\mathbf{q}$ is the reflection of the path $\mathbf{p}$ across a vertical axis (since both paths start and end at height 0). Clearly, reflecting a point across a vertical axis does not change the height of this point. Thus, the paths $\mathbf{p}$ and $\mathbf{q}$ have the same multiset of heights of vertices (although the order in which these heights appear in $\mathbf{q}$ is opposite from the order in $\mathbf{p}$). Hence, the paths $\mathbf{p}$ and $\mathbf{q}$ have the same height polynomial (since the height polynomial of a diagonal path encodes the heights of its vertices). In other words, $H(\mathbf{p}, z) = H(\mathbf{q}, z)$. Since the paths $\mathbf{p}$ and $\mathbf{q}$ have the same initial height (namely, 0) and the same final height (namely, 0), we can thus apply Lemma 5.1 and conclude that $\mathrm{w}(\mathbf{p}) \overset{\text{bal}}{\sim} \mathrm{w}(\mathbf{q})$. In other words, $u \overset{\text{bal}}{\sim} \omega(u)$ (since $\mathrm{w}(\mathbf{p}) = u$ and $\mathrm{w}(\mathbf{q}) = v = \omega(u)$). Hence, Lemma 3.10 (applied to $\omega(u)$ instead of v) yields $\phi(u) = \phi(\omega(u))$. This completes the proof of Theorem 2.3. □

5.3. Two more lemmas

For the proof of Theorem 2.1, we need two more lemmas:

Lemma 5.2. If two words $u, v \in \mathcal{M}$ satisfy $u \overset{\text{bal}}{\sim} v$, then $u \overset{\text{flip}}{\sim} v$.

Proof. Recall that $\overset{\text{flip}}{\sim}$ is an equivalence relation. Hence, it suffices to show that if a word u is obtained from a word v by a balanced commutation, then $u \overset{\text{flip}}{\sim} v$. So let us show this.

[10]The *toggle-image* of a letter is defined as follows: The toggle-image of U is D; the toggle-image of D is U.

Let a word u be obtained from a word v by a balanced commutation. Thus, we can write u and v as $u = pxyq$ and $v = pyxq$, where $p, q \in \mathcal{M}$ are two words and where $x, y \in \mathcal{M}$ are two balanced words (by the definition of a balanced commutation). Consider these p, q, x, y.

The words y and x are balanced. Hence, their concatenation yx is balanced as well (since a concatenation of balanced words is always balanced). Thus, the word $\omega(yx)$ is also balanced (since applying ω to a balanced word yields a balanced word).

Moreover, since ω is a monoid anti-morphism, we have $\omega(yx) = \omega(x)\,\omega(y)$. Thus, the word $\omega(x)\,\omega(y)$ is balanced (since $\omega(yx)$ is balanced). Furthermore, since $\omega \circ \omega = \text{id}$, we have $\omega(\omega(yx)) = yx$. In other words, $\omega(\omega(x)\,\omega(y)) = yx$ (since $\omega(yx) = \omega(x)\,\omega(y)$).

Now, we can apply a balanced flip to the word $u = pxyq$, in which we apply ω to the balanced factor x. Thus we obtain a new word $u' = p\omega(x)\,yq$. To this new word $u' = p\omega(x)\,yq$, we then apply a further balanced flip, in which we apply ω to the balanced factor y. Thus we obtain a new word $u'' = p\omega(x)\,\omega(y)\,q$. Finally, we apply one last balanced flip to this new word $u'' = p\omega(x)\,\omega(y)\,q$, in which we apply ω to the balanced factor $\omega(x)\,\omega(y)$. This produces the new word $u''' = p\underbrace{\omega(\omega(x)\,\omega(y))}_{=yx}q = pyxq = v$.

Thus we have obtained v from u by a sequence of three balanced flips (via u' and u''). Hence, $u \overset{\text{flip}}{\sim} v$. As we said, this proves Lemma 5.2. □

Lemma 5.3. If two words $u, v \in \mathcal{M}$ satisfy $u \overset{\text{flip}}{\sim} v$, then $\phi(u) = \phi(v)$.

Proof. It clearly suffices to show that if a word u is obtained from a word v by a balanced flip, then $\phi(u) = \phi(v)$.

So let us prove this. Let a word u be obtained from a word v by a balanced flip. Thus, we can write u and v as $u = pxq$ and $v = p\omega(x)\,q$, where $p, q \in \mathcal{M}$ are two words and where $x \in \mathcal{M}$ is a balanced word. Consider these p, q, x.

Theorem 2.3 (applied to x instead of u) yields $\phi(x) = \phi(\omega(x))$ and $x \overset{\text{bal}}{\sim} \omega(x)$.

Since ϕ is a monoid morphism, we have

$$\phi(pxq) = \phi(p)\,\phi(x)\,\phi(q) \qquad \text{and} \qquad \phi(p\omega(x)\,q) = \phi(p)\,\phi(\omega(x))\,\phi(q)\,.$$

The right hand sides of these two equalities are equal (since $\phi(x) = \phi(\omega(x))$). Hence, so are their left hand sides. In other words, $\phi(pxq) = \phi(p\omega(x)\,q)$. But this can be rewritten as $\phi(u) = \phi(v)$ (since $u = pxq$ and $v = p\omega(x)\,q$). This proves Lemma 5.3. □

5.4. Proof of Theorem 2.1

We are now ready to prove Theorem 2.1:

Proof of Theorem 2.1. Let $\mathbf{p}$ and $\mathbf{q}$ be the standard paths of u and v. Then, $\mathbf{p}$ and $\mathbf{q}$ are diagonal paths starting in $(0,0)$ and having reading words $\mathrm{w}(\mathbf{p}) = u$ and $\mathrm{w}(\mathbf{q}) = v$. In particular, their initial heights are 0. Furthermore, the final heights of the words u and v are (by their definitions) the final heights of the paths $\mathbf{p}$ and $\mathbf{q}$. Moreover, the height polynomials $H(u,z)$ and $H(v,z)$ are (by their definitions) the height polynomials $H(\mathbf{p},z)$ and $H(\mathbf{q},z)$, and likewise the NE-height polynomials $H_{\mathrm{NE}}(u,z)$ and $H_{\mathrm{NE}}(v,z)$ are the NE-height polynomials $H_{\mathrm{NE}}(\mathbf{p},z)$ and $H_{\mathrm{NE}}(\mathbf{q},z)$.

Having said this, let us now prove the equivalences. It suffices to show that $\mathcal{S}_1 \Longrightarrow \mathcal{S}_2$ and $\mathcal{S}_2 \Longrightarrow \mathcal{S}_1$ and $\mathcal{S}_1 \Longrightarrow \mathcal{S}_3 \Longrightarrow \mathcal{S}_4 \Longrightarrow \mathcal{S}_5 \Longrightarrow \mathcal{S}_6 \Longrightarrow \mathcal{S}_1$ and $\mathcal{S}_4 \Longleftrightarrow \mathcal{S}_3'$.

$\mathcal{S}_1 \Longrightarrow \mathcal{S}_2$: Trivial.

$\mathcal{S}_2 \Longrightarrow \mathcal{S}_1$: True since the action of $\mathcal{W}$ on $\mathbf{k}[x]$ is faithful.

$\mathcal{S}_1 \Longrightarrow \mathcal{S}_3$: Assume that $\mathcal{S}_1$ holds. Thus, $\phi(u) = \phi(v)$. In other words, $\phi(\mathrm{w}(\mathbf{p})) = \phi(\mathrm{w}(\mathbf{q}))$ (since $u = \mathrm{w}(\mathbf{p})$ and $v = \mathrm{w}(\mathbf{q})$). Moreover, the paths $\mathbf{p}$ and $\mathbf{q}$ have the same initial height (since they both start at $(0,0)$). Thus, Proposition 3.6 yields that the final heights of $\mathbf{p}$ and $\mathbf{q}$ are equal, and that we have

$$\begin{aligned}&\{\mathrm{ht}(p_i) \mid p_i \text{ is an NE-step of } \mathbf{p}\}_{\text{multiset}}\\ &= \{\mathrm{ht}(q_i) \mid q_i \text{ is an NE-step of } \mathbf{q}\}_{\text{multiset}}.\end{aligned}$$

The latter equality says that the paths $\mathbf{p}$ and $\mathbf{q}$ have the same multiset of heights of NE-steps. Equivalently, $H_{\mathrm{NE}}(\mathbf{p},z) = H_{\mathrm{NE}}(\mathbf{q},z)$ (since the NE-height polynomial of a diagonal path contains the same information as its multiset of heights of NE-steps). In other words, $H_{\mathrm{NE}}(u,z) = H_{\mathrm{NE}}(v,z)$ (since the NE-height polynomials $H_{\mathrm{NE}}(u,z)$ and $H_{\mathrm{NE}}(v,z)$ are the NE-height polynomials $H_{\mathrm{NE}}(\mathbf{p},z)$ and $H_{\mathrm{NE}}(\mathbf{q},z)$). Moreover, the final heights of the words u and v are the final heights of the paths $\mathbf{p}$ and $\mathbf{q}$, and thus are equal (since the final heights of $\mathbf{p}$ and $\mathbf{q}$ are equal). Thus, statement $\mathcal{S}_3$ holds. We have now proved the implication $\mathcal{S}_1 \Longrightarrow \mathcal{S}_3$.

$\mathcal{S}_3 \Longrightarrow \mathcal{S}_4$: Assume that $\mathcal{S}_3$ holds. That is, the words u and v have the same final height and satisfy $H_{\mathrm{NE}}(u,z) = H_{\mathrm{NE}}(v,z)$. Let b be the final height of u and v. The equality (16) from Lemma 4.7 yields

$$H(u,z) = (1+z)H_{\mathrm{NE}}(u,z) + \sum_{j \geq b} z^j - \sum_{j \geq 1} z^j$$

and

$$H(v,z) = (1+z)H_{\mathrm{NE}}(v,z) + \sum_{j \geq b} z^j - \sum_{j \geq 1} z^j.$$

The right hand sides of these two equalities are equal (since $H_{\mathrm{NE}}(u,z) = H_{\mathrm{NE}}(v,z)$). Hence, so are the left hand sides. In other words, $H(u,z) = H(v,z)$. Since we also know that the words u and v have the same final height, we thus conclude that statement $\mathcal{S}_4$ holds. Thus we have proved $\mathcal{S}_3 \Longrightarrow \mathcal{S}_4$.

$\mathcal{S}_4 \Longrightarrow \mathcal{S}_5$: Assume that statement $\mathcal{S}_4$ holds. In other words, the words u and v have the same final height and satisfy $H(u,z) = H(v,z)$.

The final heights of the words u and v are the final heights of their standard paths $\mathbf{p}$ and $\mathbf{q}$ (by definition). Thus, the final heights of $\mathbf{p}$ and $\mathbf{q}$ are equal (since the final heights of u and v are equal).

Recall that the height polynomials $H(u,z)$ and $H(v,z)$ are the height polynomials $H(\mathbf{p},z)$ and $H(\mathbf{q},z)$. Hence, $H(\mathbf{p},z) = H(\mathbf{q},z)$ (since $H(u,z) = H(v,z)$). By Lemma 5.1, this entails $\mathrm{w}(\mathbf{p}) \overset{\text{bal}}{\sim} \mathrm{w}(\mathbf{q})$ (since the paths $\mathbf{p}$ and $\mathbf{q}$ have the same initial height and the same final height). This can be rewritten as $u \overset{\text{bal}}{\sim} v$ (since $u = \mathrm{w}(\mathbf{p})$ and $v = \mathrm{w}(\mathbf{q})$). But this is exactly $\mathcal{S}_5$. Thus, the implication $\mathcal{S}_4 \Longrightarrow \mathcal{S}_5$ is proved.

$\mathcal{S}_5 \Longrightarrow \mathcal{S}_6$: This is Lemma 5.2.

$\mathcal{S}_6 \Longrightarrow \mathcal{S}_1$: This is Lemma 5.3.

$\mathcal{S}_4 \Longleftrightarrow \mathcal{S}_3'$: Next, we show the equivalence $\mathcal{S}_4 \Longleftrightarrow \mathcal{S}_3'$. This is tantamount to showing the equivalence of the two equalities $H(u,z) = H(v,z)$ and $H_{\mathrm{SE}}(u,z) = H_{\mathrm{SE}}(v,z)$ under the assumption that the words u and v have the same final height.

So let us assume that the words u and v have the same final height. Let b be this final height. The equality (17) from Lemma 4.7 yields

$$H(u,z) = (1 + z^{-1}) H_{\mathrm{SE}}(u,z) + \sum_{j \geq 0} z^j - \sum_{j \geq b+1} z^j.$$

and

$$H(v,z) = (1 + z^{-1}) H_{\mathrm{SE}}(v,z) + \sum_{j \geq 0} z^j - \sum_{j \geq b+1} z^j.$$

Clearly, the left hand sides of these two equalities are equal if and only if $H(u,z) = H(v,z)$, whereas the right hand sides are equal if and only if $H_{\mathrm{SE}}(u,z) = H_{\mathrm{SE}}(v,z)$ (because the Laurent polynomial $1 + z^{-1}$ is not a zero-divisor and thus can be cancelled). Thus, the equalities $H(u,z) = H(v,z)$ and $H_{\mathrm{SE}}(u,z) = H_{\mathrm{SE}}(v,z)$ are equivalent. As we said, this proves the equivalence $\mathcal{S}_4 \Longleftrightarrow \mathcal{S}_3'$. □

6. Enumeration

Two words $u, v \in \mathcal{M}$ will be called *Weyl-equivalent* if $\phi(u) = \phi(v)$. Obviously, Weyl equivalence is an equivalence relation. Theorem 2.1 (and, later, Theorem 8.2) provides some necessary and sufficient criteria for Weyl equivalence. In particular, the $\mathcal{S}_1 \Longleftrightarrow \mathcal{S}_5$ part of Theorem 2.1 shows that Weyl equivalence is precisely the relation $\overset{\text{bal}}{\sim}$.

In this section, we prove several enumerative results regarding the equivalence classes of Weyl equivalence (henceforth just called "equivalence classes").

6.1. Counting equivalence classes by numbers of D's and U's

First, we consider equivalence classes of words with a given number of D's and U's. For $0 \leq k \leq n$, let $a(n,k)$ be the number of equivalence classes of words with k many D's and $n-k$ many U's. One of the simplest properties of these numbers is the following symmetry:

Proposition 6.1. We have $a(n,k) = a(n,n-k)$ for any integers $0 \leq k \leq n$.

Proof. There are many easy ways to see this. For instance, Proposition 4.3 shows that the monoid anti-automorphism $\omega : \mathcal{M} \to \mathcal{M}$ sends equivalence classes to equivalence classes (since Weyl equivalence is the relation $\overset{\text{bal}}{\sim}$). But ω turns D's into U's and vice versa. Thus, the proposition follows. □

In particular, $a(n,0) = a(n,n) = 1$ (the only words in these cases are $UU\cdots U$ and $DD\cdots D$ respectively).

Our first real result about the $a(n,k)$ is the following recursion.

Lemma 6.2. For $n > 2k \geq 0$, we have

$$a(n,k) = a(n-1,k) + a(n-2,k-1).$$

Here, $a(n-2,-1)$ is interpreted as 0 when $k=0$.

Proof. Recall that $a(n,k)$ counts the equivalence classes of words with k many D's and $n-k$ many U's. Such words always have more U's than D's (since $n > 2k$), and thus (in particular) are rising. Thus, any such equivalence class has a unique up-normal representative (by Proposition 4.4). Therefore, $a(n,k)$ counts the up-normal words w with k many D's and $n-k$ many U's. Recall that every up-normal word has the form

$$D^a\,(UD)^{r_1}U\,(UD)^{r_2}U\,\cdots\,(UD)^{r_h}U\,D^b \tag{26}$$

for nonnegative integers $a, b, r_1, \ldots, r_h$ (see Proposition 4.2). An up-normal rising word w can be of the following two types:

Type 1: The standard path corresponding to w has only one vertex of maximum height. Equivalently, $r_h = 0$ in (26). In this case, we can remove the last U from w to obtain the word

$$D^a\,(UD)^{r_1}U\,(UD)^{r_2}U\,\cdots\,(UD)^{r_{h-1}}U\,D^b$$

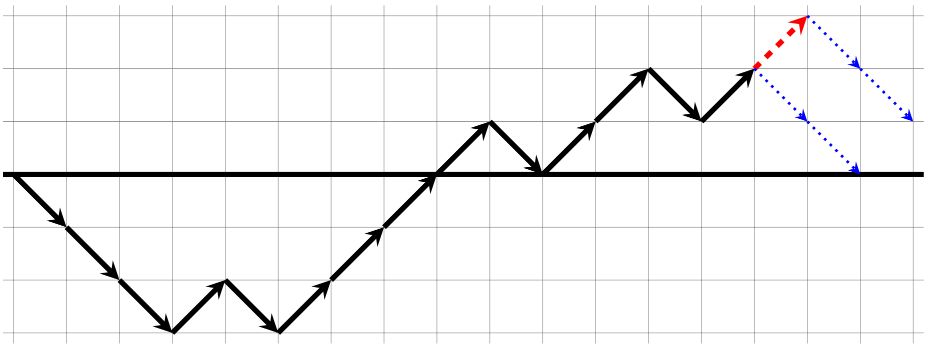
of length $n-1$ consisting of k many D's and $n-k-1$ U's. This word is still up-normal. Since $n-1 \geq 2k$, it is also still rising. It is clear that one can reverse the procedure: given a rising up-normal word of length $n-1$ with k many D's and $n-k-1$ many U's, insert a U just before the final run of D's if its last letter is D, and at the end otherwise. This gives us a bijection between equivalence classes counted by $a(n-1,k)$ and the equivalence classes of the first type. See Figure 5 for an illustration.

Type 2: The standard path corresponding to w has at least two distinct vertices of maximum height. Equivalently, $r_h > 0$ in (26). In this case, we can remove the last U and the D right before it to obtain the word

$$D^a\,(UD)^{r_1}U\,(UD)^{r_2}U\,\cdots\,(UD)^{r_h-1}U\,D^b,$$

which is still up-normal and rising. Its length is $n-2$, and it has $k-1$ many D's. Again, the process is easily reversed by inserting DU either before the final run of D's if the last letter is D, or at the end. This gives us a bijection between equivalence classes counted by $a(n-2,k-1)$ and the equivalence classes of the second type. See Figure 6 for an illustration.

Combining the two types, we obtain the desired recursion. □



$DDDUDUUUUDUUDUUDD \leftrightarrow DDDUDUUUUDUUDUDD$

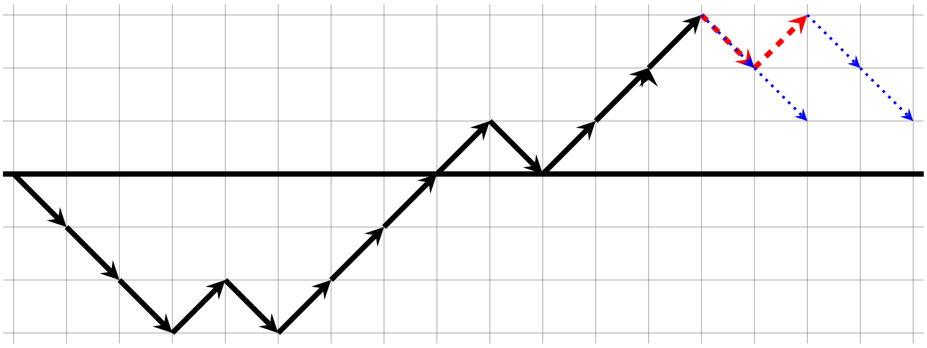
Figure 5: The bijection for words of type 1. The arc that is removed/inserted is indicated by a dashed red line. Arcs that are shifted by the procedure are indicated by dotted blue lines.

Second, we prove explicit formulas in two special cases, which together with the previous lemma characterize the numbers $a(n,k)$ for all n and k.

Lemma 6.3. For $k > 0$, we have

$$a(2k,k) = (k+3)2^{k-2} \quad \text{and} \quad a(2k+1,k) = (k+2)2^{k-1}.$$

Proof. We start with the first formula. Recall (from Theorem 2.1, equivalence $\mathcal{S}_1 \iff \mathcal{S}_3$) that the equivalence class of a word (with given numbers of U's and D's) is uniquely determined by the multiset of heights of NE-steps in its standard path. Let the minimum and maximum heights of NE-steps be $-s$ and t respectively (with $s \geq 0$ and $t \geq -1$, where the case $t = -1$ means that the path has no NE-steps). Let h_j be the number of NE-steps of height j (for each $-s \leq j \leq t$). Since our words are balanced (they have k many D's and k many U's), the equivalence class



DDDUDUUUUDUUUDUDD $\leftrightarrow$ DDDUDUUUUDUUUDD

Figure 6: The bijection for words of type 2. The arcs that are removed/inserted are indicated by dashed red lines. Arcs that are shifted by the procedure are indicated by dotted blue lines.

has an up-normal representative (by Proposition 4.3), and therefore all h_j's need to be strictly positive. Thus, $(h_{-s}, h_{-s+1}, \dots, h_t)$ is a composition of k into $s+t+1$ positive integers. For each composition of k of length ℓ (of which there are $\binom{k-1}{\ell-1}$), there are $\ell+1$ possibilities for the pair (s,t) (s can be any integer from 0 to ℓ, and $t = \ell - s - 1$). This gives us a total of

$$\sum_{\ell=1}^{k} \binom{k-1}{\ell-1}(\ell+1) = (k+3)2^{k-2}$$

possibilities.

For the second formula, we can use induction combined with the recursion in Lemma 6.2, which gives us

$$a(2k+1,k) = a(2k,k) + a(2k-1,k-1),$$

or we can apply a similar combinatorial argument with compositions of $k+1$ (the only difference is the fact that a composition of length ℓ only gives rise to ℓ possibilities, since the maximum height t of an NE-step can no longer be -1). □

The combination of these two lemmas yields an explicit formula for the bivariate generating function of $a(n,k)$.

Theorem 6.4. We have

$$\sum_{n\geq 0}\sum_{0\leq k\leq n} a(n,k)t^k x^n = \frac{(1-3tx^2+t^2x^4)(1-tx^2)^2}{(1-tx-tx^2)(1-x-tx^2)(1-2tx^2)^2} \tag{27}$$

or equivalently

$$\sum_{n\geq 0}\sum_{0\leq k\leq n/2} a(n,k)t^k x^n = \frac{(1-tx^2)^3}{(1-x-tx^2)(1-2tx^2)^2}. \tag{28}$$

Proof. We start with the second identity (28). Let us write $A(x,t)$ for the bivariate generating function on the left. Multiplying the recursion in Lemma 6.2 by $t^k x^n$ and summing over all n and k with $n > 2k \geq 0$, we obtain

$$\sum_{n\geq 1}\sum_{0\leq k<n/2} a(n,k)t^k x^n = \sum_{n\geq 1}\sum_{0\leq k<n/2} a(n-1,k)t^k x^n + \sum_{n\geq 2}\sum_{0\leq k<n/2} a(n-2,k-1)t^k x^n.$$

In terms of the generating function $A(x,t)$, this becomes

$$A(x,t) - \sum_{k\geq 0} a(2k,k)t^k x^{2k} = xA(x,t) + tx^2 A(x,t) - \sum_{k\geq 0} a(2k,k)t^{k+1}x^{2k+2}.$$

Solving for $A(x,t)$ now yields

$$A(x,t) = \frac{1-tx^2}{1-x-tx^2}\sum_{k\geq 0} a(2k,k)t^k x^{2k}. \tag{29}$$

In view of Lemma 6.3 (noting also that $a(0,0) = 1$), the sum evaluates to

$$\sum_{k\geq 0} a(2k,k)t^k x^{2k} = \frac{(1-tx^2)^2}{(1-2tx^2)^2}, \tag{30}$$

which completes the proof of (28). To prove the first identity, we write

$$\begin{aligned}\sum_{n\geq 0}\sum_{0\leq k\leq n} a(n,k)t^k x^n = &\sum_{n\geq 0}\sum_{0\leq k\leq n/2} a(n,k)t^k x^n + \sum_{n\geq 0}\sum_{0\leq k\leq n/2} a(n,n-k)t^{n-k}x^n \\ &- \sum_{k\geq 0} a(2k,k)t^k x^{2k}.\end{aligned}$$

By the symmetry property $a(n,n-k) = a(n,k)$, the first two terms are $A(x,t)$ and $A(tx,1/t)$ respectively. The final term is precisely (30) again. Now identity (27) follows upon simplification. □

Corollary 6.5. For all n and k with $0 \leq k \leq n/2$, we have

$$a(n,k) = \sum_{j=0}^{k}(k-j+1)\binom{n-k-1}{j}.$$

Proof. Let $\Sigma(n,k)$ be the sum on the right side of the equation. It is easy to verify that $\Sigma(n,0) = a(n,0) = 1$ and $\Sigma(2k,k) = a(2k,k) = (k+3)2^{k-2}$ as well as $\Sigma(2k+1,k) = a(2k+1,k) = (k+2)2^{k-1}$. Since these values together with the recursion in Lemma 6.2 characterize $a(n,k)$ uniquely, it suffices to verify that

$$\Sigma(n,k) = \Sigma(n-1,k) + \Sigma(n-2,k-1).$$

The latter is a simple consequence of the recursion for the binomial coefficients, since

$$\Sigma(n-1,k) = \sum_{j=0}^{k}(k-j+1)\binom{n-k-2}{j}$$

and

$$\Sigma(n-2,k-1) = \sum_{j=0}^{k-1}(k-j)\binom{n-k-2}{j} = \sum_{j=1}^{k}(k-j+1)\binom{n-k-2}{j-1}.$$

□

n \ k	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	1	1									
2	1	2	1								
3	1	3	3	1							
4	1	4	5	4	1						
5	1	5	8	8	5	1					
6	1	6	12	12	12	6	1				
7	1	7	17	20	20	17	7	1			
8	1	8	23	32	28	32	23	8	1		
9	1	9	30	49	48	48	49	30	9	1	
10	1	10	38	72	80	64	80	72	38	10	1

Table 1: Table of the values of $a(n,k)$.

Remark 6.6. The terms in Corollary 6.5 (see Table 1 for some explicit values) have a simple combinatorial interpretation. Recall that the up-normal representative of any equivalence class can be written as $D^a\,(UD)^{r_1}U\,(UD)^{r_2}U\,\cdots\,(UD)^{r_h}U\,D^b$ for some nonnegative integers a,b,h and $r_1,r_2,\ldots,r_h$ (see (23)). Counting U's and D's, we find that $a+b+r_1+r_2+\cdots+r_h = k$ and $(r_1+1)+(r_2+1)+\cdots+(r_h+1) = n-k$. From these, one obtains $a+b = 2k+h-n$. Given h (which is the number of distinct heights of NE-steps in the corresponding standard path), there are thus $2k+h-n+1$ possibilities for a and b. Moreover, $r_1+1, r_2+1, \ldots, r_h+1$ is a composition of $n-k$ into h positive integers, for which there are $\binom{n-k-1}{h-1}$ possibilities. Thus the total number of equivalence classes must be

$$\sum_{h=n-2k}^{n-k}(2k+h-n+1)\binom{n-k-1}{h-1} = \sum_{j=0}^{k}(k-j+1)\binom{n-k-1}{j},$$

where the second expression is obtained from the first by the simple substitution $h = n-k-j$. The numbers $r_1+1, r_2+1, \ldots, r_h+1$ are the multiplicities in the multiset of heights of NE-steps, while a and b determine at which h consecutive heights the NE-steps occur.

6.2. Counting all equivalence classes of a given length

Corollary 6.7. The total number of equivalence classes of words of length $n > 0$ is

$$\sum_{k=0}^{n} a(n,k) = 2F_{n+4} - \begin{cases} (3n+42)2^{n/2-3}, & \text{if } n \text{ even,} \\ (n+15)2^{(n-3)/2}, & \text{if } n \text{ odd,} \end{cases}$$

where F_n is the n-th Fibonacci number. See Table 2.

n	0	1	2	3	4	5	6	7	8	9	10
$\sum_k a(n,k)$	1	2	4	8	15	28	50	90	156	274	466

Table 2: Total number of equivalence classes for $n \leq 10$.

Proof. We simply plug $t = 1$ into (27) to obtain the generating function for the total number of equivalence classes, which is

$$\begin{aligned} \sum_{n\geq 0} \Big(\sum_{0\leq k\leq n} a(n,k) \Big) x^n &= \frac{(1+x-x^2)(1-x^2)^2}{(1-x-x^2)(1-2x^2)^2} \\ &= \frac{1}{4} + \frac{2(3+2x)}{1-x-x^2} - \frac{9+14x}{2(1-2x^2)} - \frac{3+4x}{4(1-2x^2)^2}. \end{aligned}$$

Reading off coefficients from the generating function yields the formula. □

6.3. c-Dyck words

Let us now turn our attention to restricted words. Fix a constant $c > 0$, and consider words with the property that every prefix has at least c times as many U's as D's. These words form a submonoid $\mathcal{M}_c$ of $\mathcal{M}$. Again, we will be interested in the number of equivalence classes of words of length n in $\mathcal{M}_c$. Note here that not all words equivalent to a word in $\mathcal{M}_c$ are necessarily also in $\mathcal{M}_c$. To give a simple example, the word $UUDUUD$ is in $\mathcal{M}_2$ while the equivalent word $UUUDDU$ is not.

Let $a_c(n,k)$ be the number of equivalence classes of words in the submonoid $\mathcal{M}_c$ that consist of k D's and $n-k$ U's. Note that we must have $n-k \geq ck$, or equivalently $n \geq (c+1)k$. We first show that we can again focus on up-normal words.

Lemma 6.8. Let $c \geq 1$ be a real constant. An equivalence class of words in $\mathcal{M}$ contains words in $\mathcal{M}_c$ if and only if it consists of rising words and the unique up-normal representative is in $\mathcal{M}_c$.

Proof. A word lies in $\mathcal{M}_c$ if and only if the associated diagonal path stays above[11] the line $y = \frac{c-1}{c+1}x$. This is because the part of the path corresponding to a prefix with a U's and b D's ends at $(a+b, a-b)$. The condition $a \geq cb$ translates to $a - b \geq \frac{c-1}{c+1}(a+b)$. In particular, for $c \geq 1$, all elements of $\mathcal{M}_c$ are rising by definition. We know that for every equivalence class of rising words, there is a unique up-normal representative (Proposition 4.4).

Now let an equivalence class C be given. All diagonal paths that correspond to a word in C have to end at the same point (a,b). If $b < \frac{c-1}{c+1}a$, then $\mathcal{M}_c$ cannot contain any elements of C, and there is nothing to prove.

So assume that $b \geq \frac{c-1}{c+1}a$, and assume also that there is a word $w \in C$ that lies in $\mathcal{M}_c$. Let $\mathbf{p}$ be the corresponding standard path, and consider any nonnegative integer $s < b$. Every vertex of $\mathbf{p}$ whose y-coordinate is less than or equal to s has to have x-coordinate less than or equal to $\frac{c+1}{c-1}s$ (we can interpret this as ∞ if $c = 1$). Therefore, there are at most $\frac{c+1}{c-1}s + 1$ arcs in $\mathbf{p}$ that have at least one end at height s or less. Since the multiset of step heights is the same for all words in C (by Theorem 2.1), this also holds for the diagonal path $\mathbf{q}$ that corresponds to the up-normal representative. By construction, all steps with an end at height s or less occur before all others in $\mathbf{q}$, and it follows that the rightmost vertex of $\mathbf{q}$ whose height is s has an x-coordinate of at most $\frac{c+1}{c-1}s$, which means that it lies above or on the line $y = \frac{c-1}{c+1}x$. For every vertex whose height is greater than or equal to b, this is automatically true since the final vertex (a,b) of $\mathbf{q}$ has this property. We have thus shown that the entire path $\mathbf{q}$ lies above the line $y = \frac{c-1}{c+1}x$, so the up-normal representative lies in $\mathcal{M}_c$. This completes the proof. $\square$

In analogy to Lemma 6.2, the following lemma holds.

Lemma 6.9. For every real constant $c \geq 1$ and every pair (n,k) of positive integers with $n - 1 \geq (c+1)k$, we have

$$a_c(n,k) = a_c(n-1,k) + a_c(n-2,k-1). \tag{31}$$

Proof. Again, we use the characterization that the standard paths corresponding to words in the submonoid $\mathcal{M}_c$ have to stay entirely above the line $y = \frac{c-1}{c+1}x$. By Lemma 6.8, it suffices to consider the unique up-normal representative of any equivalence class that is counted by $a_c(n,k)$. The same bijections as in the proof of Lemma 6.2 apply; we only need to check that removing the last U (Type 1) or the last DU (Type 2) yields a new word that is still in in $\mathcal{M}_c$. Let us consider the two different types:

[11]"Above" means "weakly above"; i.e., the path is allowed to touch this line.

Type 1: The only change in the standard path is the final descent, which is shifted one unit to the left and one unit down (or one unit to the right and one unit up in the inverse operation). The final descent is entirely above the line $y = \frac{c-1}{c+1}x$ provided that its final vertex is. This is guaranteed in both directions by the assumption that $n - 1 \geq (c+1)k$. The procedure is shown in Figure 7.

Type 2: In this case, the only change in the standard path is that the final descent is moved two units to the left (or to the right when the inverse is applied). As for Type 1, it stays entirely above the line $y = \frac{c-1}{c+1}x$ in both directions because of the assumption that $n - 1 \geq (c+1)k$. See Figure 8 for an illustration.

Combining the two types, we obtain the desired recursion again. □

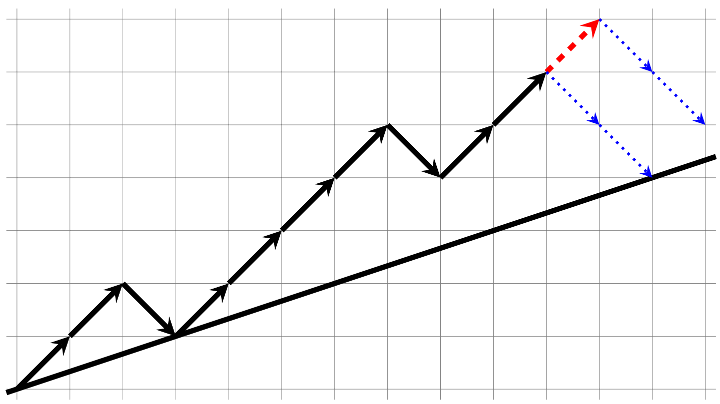


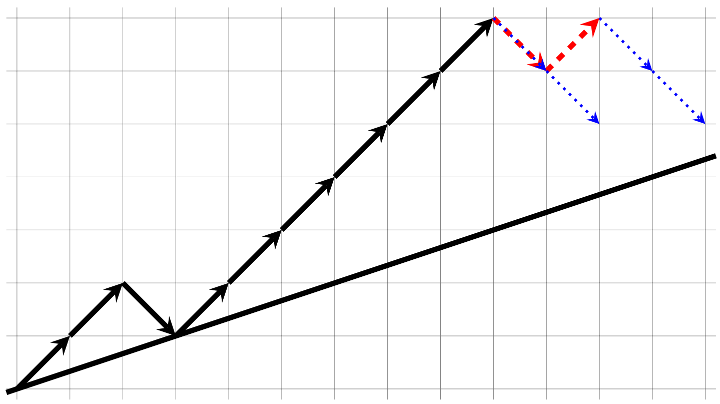
Figure 7: The bijection for words of type 1. The arc that is removed/inserted is indicated by a dashed red line. Arcs that are shifted by the procedure are indicated by dotted blue lines. The line $y = \frac{c-1}{c+1}x$ is shown as well: in this example, $c = 2$.

Remark 6.10. The recursion (31) is in general false for $c < 1$. As a counterexample, note that $a_{1/2}(4, 2) = 3$ (the three elements *UUDD*, *UDUD* and *UDDU* of $\mathcal{M}_{1/2}$ belong to three distinct equivalence classes), while $a_{1/2}(3, 2) = 1$ (the only element is *UDD*) and $a_{1/2}(2, 1) = 1$ (the only element is *UD*).

For positive integer values of c, there is in fact a fairly simple explicit formula for $a_c(n, k)$.

Theorem 6.11. If c is a positive integer and n, k are positive integers with $n \geq (c+1)k$, then we have

$$a_c(n, k) = \binom{n-k-1}{k} - (c-2)\sum_{j=0}^{k-1}\binom{n-k-1}{j}. \tag{32}$$



UUDUUUUUUDUDD $\leftrightarrow$ *UUDUUUUUUDD*

Figure 8: The bijection for words of type 2. The arcs that are removed/inserted are indicated by dashed red lines. Arcs that are shifted by the procedure are indicated by dotted blue lines. The line $y = \frac{c-1}{c+1}x$ is shown as well: in this example, $c = 2$.

Proof. The recursion (31) determines all values of $a_c(n,k)$ except for the boundary cases where $n = (c+1)k$. However, if $n = (c+1)k$, then the last letter of every valid word in $\mathcal{M}_c$ has to be a D (otherwise, the condition of the submonoid $\mathcal{M}_c$ is not satisfied for the prefix obtained by removing the last letter). This immediately implies that

$$a_c((c+1)k, k) = a_c((c+1)k - 1, k - 1). \tag{33}$$

Together with (31) and the trivial initial value $a_c(0,0) = 1$, this determines $a_c(n,k)$ uniquely for all values of n and k (with $n \geq (c+1)k \geq 0$), so it suffices to verify that the expression on the right side of (32) satisfies the recursion (31) as well as (33). The former is a simple consequence of the recursion for the binomial coefficients. The latter is (after some trivial cancellations) equivalent to

$$\binom{ck-1}{k} = (c-1)\binom{ck-1}{k-1},$$

which is readily verified. The theorem follows immediately by induction. □

Remark 6.12. In the special case $c = 1$, we obtain

$$a_1(n,k) = \sum_{j=0}^{k} \binom{n-k-1}{j},$$

see Table 3. These partial sums of binomial coefficients are the entries of *Bernoulli's triangle* (see [OEIS, A008949]) and appear famously as numbers of regions in a general-position hyperplane arrangement [Stanle06, Proposition 2.4].

$n \backslash k$	0	1	2	3	4	5	$\sum$
1	1						1
2	1	1					2
3	1	2					3
4	1	3	2				6
5	1	4	4				9
6	1	5	7	4			17
7	1	6	11	8			26
8	1	7	16	15	8		47
9	1	8	22	26	16		73
10	1	9	29	42	31	16	128

Table 3: Table of the values of $a_1(n,k)$. The final column gives the total number $\sum_k a_1(n,k)$.

The terms in the sum have a combinatorial interpretation again (compare Remark 6.6). The only difference to the unrestricted case is that the up-normal representative cannot have an initial segment of D's and can thus be written as $(UD)^{r_1}U\,(UD)^{r_2}U\,\cdots\,(UD)^{r_h}U\,D^b$ for some nonnegative integers b,h and $r_1,r_2,\ldots,r_h$. The numbers $r_1+1,r_2+1,\ldots,r_h+1$, i.e., the multiplicities in the multiset of heights of NE-steps, form a composition of $n-k$ into h positive integers. Since there are $\binom{n-k-1}{h-1}$ such compositions, the total number of equivalence classes must be

$$\sum_{h=n-2k}^{n-k}\binom{n-k-1}{h-1}=\sum_{j=0}^{k}\binom{n-k-1}{j}.$$

In this special case, we also have a simple generating function that is similar to (28). We now have

$$A_1(x,t)=\sum_{n\geq 0}\sum_{0\leq k\leq n/2}a_1(n,k)t^kx^n=\frac{(1-tx^2)^2}{(1-x-tx^2)(1-2tx^2)}. \tag{34}$$

This is proved in the same way as (28). First, since $a_1(n,k)$ satisfies the same recursion as $a(n,k)$, one obtains

$$A_1(x,t)=\frac{1-tx^2}{1-x-tx^2}\sum_{k\geq 0}a_1(2k,k)t^kx^{2k}$$

in the same way as (29). Now,

$$a_1(2k,k)=\sum_{j=0}^{k}\binom{k-1}{j}=2^{k-1}$$

for $k\geq 1$ and $a_1(0,0)=1$, thus

$$\sum_{k\geq 0}a_1(2k,k)t^kx^{2k}=\frac{1-tx^2}{1-2tx^2},$$

and (34) follows.

Furthermore, we also have an explicit formula for the total number of equivalence classes in this case: the number of equivalence classes of words of length $n > 0$ for which every prefix has at least as many U's as D's is precisely

$$F_{n+2} - 2^{\lfloor (n-1)/2 \rfloor},$$

which is [OEIS, A079289]. The final column in Table 3 gives the values of this sequence up to $n = 10$. This follows e.g. by plugging $t = 1$ into the generating function, in the same way as the formula in Corollary 6.7.

Remark 6.13. In the special case $c = 2$, the sum disappears from (32), and we obtain the remarkably simple formula

$$a_2(n,k) = \binom{n-k-1}{k}.$$

There is a connection to the famous ballot problem: consider any up-normal word in the submonoid $\mathcal{M}_2$ that consists of k D's and $n-k$ U's. Remove the last U and all D's that follow. Moreover, replace every occurrence of UD by a single D. The result is a word in $\mathcal{M}_1$, i.e., a 1-Dyck word (every prefix contains at least as many U's as D's) of length $n-k-1$ with at most k D's. Conversely, any 1-Dyck word of length $n-k-1$ with at most k many D's can be turned into an up-normal word in $\mathcal{M}_2$ with k D's and $n-k$ U's: letting r denote the number of D's ($r \le k$), replace every D by UD and add UD^{k-r} at the end. So we have a bijection between equivalence classes in $\mathcal{M}_2$ and 1-Dyck words of length $n-k-1$ with at most k D's. Since it is well-known that there are $\binom{\ell}{j} - \binom{\ell}{j-1}$ many 1-Dyck words of length ℓ with exactly j many D's ($1 \le j \le \frac{\ell}{2}$), it follows that

$$a_2(n,k) = 1 + \sum_{j=1}^{k} \left(\binom{n-k-1}{j} - \binom{n-k-1}{j-1} \right) = \binom{n-k-1}{k}.$$

See Table 4 for some values of $a_2(n,k)$.

6.4. The size of an equivalence class

Given an equivalence relation on a finite set, its equivalence classes are not the only thing that can be counted. One can also ask how large the equivalence classes are. The following theorem answers this question.

$n \backslash k$	0	1	2	3	Σ
1	1				1
2	1				1
3	1	1			2
4	1	2			3
5	1	3			4
6	1	4	3		8
7	1	5	6		12
8	1	6	10		17
9	1	7	15	10	33
10	1	8	21	20	50

Table 4: Table of the values of $a_2(n,k)$. The final column gives the total number $\sum_k a_2(n,k)$.

Theorem 6.14. Let $w \in \mathcal{M}$ be a word with NE-height polynomial $H_{\mathrm{NE}}(w,z) = \sum_{i \in \mathbb{Z}} a_i z^i$ and SE-height polynomial $H_{\mathrm{SE}}(w,z) = \sum_{i \in \mathbb{Z}} b_i z^i$. (Note that $a_i = b_i = 0$ for all but finitely many i.) Then, the size of the equivalence class containing w (that is, the number of words $u \in \mathcal{M}$ satisfying $u \overset{\mathrm{bal}}{\sim} w$) is

$$\prod_{i \geq 0} \binom{a_i + b_{i+2} - 1}{b_{i+2}} \binom{b_{-i} + a_{-i-2} - 1}{a_{-i-2}} \times \begin{cases} \binom{a_0+b_0}{a_0}, & \text{if } w \text{ is balanced;} \\ \binom{a_0+b_0-1}{b_0}, & \text{if } w \text{ is rising and non-balanced;} \\ \binom{a_0+b_0-1}{a_0}, & \text{if } w \text{ is falling and non-balanced.} \end{cases} \tag{35}$$

Proof. We prove the formula by considering the associated standard paths. Then a_i is the number of NE-arcs from height i to height $i+1$, and b_i is the number of SE-arcs from height i to height $i-1$. Our aim is to show that the number of standard paths, given all a_i and b_i, is given by the formula (35).

We first consider the special case that w is a 1-Dyck word, i.e., a word whose prefixes are all rising, so that the standard path $\mathbf{p}$ stays above the x-axis. In this case, $a_i = 0$ whenever $i < 0$, and $b_i = 0$ whenever $i \leq 0$, so the formula reduces to

$$\prod_{i \geq 0} \binom{a_i + b_{i+2} - 1}{b_{i+2}}.$$

We use induction on the maximum height d of vertices in the standard path. For $d = 0$, the word and its associated path are empty, so the statement becomes trivial.

Now we proceed with the induction step. Consider only the part $\mathbf{p}'$ of the path that lies above the line $y = 1$ (after removing all gaps, see Figure 9). By the induction hypothesis, the number of possibilities for this path is

$$\prod_{i\geq 1} \binom{a_i + b_{i+2} - 1}{b_{i+2}}.$$

Given $\mathbf{p}'$, in order to obtain a feasible path $\mathbf{p}$, we always have to add an NE-step at the beginning, an SE-step at the end if $a_0 = b_1$ (so that the path ends at height 0), and insert a total of $a_0 - 1$ copies of an SE-step followed by an NE-step at vertices of $\mathbf{p}'$ that lie at height 0. There are $b_2 + 1$ such places (at the beginning and after each of the b_2 SE-steps that end at height 0), so the possibilities for $\mathbf{p}$, given $\mathbf{p}'$, correspond to the weak compositions of $a_0 - 1$ into $b_2 + 1$ nonnegative integers. Since there are $\binom{a_0+b_2-1}{b_2}$ such compositions, the desired formula follows, completing the induction.

Now we consider the general case: every diagonal path can be decomposed into the part above the x-axis and the part below the x-axis (see Figure 10). The number of possibilities for these two parts is

$$\prod_{i\geq 0} \binom{a_i + b_{i+2} - 1}{b_{i+2}} \binom{b_{-i} + a_{-i-2} - 1}{a_{-i-2}}$$

in view of what has already been shown. It remains to multiply by the number of ways to combine them: each time the path is at height 0 (but not completed yet), we have to decide whether to continue with an NE-step (thus a piece of the path that lies above the x-axis) or an SE-step (thus a piece of the path that lies below the x-axis). The only exception is the final return to the x-axis in the non-balanced case: if the path ends above the x-axis, the final step from the x-axis must be an NE-step; if it ends below the x-axis, it must be an SE-step.

The total number of steps to be chosen in this way is $a_0 + b_0$. In the balanced case, we have $\binom{a_0+b_0}{a_0}$ possibilities. In the non-balanced case, the number of choices is $\binom{a_0+b_0-1}{b_0}$ (rising) or $\binom{a_0+b_0-1}{a_0}$ (falling), respectively. Combining this with the number of possibilities for the two parts above and below the x-axis, we reach the desired formula. □

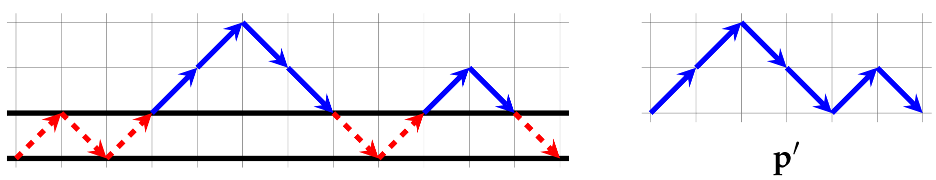


Figure 9: Decomposition into the part above the line $y = 1$ (blue, solid) and the part between $y = 0$ and $y = 1$ (red, dashed).

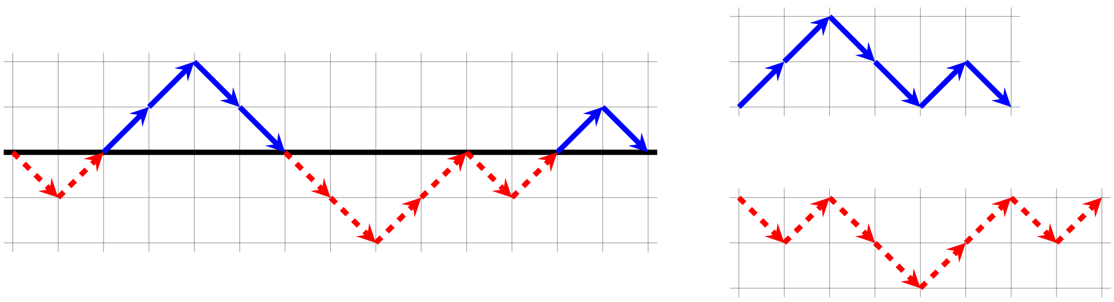


Figure 10: Decomposition into the part above (blue, solid) and below (red, dashed) the x-axis.

Remark 6.15. The special case of balanced 1-Dyck words (balanced words for which every prefix is rising) appears in different places in the literature. See [GJW75, Theorem 6] (in the context of rook theory) or [Flajol80, Proposition 3A and 3B] (in the context of continued fractions and lattice paths).

7. Bond percolation

The diagonal lattice interpretation of the Weyl algebra brings forward an intriguing connection with bond percolation on a directed square lattice. In this section we explore this connection in depth.

Percolation is one of the fundamental problems in statistical physics [Grimme99] [StaAha94], and is of great theoretical interest in its own right as well as being applicable to a wide variety of problems in physics, biology, chemistry, and many other areas of science. Bond percolation, the phenomenon of interest here, was introduced in the mathematics literature by Broadbent and Hammersley in 1957 [BroHam57], and has been studied extensively by mathematicians since then.

The prototype setting for bond percolation is a directed square lattice, whose vertices (called *sites* in this context) are the points in the Cartesian t-x-plane with integer coordinates such that $t \geq 0$ and $t + x$ is even. See Figure 11. Here t is commonly thought of as the time (or stage) of the percolation process. We regard the lattice as originally consisting of dry sites except for the origin, which is the source of fluid and wet at stage 0. There are two *bonds* (i.e., arcs) leading from each site (t, x); they terminate at the sites $(t + 1, x + 1)$ and $(t + 1, x - 1)$. (In our language, they are the NE-arcs and the SE-arcs.) All bonds have probability p of being open to the passage of fluid and probability $1 - p$ of being closed. Fluid flows from a wet site along an unblocked bond to wet another site (in the forward direction, i.e., from source to target). Thus a site is wetted if there is a path of unblocked directed bonds (and wet sites) from the origin to that site. See Figure 12 for an illustration of all possible scenarios of the percolation process from the origin $(0, 0)$ to the site $(2, 0)$ in two time steps. *Clusters* are sets of connected bonds, where

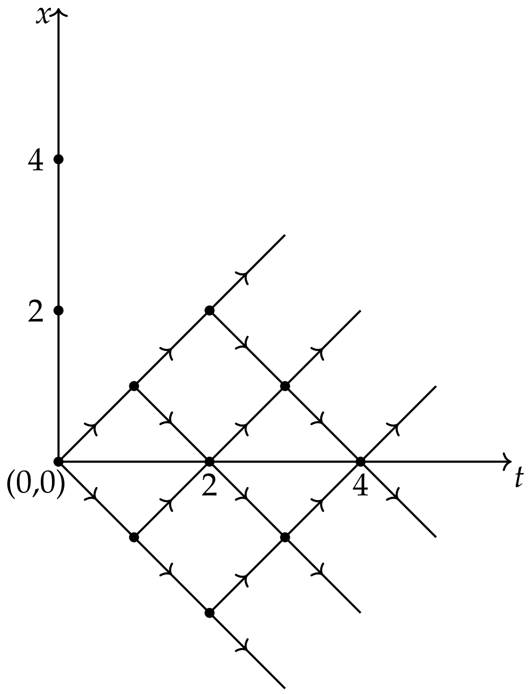


Figure 11: Acyclic directed square lattice.

two bonds are said to be *adjacent* if they have a vertex in common. Sometimes, a *wall* parallel to the t-axis ("growth direction") is present to restrict the lateral growth of the percolation clusters. In particular, we consider the situation where the t-axis itself is the wall, so that the bonds leading to sites with $x < 0$ are always closed [EGJT96]. See Figure 13.

The mean size $S(p)$ of the clusters is a quantity that has captured a lot of interest:

$$S(p) = \sum_{\text{sites } (t,x)} C(t, x; p),$$

where $C(t, x; p)$ is the probability that there is an open path from the origin to the site (t, x). For example, we may readily calculate that $C(2, 0; p) = 2p^2 - p^4$ for percolation (without a wall) from Figure 12. We may also easily calculate that $C(2, 0; p) = p^2$ for percolation (with a wall). For large t and x, however, calculating the probability $C(t, x; p)$ becomes a tedious matter and is usually done with the help of a computer. There is a vast body of literature in statistical physics regarding the implementation of the computational procedure, commonly referred to as a *transfer matrix method*. See [Blease77] for the setup in physics. The main idea is the following: The state of time step t is a specification of which sites in column t of the directed square lattice are wet and which sites are dry. Essentially the state vector of a given column is completely determined by that of the previous column and only one state vector need be held in the computer at any stage and all other state vectors overwritten, although some care is necessary for the execution.

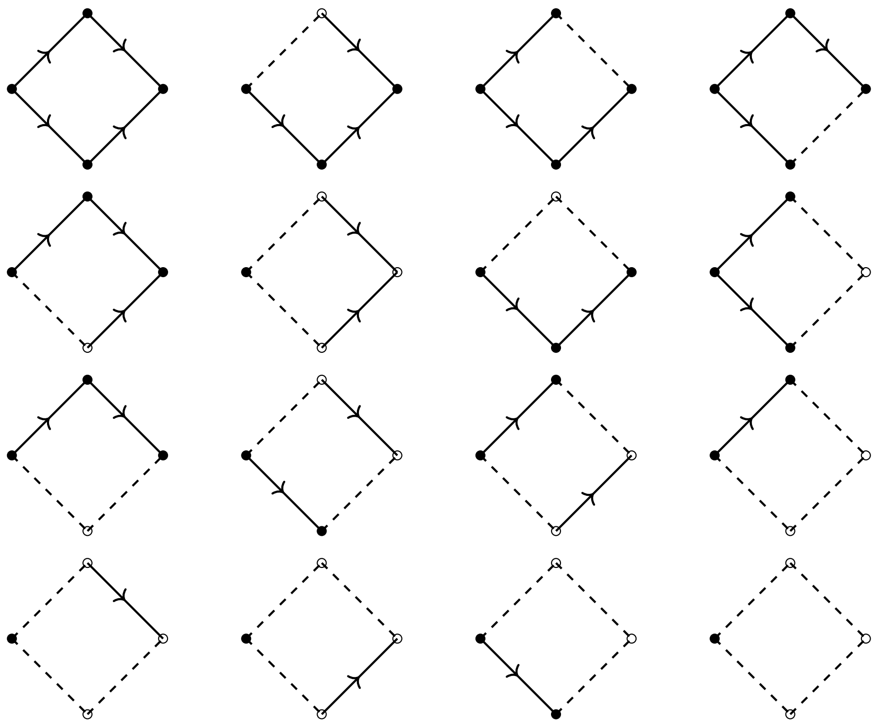


Figure 12: Directed bond percolation from the origin to the site $(2, 0)$. Open (closed) bonds are indicated by solid (dashed) lines. Filled (hollow) circles denote wet (dry) sites.

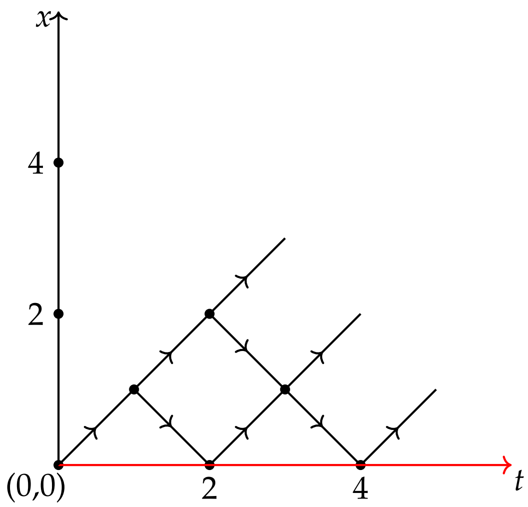


Figure 13: Acyclic directed square lattice with a wall at $x = 0$.

The low-density series expansion of $S(p)$ for bond percolation (both with and without a wall) may be performed to order p^n (for varying values of n) by calculating $C(t,x;p)$ to order p^n of all sites which may be reached in a walk of n or fewer steps from the origin, i.e., summing up $C(t,x;p)$ of all those reachable sites before or at column n. So for example, when a wall is not present, to obtain $S(p)$ to order p, we compute

$$C(0,0;p)+C(1,1;p)+C(1,-1;p)=1+p+p=1+2p.$$

To obtain $S(p)$ to order p^2, we compute

$$\begin{aligned}C(0,0;p)&+C(1,1;p)+C(1,-1;p)+C(2,2;p)+C(2,0;p)+C(2,-2;p)\\&=1+p+p+p^2+(2p^2-p^4)+p^2=1+2p+4p^2-p^4,\end{aligned}$$

which gives $1+2p+4p^2$ when kept to order p^2. And when a wall is present, to obtain $S(p)$ to order p, we compute

$$C(0,0;p)+C(1,1;p)=1+p.$$

To obtain $S(p)$ to order p^2, we compute

$$C(0,0;p)+C(1,1;p)+C(2,2;p)+C(2,0;p)=1+p+p^2+p^2=1+p+2p^2.$$

The coefficients of the series expansion of $S(p)$ (without a wall/with a wall) are respectively given in [OEIS, A006727] and [OEIS, A056532]. We note in particular that negative terms appear starting from $n=50$ for [OEIS, A006727] and from $n=39$ for [OEIS, A056532], as for large n, the negative higher-order terms from columns before column n might dominate the positive p^n term from column n.

Compared with our results from earlier, we see that $a(n):=\sum_k a(n,k)$ (see Table 2) gives the total number of equivalence classes of diagonal paths from the origin to all sites in column n on a directed square lattice without a wall in Figure 11 while $a_1(n):=\sum_k a_1(n,k)$ (see Table 3) gives the analogous number for a directed square lattice with a wall in Figure 13. It is thus not entirely surprising that $a(n)$ agrees with [OEIS, A006727] up to $n=11$, and that $a_1(n)$ agrees with [OEIS, A056532] up to $n=8$. After all, recall from Lemma 6.2 that the $a(n,k)$ satisfy the recursion $a(n,k)=a(n-1,k)+a(n-2,k-1)$. Due to the transfer matrix method that was explained briefly earlier, this recursive feature appears again in calculating the bond percolation on a directed square lattice. To derive $C(t,x;p)$, the probability that there is an open path from the origin to the site (t,x), we only need to keep track of the corresponding C values on the one square to the left of the site (t,x), consisting of sites $(t-2,x),(t-1,x+1),(t-1,x-1),(t,x)$.

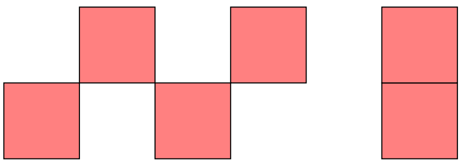
Nevertheless, the Weyl algebra problem and the bond percolation problem are different in nature: One is deterministic while the other is probabilistic, and more importantly, the contribution to the nth term only comes from the nth column for one but might involve some columns before column n for the other. In detail, the coefficient of p^n term in the low-density series expansion of $S(p)$ is calculated by summing up $C(t, x; p)$ of all those reachable sites before or at column n, not just at column n. (In our earlier calculation for $S(p)$, summing up column 2 sites contributes a higher-order term $-p^4$ in addition to $4p^2$. For larger n, the difference is more evident.)

8. The rook theory connection

Theorem 2.1 classifies equalities between products of D's and U's in the Weyl algebra $\mathcal{W}$. Another approach to this classification problem is to expand any such product in one of the bases $\left(D^i U^j\right)_{i,j \in \mathbb{N}}$ and $\left(U^j D^i\right)_{i,j \in \mathbb{N}}$ of $\mathcal{W}$; the uniqueness of this expansion then allows us to compare two such products by comparing their respective coefficients.

It turns out that this expansion can be done in explicit combinatorial terms using *rook theory*. We do not give a detailed introduction to this subject (see [BCHR11] and [ManSch16, §2.4.4] for that), but quickly recall the basics we need.

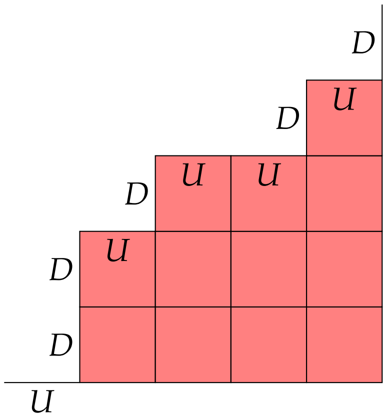
A *cell* means a pair (i, j) of two positive integers. Each cell (i, j) will be drawn as a 1×1-square, situated in the Cartesian plane with center at the point (i, j), with its sides parallel to the axes. A *board* means a finite set of cells. For instance, the board $\{(1,1), (2,2), (3,1), (4,2), (6,1), (6,2)\}$ looks as follows:


. (36)

A *rook placement* of a board B is a subset S of B such that no two cells in S lie in the same row or column. If B is a board, and if $k \in \mathbb{N}$, then the *rook number* $r_k(B)$ is the number of k-element rook placements of B. For instance, if B is the board in (36), then its rook numbers are $r_0(B) = 1$ and $r_1(B) = |B| = 6$ and $r_2(B) = 8$ and $r_k(B) = 0$ for all $k > 2$.

Two boards B and C are said to be *rook-equivalent* if they share the same rook numbers (i.e., if $r_k(B) = r_k(C)$ for all $k \in \mathbb{N}$).

If w is a word in $\mathcal{M}$, then the *Ferrers board* B_w is a special board defined as follows: It is a contiguous set of cells, whose bottom and right boundaries are straight lines, whereas the rest of its boundary is a jagged path that (when walked from southwest to northeast) takes a north-step for each D in w and an east-step for each U in w (reading the word w from left to right). For instance, if $w = UDDUDUUDUD$, then B_w looks as follows:



(where the D and U labels are signaling the correspondence between the letters of w and the steps of the jagged boundary).[12]

Now a classical result of Navon (originally [Navon73, §2], but see [BCHR11, Theorem 20] or [ManSch16, Theorem 6.11 for $h = 1$] or [Varvak04, Theorem 3.2] for a modern treatment[13]) says the following:

Theorem 8.1. Let $w \in \mathcal{M}$ be any word that contains n many D's and m many U's. Then, in $\mathcal{W}$, we have

$$\phi(w) = \sum_{k=0}^{\min\{m,n\}} r_k(B_w) U^{m-k} D^{n-k}.$$

As a consequence, we obtain the following:

Theorem 8.2. Let u and v be two words in $\mathcal{M}$ that have the same number of D's and the same number of U's. Then, the following statements are equivalent: the statements $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3$, $\mathcal{S}'_3$, $\mathcal{S}_4$, $\mathcal{S}_5$ and $\mathcal{S}_6$ from Theorem 2.1, and the additional statement

- $\mathcal{R}_1$: The boards B_u and B_v are rook-equivalent.

Proof. It suffices to prove the equivalence $\mathcal{S}_1 \iff \mathcal{R}_1$.

Let n be the # of D's in u (or, equivalently, in v), and let m be the # of U's in u (or, equivalently, in v). Then, the board B_u has $\leq n$ nonempty rows (since u has n many D's). Hence, any rook placement of B_u has size $\leq n$ (since otherwise, it

[12] Note that the U at the beginning of w, and the D at the end, do not affect the Ferrers board.

[13] See also [BHDPS08] for a brief introduction with physics in mind, and [BlaFla11] for combinatorial applications. The sources differ slightly in their notation, but are easily seen to be equivalent (e.g., by reflecting the Ferrers boards across a diagonal line).

would contain two cells in the same row, by the pigeonhole principle). In other words, $r_k(B_u) = 0$ for all $k > n$. Similarly, $r_k(B_u) = 0$ for all $k > m$. Combining these two observations, we obtain

$$r_k(B_u) = 0 \qquad \text{for all } k > \min\{m,n\}. \tag{37}$$

Similarly,

$$r_k(B_v) = 0 \qquad \text{for all } k > \min\{m,n\}. \tag{38}$$

Comparing these two equalities, we conclude that

$$r_k(B_u) = r_k(B_v) \qquad \text{for all } k > \min\{m,n\}. \tag{39}$$

Recall that the family $(U^j D^i)_{i,j\in\mathbb{N}}$ is a basis of $\mathcal{W}$ (by [EGHetc11, Proposition 2.7.1 (i)]), therefore **k**-linearly independent.

Now, we have the following chain of equivalences:

$$\begin{aligned}
\mathcal{S}_1 &\iff (\phi(u) = \phi(v)) \\
&\iff \left(\sum_{k=0}^{\min\{m,n\}} r_k(B_u) U^{m-k} D^{n-k} = \sum_{k=0}^{\min\{m,n\}} r_k(B_v) U^{m-k} D^{n-k} \right) \\
&\qquad\qquad (\text{here, we rewrote } \phi(u) \text{ and } \phi(v) \text{ using Theorem 8.1}) \\
&\iff (r_k(B_u) = r_k(B_v) \text{ for all } k \in \{0,1,\ldots,\min\{m,n\}\}) \\
&\qquad\qquad \left(\text{since the family } \left(U^j D^i\right)_{i,j\in\mathbb{N}} \text{ is } \mathbf{k}\text{-linearly independent} \right) \\
&\iff (r_k(B_u) = r_k(B_v) \text{ for all } k \in \mathbb{N}) \qquad (\text{by (39)}) \\
&\iff (\text{the boards } B_u \text{ and } B_v \text{ are rook-equivalent}) \iff \mathcal{R}_1.
\end{aligned}$$

This completes the proof of $\mathcal{S}_1 \iff \mathcal{R}_1$ and thus the proof of Theorem 8.2. □

The implication $\mathcal{R}_1 \implies \mathcal{S}_1$ in Theorem 8.2 has been implicitly observed in [BCHR11, bottom of p. 40]. Note that this implication really requires the assumption about equal numbers of D's and of U's in Theorem 8.2, since (e.g.) the Ferrers boards B_{DUUDU} and B_{DDUU} are rook-equivalent without $\phi(DUUDU)$ equalling $\phi(DDUU)$. For an even starker example, if $w \in \mathcal{M}$ is any word, then the Ferrers boards B_w and $B_{\omega(w)}$ are rook-equivalent (being each other's reflection across a diagonal), but $\phi(w)$ is usually not $\phi(\omega(w))$. We note that this reasoning leads to a new proof of Theorem 2.3.

Theorem 8.2 connects our study of the kernel of ϕ to a classical question, namely: When are two Ferrers boards rook-equivalent? A classical result of Foata and Schützenberger ([FoaSch70, Theorem 6] or [BCHR11, Theorem 7]) shows that each Ferrers board is rook-equivalent to a unique "increasing Ferrers board". These "increasing Ferrers boards" are somewhat similar to our up-normal words, but not quite in bijection, since (as we said) the rook equivalence of B_u and B_v implies $\phi(u) = \phi(v)$ only when we know that u and v have the same number of U's and the same number of D's.

Interestingly, Foata and Schützenberger have their own kind of moves that they use to normalize a Ferrers board modulo rook equivalence: the "(k,k')-transforms" (see [FoaSch70, Definition 8 bis on page 9]). These appear to be close relatives of our balanced flips.

A recent preprint by Cotardo, Gruica and Ravagnani [CoGrRa23] proves another set of equivalent criteria for the rook-equivalence of two Ferrers boards [CoGrRa23, Corollary 3.2]. It lists six equivalent conditions, one of which (condition 6) is rook-equivalence, whereas another (condition 1) is equivalent to our statement $\mathcal{S}_3$ (albeit this equivalence takes some work to prove). The other four conditions are not found in our lists so far. In the following, we state two of these four conditions (4 and 5), as they are rather surprising and reveal an unexpected connection to the theory of finite fields. (Arguably, at least one of them has been foreseen, to some extent, in Haglund's [Haglun98].)

First, we introduce the necessary notations. For any finite field F, any non-negative integers n and k, and any board $B \subseteq \{1,2,\ldots,n\}^2$, we define $P_k(B/F)$ to be the number of $n \times n$-matrices $A \in F^{n\times n}$ of rank k such that all entries of A in cells outside of B are zero. This is called $P_k(B)$ in [Haglun98, Definition 1], where $F = \mathbb{F}_q$; but we include F in the notation since $P_k(B/F)$ depends on F. It is easy to see, however, that $P_k(B/F)$ does not depend on n, as long as n is large enough that $B \subseteq \{1,2,\ldots,n\}^2$. In [CoGrRa23], our $P_k(B/F)$ is called $W_k\left(\operatorname{Mat}_q^{n\times m}(B)\right)$, where $F = \mathbb{F}_q$, and where n and m are chosen large enough that $B \subseteq \{1,2,\ldots,n\} \times \{1,2,\ldots,m\}$. Now, we claim the following:

Theorem 8.3. Let u and v be two words in $\mathcal{M}$ that have the same number of D's and the same number of U's. Then, the following statements are equivalent: the statements $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3$, $\mathcal{S}_3'$, $\mathcal{S}_4$, $\mathcal{S}_5$ and $\mathcal{S}_6$ from Theorem 2.1, the statement $\mathcal{R}_1$ from Theorem 8.2, and the following two additional statements:

- $\mathcal{R}_2$: For any finite field F and any $k \in \mathbb{N}$, we have $P_k(B_u/F) = P_k(B_v/F)$.
- $\mathcal{R}_3$: For any finite field F, we have $P_1(B_u/F) = P_1(B_v/F)$.

Proof. Let $\mathcal{F}$ and $\mathcal{F}'$ be the Ferrers boards B_u and B_v. Then, our statements $\mathcal{R}_1$, $\mathcal{R}_2$ and $\mathcal{R}_3$ are (respectively) the conditions 6, 5 and 4 of [CoGrRa23, Corollary 3.2]. Thus, the former three statements are equivalent (since [CoGrRa23, Corollary 3.2] shows that the latter three conditions are equivalent). In other words, $\mathcal{R}_1 \iff \mathcal{R}_2 \iff \mathcal{R}_3$. Combined with Theorem 8.2, this proves Theorem 8.3.

However, let us also give an alternative proof of the equivalence $\mathcal{R}_1 \iff \mathcal{R}_2$ using some results of Haglund [Haglun98] and Garsia and Remmel [GarRem86]. We should warn that Haglund's paper [Haglun98] differs from us (and from [GarRem86]) in how it defines Ferrers boards: Haglund's Ferrers boards are the reflections of ours across a horizontal axis. Fortunately, this reflection does not affect the theory in any significant way (since rook placements and matrices can be reflected along with the boards), and thus any result can be easily translated from one convention to another.

Any Ferrers board B and any number $k \in \mathbb{N}$ give rise to a *q-rook polynomial* (aka *q-rook number*) $R_k(B,q) \in \mathbb{Z}[q]$, which is defined in [GarRem86, (I.4)] or in [Haglun98, (1)]. (In [Haglun98, (1)], it is denoted by $R_k(B)$, and the variable q is renamed x.) A result of Garsia and Remmel ([GarRem86, last paragraph of §1], [Haglun98, between (2) and (3)]) says that the two Ferrers boards B_u and B_v have the same rook numbers (i.e., are rook-equivalent) if and only if they have the same q-rook numbers (i.e., if $R_k(B_u,q) = R_k(B_v,q)$ for all $k \in \mathbb{N}$). In other words, the statement $\mathcal{R}_1$ is equivalent to the following statement:

- $\mathcal{R}_5$: We have $R_k(B_u,q) = R_k(B_v,q)$ for all $k \in \mathbb{N}$.

Next we recall a result of Haglund [Haglun98, Theorem 1], which says that any Ferrers board B, any finite field F and any $k \in \mathbb{N}$ satisfy

$$P_k(B/F) = (|F|-1)^k |F|^{|B|-k} R_k\left(B,\ |F|^{-1}\right). \tag{40}$$

(Note that [Haglun98] denotes $|F|$ by q and denotes $|B|$ by $\operatorname{Area}(B)$, and also abbreviates $R_k\left(B,q^{-1}\right)$ by $R_k\left(q^{-1}\right)$.)

We are now ready to prove the equivalence $\mathcal{R}_1 \iff \mathcal{R}_2$:

$\mathcal{R}_1 \Longrightarrow \mathcal{R}_2$: Assume that $\mathcal{R}_1$ holds. Thus, $r_k(B_u) = r_k(B_v)$ for all $k \in \mathbb{N}$. In particular, $r_1(B_u) = r_1(B_v)$. In other words,

$$|B_u| = |B_v| \tag{41}$$

(since $r_1(B) = |B|$ for any board B). Moreover, we have assumed that $\mathcal{R}_1$ holds; thus, $\mathcal{R}_5$ holds as well (since we have already observed that $\mathcal{R}_1$ is equivalent to $\mathcal{R}_5$). In other words,

$$R_k(B_u,q) = R_k(B_v,q) \tag{42}$$

for all $k \in \mathbb{N}$. Now, if F is any finite field, and if $k \in \mathbb{N}$, then

$$\begin{aligned} P_k(B_u/F) &= (|F|-1)^k |F|^{|B_u|-k} R_k\left(B_u,\ |F|^{-1}\right) && \text{(by (40))} \\ &= (|F|-1)^k |F|^{|B_v|-k} R_k\left(B_v,\ |F|^{-1}\right) && \text{(by (41) and (42))} \\ &= P_k(B_v/F) \qquad \text{(by (40))}. \end{aligned}$$

In other words, statement $\mathcal{R}_2$ holds. This proves the implication $\mathcal{R}_1 \Longrightarrow \mathcal{R}_2$.

$\mathcal{R}_2 \Longrightarrow \mathcal{R}_1$: Assume that $\mathcal{R}_2$ holds. In other words, for any finite field F and any $k \in \mathbb{N}$, we have

$$P_k(B_u/F) = P_k(B_v/F). \tag{43}$$

Summing this equality over all $k \in \mathbb{N}$, we obtain

$$\sum_{k\in\mathbb{N}} P_k(B_u/F) = \sum_{k\in\mathbb{N}} P_k(B_v/F) \tag{44}$$

for any finite field F. However, if F is a finite field and $B \subseteq \{1, 2, \ldots, n\}^2$ is any board, then $\sum_{k \in \mathbb{N}} P_k(B/F)$ is the number of **all** $n \times n$-matrices $A \in F^{n \times n}$ (of all possible ranks) such that all entries of A in cells outside of B are zero (by the definition of $P_k(B/F)$). This number is $|F|^{|B|}$, since the $|B|$ entries of such a matrix A that lie in the cells of B can be chosen freely from the field F (while the remaining entries are determined to be 0). Thus, for any finite field F and any board $B \subseteq \{1, 2, \ldots, n\}^2$, we obtain

$$\sum_{k \in \mathbb{N}} P_k(B/F) = |F|^{|B|}.$$

Therefore, (44) can be rewritten as

$$|F|^{|B_u|} = |F|^{|B_v|}.$$

Applying this to $F = \mathbb{F}_2$ (in which case $|F| = 2$), we obtain $2^{|B_u|} = 2^{|B_v|}$. Therefore, $|B_u| = |B_v|$.

Now, let $k \in \mathbb{N}$. If F is any finite field, then

$$P_k(B_u/F) = (|F| - 1)^k |F|^{|B_u| - k} R_k\left(B_u,\ |F|^{-1}\right) \qquad \text{(by (40))}$$

and

$$P_k(B_v/F) = (|F| - 1)^k |F|^{|B_v| - k} R_k\left(B_v,\ |F|^{-1}\right) \qquad \text{(by (40))}.$$

The left hand sides of these two equalities are equal (by (43)). Thus, so are their right hand sides. In other words, for any finite field F, we have

$$(|F| - 1)^k |F|^{|B_u| - k} R_k\left(B_u,\ |F|^{-1}\right) = (|F| - 1)^k |F|^{|B_v| - k} R_k\left(B_v,\ |F|^{-1}\right).$$

The factors $(|F| - 1)^k |F|^{|B_u| - k}$ and $(|F| - 1)^k |F|^{|B_v| - k}$ in this equality are equal (since $|B_u| = |B_v|$) and nonzero (since $|F| > 1$); thus, we can cancel them and obtain

$$R_k\left(B_u,\ |F|^{-1}\right) = R_k\left(B_v,\ |F|^{-1}\right). \tag{45}$$

In particular, for any prime number p, we obtain

$$R_k\left(B_u,\ p^{-1}\right) = R_k\left(B_v,\ p^{-1}\right). \tag{46}$$

(by applying (45) to $F = \mathbb{F}_p$). But $R_k(B_u, q)$ and $R_k(B_v, q)$ are two polynomials in q. The equality (46) shows that these two polynomials agree at infinitely many rational inputs (namely, at the reciprocals of all prime numbers). Hence, these two polynomials must be equal. In other words, $R_k(B_u, q) = R_k(B_v, q)$.

Since we have proved this for all $k \in \mathbb{N}$, we thus conclude that statement $\mathcal{R}_5$ holds. Hence, $\mathcal{R}_1$ holds as well (since we have already observed that $\mathcal{R}_1$ is equivalent to $\mathcal{R}_5$). The implication $\mathcal{R}_2 \Longrightarrow \mathcal{R}_1$ is thus proved.

Combining the implications $\mathcal{R}_1 \Longrightarrow \mathcal{R}_2$ and $\mathcal{R}_2 \Longrightarrow \mathcal{R}_1$ yields the equivalence $\mathcal{R}_1 \Longleftrightarrow \mathcal{R}_2$. This completes our proof. □

Remark 8.4. Any word $w \in \mathcal{M}$ satisfies $B_w = B_{Uw} = B_{wD}$. That is, the Ferrers board B_w of a word $w \in \mathcal{M}$ does not change if we insert a U at the beginning of w or a D at the end of w. Thus, we can use Theorem 8.3 to tell whether two Ferrers boards are rook-equivalent: Namely, we write the two Ferrers boards as B_u and B_v, where u and v are two words with the same number of U's and the same number of D's (this can be ensured by inserting an appropriate number of U's at the beginning and an appropriate number of D's at the end of either word), and then Theorem 8.3 provides us several equivalent criteria for the rook-equivalence of B_u and B_v.

9. Balanced commutations revisited: irreducible balanced words

In our definition of balanced commutations (which underlay the definition of the equivalence relation $\overset{\text{bal}}{\sim}$), we allowed two arbitrary balanced factors of our word to trade places, as long as they were adjacent in the word. Now, one may wonder whether we can get by with a smaller set of allowed swaps: Is there a more restrictive subset of balanced commutations that generates the same equivalence relation $\overset{\text{bal}}{\sim}$?

The answer is "yes", and in fact there are likely several reasonable choices. We here present one, which is not minimal but still far more parsimonious than the set of all balanced commutations.

To define it, we begin with a simple notion: A balanced word w is said to be *irreducible* if it is nonempty and cannot be written as a concatenation $w = uv$ of two nonempty balanced words u and v. For instance, the balanced word $DDUDUU$ is irreducible, whereas the balanced word $DUUUDD$ is not (since it is the concatenation $DU \cdot UUDD$). In terms of diagonal paths, this notion can be restated as follows: Given a nontrivial diagonal path $\mathbf{p}$ with initial height 0 and final height 0, its reading word $\mathrm{w}(\mathbf{p})$ is irreducible if and only if $\mathbf{p}$ intersects the x-axis only in its first and last vertices.

Given two words $v, w \in \mathcal{M}$, we say that v is obtained from w by an *irreducible balanced commutation* if and only if we can write v and w as $v = pxyq$ and $w = pyxq$, where $p, q \in \mathcal{M}$ are two arbitrary words and where $x, y \in \mathcal{M}$ are two irreducible balanced words with different first letters. Clearly, this condition implies that v is obtained from w by a balanced commutation, but the converse is not true. For example, the word $UUDDDUUD$ is obtained from $UDDUUUDD$ by an irreducible balanced commutation (swapping the $DDUU$ with the UD in the middle), but the word $UUDDUDDU$ is not (indeed, it is obtained by swapping the balanced factors $UDDU$ and $UUDD$, but these factors don't have different first letters, and the first of them is not irreducible).

We define an equivalence relation $\overset{\text{irr}}{\sim}$ on $\mathcal{M}$ by stipulating that two words $w, v \in$

$\mathcal{M}$ satisfy $w \overset{\text{irr}}{\sim} v$ if and only if v can be obtained from w by a sequence (possibly empty) of irreducible balanced commutations. Even though not every balanced commutation is irreducible, we claim the following:

Theorem 9.1. The relations $\overset{\text{irr}}{\sim}$ and $\overset{\text{bal}}{\sim}$ are the same. That is, if a word v can be obtained from a word w by a sequence of balanced commutations, then we can also obtain v from w by a (possibly longer) sequence of irreducible balanced commutations.

The proof of this theorem needs a few lemmas. The first one is nearly obvious:

Lemma 9.2. Any balanced word $w \in \mathcal{M}$ can be decomposed into a product $w = v_1 v_2 \cdots v_k$ of irreducible balanced words $v_1, v_2, \ldots, v_k \in \mathcal{M}$. (If w is empty, we will have $k = 0$ here.)

Proof. This is shown in the same way as the existence of a factorization of a positive integer into primes:

Let $w \in \mathcal{M}$ be a balanced word. If w is irreducible or empty, then we are done. If not, then w can be written as a product of two shorter nonempty balanced words. If these two shorter words are irreducible, then we are done. If not, then at least one of them can itself be written as a product of two shorter nonempty balanced words, so that w becomes a product of three nonempty balanced words. Thus, we obtain longer and longer factorizations of w into shorter and shorter nonempty balanced words. Obviously, this process will eventually have to stop, and at that point we will have a factorization of w into irreducible balanced words in front of us. □

The decomposition in Lemma 9.2 is furthermore unique (and this is easy to see using diagonal paths), but we do not need this.

Next we introduce some shorthand terminology: A *UIB word* will mean an irreducible balanced word that begins with a U. A *DIB word* will mean an irreducible balanced word that begins with a D. Note that any irreducible balanced word is nonempty, and thus is either UIB or DIB (but not both). The following is another easy observation:

Lemma 9.3. (a) Any UIB word ends with a D.

(b) Any DIB word ends with a U.

(c) If $u \in \mathcal{M}$ is a UIB word, then $\omega(u)$ is a UIB word as well.

(d) If $u \in \mathcal{M}$ is a DIB word, then $\omega(u)$ is a DIB word as well.

Proof. **(a)** Let w be a UIB word. We need to prove that w ends with a D.

Assume the contrary. Thus, w ends with a U. But w also starts with a U (since w is UIB) and is balanced (for the same reason). Hence, Lemma 4.13 shows that we can write w as a concatenation $w = pq$, where p is a balanced word starting with a U, and where q is a balanced word starting with a D. Consider these p and q. Thus, both p and q are nonempty balanced words. Hence, $w = pq$ shows that w is not irreducible. But this contradicts our assumption that w be UIB. This contradiction shows that our assumption was false. Lemma 9.3 **(a)** is thus proved.

(b) This is analogous to part **(a)**; we just need to interchange the roles of U and D.

(c) Let $u \in \mathcal{M}$ be a UIB word. Then, u ends with a D (by part **(a)**). Hence, $\omega(u)$ starts with a U (since the anti-automorphism ω reverses a word and replaces each D by a U and each U by a D). Moreover, u is irreducible balanced (since u is UIB). Therefore, $\omega(u)$ is irreducible balanced as well (indeed, the irreducibility follows from the fact that any factorization of $\omega(u)$ into two nonempty balanced factors could be turned back into a factorization of u by applying ω again[14]). Hence, $\omega(u)$ is a UIB word (since $\omega(u)$ starts with a U). This proves Lemma 9.3 **(c)**.

(d) This is analogous to part **(c)**; we just need to interchange the roles of U and D. □

It is not hard to see that each UIB word has the form UsD where s is a balanced 1-Dyck word (i.e., a balanced word whose each prefix is rising). Likewise, each DIB word has the form DtU where t is a balanced anti-1-Dyck word (i.e., a balanced word whose each prefix is falling). Obviously, UIB words can be distinguished from DIB words by their first letter.

Now we claim the following variant of Lemma 4.13:

Lemma 9.4. Let $w \in \mathcal{M}$ be a balanced word that starts with a U and ends with a U. Then, we can write w as a concatenation $w = spqt$, where $s, p, q, t \in \mathcal{M}$ are four balanced words such that p is UIB and q is DIB.

Proof. Lemma 4.13 shows that we can write w as a concatenation $w = p'q'$, where p' is a balanced word starting with a U, and where q' is a balanced word starting with a D. Consider these p' and q'.

Lemma 9.2 shows that p' can be decomposed into a product $p' = p_1 p_2 \cdots p_k$ of irreducible balanced words $p_1, p_2, \ldots, p_k \in \mathcal{M}$. Likewise, q' can be decomposed into a product $q' = p_{k+1} p_{k+2} \cdots p_\ell$ of irreducible balanced words $p_{k+1}, p_{k+2}, \ldots, p_\ell \in \mathcal{M}$. Consider these decompositions. Thus, $p_1, p_2, \ldots, p_\ell$ are ℓ irreducible balanced words such that $p' = p_1 p_2 \cdots p_k$ and $q' = p_{k+1} p_{k+2} \cdots p_\ell$. Hence,

$$p'q' = (p_1 p_2 \cdots p_k)(p_{k+1} p_{k+2} \cdots p_\ell) = p_1 p_2 \cdots p_\ell.$$

[14] In more detail: Assume that $\omega(u)$ is not irreducible. Thus, $\omega(u)$ can be factored as $\omega(u) = pq$ for two nonempty balanced words p and q. Consider these p and q. Then, $\omega(p)$ and $\omega(q)$ are nonempty balanced words (since p and q are nonempty balanced words). However, from $\omega \circ \omega = \mathrm{id}$, we obtain $\omega(\omega(u)) = u$. Thus, $u = \omega\left(\underbrace{\omega(u)}_{=pq}\right) = \omega(pq) = \omega(q)\,\omega(p)$. Thus, u is the product of two nonempty balanced words (namely, $\omega(q)$ and $\omega(p)$). But this contradicts the irreducibility of u. This contradiction shows that our assumption was false. Hence, $\omega(u)$ is irreducible.

The word p_1 is nonempty (since it is irreducible) and is a prefix of p' (since $p' = p_1 p_2 \cdots p_k$). Thus, it starts with a U (since p' starts with a U). Similarly, p_{k+1} starts with a D.

Now, consider the **smallest** number $i \in \{1, 2, \ldots, \ell\}$ for which the word p_i starts with a D. (Such an i exists, since p_{k+1} starts with a D.) Then, i cannot be 1 (since p_1 starts with a U, not with a D), and thus must be ≥ 2. Hence, $i - 1 \in \{1, 2, \ldots, \ell\}$. The word p_{i-1} cannot start with a D (because then, i would not be the **smallest** number for which p_i starts with a D), and thus must start with a U. Hence, the word p_{i-1} is UIB (since it is irreducible balanced). Meanwhile, the word p_i is DIB (since it is irreducible balanced and starts with a D). All the words $p_1, p_2, \ldots, p_\ell$ are balanced; hence, their concatenations $p_1 p_2 \cdots p_{i-2}$ and $p_{i+1} p_{i+2} \cdots p_\ell$ are balanced as well (since any concatenation of balanced words is balanced).

Now,

$$w = p'q' = p_1 p_2 \cdots p_\ell = \underbrace{(p_1 p_2 \cdots p_{i-2})}_{\text{balanced word}} \underbrace{p_{i-1}}_{\text{UIB word}} \underbrace{p_i}_{\text{DIB word}} \underbrace{(p_{i+1} p_{i+2} \cdots p_\ell)}_{\text{balanced word}}.$$

Hence, we can write w as a concatenation $w = spqt$, where $s, p, q, t \in \mathcal{M}$ are four balanced words such that p is UIB and q is DIB (namely, we take $s = p_1 p_2 \cdots p_{i-2}$ and $p = p_{i-1}$ and $q = p_i$ and $t = p_{i+1} p_{i+2} \cdots p_\ell$). This proves Lemma 9.4. □

This, in turn, allows us to improve Lemma 4.14 as follows:

Lemma 9.5. Let $w \in \mathcal{M}$ be a rising word that is not up-normal. Then, we can write w in the form $w = upqv$, where u and v are two words, where p is a UIB word, and where q is a DIB word.

Proof. Write w as $w = w_1 w_2 \cdots w_\ell$, where $w_1, w_2, \ldots, w_\ell \in \{U, D\}$ are the letters of w. In the proof of Lemma 4.14, we have found a balanced factor $w_{a+1} w_{a+2} \cdots w_b$ of w that starts with a U and ends with a U. Consider this balanced factor. Lemma 9.4 (applied to $w_{a+1} w_{a+2} \cdots w_b$ instead of w) then shows that we can write $w_{a+1} w_{a+2} \cdots w_b$ as a concatenation $w_{a+1} w_{a+2} \cdots w_b = spqt$, where $s, p, q, t \in \mathcal{M}$ are four balanced words such that p is UIB and q is DIB. Consider these four words s, p, q, t.

Now,

$$\begin{aligned} w = w_1 w_2 \cdots w_\ell &= (w_1 w_2 \cdots w_a) \underbrace{(w_{a+1} w_{a+2} \cdots w_b)}_{= spqt} (w_{b+1} w_{b+2} \cdots w_\ell) \\ &= (w_1 w_2 \cdots w_a)\, spqt\, (w_{b+1} w_{b+2} \cdots w_\ell). \end{aligned}$$

Hence, we can write w in the form $w = upqv$, where u and v are two words, where p is a UIB word, and where q is a DIB word (namely, we set $u = (w_1 w_2 \cdots w_a)\, s$ and $p = p$ and $q = q$ and $v = t\, (w_{b+1} w_{b+2} \cdots w_\ell)$). This proves Lemma 9.5. □

Next, we prove an analogue of Proposition 4.4:

Lemma 9.6. Let $w \in \mathcal{M}$ be a rising word. Then, there exists a unique up-normal word $t \in \mathcal{M}$ such that $t \overset{\text{irr}}{\sim} w$.

Proof. The existence of t can be proved in the same way as the existence part of Proposition 4.4, but using Lemma 9.5 instead of Lemma 4.14 (since UIB words start with a U, while DIB words start with a D).

The uniqueness of t follows from the uniqueness of t in Proposition 4.4, since the relation $t \overset{\text{irr}}{\sim} w$ implies $t \overset{\text{bal}}{\sim} w$. □

The following is an analogue of Proposition 4.3:

Proposition 9.7. Let u and v be two words in $\mathcal{M}$. Then, $u \overset{\text{irr}}{\sim} v$ if and only if $\omega(u) \overset{\text{irr}}{\sim} \omega(v)$.

Proof. This is similar to the proof of Proposition 4.3, with a slight twist: We need to show that if a and b are two irreducible balanced words that have different first letters, then their images $\omega(a)$ and $\omega(b)$ are again two irreducible balanced words that have different first letters. But this follows from parts **(c)** and **(d)** of Lemma 9.3. □

We have an analogue of Lemma 5.1 as well:

Lemma 9.8. Let $\mathbf{p}$ and $\mathbf{q}$ be two diagonal paths with the same initial height and the same final height. Assume that $H(\mathbf{p}, z) = H(\mathbf{q}, z)$. Then, $\mathrm{w}(\mathbf{p}) \overset{\text{irr}}{\sim} \mathrm{w}(\mathbf{q})$.

Proof. Analogous to the proof of Lemma 5.1, but using Proposition 9.7 and Lemma 9.6 instead of Proposition 4.3 and Proposition 4.4. (Use the obvious fact that $u \overset{\text{irr}}{\sim} v$ implies $u \overset{\text{bal}}{\sim} v$.) □

We can now prove Theorem 9.1:

Proof of Theorem 9.1. Let us modify Theorem 2.1 by adding the following extra statement:

- $\mathcal{S}'_5$: We have $u \overset{\text{irr}}{\sim} v$.

Clearly, this statement $\mathcal{S}'_5$ implies $\mathcal{S}_5$, since $u \overset{\text{irr}}{\sim} v$ implies $u \overset{\text{bal}}{\sim} v$. But the implication $\mathcal{S}_4 \Longrightarrow \mathcal{S}'_5$ holds as well, and can be proved just as the implication $\mathcal{S}_4 \Longrightarrow \mathcal{S}_5$ in Theorem 2.1 was proved (but using Lemma 9.8 instead of Lemma 5.1). Hence, the statement $\mathcal{S}'_5$ is equivalent to all the seven statements $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{S}'_3, \mathcal{S}_4, \mathcal{S}_5, \mathcal{S}_6$ from Theorem 2.1. In particular, $\mathcal{S}'_5$ is equivalent to $\mathcal{S}_5$. In other words, $u \overset{\text{irr}}{\sim} v$ is equivalent to $u \overset{\text{bal}}{\sim} v$. In other words, the relations $\overset{\text{irr}}{\sim}$ and $\overset{\text{bal}}{\sim}$ are the same. This proves Theorem 9.1. □

10. Other algebras

Everything we have done so far concerned the "rank-1" Weyl algebra

$$\mathcal{W} = \mathbf{k} \langle D, U \mid DU - UD = 1 \rangle .$$

But the main question we addressed – to classify equal products of generators – can be posed for any $\mathbf{k}$-algebra given by generators and relations. In particular, several analogues and variants of $\mathcal{W}$ are natural candidates for a similar study. In this section, we briefly discuss some of them, giving some answers and posing some questions. (There are many more – see, e.g., [Gaddis23] for a recent survey.)

10.1. Multivariate Weyl algebras

For any $n \in \mathbb{N}$, there is an "n-Weyl algebra" $\mathcal{W}_n$, defined as the $\mathbf{k}$-algebra given by $2n$ generators $D_1, D_2, \ldots, D_n, U_1, U_2, \ldots, U_n$ and relations

$$\begin{aligned} D_i U_j &= U_j D_i && \text{for all } i \neq j; \\ D_i U_i &= U_i D_i + 1 && \text{for all } i; \\ D_i D_j &= D_j D_i && \text{for all } i, j; \\ U_i U_j &= U_j U_i && \text{for all } i, j. \end{aligned}$$

It is isomorphic to the $\mathbf{k}$-algebra of differential operators on the polynomial ring $\mathbf{k}[x_1, x_2, \ldots, x_n]$.

However, this algebra $\mathcal{W}_n$ can also be seen as the n-fold tensor power[15] $\mathcal{W}^{\otimes n}$ of the original Weyl algebra $\mathcal{W}$, via the $\mathbf{k}$-algebra isomorphism $\mathcal{W}_n \to \mathcal{W}^{\otimes n}$ that sends each generator D_i to $\underbrace{1 \otimes 1 \otimes \cdots \otimes 1}_{i-1 \text{ times}} \otimes D \otimes \underbrace{1 \otimes 1 \otimes \cdots \otimes 1}_{n-i \text{ times}}$ and each generator U_i to $\underbrace{1 \otimes 1 \otimes \cdots \otimes 1}_{i-1 \text{ times}} \otimes U \otimes \underbrace{1 \otimes 1 \otimes \cdots \otimes 1}_{n-i \text{ times}}$. From this point of view, products of generators of $\mathcal{W}_n$ are just elements of the form $\phi(w_1) \otimes \phi(w_2) \otimes \cdots \otimes \phi(w_n) \in \mathcal{W}^{\otimes n}$, where $w_1, w_2, \ldots, w_n \in \mathcal{M}$ are some words. Which of these products are equal? The answer turns out to boil down to the answer for $n = 1$ (which we know from Theorems 2.1, 8.2 and 8.3):

Theorem 10.1. Let $u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n$ be $2n$ words in $\mathcal{M}$. Then,

$$\phi(u_1) \otimes \phi(u_2) \otimes \cdots \otimes \phi(u_n) = \phi(v_1) \otimes \phi(v_2) \otimes \cdots \otimes \phi(v_n) \text{ in } \mathcal{W}^{\otimes n}$$

if and only if

$$\text{each } i \in \{1, 2, \ldots, n\} \text{ satisfies } \phi(u_i) = \phi(v_i).$$

[15] All tensor products and tensor powers in this paper are taken over the field $\mathbf{k}$.

In other words, we don't get any "new" equalities by tensoring n copies of $\mathcal{W}$.

The "if" part of Theorem 10.1 is obvious, while the "only if" part of the theorem follows immediately from two lemmas. The first is a general fact from linear algebra ([Conrad24, Theorem 5.15]):

Lemma 10.2. Let $V_1, V_2, \ldots, V_n$ be any $\mathbf{k}$-vector spaces. For each $i \in \{1, 2, \ldots, n\}$, let x_i and y_i be two nonzero vectors in V_i. Then,

$$x_1 \otimes x_2 \otimes \cdots \otimes x_n = y_1 \otimes y_2 \otimes \cdots \otimes y_n \text{ in } V_1 \otimes V_2 \otimes \cdots \otimes V_n$$

if and only if there exist some scalars $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbf{k}$ such that $\lambda_1 \lambda_2 \cdots \lambda_n = 1$ and such that

$$\text{each } i \in \{1, 2, \ldots, n\} \text{ satisfies } x_i = \lambda_i y_i.$$

The next lemma ensures that the conditions of Lemma 10.2 are met in the appropriate case:

Lemma 10.3. (a) For any $w \in \mathcal{M}$, we have $\phi(w) \neq 0$.

(b) If $u, v \in \mathcal{M}$ and $\lambda \in \mathbf{k}$ satisfy $\phi(u) = \lambda \phi(v)$, then $\lambda = 1$ and $\phi(u) = \phi(v)$.

First proof. The family $\left(U^j D^i\right)_{i,j \in \mathbb{N}}$ is a basis of $\mathcal{W}$ (by [EGHetc11, Proposition 2.7.1 (i)]). Hence, each element a of $\mathcal{W}$ can be uniquely written as a $\mathbf{k}$-linear combination $\sum\limits_{i,j \in \mathbb{N}} a_{i,j} U^j D^i$ of the elements of this family. When this a is nonzero, we define the *leading monomial* of a to be the lexicographically highest pair $(i, j) \in \mathbb{N}^2$ for which $a_{i,j} \neq 0$, and we define the *leading coefficient* of a to be the coefficient $a_{i,j}$ corresponding to this pair (i, j).

(a) Let $w \in \mathcal{M}$. Then, Theorem 8.1 yields

$$\phi(w) = \sum_{k=0}^{\min\{m,n\}} r_k(B_w)\, U^{m-k} D^{n-k} \tag{47}$$

for appropriate $n, m \in \mathbb{N}$. The elements $U^{m-k} D^{n-k}$ on the right hand side of this equality are $\mathbf{k}$-linearly independent (since the family $\left(U^j D^i\right)_{i,j \in \mathbb{N}}$ is a basis of $\mathcal{W}$), and at least one of the coefficients $r_k(B_w)$ is nonzero (indeed, we have $r_0(B_w) = 1$, since any board B satisfies $r_0(B) = 1$). Thus, the entire right hand side is nonzero. Hence, $\phi(w) \neq 0$. This proves Lemma 10.3 **(a)**.

(b) Let $u, v \in \mathcal{M}$ and $\lambda \in \mathbf{k}$ satisfy $\phi(u) = \lambda \phi(v)$. Recall that for each $w \in \mathcal{M}$, the element $\phi(w) \in \mathcal{W}$ is nonzero (by part **(a)**) and has leading coefficient 1 (indeed, the equality (47) shows that the leading coefficient of $\phi(w)$ is $r_0(B_w) = 1$). Hence, the element $\phi(u)$ has leading coefficient 1. Similarly, $\phi(v)$ also has leading coefficient 1. Thus, $\lambda \phi(v)$ has leading coefficient $\lambda \cdot 1 = \lambda$. In other words, $\phi(u)$ has leading coefficient λ (since $\phi(u) = \lambda \phi(v)$). Since we also know that $\phi(u)$ has leading coefficient 1, we thus conclude that $\lambda = 1$. Hence, $\phi(u) = \underbrace{\lambda}_{=1} \phi(v) = \phi(v)$. Lemma 10.3 **(b)** is thus proved. □

Second proof. **(b)** First, we generalize Proposition 3.6 by replacing the assumption "$\phi(\mathrm{w}(\mathbf{p})) = \phi(\mathrm{w}(\mathbf{q}))$" by "$\phi(\mathrm{w}(\mathbf{p})) = \lambda\phi(\mathrm{w}(\mathbf{q}))$" and adding the claim "$\lambda = 1$" to the conclusion. The proof of Proposition 3.6 that we gave above still applies to this generalization, once a few trivial changes are made. In particular, the polynomial identity

$$\prod_{p_i \text{ is an NE-step of } \mathbf{p}} (x - h_i) = \prod_{q_i \text{ is an NE-step of } \mathbf{q}} (x - g_i)$$

must be replaced by

$$\prod_{p_i \text{ is an NE-step of } \mathbf{p}} (x - h_i) = \lambda \prod_{q_i \text{ is an NE-step of } \mathbf{q}} (x - g_i),$$

which of course entails not only

$$\begin{aligned} &\{h_i \mid p_i \text{ is an NE-step of } \mathbf{p}\}_{\text{multiset}} \\ &= \{g_i \mid q_i \text{ is an NE-step of } \mathbf{q}\}_{\text{multiset}} \end{aligned}$$

but also $\lambda = 1$ (by comparing leading coefficients).

Now, let us apply this to our situation. Let $u, v \in \mathcal{M}$ and $\lambda \in \mathbf{k}$ satisfy $\phi(u) = \lambda\phi(v)$. Let $\mathbf{p} = (p_0, p_1, \ldots, p_k)$ be the diagonal path starting at $(0, 0)$ that satisfies $u = \mathrm{w}(\mathbf{p})$. Similarly, let $\mathbf{q} = (q_0, q_1, \ldots, q_m)$ be the diagonal path starting at $(0, 0)$ that satisfies $v = \mathrm{w}(\mathbf{q})$. The paths $\mathbf{p}$ and $\mathbf{q}$ have the same initial height (namely, 0). Moreover, we have $\phi(u) = \lambda\phi(v)$, so that $\phi(\mathrm{w}(\mathbf{p})) = \lambda\phi(\mathrm{w}(\mathbf{q}))$ (since $u = \mathrm{w}(\mathbf{p})$ and $v = \mathrm{w}(\mathbf{q})$). Hence, the generalized version of Proposition 3.6 that we have just proposed yields (among other things) that $\lambda = 1$. Thus, $\phi(u) = \underbrace{\lambda}_{=1} \phi(v) = \phi(v)$. This proves Lemma 10.3 **(b)**.

(a) Let $w \in \mathcal{M}$. We must prove that $\phi(w) \neq 0$. Assume the contrary. Thus, $\phi(w) = 0 = 0\phi(w)$. Hence, part **(b)** (applied to $u = w$ and $v = w$ and $\lambda = 0$) yields $0 = 1$ and $\phi(w) = \phi(w)$. Obviously, $0 = 1$ is absurd, so we found a contradiction. This proves Lemma 10.3 **(a)**. □

Proof of Theorem 10.1. The "if" part is obvious, so let us prove the "only if" part.

Lemma 10.3 **(a)** shows that the vectors $\phi(u_i)$ and $\phi(v_i)$ in $\mathcal{W}$ are nonzero for all $i \in \{1, 2, \ldots, n\}$.

Assume that

$$\phi(u_1) \otimes \phi(u_2) \otimes \cdots \otimes \phi(u_n) = \phi(v_1) \otimes \phi(v_2) \otimes \cdots \otimes \phi(v_n) \text{ in } \mathcal{W}^{\otimes n}.$$

Then, Lemma 10.2 (applied to $V_i = \mathcal{W}$ and $x_i = \phi(u_i)$ and $y_i = \phi(v_i)$) yields that there exist some scalars $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbf{k}$ such that $\lambda_1\lambda_2 \cdots \lambda_n = 1$ and such that

$$\text{each } i \in \{1, 2, \ldots, n\} \text{ satisfies } \phi(u_i) = \lambda_i\phi(v_i).$$

Consider these $\lambda_1, \lambda_2, \ldots, \lambda_n$. For each $i \in \{1, 2, \ldots, n\}$, we have $\phi(u_i) = \lambda_i \phi(v_i)$ and therefore $\phi(u_i) = \phi(v_i)$ (by Lemma 10.3 **(b)**, applied to $u = u_i$ and $v = v_i$ and $\lambda = \lambda_i$). In other words,

$$\text{each } i \in \{1, 2, \ldots, n\} \text{ satisfies } \phi(u_i) = \phi(v_i).$$

This proves the "only if" part of Theorem 10.1. □

10.2. Characteristic p

We have hitherto assumed that the field $\mathbf{k}$ has characteristic 0. If $\mathbf{k}$ has characteristic $p \neq 0$ instead, things change significantly: Purely identity-type results such as Proposition 3.3, Proposition 3.5 and Lemma 3.8 remain valid (indeed, they hold whenever $\mathbf{k}$ is merely a commutative ring), but the action of $\mathcal{W}$ on $\mathbf{k}[x]$ is no longer faithful (i.e., Lemma 3.1 fails), and various other results that build on the tacit identification of integers with elements of $\mathbf{k}$ become false as well (e.g., Proposition 3.6). Lemma 3.9 remains true, but the first proof we gave above no longer works (although it is not hard to derive it from the characteristic-0 case, since it is an identity in the free $\mathbf{k}$-module $\mathcal{W}$). Thus, Lemma 3.10 remains true as well.

The proof of Proposition 3.7 also falls flat in characteristic p, but the proposition itself survives. Indeed, it holds for any nontrivial ring $\mathbf{k}$, and can be proved by comparing leading terms in Theorem 8.1.

The most interesting question is when two words $u, v \in \mathcal{M}$ satisfy $\phi(u) = \phi(v)$ for a field $\mathbf{k}$ of characteristic $p \neq 0$. The equivalence $\mathcal{S}_1 \iff \mathcal{S}_5$ in Theorem 2.1 no longer holds in this case, as (e.g.) we have $\phi\left(U^{p+1}D\right) = \phi(UDU^p)$ (indeed, U^p is a central element of $\mathcal{W}$ when $\operatorname{char} \mathbf{k} = p$) but we don't have $U^{p+1}D \overset{\text{bal}}{\sim} UDU^p$. One might try to salvage the equivalence by loosening the notion of balanced commutations, e.g., by allowing both U^p and D^p to be swapped with any (neighboring) factor of the word; the resulting equivalence relation $\overset{p-\text{bal}}{\sim}$ might satisfy the equivalence $(\phi(u) = \phi(v)) \iff \left(u \overset{p-\text{bal}}{\sim} v\right)$, but we don't know if it does.

Question 10.4. Does it?

The Weyl algebra $\mathcal{W}$ in characteristic $p \neq 0$ has some quotients (unlike in characteristic 0, where it is famously a simple algebra). Indeed, both elements U^p and D^p are known to lie in its center, and thus generate two-sided ideals $\mathcal{W}U^p\mathcal{W}$ and $\mathcal{W}D^p\mathcal{W}$ that can be quotiented out. We can thus form the three quotients

$$\mathcal{W}^- := \mathcal{W} / (\mathcal{W}U^p\mathcal{W}), \qquad \mathcal{W}_- := \mathcal{W} / (\mathcal{W}D^p\mathcal{W}),$$
$$\mathcal{W}^-_- := \mathcal{W} / (\mathcal{W}U^p\mathcal{W} + \mathcal{W}D^p\mathcal{W}).$$

The third quotient, $\mathcal{W}^-_-$, is actually a finite-dimensional $\mathbf{k}$-vector space, of dimension p^2 and with basis $\left(U^j D^i\right)_{i,j \in \{0,1,\ldots,p-1\}}$. It acts faithfully and densely on the

k-algebra $\mathbf{k}[x] / (x^p)$. The quotient $\mathcal{W}_-$ acts faithfully on the full polynomial ring $\mathbf{k}[x]$. The quotients $\mathcal{W}_-$ and $\mathcal{W}^-$ are isomorphic via the isomorphism sending $U \mapsto D$ and $D \mapsto -U$.

The question of when two words $u, v \in \mathcal{M}$ give rise to equal monomials can now be asked not only for $\mathcal{W}$, but also for any of its quotient algebras $\mathcal{W}^-$, $\mathcal{W}_-$ and $\mathcal{W}_-^-$. For instance, the words $U^{p-1}DUD^{p-1}$ and $U^{p-1}D^{p-1}$ do not yield equal monomials in $\mathcal{W}$ (by Proposition 3.7), but yield equal monomials in each of the quotients $\mathcal{W}^-$, $\mathcal{W}_-$ and $\mathcal{W}_-^-$ (since their difference in $\mathcal{W}$ is $U^{p-1}DUD^{p-1} - U^{p-1}D^{p-1} = U^{p-1}\underbrace{(DU-1)}_{=UD}D^{p-1} = U^{p-1}UDDD^{p-1} = U^pD^p$, which becomes 0 in each of the quotients). What combinatorial condition is responsible for this equality? We don't know; there are neither any balanced commutations that can be applied to them to produce new words, nor any U^p or D^p factors that can be annihilated. Thus, we ask the following rather open-ended question (actually three questions in disguise):

Question 10.5. Characterize equalities between monomials in $\mathcal{W}^-$, $\mathcal{W}_-$ and $\mathcal{W}_-^-$ combinatorially.

10.3. Down-up algebras

We now return to the case when **k** is a field of characteristic 0.

There are several deformations and other variations of the Weyl algebra $\mathcal{W}$, and our question about equal monomials can be asked for each of them. We here discuss one of the most recent such variations: the *down-up algebra*, actually a family of algebras depending on three scalar parameters α, β, γ.

We fix three scalars $\alpha, \beta, \gamma \in \mathbf{k}$. The *down-up algebra* $\mathcal{A}(\alpha, \beta, \gamma)$ is defined to be the **k**-algebra with generators D and U and the two relations

$$D^2U = \alpha DUD + \beta UD^2 + \gamma D \qquad \text{and}$$
$$DU^2 = \alpha UDU + \beta U^2D + \gamma U.$$

Clearly, this algebra $\mathcal{A}(\alpha, \beta, \gamma)$ has an algebra anti-automorphism ω sending D and U to U and D. Also, it is easy to check that the above two relations of $\mathcal{A}(\alpha, \beta, \gamma)$ are satisfied in the Weyl algebra $\mathcal{W}$ whenever $\alpha + \beta = \gamma - \beta = 1$; therefore, the Weyl algebra $\mathcal{W}$ is a quotient of $\mathcal{A}(\alpha, \beta, \gamma)$ in this case. Down-up algebras originate in [BenRob98, Proposition 3.5] and have since found uses in noncommutative algebraic geometry and combinatorics.

We can define a map $\phi : \mathcal{M} \to \mathcal{A}(\alpha, \beta, \gamma)$ in the same way as we defined $\phi : \mathcal{M} \to \mathcal{W}$, but using the down-up algebra $\mathcal{A}(\alpha, \beta, \gamma)$ instead of $\mathcal{W}$. (Thus, ϕ is a morphism of multiplicative monoids and sends D and U to D and U.) Surprisingly, we have:

Theorem 10.6. Assume that $\alpha + \beta = \gamma - \beta = 1$. Then, the equivalence of the six statements $\mathcal{S}_1$, $\mathcal{S}_3$, $\mathcal{S}_3'$, $\mathcal{S}_4$, $\mathcal{S}_5$ and $\mathcal{S}_6$ in Theorem 2.1 still holds if we replace $\mathcal{W}$ by $\mathcal{A}(\alpha, \beta, \gamma)$.

Proof. The statements $\mathcal{S}_3$, $\mathcal{S}_3'$, $\mathcal{S}_4$, $\mathcal{S}_5$ and $\mathcal{S}_6$ are unchanged from Theorem 2.1, so they are still equivalent. It thus remains to prove that $\mathcal{S}_1 \iff \mathcal{S}_5$.

$\mathcal{S}_5 \implies \mathcal{S}_1$: This relies on a version of Lemma 3.10 for the algebra $\mathcal{A}(\alpha, \beta, \gamma)$ instead of $\mathcal{W}$. This version, in turn, relies on a version of Lemma 3.9 for the algebra $\mathcal{A}(\alpha, \beta, \gamma)$ instead of $\mathcal{W}$. But the latter has already been established in [BenRob98]. In fact, the **k**-algebra $\mathcal{A}(\alpha, \beta, \gamma)$ is graded (just like $\mathcal{W}$: the generators U and D are homogeneous of degrees 1 and -1). Its 0-th graded component is commutative, by [BenRob98, Proposition 3.5]. As in the second proof of Lemma 3.9, this entails the validity of Lemma 3.9 for $\mathcal{A}(\alpha, \beta, \gamma)$, and this in turn yields that Lemma 3.10 holds for $\mathcal{A}(\alpha, \beta, \gamma)$ as well. Hence, $\mathcal{S}_5$ implies $\mathcal{S}_1$.

$\mathcal{S}_1 \implies \mathcal{S}_5$: The condition $\alpha + \beta = \gamma - \beta = 1$ ensures that the Weyl algebra $\mathcal{W}$ is a quotient of $\mathcal{A}(\alpha, \beta, \gamma)$ (with its generators D and U being the projections of the generators D and U). Thus, $\phi(u) = \phi(v)$ in $\mathcal{A}(\alpha, \beta, \gamma)$ implies $\phi(u) = \phi(v)$ in $\mathcal{W}$. In other words, our new statement $\mathcal{S}_1$ implies the old statement $\mathcal{S}_1$ from Theorem 2.1. But that old statement implies $\mathcal{S}_5$, as we already know. Hence, our new $\mathcal{S}_1$ also implies $\mathcal{S}_5$. □

Example 10.7. One of the situations to which Theorem 10.6 applies is the case when $\alpha = 2$ and $\beta = -1$ and $\gamma = 0$. In this case, the down-up algebra $\mathcal{A}(\alpha, \beta, \gamma) = \mathcal{A}(2, -1, 0)$ is the *homogenized Weyl algebra*, since its two defining relations

$$D^2U = 2DUD - UD^2 \qquad \text{and} \qquad DU^2 = 2UDU - U^2D$$

can be rewritten (in terms of commutators $[x, y] := xy - yx$) as

$$[D, \ DU - UD] = 0 \qquad \text{and} \qquad [U, \ DU - UD] = 0$$

and thus mean that the commutator $DU - UD$ is a central element. This algebra appears in [BDSHP10, §2.3] under the name of $\mathcal{U}(\mathcal{L}_{\mathcal{H}})$, with the generators a, $a^\dagger$ and e corresponding to our D, U and $DU - UD$.

Question 10.8. To what extent can the condition $\alpha + \beta = \gamma - \beta = 1$ be lifted in Theorem 10.6?

This condition is unnecessary for $\mathcal{S}_5 \implies \mathcal{S}_1$, but needed for $\mathcal{S}_1 \implies \mathcal{S}_5$. For example, $\mathcal{S}_1 \implies \mathcal{S}_5$ would fail in the following five cases:

1. the case $(\alpha, \beta, \gamma) = (0, 1, 0)$ (here, we have $\phi(DU^2) = \phi(U^2D)$);
2. more generally, the case $\alpha = \gamma = 0$ and arbitrary β (here we have $\phi(DU^4D) = \phi(U^2D^2U^2)$);

3. the case $(\alpha, \beta) = (0, -1)$ and arbitrary γ (here we have $\phi\left(DU^4\right) = \phi\left(U^4 D\right)$);
4. the case $(\alpha, \beta) = (-1, -1)$ and arbitrary γ (here we have $\phi\left(DU^3\right) = \phi\left(U^3 D\right)$);
5. the case $(\alpha, \beta) = (1, -1)$ and arbitrary γ (here we have $\phi\left(DU^6\right) = \phi\left(U^6 D\right)$).

However, such failures seem to be the exception, not the rule. The last three are explained by [Zhao99, Theorem 1.3 (f)], and correspond to the only roots of unity that are quadratic over $\mathbb{Q}$. There are more exceptions with irrational α, β, γ, for instance $(\alpha, \beta) = (0, i)$ with $i = \sqrt{-1}$, which satisfies $\phi\left(DU^8\right) = \phi\left(U^8 D\right)$.

With some additional work, we can adapt the above proof of Theorem 10.6 to replace the condition "$\alpha + \beta = \gamma - \beta = 1$" by "$\alpha + \beta = 1$ and $(\gamma = 0) \iff (\beta = -1)$". Indeed, under this condition, we can find a nonzero scalar $\zeta \in \mathbf{k}$ such that $\gamma = (1 + \beta)\zeta$. Then, there is a $\mathbf{k}$-algebra morphism from $\mathcal{A}(\alpha, \beta, \gamma)$ to $\mathcal{W}$ that sends D and U to D and ζU. This morphism sends equal monomials in $\mathcal{A}(\alpha, \beta, \gamma)$ to proportional monomials in $\mathcal{W}$; but Lemma 10.3 **(b)** says that proportional monomials in $\mathcal{W}$ must actually be equal, and so we can argue as in the proof of Theorem 10.6. However, the condition $\alpha + \beta = 1$ cannot be lifted in this way. Instead, we suspect that the linear-recurrence highest weight modules of [BenRob98, Proposition 2.2] should be used in the general case in lieu of $\mathbf{k}[x]$ (certainly, this would explain the above exceptions as coming from the periodic linearly recurrent sequences).

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