Noncommutative birational rowmotion on a rectangle

A case study in noncommutative dynamics

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Massachusetts Institute of Technology


paper: https://arxiv.org/abs/2208.11156
A poset (= partially ordered set) is a set $P$ with a reflexive, transitive and antisymmetric relation.

We use the symbols $<, \leq, >$ and $\geq$ accordingly.

We draw posets as Hasse diagrams:

We only care about finite posets here.

We say that $u \in P$ is covered by $v \in P$ (written $u \lessdot v$) if we have $u < v$ and there is no $w \in P$ satisfying $u < w < v$.

We say that $u \in P$ covers $v \in P$ (written $u \gtrdot v$) if we have $u > v$ and there is no $w \in P$ satisfying $u > w > v$. 
Let $P$ be a finite poset. We define $\hat{P}$ to be the poset obtained by adjoining two new elements 0 and 1 to $P$ and forcing

- 0 to be less than every other element, and
- 1 to be greater than every other element.

**Example:**

\[
P = \begin{array}{c}
\delta \\
\alpha \\
\gamma \\
\beta
\end{array}
\quad \Rightarrow \quad \hat{P} = \begin{array}{c}
1 \\
\delta \\
\gamma \\
\alpha \\
\beta \\
0
\end{array}
\]
A **linear extension** of $P$ means a list $(v_1, v_2, \ldots, v_n)$ of all elements of $P$ (each only once) such that $i < j$ whenever $v_i < v_j$.

For instance,

![Diagram](image)

has two linear extensions $(\alpha, \beta, \gamma, \delta)$ and $(\beta, \alpha, \gamma, \delta)$.

Every finite poset has at least one linear extension.
More poset basics: order ideals

- An **order ideal** of a poset $P$ is a subset $S$ of $P$ such that if $v \in S$ and $w \leq v$, then $w \in S$.
- Examples (the elements of the order ideal are marked in red):

![Diagram of order ideals]

- We let $J(P)$ denote the set of all order ideals of $P$. 
Classical rowmotion is the rowmotion studied by Striker/Williams (arXiv:1108.1172). It has appeared many times before, under different guises:

- Brouwer/Schrijver (1974) (as a permutation of the antichains),
- Fon-der-Flaass (1993) (as a permutation of the antichains),
- Cameron/Fon-der-Flaass (1995) (as a permutation of the monotone Boolean functions),
- Panyushev (2008), Armstrong-Stump-Thomas (2011) (as a permutation of the antichains or “nonnesting partitions”, with relations to Lie theory).
Let $P$ be a finite poset. **Classical rowmotion** is the map $r : \mathcal{J}(P) \to \mathcal{J}(P)$ which sends every order ideal $S$ to a new order ideal $r(S)$ defined as follows:

- **Invert colors** (i.e., take the complement $P \setminus S$).
- **Boil down to generators** (i.e., take the set $M$ of minimal elements of this complement).
- **Complete downwards** (i.e., take the set $J$ of all $w \in P$ such that there exists an $m \in M$ such that $w \leq m$).

Then, $r(S) = J$.

**Example:**
Let $S$ be the following order ideal ($\bullet = \text{inside order ideal}$):

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```
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- **Invert colors** (i.e., take the complement $P \setminus S$).
- **Boil down to generators** (i.e., take the set $M$ of minimal elements of this complement).
- **Complete downwards** (i.e., take the set $J$ of all $w \in P$ such that there exists an $m \in M$ such that $w \leq m$).

Then, $r(S) = J$.

**Example:**
Mark the elements of the complement **blue**.
Let $P$ be a finite poset. **Classical rowmotion** is the map $r : J(P) \to J(P)$ which sends every order ideal $S$ to a new order ideal $r(S)$ defined as follows:

- **Invert colors** (i.e., take the complement $P \setminus S$).
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Then, $r(S) = J$.

**Example:**
Leave only the minimal elements:

![Diagram of a poset with minimal elements highlighted]
Let $P$ be a finite poset. **Classical rowmotion** is the map $r : J(P) \rightarrow J(P)$ which sends every order ideal $S$ to a new order ideal $r(S)$ defined as follows:

- **Invert colors** (i.e., take the complement $P \setminus S$).
- **Boil down to generators** (i.e., take the set $M$ of minimal elements of this complement).
- **Complete downwards** (i.e., take the set $J$ of all $w \in P$ such that there exists an $m \in M$ such that $w \leq m$).

Then, $r(S) = J$.

**Example:**

$r(S)$ is the order ideal generated by $M$ (“everything below $M$”):

![Diagram of a poset with order ideals and rowmotion operations]
Classical rowmotion is a permutation of $J(P)$, hence has finite order. This order can be fairly large.
Classical rowmotion is a permutation of $J(P)$, hence has finite order. This order can be fairly large. However, for some types of $P$, the order can be explicitly computed or bounded from above. See Striker/Williams (arXiv:1108.1172) for an exposition of known results.

- If $P$ is a $p \times q$-rectangle:

\[
\begin{array}{c}
(2,3) \\
(2,2) & (1,3) \\
(2,1) & (1,2) \\
(1,1)
\end{array}
\]

(shown here for $p = 2$ and $q = 3$), then $\text{ord}(r) = p + q$. 
Example:
Let $S$ be the order ideal of the $2 \times 3$-rectangle given by:
Example:
\( r(S) \) is

\[
\begin{array}{c}
(1, 1) \\
(2, 1) \\
(2, 2) \\
(2, 3)
\end{array}
\begin{array}{c}
(1, 2) \\
(1, 3) \\
(2, 2) \\
(2, 3)
\end{array}
\]
Example:

$r^2(S)$ is

```
(2,3)

(2,2)   (1,3)

(2,1)   (1,2)

(1,1)
```
Example:
$r^3(S)$ is

```
(2, 3)
/
(2, 2)        (1, 3)
/
(2, 1)        (1, 2)
/
(1, 1)
```
Example:
$r^4(S)$ is

```
(2, 3)
(2, 2)   (1, 3)
(2, 1)   (1, 2)
(1, 1)
```
Example:
\( r^5(S) \) is

\[
\begin{array}{c}
(2, 3) \\
(2, 2) & (1, 3) \\
(2, 1) & (1, 2) \\
(1, 1)
\end{array}
\]

which is precisely the \( S \) we started with.

\( \text{ord}(r) = p + q = 2 + 3 = 5. \)
Further posets for which classical rowmotion has small order:

- If $P$ is a $\Delta$-shaped triangle with sidelength $p - 1$:

  (shown here for $p = 4$), then $\text{ord}(r) = 2p$ (if $p > 2$).

- In this case, $r^p$ is “reflection in the $y$-axis” (i.e., the central vertical axis).
Further posets for which classical rowmotion has small order:

- If $P$ is a $\Delta$-shaped triangle with sidelength $p - 1$:

  (shown here for $p = 4$), then $\text{ord}(r) = 2p$ (if $p > 2$).

- In this case, $r^p$ is “reflection in the $y$-axis” (i.e., the central vertical axis).

Classical rowmotion: the toggling definition

There is an alternative definition of classical rowmotion, which splits it into many little steps.

- If $P$ is a poset and $v \in P$, then the $v$-toggle is the map $t_v : J(P) \to J(P)$ which takes every order ideal $S$ to:
  - $S \cup \{v\}$, if $v$ is not in $S$ but all elements of $P$ covered by $v$ are in $S$ already;
  - $S \setminus \{v\}$, if $v$ is in $S$ but none of the elements of $P$ covering $v$ is in $S$;
  - $S$ otherwise.

- Simpler way to state this: $t_v(S)$ is:
  - $S \triangle \{v\}$ (symmetric difference) if this is an order ideal;
  - $S$ otherwise.

("Try to add or remove $v$ from $S$; if this breaks the order ideal axiom, leave $S$ fixed.")
Classical rowmotion: the toggling definition

- Let \((v_1, v_2, \ldots, v_n)\) be a **linear extension** of \(P\); this means a list of all elements of \(P\) (each only once) such that \(i < j\) whenever \(v_i < v_j\).

- Cameron and Fon-der-Flaass showed that

\[
\mathbf{r} = t_{v_1} \circ t_{v_2} \circ \ldots \circ t_{v_n}.
\]

**Example:**
Start with this order ideal \(S\):

![Diagram](image)

Start with this order ideal \(S\):
Classical rowmotion: the toggling definition

- Let \((v_1, v_2, ..., v_n)\) be a **linear extension** of \(P\); this means a list of all elements of \(P\) (each only once) such that \(i < j\) whenever \(v_i < v_j\).
- Cameron and Fon-der-Flaass showed that

\[
\mathbf{r} = t_{v_1} \circ t_{v_2} \circ ... \circ t_{v_n}.
\]

**Example:**
First apply \(t_{(2,2)}\), which changes nothing:

\[
\begin{array}{ccc}
(2, 2) & & \\
\downarrow & & \downarrow \\
(2, 1) & & (1, 2) \\
\downarrow & & \downarrow \\
(1, 1) & & \\
\end{array}
\]
Classical rowmotion: the toggling definition

- Let \((v_1, v_2, \ldots, v_n)\) be a **linear extension** of \(P\); this means a list of all elements of \(P\) (each only once) such that \(i < j\) whenever \(v_i < v_j\).

- Cameron and Fon-der-Flaass showed that

\[
  r = t_{v_1} \circ t_{v_2} \circ \ldots \circ t_{v_n}.
\]

**Example:**
Then apply \(t_{(1,2)}\), which adds \((1, 2)\) to the order ideal:

```
(2, 2)
```
```
(2, 1)   (1, 2)
```
```
(1, 1)
```
Let \((\nu_1, \nu_2, \ldots, \nu_n)\) be a **linear extension** of \(P\); this means a list of all elements of \(P\) (each only once) such that \(i < j\) whenever \(\nu_i < \nu_j\).

Cameron and Fon-der-Flaass showed that

\[ r = t_{\nu_1} \circ t_{\nu_2} \circ \ldots \circ t_{\nu_n}. \]

**Example:**
Then apply \(t_{(2,1)}\), which removes \((2, 1)\) from the order ideal:

```
(2, 2)
(2, 1)       (1, 2)
(1, 1)
```
Classical rowmotion: the toggling definition

- Let \((v_1, v_2, ..., v_n)\) be a **linear extension** of \(P\); this means a list of all elements of \(P\) (each only once) such that \(i < j\) whenever \(v_i < v_j\).

- Cameron and Fon-der-Flaass showed that

\[ r = t_{v_1} \circ t_{v_2} \circ ... \circ t_{v_n}. \]

**Example:**
Finally apply \(t_{(1,1)}\), which changes nothing:

```
(2, 2)
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```
(2, 1)  (1, 2)
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```
(1, 1)
```
Let \((v_1, v_2, \ldots, v_n)\) be a **linear extension** of \(P\); this means a list of all elements of \(P\) (each only once) such that \(i < j\) whenever \(v_i < v_j\).

Cameron and Fon-der-Flaass showed that

\[
\mathbf{r} = t_{v_1} \circ t_{v_2} \circ \ldots \circ t_{v_n}.
\]

**Example:**
So this is \(r(S)\):

```
  (2, 2)
   /   \
(2, 1)  (1, 2)
   \   /
    (1, 1)
```
Goals of the talk

- define **noncommutative birational rowmotion**: a generalization of classical rowmotion on several levels, due to David Einstein, James Propp, Tom Roby and myself, based on ideas of Anatol Kirillov and Arkady Berenstein.

- extend the “order $p + q$” theorem for rectangles to this generalization.

- ask some questions.
Let $\mathbb{K}$ be a ring (not necessarily commutative).

A $\mathbb{K}$-labelling of $P$ will mean a function $\hat{P} \to \mathbb{K}$.

The values of such a function will be called the labels of the labelling.

We will represent labellings by drawing the labels on the vertices of the Hasse diagram of $\hat{P}$.

**Example:** This is a $\mathbb{Q}$-labelling of the $2 \times 2$-rectangle:

```
14
  |
  10
```

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  |
-2 7
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  |
1/3
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  |
12
```
For any $v \in P$, define the **birational $v$-toggle** as the partial map $T_v : \mathbb{K}^\hat{P} \rightarrow \mathbb{K}^\hat{P}$ defined by

$$
(T_v f)(w) = \begin{cases} 
    f(w), & \text{if } w \neq v; \\
    \left( \sum_{u \in \hat{P} : \ u \leq v} f(u) \right) \cdot \frac{1}{f(v)} \cdot \left( \sum_{u \in \hat{P} : \ u \geq v} f(u) \right), & \text{if } w = v
\end{cases}
$$

for all $w \in \hat{P}$.

Here (and in the following), $\overline{m}$ means $m^{-1}$ whenever $m \in \mathbb{K}$. 

Notice that this is a **local change** to the label at $v$; all other labels stay the same.

If $K$ is commutative, then $T_2 v = id$ (on the range of $T_v$).
For any \( v \in P \), define the **birational \( v \)-toggle** as the partial map \( T_v : \mathbb{K}^{\hat{\mathcal{P}}} \rightarrow \mathbb{K}^{\hat{\mathcal{P}}} \) defined by

\[
(T_v f)(w) = \begin{cases} 
  f(w), & \text{if } w \neq v; \\
  \left( \sum_{u \in \hat{\mathcal{P}}; u \leq v} f(u) \right) \cdot \frac{f(v)}{m} \cdot \sum_{u \in \hat{\mathcal{P}}; u > v} f(u), & \text{if } w = v 
\end{cases}
\]

for all \( w \in \hat{\mathcal{P}} \).

Here (and in the following), \( m \) means \( m^{-1} \) whenever \( m \in \mathbb{K} \).

This is a **partial** map. If any of the inverses does not exist in \( \mathbb{K} \), then \( T_v f \) is undefined!
For any \( v \in P \), define the **birational** \( v \)-**toggle** as the partial map \( T_v : \mathbb{K}^\hat{P} \to \mathbb{K}^\hat{P} \) defined by

\[
(T_v f)(w) = \begin{cases} 
  f(w), & \text{if } w \neq v; \\
  \left( \sum_{u \in \hat{P}; u \leq v} f(u) \right) \cdot \overline{f(v)} \cdot \sum_{u \in \hat{P}; u \geq v} \overline{f(u)}, & \text{if } w = v
\end{cases}
\]

for all \( w \in \hat{P} \).

Here (and in the following), \( \overline{m} \) means \( m^{-1} \) whenever \( m \in \mathbb{K} \).

This is a **partial** map. If any of the inverses does not exist in \( \mathbb{K} \), then \( T_v f \) is undefined!

Notice that this is a **local change** to the label at \( v \); all other labels stay the same.
For any $v \in P$, define the **birational $v$-toggle** as the partial map $T_v : \overline{K}P \rightarrow \overline{K}P$ defined by

$$(T_v f)(w) = \begin{cases} 
  f(w), & \text{if } w \neq v; \\
  \left( \sum_{u \in \widehat{P}; \ u \prec v} f(u) \right) \cdot \overline{f(v)} \cdot \sum_{u \in \widehat{P}; \ u \succeq v} \overline{f(u)}, & \text{if } w = v
\end{cases}$$

for all $w \in \widehat{P}$.

Here (and in the following), $\overline{m}$ means $m^{-1}$ whenever $m \in K$.

This is a **partial** map. If any of the inverses does not exist in $K$, then $T_v f$ is undefined!

Notice that this is a **local change** to the label at $v$; all other labels stay the same.

If $K$ is commutative, then $T_v^2 = id$ (on the range of $T_v$).
We define **(noncommutative) birational rowmotion** as the partial map

\[ R := T_{v_1} \circ T_{v_2} \circ \cdots \circ T_{v_n} : \mathbb{K}^{\hat{P}} \rightarrow \mathbb{K}^{\hat{P}}, \]

where \((v_1, v_2, \ldots, v_n)\) is a linear extension of \(P\).

This is indeed independent on the linear extension, because:
We define (noncommutative) birational rowmotion as the partial map

\[ R := T_{v_1} \circ T_{v_2} \circ \cdots \circ T_{v_n} : K^\hat{P} \rightarrow K^\hat{P}, \]

where \((v_1, v_2, \ldots, v_n)\) is a linear extension of \(P\).

This is indeed independent on the linear extension, because:

- \(T_v\) and \(T_w\) commute whenever \(v\) and \(w\) are incomparable (or just don’t cover each other);
- we can get from any linear extension to any other by switching incomparable adjacent elements.
Example:
Let us “rowmote” a (generic) \( \mathbb{K} \)-labelling of the \( 2 \times 2 \)-rectangle:

<table>
<thead>
<tr>
<th>poset</th>
<th>labelling</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( b )</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>(2,2)</td>
<td>( z )</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>(2,1)</td>
<td>( x )</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>(1,2)</td>
<td>( y )</td>
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<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>(1,1)</td>
<td>( w )</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>( a )</td>
</tr>
</tbody>
</table>
Example:
Let us “rowmote” a (generic) $\mathbb{K}$-labelling of the $2 \times 2$-rectangle:

<table>
<thead>
<tr>
<th>poset</th>
<th>labelling</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$b$</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>$(2, 2)$</td>
<td>$z$</td>
</tr>
<tr>
<td>$(2, 1)$</td>
<td>$x$</td>
</tr>
<tr>
<td>$(1, 2)$</td>
<td>$y$</td>
</tr>
<tr>
<td>$(1, 1)$</td>
<td>$w$</td>
</tr>
<tr>
<td></td>
<td>$a$</td>
</tr>
</tbody>
</table>

We have $R = T_{(1,1)} \circ T_{(1,2)} \circ T_{(2,1)} \circ T_{(2,2)}$ (using the linear extension $((1, 1), (1, 2), (2, 1), (2, 2))$).
That is, toggle in the order “top, left, right, bottom”.
Example:
Let us “rowmote” a (generic) \( \mathbb{K} \)-labelling of the \( 2 \times 2 \)-rectangle:

<table>
<thead>
<tr>
<th>original labelling ( f )</th>
<th>labelling ( T_{(2,2)}f )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( b )</td>
<td>( b )</td>
</tr>
<tr>
<td>( z )</td>
<td>( (x + y)zb )</td>
</tr>
<tr>
<td>( x )</td>
<td></td>
</tr>
<tr>
<td>( y )</td>
<td></td>
</tr>
<tr>
<td>( w )</td>
<td></td>
</tr>
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Let us “rowmote” a (generic) \( \mathbb{K} \)-labelling of the \( 2 \times 2 \)-rectangle:

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<tbody>
<tr>
<td>( b )</td>
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</tr>
<tr>
<td>( z )</td>
<td>( (x + y) \bar{z}b )</td>
</tr>
<tr>
<td>( x )</td>
<td>( w \bar{x}(x + y) \bar{z}b )</td>
</tr>
<tr>
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<td></td>
</tr>
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<tr>
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</tr>
<tr>
<td>$z$</td>
<td>$(x + y)z b$</td>
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<tr>
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<td></td>
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</tr>
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<td></td>
</tr>
<tr>
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</tr>
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</tr>
<tr>
<td>$w$</td>
<td>$a\overline{w} \cdot w\overline{x}(x + y)\overline{z}b + w\overline{y}(x + y)\overline{z}b$</td>
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Let us “rowmote” a (generic) $\mathbb{K}$-labelling of the $2 \times 2$-rectangle:

original labelling $f$ | labelling $T_{(1,1)} T_{(1,2)} T_{(2,1)} T_{(2,2)} f = Rf$
---|---
$b$
$z$
$x$
$y$
$w$
$a$

We have used $R = T_{(1,1)} \circ T_{(1,2)} \circ T_{(2,1)} \circ T_{(2,2)}$ and simplified the result.
Birational rowmotion: motivation

- Why is this called birational rowmotion?
- Indeed, it generalizes classical rowmotion of order ideals:
  - Let \( \text{Trop} \mathbb{Z} \) be the **tropical semiring** over \( \mathbb{Z} \). This is the set \( \mathbb{Z} \cup \{-\infty\} \) with “addition” \( (a, b) \mapsto \max \{a, b\} \) and “multiplication” \( (a, b) \mapsto a + b \). This is a semifield.
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  - To every order ideal $S \in J(P)$, assign a $\text{Trop } \mathbb{Z}$-labelling $t_{\text{lab}} S$ defined by
    $$(t_{\text{lab}} S)(v) = \begin{cases} 1, & \text{if } v \notin S \cup \{0\}; \\ 0, & \text{if } v \in S \cup \{0\}. \end{cases}$$

This map $t_{\text{lab}} : J(P) \to (\text{Trop } \mathbb{Z})^\hat{P}$ is injective.
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- Let $t_v$ be the order ideal $v$-toggle, and let $r$ be order ideal rowmotion. Then:

$$
T_v \circ t_{\text{lab}} = t_{\text{lab}} \circ t_v, \quad R \circ t_{\text{lab}} = t_{\text{lab}} \circ r.
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  - Let \( t_v \) be the order ideal \( v \)-toggle, and let \( r \) be order ideal rowmotion. Then:
    \[
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    \]
  - Don’t like semifields? Use \( \mathbb{Q} \) and take the “tropical limit”.  

If $\mathbb{K}$ is commutative, then birational rowmotion $R$ has nice orders for nice posets (mostly Grinberg/Roby 2014):

- If $P$ is a rectangle $[p] \times [q]$, then $R^{p+q} = \text{id}$.

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- More generally, if $P$ is the minuscule poset associated to a minuscule weight $\lambda$ of a finite-dimensional simple Lie algebra $g$, then $R^h = \text{id}$, where $h$ is the Coxeter number of $g$. (Soichi Okada, doi:10.37236/9557.)

- If $P$ is an "$n$-graded forest" (a forest with all leaves having rank $n$), then $R^{\ell} = \text{id}$ for $\ell = \text{lcm}(1, 2, \ldots, n+1)$.
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- If $P$ is a “top half” $\Delta$ or “bottom half” $\nabla$ of the square $[p] \times [p]$, then $R^{2p} = \text{id}$, and moreover $R^p$ is reflection across the vertical axis.
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Take this poset:

This satisfies $R^6 = \text{id}$ if $\mathbb{K}$ is commutative, but nothing like that in general.
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However, not all is lost!
Let $p$ and $q$ be two positive integers. Let $\mathbb{K}$ be a ring. Let $P$ be the $p \times q$-rectangle poset: i.e.,

$$P := [p] \times [q], \quad \text{where } [m] := \{1, 2, \ldots, m\}.$$  

(The order on $P$ is entrywise.)

**Example:** For $p = 3$ and $q = 4$, this is

![Rectangle Poset Diagram](image-url)
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Let $f \in \mathbb{K}^\hat{P}$ be a $\mathbb{K}$-labelling. Let $a = f(0)$ and $b = f(1)$.
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Let $f \in \mathbb{K}\hat{P}$ be a $\mathbb{K}$-labelling. Let $a = f(0)$ and $b = f(1)$.

### Periodicity theorem (* 2015, † 2021+ G & Roby):

If $a$ and $b$ are invertible and $R^{p+q}f$ is well-defined, then

$$(R^{p+q}f)(x) = ab \cdot f(x) \cdot \overline{ab} \quad \text{for each} \ x \in \hat{P}.$$  

Note that $ab \cdot f(x) \cdot \overline{ab}$ is not generally conjugate to $f(x)$.
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**Reciprocity theorem (** 2015, † 2021+ G & Roby):**

Let $\ell \in \mathbb{N}$. If $R^\ell f$ is well-defined and $\ell \geq i + j - 1$, then

$$\left(R^\ell f\right)(i, j) = a \cdot \left(R^{\ell-i-j+1}f\right)(p + 1 - i, q + 1 - j) \cdot b$$

$$= \text{antipode of } (i, j) \text{ in } P$$

for each $(i, j) \in P$. 
**Example:** Iteratively apply $R$ to a labelling of the $2 \times 2$-rectangle.

Here is $R^0 f$:

```
  b
 /|
/  |
/   z
/    |
/     x
/      |
/       y
/         |
/           w
/             a
```
**Example:** Iteratively apply $R$ to a labelling of the $2 \times 2$-rectangle.

Here is $R^1 f$:

\[
\begin{array}{c}
\text{b} \\
\downarrow \\
(x + y) zb \\
\downarrow \\
wx(x + y) zb \\
\downarrow \\
a zb \\
\downarrow \\
a
\end{array}
\]

\[
\begin{array}{c}
\text{wz}(x + y) zb \\
\downarrow \\
wz(x + y) zb \\
\downarrow \\
a zb \\
\downarrow \\
a
\end{array}
\]

This confirms the periodicity theorem for $p = q = 2$.

Note that this is similar to Kontsevich's periodicity conjecture, proved by Iyudu/Shkarin ([arXiv:1305.1965](https://arxiv.org/abs/1305.1965)).
Example: Iteratively apply $R$ to a labelling of the $2 \times 2$-rectangle.

Here is $R^2f$:

\[
\begin{array}{c}
\quad b \\
\quad w (\bar{x} + \bar{y}) b \\
\quad a \cdot x + y \cdot x (\bar{x} + \bar{y}) b & \quad a \cdot x + y \cdot y (\bar{x} + \bar{y}) b \\
\quad \quad abz \cdot x + y \cdot b \\
\quad \quad a \\
\end{array}
\]
Example: Iteratively apply $R$ to a labelling of the $2 \times 2$-rectangle.
Here is $R^3 f$:

$$
\begin{align*}
\text{b} \\
\text{a} & \quad \text{w} & \text{b} \\
\text{a} & \quad \text{w} & \text{b} \\
\ldots & \quad \text{a} & \quad \text{b} & \quad \text{x} & \quad \text{y} & \quad \text{w} & \quad \text{b} \\
\text{a} & \quad \text{b} & \quad \text{w} & \quad \text{b} \\
\text{a} \\
\end{align*}
$$

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Example: Iteratively apply $R$ to a labelling of the $2 \times 2$-rectangle.
Here is $R^4 f$:

\[
\begin{array}{c}
\text{b} \\
\bar{a} \bar{b} z \bar{a} b \\
\bar{a} b \bar{w} \bar{a} b \\
\text{a}
\end{array}
\]

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**Example:** Iteratively apply $R$ to a labelling of the $2 \times 2$-rectangle.
Here is $R^4 f$:

$$
\begin{array}{c}
\text{b} \\
\text{abz\bar{a}b} \\
\text{\bar{a}bx\bar{a}b} & \text{\bar{a}by\bar{a}b} \\
\text{\bar{a}bw\bar{a}b} \\
\text{a}
\end{array}
$$

(after nontrivial simplifications).

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Example: Iteratively apply $R$ to a labelling of the $2 \times 2$-rectangle. Here is $R^4f$:

```
  b
 /\  
abz\abar
 / \   /
/  \ /  
/   \ /
/    \ 
/     |
/      a
```

This confirms the periodicity theorem for $p = q = 2$.

Note that this is similar to Kontsevich’s periodicity conjecture, proved by Iyudu/Shkarin (arXiv:1305.1965).
Here are $R^0 f, R^1 f, \ldots, R^4 f$ for a generic $f \in \mathbb{K}^{[2] \times [2]}$ again, this time fully simplified and with the $f(0) = a$ and $f(1) = b$ labels removed:

$$R^0 f = x \quad ; \quad R^1 f = w \bar{x} (x + y) \bar{z} b \quad ; \quad R^2 f = a \bar{y} b \quad ; \quad R^3 f = \bar{a} \bar{b} z x + y y \bar{w} b$$

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$$R^0 f = x \rightarrow z \rightarrow y ; \quad R^1 f = w \bar{x} (x + y) \bar{z} b$$

$$R^2 f = a \bar{y} b \rightarrow a \bar{x} b ; \quad R^3 f = a \bar{w} b \rightarrow a b z x + y y \overline{w} b$$

Equally colored labels are related by reciprocity. Can you spot some more?
Here are $R^0 f, R^1 f, \ldots, R^4 f$ for a generic $f \in \mathbb{K}^{[2] \times [2]}$ again, this time fully simplified and with the $f(0) = a$ and $f(1) = b$ labels removed:

$$R^0 f = x \quad R^1 f = w(x+y)zb\quad R^2 f = a\bar{y}b\quad R^3 f = abzx + y\bar{y}\bar{w}b$$

Here are some more instances of reciprocity. (There are more.)
The commutative case

- In 2014, we proved both theorems for commutative $\mathbb{K}$. 


WLOG assume $\mathbb{K}$ is a field (because our claims boil down to polynomial identities).

Show that "almost all" labellings of $P$ are in the image of a certain map $\text{Grasp}_0$ from the matrix space $\mathbb{K}^p \times (p + q)$ to $\mathbb{K}^b_P$.

Construct a commutative diagram $\mathbb{K}^p \times (p + q) \xrightarrow{\text{Grasp}_0} \mathbb{K}^b_P$, where $\rho$ is (more or less) rotating the matrix horizontally (last column to front).

Conclude that $R^{p + q} = \text{id}$ because $\rho^{p + q} = \text{id}$.

Reciprocity also easy using $\text{Grasp}_0$. 


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Explicitly, if $A \in \mathbb{K}^{p \times (p+q)}$ is any matrix, then

$$(\text{Grasp}_0 A)(0) = (\text{Grasp}_0 A)(1) = 1$$

and

$$(\text{Grasp}_0 A)(i,j) = \frac{\det(A[1 : i \mid i + j - 1 : p + j])}{\det(A[0 : i \mid i + j : p + j])}$$

for all $(i,j) \in P$, where the $A[a : b \mid c : d]$s are certain submatrices of $A$. (Note that this map $\text{Grasp}_0$ actually factors through the Grassmannian.)
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- Construct a commutative diagram

$$
\begin{array}{c}
\mathbb{K}^{p \times (p+q)} \xrightarrow{\text{Grasp}_0} \mathbb{K}^\hat{P}

\downarrow \rho \\
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\end{array}
$$

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\[
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\mathbb{K}^{p \times (p+q)} & \xrightarrow{\text{Grasp}_0} & \mathbb{K}^\hat{P} \\
\downarrow \rho & & \downarrow R \\
\mathbb{K}^{p \times (p+q)} & \xrightarrow{\text{Grasp}_0} & \mathbb{K}^\hat{P}
\end{array}
\]

where $\rho$ is (more or less) rotating the matrix horizontally (last column to front).
- Conclude that $R^{p+q} = \text{id}$ because $\rho^{p+q} = \text{id}$.
The commutative case

- In 2014, we proved both theorems for commutative $\mathbb{K}$.
- **Proof outline (inspired by A. Y. Volkov, arXiv:hep-th/0606094):**
  - WLOG assume $\mathbb{K}$ is a field (because our claims boil down to polynomial identities).
  - Show that “almost all” labellings of $P$ are in the image of a certain map $\text{Grasp}_0$ from the matrix space $\mathbb{K}^{p \times (p+q)}$ to $\mathbb{K}\hat{P}$.
  - Construct a commutative diagram

\[
\begin{array}{ccc}
\mathbb{K}^{p \times (p+q)} & \xrightarrow{\text{Grasp}_0} & \mathbb{K}\hat{P} \\
\downarrow{\rho} & & \downarrow{R} \\
\mathbb{K}^{p \times (p+q)} & \xrightarrow{\text{Grasp}_0} & \mathbb{K}\hat{P}
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where $\rho$ is (more or less) rotating the matrix horizontally (last column to front).
- Conclude that $R^{p+q} = \text{id}$ because $\rho^{p+q} = \text{id}$.
- Reciprocity also easy using $\text{Grasp}_0$. 


This looks easy; the devil is in the details (particularly the “almost all” part: not just Zariski density but also some rescaling required).

In some sense, yes: Replace determinants by quasideterminants (Gelfand/Retakh, arXiv:q-alg/9705026; see also arXiv:math/0208146).

Specifically, redefine Grasp \( 0 \) by

\[
(\text{Grasp}_0 A)_{ij} = (-1)^{i} q^{\{1:i, i+j:p+j\}}_{0, i+j-1}(A).
\]

The “algebra” works!

Unfortunately, the technical parts no longer work: What does “almost all” mean for noncommutative \( K \)? Can we WLOG assume that \( K \) is a skew field? No: e.g., the identity \( xyxy = 1 \) holds in all skew fields but not in all rings.

We now believe this approach is a dead end.
First attempts at general proof

- This looks easy; the devil is in the details (particularly the “almost all” part: not just Zariski density but also some rescaling required).
- Can this be generalized to arbitrary $\mathbb{K}$?
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- Unfortunately, the technical parts no longer work:
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    - No: e.g., the identity $xyyx = 1$ holds in all skew fields but not in all rings.
- We now believe this approach is a dead end.
New proofs of periodicity and reciprocity in the commutative-$K$ case were found by Gregg Musiker and Tom Roby in arXiv:1801.03877. They proceed by giving an explicit formula for $(R^k f)(i,j)$. For instance, $(R^3 f)(3, 2)$

\[
\frac{1}{A_{02} + A_{11} + A_{20}} \left( A_{01} A_{02} A_{11} A_{12} + A_{01} A_{02} A_{12} A_{20} + A_{01} A_{02} A_{20} A_{21} + A_{02} A_{10} A_{12} A_{20} + A_{02} A_{10} A_{20} A_{21} + A_{10} A_{11} A_{20} A_{21} \right),
\]

where

\[A_{ij} := \left( f(i, j + 1) + f(i + 1, j) \right) / f(i + 1, j + 1).\]
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$$A_{ij} := (f(i, j + 1) + f(i + 1, j))/f(i + 1, j + 1).$$

General formula for $(R^k f)(i,j)$ involves sums over NILPs (non-intersecting lattice path families) in numerator and denominator, as well as index shifting and a case split (“small” $k$ and “large” $k$ behave differently).
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where

$$A_{ij} := \frac{(f(i, j + 1) + f(i + 1, j)) \div f(i + 1, j + 1)}{f(i, j + 1) + f(i + 1, j)}.$$

General formula for $(R^k f)(i,j)$ involves sums over NILPs (non-intersecting lattice path families) in numerator and denominator, as well as index shifting and a case split (“small” $k$ and “large” $k$ behave differently).

Lattice paths can be generalized to noncommutative $\mathbb{K}$, but NILPs? Unclear in what order to multiply different paths.
What now?

- We are back at square 1: no known theory available.
What now?

- We are back at square 1: no known theory available.
- Let’s play around with the setting.
  Step 1: Introduce notations...
Fix $p, q, P$ and $f$. Assume that $R^\ell f$ is well-defined for all necessary $\ell$. Let $a = f(0)$ and $b = f(1)$.
Fix \( p, q, P \) and \( f \). Assume that \( R^\ell f \) is well-defined for all necessary \( \ell \). Let \( a = f(0) \) and \( b = f(1) \).

For any \( x \in \hat{P} \) and \( \ell \in \mathbb{N} \), write

\[
x_\ell := (R^\ell f)(x).
\]

Thus, \( x_0 = f(x) \) and \( 0_\ell = a \) and \( 1_\ell = b \).
Fix \( p, q, P \) and \( f \). Assume that \( R^\ell f \) is well-defined for all necessary \( \ell \). Let \( a = f(0) \) and \( b = f(1) \).

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Thus, \( x_0 = f(x) \) and \( 0_\ell = a \) and \( 1_\ell = b \).

The definition of \( R \) yields

\[
(Rf)(v) = \left( \sum_{u \leq v} f(u) \right) \cdot f(v) \cdot \sum_{u \geq v} (Rf)(u) \quad \text{for each } v \in P.
\]

(In both sums, \( u \) ranges over \( \hat{P} \); this is implied from now on.)
A new beginning

- Fix \( p, q, P \) and \( f \). Assume that \( R^\ell f \) is well-defined for all necessary \( \ell \). Let \( a = f(0) \) and \( b = f(1) \).
- For any \( x \in \hat{P} \) and \( \ell \in \mathbb{N} \), write

\[
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Thus, \( x_0 = f(x) \) and \( 0_\ell = a \) and \( 1_\ell = b \).
- The definition of \( R \) yields

\[
(Rf)(v) = \left( \sum_{u < v} f(u) \right) \cdot f(v) \cdot \sum_{u > v} (Rf)(u) \quad \text{for each } v \in P.
\]

(In both sums, \( u \) ranges over \( \hat{P} \); this is implied from now on.)
- In other words,

\[
v_1 = \left( \sum_{u \leq v} u_0 \right) \cdot \overline{v_0} \cdot \sum_{u > v} \overline{u_1} \quad \text{for each } v \in P.
\]
We have just shown that

\[ v_1 = \left( \sum_{u < v} u_0 \right) \cdot \overline{v_0} \cdot \sum_{u > v} u_1 \]  

for each \( v \in P \).
We have just shown that

\[ v_1 = \left( \sum_{u \prec v} u_0 \right) \cdot \overline{v_0} \cdot \sum_{u \succ v} u_1 \]

for each \( v \in P \).

Similarly,

\[ v_{\ell+1} = \left( \sum_{u \prec v} u_\ell \right) \cdot \overline{v_\ell} \cdot \sum_{u \succ v} u_{\ell+1} \]

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Similarly,

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So far, we have just rewritten our setup using the (more convenient) \( x_\ell := (R^\ell f)(x) \) notation.
We must prove:

**periodicity:** \( x_{p+q} = ab \cdot x_0 \cdot ab; \)

**reciprocity:** \( x_\ell = a \cdot y_{\ell-i-j+1} \cdot b \)

if \( x = (i, j) \) and \( y = (p + 1 - i, q + 1 - j) \).
Simplifying the goal

- We must prove:
  
  **periodicity**: \( x_{p+q} = a \overline{b} \cdot x_0 \cdot \overline{ab} \);
  
  **reciprocity**: \( x_{\ell} = a \cdot y_{\ell-i-j+1} \cdot b \)
  
  if \( x = (i, j) \) and \( y = (p + 1 - i, q + 1 - j) \).

- Periodicity follows from reciprocity: Indeed, if \( x = (i, j) \) and \( x' = (p + 1 - i, q + 1 - j) \), then

\[
\begin{align*}
  x_{p+q} &= a \cdot x'_{p+q-i-j+1} \cdot b \\
  &= a \cdot a \cdot x_0 \cdot b \cdot b \\
  &= a \overline{b} \cdot x_0 \cdot \overline{ab}.
\end{align*}
\]
Simplifying the goal

We must prove:

- **periodicity**: \( x_{p+q} = ab \cdot x_0 \cdot \overline{ab} \);

- **reciprocity**: \( x_\ell = a \cdot y_{\ell-i-j+1} \cdot b \)
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Periodicity follows from reciprocity: Indeed, if \( x = (i,j) \) and \( x' = (p+1-i, q+1-j) \), then

\[
x_{p+q} = a \cdot x'_{p+q-i-j+1} \cdot b \quad \text{(by reciprocity)}
\]
\[
= a \cdot a \cdot x_0 \cdot b \cdot b \quad \text{(by reciprocity again)}
\]
\[
= ab \cdot x_0 \cdot \overline{ab}.
\]

Thus, it suffices to prove reciprocity.
We must prove:

**periodicity**: \( x_{p+q} = \overline{ab} \cdot x_0 \cdot \overline{ab} \);

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Periodicity follows from reciprocity: Indeed, if \( x = (i, j) \) and \( x' = (p + 1 - i, q + 1 - j) \), then

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\begin{align*}
x_{p+q} &= a \cdot \overline{x'_{p+q-i-j+1}} \cdot b \\
&= a \cdot a \cdot \overline{x_0} \cdot b \cdot b \\
&= \overline{ab} \cdot x_0 \cdot \overline{ab}.
\end{align*}
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Thus, it suffices to prove reciprocity.

Moreover, reciprocity in general follows from reciprocity for \( \ell = i + j - 1 \) (just apply it to \( R^k f \) instead of \( f \) otherwise).
A **path** shall mean a sequence \((v_0 \succ v_1 \succ \cdots \succ v_k)\) of elements of \(\hat{P}\). We call it a path from \(v_0\) to \(v_k\).
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For each \(v \in P\) and \(\ell \in \mathbb{N}\), set

\[
A^v_\ell := v_\ell \cdot \sum_{u < v} u_\ell \quad \text{and} \quad \forall^v_\ell := \sum_{u > v} u_\ell \cdot \overline{v}_\ell.
\]

Also, set \(A^v_\ell = \forall^v_\ell = 1\) when \(v \in \{0, 1\}\).
A path shall mean a sequence \((v_0 > v_1 > \cdots > v_k)\) of elements of \(\hat{P}\). We call it a path from \(v_0\) to \(v_k\).

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For any path \(p = (v_0 > v_1 > \cdots > v_k)\), set

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A^p_\ell := A^{v_0}_\ell A^{v_1}_\ell \cdots A^{v_k}_\ell \quad \text{and} \quad \forall^p_\ell := \forall^{v_0}_\ell \forall^{v_1}_\ell \cdots \forall^{v_k}_\ell.
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\]

If \(u\) and \(v\) are elements of \(\widehat{P}\), set

\[
A^{u \rightarrow v}_\ell := \sum_{p \text{ is a path from } u \text{ to } v} A^p_\ell \quad \text{and} \quad \forall^{u \rightarrow v}_\ell := \sum_{p \text{ is a path from } u \text{ to } v} \forall^p_\ell.
\]
Path formulas

(a) We have

\[ u_\ell = \bigwedge_{\ell}^{\mathbb{1} \rightarrow u} \cdot b \quad \text{for each } u \in P. \]

(b) We have

\[ u_\ell = A_{\ell}^{u \rightarrow 0} \cdot a \quad \text{for each } u \in P. \]
Path formulas

Path formulas:

(a) We have

\[ u_\ell = \mathbb{A}_\ell^{1 \to u} \cdot b \quad \text{for each } u \in P. \]

(b) We have

\[ u_\ell = A_\ell^{u \to 0} \cdot a \quad \text{for each } u \in P. \]

Proof idea: The \( \ell \) is constant. Hence, we omit it, writing \( \mathbb{A} \) for \( \mathbb{A}_\ell \).
Path formulas

(a) We have

\[ u_\ell = \overrightarrow{A_1 \to u} \cdot b \]

for each \( u \in P \).

(b) We have

\[ u_\ell = A_{\ell \to 0}^u \cdot a \]

for each \( u \in P \).

Proof idea: The \( \ell \) is constant. Hence, we omit it, writing \( \forall^v \) for \( \forall^\ell \).

(a) Rewrite the claim as \( \forall^1 \to u = b_\ell u_\ell \).
Path formulas

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\[ u_\ell = \forall^{1 \to u}_\ell \cdot b \quad \text{for each } u \in P. \]

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(a) Rewrite the claim as \( \forall^{1 \to u} = b u_\ell \).
Prove this by downwards induction on \( u \).
Path formulas

- **Path formulas:**
  - (a) We have
    \[ u_\ell = \mathcal{A}^{1 \rightarrow u}_\ell \cdot b \quad \text{for each } u \in P. \]
  - (b) We have
    \[ u_\ell = \mathcal{A}^{u \rightarrow 0}_\ell \cdot a \quad \text{for each } u \in P. \]

- **Proof idea:** The \( \ell \) is constant. Hence, we omit it, writing \( \mathcal{A}^v \) for \( \mathcal{A}^{v}_\ell \).
  - (a) Rewrite the claim as \( \mathcal{A}^{1 \rightarrow u} = b u_\ell \).
    Prove this by downwards induction on \( u \).
    Induction step: Given \( v \in P \) such that \( \mathcal{A}^{1 \rightarrow u} = b u_\ell \) for all \( u \succ v \). Since any path \( 1 \rightarrow v \) passes through a unique \( u \succ v \), we have
    \[ \mathcal{A}^{1 \rightarrow v} = \sum_{u \succ v} \mathcal{A}^{1 \rightarrow u} \mathcal{A}^v = \sum_{u \succ v} b u_\ell \mathcal{A}^v \quad \text{(by induction hypothesis)} \]
    \[ = b v_\ell \quad \text{(by definition of } \mathcal{A}^v), \quad \text{qed}. \]
Path formulas:

(a) We have
\[ u_\ell = \forall_{\ell}^{1\rightarrow u} \cdot b \quad \text{for each } u \in P. \]

(b) We have
\[ u_\ell = A_{\ell}^{u\rightarrow 0} \cdot a \quad \text{for each } u \in P. \]

Proof idea: The \( \ell \) is constant. Hence, we omit it, writing \( \forall^v \) for \( \forall^v_{\ell} \).

(b) Analogous, but use upwards induction instead.
Path formulas:

(a) We have

\[ u_\ell = \bigvee_{\ell}^1 \cdot b \quad \text{for each } u \in P. \]

(b) We have

\[ u_\ell = A_{\ell}^{u \rightarrow 0} \cdot a \quad \text{for each } u \in P. \]

(c) We have

\[ u_\ell = \bigvee_{\ell}^{(p,q) \rightarrow u} \cdot b \quad \text{for each } u \in P. \]

(d) We have

\[ u_\ell = A_{\ell}^{u \rightarrow (1,1)} \cdot a \quad \text{for each } u \in P. \]
Path formulas:

(a) We have

\[ u_\ell = \overline{\forall^{1 \rightarrow u}_\ell} \cdot b \quad \text{for each } u \in P. \]

(b) We have

\[ u_\ell = A^{u \rightarrow 0}_\ell \cdot a \quad \text{for each } u \in P. \]

(c) We have

\[ u_\ell = \overline{\forall^{(p,q) \rightarrow u}_\ell} \cdot b \quad \text{for each } u \in P. \]

(d) We have

\[ u_\ell = A^{u \rightarrow (1,1)}_\ell \cdot a \quad \text{for each } u \in P. \]

Proof idea: Each path \( 1 \rightarrow u \) begins with the step \( 1 \triangleright (p, q) \).

Thus, \( \overline{\forall^{1 \rightarrow u}_\ell} = \overline{\forall^{(p,q) \rightarrow u}_\ell} \) (since \( \forall^{1}_\ell = 1 \)). Hence, (c) follows from (a).
Path formulas:

(a) We have

\[ u_\ell = A^1_{u \rightarrow u} \cdot b \quad \text{for each } u \in P. \]

(b) We have

\[ u_\ell = A^{u \rightarrow 0}_\ell \cdot a \quad \text{for each } u \in P. \]

(c) We have

\[ u_\ell = A^{(p, q) \rightarrow u}_\ell \cdot b \quad \text{for each } u \in P. \]

(d) We have

\[ u_\ell = A^{u \rightarrow (1, 1)}_\ell \cdot a \quad \text{for each } u \in P. \]

Proof idea: Each path \(1 \rightarrow u\) begins with the step \(1 \geq (p, q)\). Thus, \(A^1_{u \rightarrow u} = A^{(p, q) \rightarrow u}_\ell \) (since \(A^1_\ell = 1\)). Hence, (c) follows from (a).

Similarly, (d) follows from (b).
Transition equation in $A$-$\forall$-form:

$$A_{\ell+1}^\mathcal{V} = A_\ell^\mathcal{V}$$

for each $\mathcal{v} \in \widehat{P}$ and $\ell \in \mathbb{N}$.
Transition equation in $A$-$\forall$-form:

\[ \forall^\nu_{\ell+1} = A^\nu_\ell \quad \text{for each } \nu \in \hat{\mathcal{P}} \text{ and } \ell \in \mathbb{N}. \]

**Proof idea:** Above we showed that

\[ v_{\ell+1} = \left( \sum_{u < v} u_\ell \right) \cdot \overline{v_\ell} \cdot \sum_{u > v} \overline{u_{\ell+1}}. \]

Take reciprocals on both sides, multiply by $\sum_{u > v} \overline{u_{\ell+1}}$ and rewrite using $\forall^\nu_{\ell+1}$ and $A^\nu_\ell$. 
Transition equation in $A$-$\forall$-form:

$$\forall^\nu_{\ell+1} = A^\nu_{\ell}$$
for each $\nu \in \hat{P}$ and $\ell \in \mathbb{N}$.

Proof idea: Above we showed that

$$\nu_{\ell+1} = \left( \sum_{u < \nu} u_{\ell} \right) \cdot \overline{\nu_{\ell}} \cdot \sum_{u > \nu} \overline{u_{\ell+1}}.$$

Take reciprocals on both sides, multiply by $\sum_{u > \nu} \overline{u_{\ell+1}}$ and rewrite using $\forall^\nu_{\ell+1}$ and $A^\nu_{\ell}$.

As a consequence of $\forall^\nu_{\ell+1} = A^\nu_{\ell}$, we have

$$\forall^p_{\ell+1} = A^p_{\ell}$$
for each path $p$ and each $\ell \in \mathbb{N}$.
Transition equation in $A$-$\forall$-form:

$$\forall^\nu_{\ell+1} = A^\nu_\ell$$ for each $\nu \in \hat{P}$ and $\ell \in \mathbb{N}$.

**Proof idea:** Above we showed that

$$v_{\ell+1} = \left( \sum_{u < \nu} u_\ell \right) \cdot \overline{v_\ell} \cdot \sum_{u > \nu} \overline{u_{\ell+1}}.$$

Take reciprocals on both sides, multiply by $\sum_{u > \nu} \overline{u_{\ell+1}}$ and rewrite using $\forall^\nu_{\ell+1}$ and $A^\nu_\ell$.

As a consequence of $\forall^\nu_{\ell+1} = A^\nu_\ell$, we have

$$\forall^p_{\ell+1} = A^p_\ell$$ for each path $p$ and each $\ell \in \mathbb{N}$.

Hence, $\forall^{u \rightarrow \nu}_{\ell+1} = A^{u \rightarrow \nu}_\ell$ for any $u, \nu \in \hat{P}$.
Now, for the bottommost element \((1,1)\) of \(P\), we have

\[
(1, 1)_1 = \frac{\forall^{(p, q) \rightarrow (1, 1)}_1}{b} \quad \text{(by path formula (c))}
\]

\[
= \frac{A_0^{(p, q) \rightarrow (1, 1)}}{b} \quad \text{(since } \forall^{u \rightarrow v}_{\ell+1} = A^{u \rightarrow v}_\ell)\]

\[
= a \cdot (p, q)_0 \cdot b \quad \text{(by path formula (d))}.
\]

Thus, reciprocity is proved for \(i = j = 1\).
Now, for the bottommost element \((1, 1)\) of \(P\), we have

\[
(1, 1)_1 = \frac{\forall_1^{(p,q)\rightarrow(1,1)} \cdot b}{A_0^{(p,q)\rightarrow(1,1)} \cdot b} \quad \text{(by path formula (c))}
\]

\[
= \frac{a \cdot (p, q)_{0} \cdot b}{A_0^{(p,q)\rightarrow(1,1)} \cdot b} \quad \text{(since } \forall_{\ell+1}^{u\rightarrow v} = A_0^{u\rightarrow v})
\]

Thus, reciprocity is proved for \(i = j = 1\).

What now?
We can simplify our goal one bit further. Consider the “neighborhood” of an element of our rectangle $P$:

![Diagram of a rectangle with elements $u$, $v$, $m$, $s$, $t$ and ranks](image)

(Where the rank of an $(i, j) \in P$ is defined to be $i + j - 1$).

Say we have shown (our “induction hypotheses”) that reciprocity holds for each of $s$, $t$, $m$, $u$; that is, we have

\[
\begin{align*}
    s_\ell &= a \cdot s'_{\ell-(k-1)} \cdot b, \\
    t_\ell &= a \cdot t'_{\ell-(k-1)} \cdot b, \\
    m_\ell &= a \cdot m'_{\ell-k} \cdot b, \\
    u_\ell &= a \cdot u'_{\ell-(k+1)} \cdot b
\end{align*}
\]

for all sufficiently high $\ell$, where $x'$ denotes the antipode of $x$ (that is, if $x = (i, j)$, then $x' = (p + 1 - i, q + 1 - j)$).
We can simplify our goal one bit further. Consider the “neighborhood” of an element of our rectangle \( P \):

\[
\begin{array}{ccc}
  & u & \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ 
\end{array}
\]

(where the \textbf{rank} of an \((i, j) \in P\) is defined to be \(i + j - 1\)).

Say we have shown (our “induction hypotheses”) that reciprocity holds for each of \(s, t, m, u\); that is, we have

\[
\begin{align*}
  s_\ell &= a \cdot s'_{\ell-(k-1)} \cdot b, \\
  t_\ell &= a \cdot t'_{\ell-(k-1)} \cdot b, \\
  m_\ell &= a \cdot m'_{\ell-k} \cdot b, \\
  u_\ell &= a \cdot u'_{\ell-(k+1)} \cdot b
\end{align*}
\]

for all sufficiently high \(\ell\), where \(x'\) denotes the antipode of \(x\) (that is, if \(x = (i, j)\), then \(x' = (p + 1 - i, q + 1 - j)\)).

\textbf{Claim:} Then, reciprocity also holds for \(v\); that is, we have \(v_\ell = a \cdot v'_{\ell-(k+1)} \cdot b\) for all \(\ell \geq k + 1\).
Proof idea. Fix $\ell \geq k + 1$, and compare the transition equations

\[
m_\ell = (s_{\ell-1} + t_{\ell-1}) \cdot \overline{m_{\ell-1}} \cdot \overline{u_\ell + v_\ell} \quad \text{and} \quad m'_{\ell-k} = (u'_{\ell-k-1} + v'_{\ell-k-1}) \cdot \overline{m'_{\ell-k-1}} \cdot \overline{s'_{\ell-k} + t'_{\ell-k}}
\]

using the induction hypotheses $m_\ell = a \cdot m'_{\ell-k} \cdot b$, $s_{\ell-1} = a \cdot s'_{\ell-k} \cdot b$, $t_{\ell-1} = a \cdot t'_{\ell-k} \cdot b$, $m_{\ell-1} = a \cdot m'_{\ell-1-k} \cdot b$, $u_\ell = a \cdot u'_{\ell-(k+1)} \cdot b$, noting that

\[
\begin{array}{c}
u \\
\downarrow \quad m \\
\downarrow \\
\downarrow \\
s & t \\
\end{array} \quad \Rightarrow \quad \begin{array}{c}
t' \\
\downarrow \quad m' \\
\downarrow \\
\downarrow \\
v' & u' \\
\end{array}
\]
Proof idea. Fix \( \ell \geq k + 1 \), and compare the transition equations

\[
m_\ell = (s_{\ell-1} + t_{\ell-1}) \cdot m_{\ell-1} \cdot u_\ell + v_\ell \quad \text{and} \quad m'_{\ell-k} = (u'_{\ell-k-1} + v'_{\ell-k-1}) \cdot m'_{\ell-k-1} \cdot s'_{\ell-k} + t'_{\ell-k}
\]

using the induction hypotheses

\[
m_\ell = a \cdot m'_{\ell-k} \cdot b, \\
\ell-1 = a \cdot s'_{\ell-k} \cdot b, \\
m_{\ell-1} = a \cdot m'_{\ell-1-k} \cdot b, \\
u_\ell = a \cdot u'_{\ell-(k+1)} \cdot b,
\]

noting that

\[
\begin{align*}
& u \quad v \\ \\ & m \\ & s \\ & t
\end{align*} \quad \Rightarrow \quad
\begin{align*}
& t' \quad s' \\ \\ & m' \\ & v' \\ & u'
\end{align*}
\]

After subtracting \( u_\ell = a \cdot u'_{\ell-(k+1)} \cdot b \), out comes

\[
v_\ell = a \cdot v'_{\ell-(k+1)} \cdot b.
\]
The case $j = 1$ suffices: part 2

- **Proof idea.** Fix $\ell \geq k + 1$, and compare the transition equations

  $$m_\ell = (s_{\ell-1} + t_{\ell-1}) \cdot \overline{m_{\ell-1}} \cdot \overline{u_\ell + v_\ell} \quad \text{and} \quad m'_{\ell-k} = (u'_{\ell-k-1} + v'_{\ell-k-1}) \cdot \overline{m'_{\ell-k-1}} \cdot \overline{s'_{\ell-k} + t'_{\ell-k}}$$

  using the induction hypotheses $m_\ell = a \cdot \overline{m'_{\ell-k}} \cdot b,$

  $$s_{\ell-1} = a \cdot \overline{s'_{\ell-k}} \cdot b, \quad t_{\ell-1} = a \cdot \overline{t'_{\ell-k}} \cdot b,$$

  $$m_{\ell-1} = a \cdot \overline{m'_{\ell-1-k}} \cdot b, \quad u_\ell = a \cdot \overline{u'_{\ell-(k+1)}} \cdot b,$$

  noting that

  $$\begin{array}{ccc}
  u & v \\
  \downarrow & \downarrow \\
  m & s \\
  \downarrow & \downarrow \\
  s & t \\
  \end{array} \quad \Longrightarrow \quad \begin{array}{ccc}
  t' & s' \\
  \downarrow & \downarrow \\
  m' & v' \\
  \downarrow & \downarrow \\
  u' \\
  \end{array}.$$

- This argument still works if $s$, $t$ or $u$ does not exist.
The case $j = 1$ suffices: part 2

- **Proof idea.** Fix $\ell \geq k + 1$, and compare the transition equations

\[
\begin{align*}
    m_\ell &= (s_{\ell-1} + t_{\ell-1}) \cdot \overline{m_{\ell-1}} \cdot \overline{u_\ell + v_\ell} \\
    m'_{\ell-k} &= (u'_\ell - k + 1 + v'_\ell - k + 1) \cdot \overline{m'_{\ell-k-1}} \cdot \overline{s'_{\ell-k} + t'_{\ell-k}}
\end{align*}
\]

using the induction hypotheses $m_\ell = a \cdot \overline{m'_{\ell-k}} \cdot b,$ $s_{\ell-1} = a \cdot \overline{s'_{\ell-k}} \cdot b,$ $t_{\ell-1} = a \cdot \overline{t'_{\ell-k}} \cdot b,$ $m_{\ell-1} = a \cdot \overline{m'_{\ell-1-k}} \cdot b,$ $u_\ell = a \cdot \overline{u'_{\ell-(k+1)}} \cdot b,$ noting that

\[
\begin{array}{c}
  u \\
  m \quad v
\end{array} \quad \Longrightarrow \quad
\begin{array}{c}
  t' \\
  m' \quad s'
\end{array}
\]

\[
\begin{array}{c}
  s \\
  m \quad t
\end{array} \quad \Longrightarrow \quad
\begin{array}{c}
  v' \\
  m' \quad u'
\end{array}.
\]

- This argument still works if $s$, $t$ or $u$ does not exist.

- Thus, in order to prove reciprocity for all $(i, j)$, it suffices (by induction) to prove it in the case when $j = 1$. 
So we have proved reciprocity for $i = j = 1$, and we need to prove it for $j = 1$. 

Note the lack of rowmotion in this formula! The $\ell$ here is constantly 1, so it is a property of a single labeling. Thus, we drop the subscripts.

Our new goal: Prove that 

$$A(p, q) \rightarrow (2, 1) = A(p - 1, q) \rightarrow (1, 1).$$
So we have proved reciprocity for $i = j = 1$, and we need to prove it for $j = 1$.

The next case to try is $(i, j) = (2, 1)$. We need to show that

$$(2, 1)_2 = a \cdot (p - 1, q)_0 \cdot b.$$
So we have proved reciprocity for \( i = j = 1 \), and we need to prove it for \( j = 1 \).

The next case to try is \((i,j) = (2,1)\). We need to show that

\[
(2,1)_2 = a \cdot (p-1,q)_0 \cdot b.
\]

Using the path formulas (as in the case \( i = j = 1 \)), we can boil this down to

\[
\mathcal{A}_{1 \rightarrow (2,1)}^{(p,q)} = \mathcal{A}_{1 \rightarrow (1,1)}^{(p-1,q)}.
\]
So we have proved reciprocity for $i = j = 1$, and we need to prove it for $j = 1$.

The next case to try is $(i, j) = (2, 1)$. We need to show that

$$(2, 1)_2 = a \cdot (p - 1, q)_0 \cdot b.$$ 

Using the path formulas (as in the case $i = j = 1$), we can boil this down to

$$A_{(p, q) \to (2, 1)}^{(p, q) \to (2, 1)} = A_{(p-1, q) \to (1, 1)}^{(p-1, q) \to (1, 1)}.$$ 

Note the lack of rowmotion in this formula! The $\ell$ here is constantly 1, so it is a property of a single labeling. Thus, we drop the subscripts.

**Our new goal:** Prove that

$$A_{(p, q) \to (2, 1)}^{(p, q) \to (2, 1)} = A_{(p-1, q) \to (1, 1)}^{(p-1, q) \to (1, 1)}.$$
The conversion lemma

- More generally:
- **Conversion lemma:**
  Let \( u \) and \( u' \) be two adjacent elements on the top-right edge of \( P \) (that is, \( u = (k, q) \) and \( u' = (k - 1, q) \)). Let \( d \) and \( d' \) be two adjacent elements on the bottom-left edge of \( P \) (that is, \( d = (i, 1) \) and \( d' = (i - 1, 1) \)). Then,

\[
A_{u \rightarrow d} = \bigvee_{\ell} A_{u' \rightarrow d'}
\]

for each \( \ell \in \mathbb{N} \).

In short:

\[
A^{u \rightarrow d} = \bigvee u' \rightarrow d'.
\]
If we can prove the conversion lemma, we will obtain reciprocity not only for \((i, j) = (2, 1)\), but also for all \((i, j)\) on the bottom-left edge of \(P\) (that is, for the entire case \(j = 1\)), because we can argue as follows:
Rowmotion begone, part 2

\[(i, 1)_i = \overline{\land_{\rightarrow (i, 1)}^{(p, q)}} \cdot b = A_{i-1}^{(p, q) \rightarrow (i, 1)} \cdot b \]

(by path formula (c))

\[(\text{since } \land_{\rightarrow \ell+1}^{u \rightarrow v} = A_{\ell}^{u \rightarrow v})

= \overline{\land_{\rightarrow (i-1, 1)}^{(p-1, q)}} \cdot b

= A_{i-2}^{(p-1, q) \rightarrow (i-1, 1)} \cdot b

(\text{by the conversion lemma})

= \overline{\land_{\rightarrow (i-2, 1)}^{(p-2, q)}} \cdot b

= A_{i-2}^{(p-2, q) \rightarrow (i-2, 1)} \cdot b

(\text{by the conversion lemma})

= \ldots

= \overline{\land_{\rightarrow (1, 1)}^{(p-i+1, q)}} \cdot b

= A_1^{(p-i+1, q) \rightarrow (1, 1)} \cdot b

(\text{by the conversion lemma})

= A_0^{(p-i+1, q) \rightarrow (1, 1)} \cdot b

(\text{since } \land_{\rightarrow \ell+1}^{u \rightarrow v} = A_{\ell}^{u \rightarrow v})

= a \cdot (p - i + 1, q)_0 \cdot b

(by path formula (d)).
This proves reciprocity

\[(i, 1)_\ell = a \cdot (p - i + 1, q)_{\ell - i} \cdot b\]

for \(\ell = i\).
This proves reciprocity

\[(i, 1)_\ell = a \cdot (p - i + 1, q)_{\ell - i} \cdot b\]

for \(\ell = i\).

The case \(\ell > i\) follows by applying this to \(R^{\ell - i}f\) instead of \(f\).
This proves reciprocity

\[(i, 1)_\ell = a \cdot (p - i + 1, q)_{\ell-i} \cdot b\]

for \(\ell = i\).

The case \(\ell > i\) follows by applying this to \(R^{\ell-i}f\) instead of \(f\).

Thus, we only need to prove the conversion lemma. We can now drop all subscripts forever!
Let us again look at the picture:

We must prove $A^{u \rightarrow d} = \forall^{u' \rightarrow d'}$.
Let us again look at the picture:

We must prove \( A^{u \to d} = \forall^{u' \to d'} \).

How do we interpolate between paths \( u \to d \) and paths \( u' \to d' \)?
We define a **path-jump-path** to be a sequence

\[ p = (v_0 > v_1 > \cdots > v_i \triangleright v_{i+1} > v_{i+2} > \cdots > v_k) \]

of elements of \( P \), where the relation \( x \triangleright y \) means “\( y \) is one step down and some steps to the right of \( x \)” (that is, if \( x = (r, s) \), then \( y = (r - k, s + k - 1) \) for some \( k > 0 \)). We say that this path-jump-path \( p \) has **jump at** \( i \).
We define a **path-jump-path** to be a sequence

\[ p = (v_0 \triangleright v_1 \triangleright \cdots \triangleright v_i \triangleright v_{i+1} \triangleright v_{i+2} \triangleright \cdots \triangleright v_k) \]

of elements of \( P \), where the relation \( x \triangleright y \) means “\( y \) is one step down and some steps to the right of \( x \)” (that is, if \( x = (r, s) \), then \( y = (r - k, s + k - 1) \) for some \( k > 0 \)). We say that this path-jump-path \( p \) has **jump at** \( i \).

**Example** of a path-jump-path:

(The red edge is the jump.)
We define a \textbf{path-jump-path} to be a sequence
\[ p = (v_0 \succ v_1 \succ \cdots \succ v_i \Rightarrow v_{i+1} \succ v_{i+2} \succ \cdots \succ v_k) \]
of elements of $P$, where the relation $x \Rightarrow y$ means “$y$ is one step down and some steps to the right of $x$” (that is, if $x = (r, s)$, then $y = (r - k, s + k - 1)$ for some $k > 0$). We say that this path-jump-path $p$ has \textbf{jump at} $i$.

For any such path-jump-path $p$, we set
\[ E_p := A^{v_0} A^{v_1} \cdots A^{v_{i-1}} v_i \overline{v_{i+1}} \forall v_{i+2} \forall v_{i+3} \cdots \forall v_k. \]
(Here, we are omitting the $\ell$ subscripts – so $v_i$ means $(v_i)_\ell$ and $v_{i+1}$ means $(v_{i+1})_\ell$.)
We define a **path-jump-path** to be a sequence

\[ p = (v_0 \triangleright v_1 \triangleright \cdots \triangleright v_i \triangleright v_{i+1} \triangleright v_{i+2} \triangleright \cdots \triangleright v_k) \]

of elements of \( P \), where the relation \( x \triangleright y \) means "\( y \) is one step down and some steps to the right of \( x \)" (that is, if \( x = (r, s) \), then \( y = (r - k, s + k - 1) \) for some \( k > 0 \)).

We say that this path-jump-path \( p \) has **jump at** \( i \).

For any such path-jump-path \( p \), we set

\[ E_p := A^{v_0} A^{v_1} \cdots A^{v_{i-1}} v_i \overline{v_{i+1}} \forall v_{i+2} \forall v_{i+3} \cdots \forall v_k. \]

Now, if \( k = \text{rank } u - \text{rank } (d') \), then

\[ A^{u \rightarrow d} = \sum_{p \text{ is a path-jump-path } u \rightarrow d' \text{ with jump at } k-1} E_p, \]

since \( A^d = d \overline{d'} \), and similarly

\[ \forall^{u' \rightarrow d'} = \sum_{p \text{ is a path-jump-path } u \rightarrow d' \text{ with jump at } 0} E_p. \]
So we need to show that

\[ \sum_{\text{\(p\) is a path-jump-path} \ u \rightarrow d'} E_p = \sum_{\text{\(p\) is a path-jump-path} \ u \rightarrow d'} E_p. \]

for each \(0 \leq i < k - 1\). And yes, this is true and can be proved by a “local” argument (rewriting two consecutive steps of the path). This is similar to the “zipper argument” in lattice models. (Is there a Yang–Baxter equation lurking?)
So we need to show that

\[ E_p = \sum_{\text{p is a path-jump-path } u \to d' \text{ with jump at } k-1} E_p \]

Reasonable to expect that

\[ E_p = \sum_{\text{p is a path-jump-path } u \to d' \text{ with jump at } i} E_p \]

for each \( 0 \leq i < k - 1 \).
Proving the conversion lemma: moving the jump

So we need to show that

\[ \sum_{p \text{ is a path-jump-path } u \to d'} E_p = \sum_{p \text{ is a path-jump-path } u \to d'} E_p. \]

Reasonable to expect that

\[ \sum_{p \text{ is a path-jump-path } u \to d'} E_p = \sum_{p \text{ is a path-jump-path } u \to d'} E_p \]

for each \( 0 \leq i < k - 1 \).

And yes, this is true and can be proved by a “local” argument (rewriting two consecutive steps of the path).
So we need to show that

\[ E_p = \sum_{\text{p is a path-jump-path } u \rightarrow d' \text{ with jump at } k-1} E_p = \sum_{\text{p is a path-jump-path } u \rightarrow d' \text{ with jump at } 0} E_p. \]

Reasonable to expect that

\[ E_p = \sum_{\text{p is a path-jump-path } u \rightarrow d' \text{ with jump at } i} E_p = \sum_{\text{p is a path-jump-path } u \rightarrow d' \text{ with jump at } i+1} E_p \]

for each \(0 \leq i < k - 1\).

And yes, this is true and can be proved by a “local” argument (rewriting two consecutive steps of the path).

This is similar to the “zipper argument” in lattice models. (Is there a Yang–Baxter equation lurking?)
Modulo the details omitted, this finishes the proof of the reciprocity theorem.
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However, the path-jump-path argument is somewhat messy. We can make it slicker by rewriting it in matrix notation:
Modulo the details omitted, this finishes the proof of the reciprocity theorem.

However, the path-jump-path argument is somewhat messy. We can make it slicker by rewriting it in matrix notation:

Define three \( P \times P \)-matrices \( \mathbf{A}, \mathbf{V}, \) and \( \mathbf{U} \) by

\[
\begin{align*}
\mathbf{A}_{x,y} &:= A^x [x \triangleright y], \\
\mathbf{V}_{x,y} &:= \forall^y [x \triangleright y], \\
\mathbf{U}_{x,y} &:= x\bar{y} [x \uparrow y]
\end{align*}
\]

for all \( x, y \in P \).

Here, \([A]\) is the Iverson bracket (i.e., truth value) of a statement \( A \); the relation \( x \uparrow y \) means “\( y \) is one step down and some steps to the right of \( x \)” as before. And again, we are omitting the \( \ell \) subscripts, so \( x\bar{y} \) actually means \( x_{\ell \bar{y}_{\ell}} \).

Now, we claim that

\[
\mathbf{A}\mathbf{U} = \mathbf{U}\mathbf{V}.
\]
Now, we claim that \( AU = U A \).

Indeed, this follows easily from the following neat lemma: If

\[
\begin{array}{c}
\text{\( u \)} \\
\text{\( v \)} \\
\text{\( w \)} \\
\text{\( d \)}
\end{array}
\]

are four adjacent elements of \( P \), then

\[
\overline{w} \cdot A^d \cdot d = \overline{u} \cdot A^u \cdot v \quad \text{and} \quad \overline{v} \cdot A^d \cdot d = \overline{u} \cdot A^u \cdot w.
\]

(The \( u \) and \( d \) here are unrelated to the \( u \) and \( d \) from the conversion lemma!)
Now, we claim that $AU = UA$.
Indeed, this follows easily from the following neat lemma: If
\[
\begin{array}{ccc}
  u & \leftarrow & v \\
  v & \leftrightarrow & w \\
  v & \leftrightarrow & d \\
\end{array}
\]
are four adjacent elements of $P$, then
\[
\overline{w} \cdot 
\begin{array}{cc}
  \forall^d \\
\end{array} 
\cdot d = \overline{u} \cdot A^u \cdot v \\
\text{and} \\
\overline{v} \cdot 
\begin{array}{cc}
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\end{array} 
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\]

From $AU = UA$, we easily obtain
\[
A^{\circ k} U = U A^{\circ k}
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for any $k \in \mathbb{N},$
where $A^{\circ k}$ means the $k$-th power of a matrix $A$. 
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Setting $k = \text{rank } u - \text{rank } d$ and comparing the $(u, d')$-entries of both sides, we quickly obtain $A^{u \rightarrow d} = \forall^{u' \rightarrow d'}$ (since $x \uparrow d'$ holds only for $x = d$, and since $u \uparrow x$ holds only for $x = u'$).

This proves the conversion lemma again.
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In the commutative case, the theorems hold for semifields (and, more generally, commutative semirings) because they hold for fields and because they are “essentially” polynomial identities (once you clear denominators).
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Recall: Classical rowmotion is (a restriction of) birational rowmotion on the tropical semifield. Semifields are not rings! (No subtraction.) In the commutative case, the theorems hold for semifields (and, more generally, commutative semirings) because they hold for fields and because they are “essentially” polynomial identities (once you clear denominators). This fails for noncommutative \( \mathbb{K} \)!

Scary example (David Speyer, MathOverflow #401273): If \( x \) and \( y \) are two elements of a ring such that \( x + y \) is invertible, then

\[
x \cdot x + y \cdot y = y \cdot x + y \cdot x.
\]

But this is not true if “ring” is replaced by “semiring”!
Thus, we are left with a

**Question:**
Are the periodicity and reciprocity theorems still true if “ring” is replaced by “semiring”? (I.e., we no longer require $\mathbb{K}$ to have a subtraction.)
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- Note that the main hurdle is the argument that reduced the general case to the $j = 1$ case. That argument used subtraction!
- We have partial results, e.g., for $p = q = 3$ and for $p = 2$. 
Other posets remain to be studied.

Conjecture:

Let $P$ be the triangle-shaped poset $\Delta(p)$ or its reflection $\nabla(p)$. Let $f \in \mathbb{K}P$ be a labelling such that $R^p f$ exists. Let $a = f(0)$ and $b = f(1)$. Then, for each $x \in \hat{P}$, we have

$$(R^p f)(x) = ab \cdot f(x') \cdot \bar{ab},$$

where $x'$ is the reflection of $x$ across the y-axis.
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We have a similar conjecture for other kinds of triangles and (still unproved even in the commutative case!) for trapezoids.
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- We have a similar conjecture for other kinds of triangles and (still unproved even in the commutative case!) for trapezoids.
- As already mentioned, other simple posets such as

  $\circ \quad \circ \quad \circ$

  $\quad \quad \quad \quad \downarrow \quad \quad \downarrow$

  $\circ$

  do not have periodic behavior for noncommutative $K$. 
Other posets remain to be studied.

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Let $P$ be the triangle-shaped poset $\Delta (p)$ or its reflection $\nabla (p)$. Let $f \in \mathbb{K}\hat{P}$ be a labelling such that $R^p f$ exists. Let $a = f(0)$ and $b = f(1)$. Then, for each $x \in \hat{P}$, we have

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**Question:**

What other results like ours are known in the noncommutative case?
Acknowledgments

- **Tom Roby**: collaboration
- **Mathematisches Forschungsinstitut Oberwolfach**: hospitality in July/August 2021
- **Banff International Research Station**: 2021 conference where this was first presented
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- **Sage and Sage-combinat**: computations
- **the birational combinatorics community**: keeping the subject exciting since 2013
- **you**: your patience
Some references

Zamolodchikov periodicity conjecture in type AA (proved by A. Yu. Volkov, arXiv:hep-th/0606094v1): Let $r$ and $s$ be positive integers. Let $Y_{i, j, k}$ be elements of a commutative ring for $i \in [r]$ and $j \in [s]$ and $k \in \mathbb{Z}$. Assume that

$$Y_{i, j, k+1} Y_{i, j, k-1} = \frac{(1 + Y_{i+1, j, k})(1 + Y_{i-1, j, k})}{(1 + 1/Y_{i, j+1, k})(1 + 1/Y_{i, j-1, k})}$$

for all $i, j, k$, where sums involving “off-grid” points (e.g., $1 + Y_{0, j, k}$) are understood as 1.

Then, $Y_{i, j, k+2(r+s+2)} = Y_{i, j, k}$ for all $i, j, k$. 

Disappointment: Zamolodchikov periodicity does not generalize to noncommutative rings (no matter how we order the five factors). 

Observation (Max Glick and others, ca. 2015?): This is equivalent to periodicity of birational rowmotion ($R_p + q = 1$) for $[p] \times [q]$, where $p = r + 1$ and $q = s + 1$, when the ring is commutative.
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\[
Y_{i, j, i+j-2k} = (R^k f)(i, j + 1) \big/ (R^k f)(i + 1, j).
\]

(Fine points omitted.)
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The main idea of their proof is to reduce birational rowmotion to the octahedron recurrence, and prove the latter is periodic using lattice paths and LGV.

Lemma 4.1 in the Johnson-Liu preprint generalizes our conversion lemma in the commutative case from single paths to $k$-tuples of nonintersecting paths. We don't know how this could be done in the noncommutative case; it is unclear in what order to multiply labels from different paths.
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One more little result

**Proposition (2022, G & Roby):**

Let $P$ be any finite poset. Let $f \in \mathbb{K}^{\hat{P}}$. Then,

\[
f(1) \cdot \sum_{u \in \hat{P}; u \geq 0} (Rf)(u) \cdot f(0) = \sum_{u \in \hat{P}; u \leq 1} f(u),
\]

assuming that the inverses $(Rf)(u)$ are well-defined.
One more little result

**Proposition (2022, G & Roby):**

Let $P$ be any finite poset. Let $f \in \mathbb{K}\hat{P}$. Then,

$$f(1) \cdot \sum_{\substack{u \in \hat{P}; \quad u > 0}} (Rf)(u) \cdot f(0) = \sum_{\substack{u \in \hat{P}; \quad u \leq 1}} f(u),$$

assuming that the inverses $(Rf)(u)$ are well-defined.

**Corollary (2022+, G & Roby):**

Let $P$ be any finite poset. Let $f \in \mathbb{K}\hat{P}$ with $f(0) = f(1) = 1$. Then, the quantity

$$\sum_{\substack{u, v \in \hat{P}; \quad u \leq v}} f(u) \cdot f(v)$$

is unchanged under birational rowmotion (i.e., when we replace $f$ by $Rf$).