Schur functions: Theme and variations
I. G. Macdonald
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Actes 28$^{\text{e}}$ Séminaire Lotharingien, p. 5-39.
Errata and addenda by Darij Grinberg

I will refer to the results appearing in the paper "Schur functions: Theme and variations" by the numbers under which they appear in this paper. Page numbering goes from 5 to 39.

I have read the "6th variation" section of the paper so far (minus (6.11), (6.19) and (6.21)).

The list below contains both actual corrections and what I believe to be clarifications and pertinent comments. I have not tried to separate the former from the latter, as I suspect that the precise boundary is in the eyes of the beholder.

B. Errata and addenda

I shall use the following notations:

- If p and q are two integers such that $p \leq q+1$, then $[p,q]$ shall denote the set of all integers m such that $p \leq m \leq q$. We call this set an *integer interval*. It has size $q-p+1$ (so it is empty if $p = q+1$).

- For a given $r \in \mathbb{N}$, we shall denote the integer interval $[1,r] = \{1,2,\ldots,r\}$ by $[r]$.

- If $A = \left(a_{i,j}\right)_{i \in S,\ j \in T}$ is an arbitrary matrix (where the sets S and T may be finite or infinite), and if $P = \{p_1 < p_2 < \cdots < p_\alpha\}$ is a finite subset of S, and if $Q = \left\{q_1 < q_2 < \cdots < q_\beta\right\}$ is a finite subsets of T, then $\operatorname{sub}_P^Q A$ shall denote the submatrix $\left(a_{p_i,q_j}\right)_{i \in [\alpha],\ j \in [\beta]}$ of A. For instance, if $A = \left(a_{i,j}\right)_{i,j \in [4]}$ is a 4×4-matrix, then $\operatorname{sub}_{\{2,3\}}^{\{2,4\}} A = \begin{pmatrix} a_{2,2} & a_{2,4} \\ a_{3,2} & a_{3,4} \end{pmatrix}$.

- If $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$ and $\mu = (\mu_1, \mu_2, \mu_3, \ldots)$ are two partitions, then the notation "$\lambda \subseteq \mu$" shall mean that the Young diagram of λ is a subset of the Young diagram of μ, or, equivalently, that each $i \geq 1$ satisfies $\lambda_i \leq \mu_i$. This is denoted by "$\lambda \subset \mu$" in Macdonald's paper, but I shall use the notation "$\lambda \subseteq \mu$" instead, since it corresponds better to my use of "$\subseteq$" for subsets.

Now, the actual corrections:

1. **page 15, line 2:** "for each $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{N}^n$" should be "for each $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$".

2. **page 15, after (6.3):** For a detailed proof of the fact that the quotient

$$s_\lambda (x \mid a) = A_{\lambda+\delta} (x \mid a) \diagup A_\delta (x \mid a)$$

   is a symmetric polynomial in $x_1, \ldots, x_n$ with coefficients in R, see [Grinbe18, Corollary 9.14]. (Apply this corollary to $X_i = x_i$ and $P_j (T) = (T \mid a)^{\lambda_j + n - j}$.)

3. **page 16, proof of (6.6):** This argument only shows that $\mathbf{E} (x \mid a) \cdot \mathbf{H} (x \mid a) = I_\mathbb{Z}$ (where $I_\mathbb{Z}$ is the identity matrix with rows and columns indexed by all integers). In order to prove that the two matrices $\mathbf{E} (x \mid a)$ and $\mathbf{E} (x \mid a)$ are inverse to each other (i.e., in order to prove (6.6)), it must also be shown that $\mathbf{H} (x \mid a) \cdot \mathbf{E} (x \mid a) = I_\mathbb{Z}$.

   Fortunately, there is a simple shortcut for this: Let $\mathrm{UT}_\mathbb{Z}$ be the set of all upper unitriangular matrices with rows and columns indexed by all integers (and with entries in a given base ring, which in our case is the polynomial ring over $\mathbb{Z}$ in the variables x_i for $i \in [n]$ and a_j for $j \in \mathbb{Z}$). This set $\mathrm{UT}_\mathbb{Z}$ is closed under matrix multiplication, thus is a monoid. Moreover, each matrix $A \in \mathrm{UT}_\mathbb{Z}$ can be written as $I + M$ for some strictly upper-triangular matrix M, and thus has an inverse $A^{-1} = (I + M)^{-1}$, which can be computed by the formula $(I + M)^{-1} = I - M + M^2 - M^3 + M^4 \pm \cdots$ (this infinite sum makes sense, since the nonzero entries of each power M^i start no earlier than i steps above the main diagonal[1]). This inverse $A^{-1} = I - M + M^2 - M^3 + M^4 \pm \cdots$ again belongs to $\mathrm{UT}_\mathbb{Z}$. Hence, each element of the monoid $\mathrm{UT}_\mathbb{Z}$ has an inverse. Thus, $\mathrm{UT}_\mathbb{Z}$ is a group with respect to matrix multiplication. Since both matrices $\mathbf{E} (x \mid a)$ and $\mathbf{H} (x \mid a)$ belong to this group $\mathrm{UT}_\mathbb{Z}$, we can thus conclude $\mathbf{E} (x \mid a) = \mathbf{H} (x \mid a)^{-1}$ from $\mathbf{E} (x \mid a) \cdot \mathbf{H} (x \mid a) = I_\mathbb{Z}$.

4. **page 17, proof of (6.7):** It is worth saying that all the matrices that appear in the proof of the first of the formulas (6.7) are understood to be $n \times n$-matrices.

5. **page 17, proof of (6.7):** Let me give a proof of the second of the formulas (6.7) along with the more general formula (6.9). We will need some notations and some lemmas.

   Let $\mathrm{UT}_\mathbb{Z}$ be the set of all upper unitriangular matrices with rows and columns indexed by all integers (and with entries in a given commutative ring). For any $m \in \mathbb{N}$, we let UT_m be the set of all upper unitriangular $m \times m$-matrices (again, with entries in our given commutative ring). Both of these sets $\mathrm{UT}_\mathbb{Z}$ and UT_m are groups (under matrix multiplication). The following fact is easy:

[1] That is: For any $u, v \in \mathbb{Z}$ and any $i \in \mathbb{N}$, the (u, v)-th entry of M^i is 0 whenever $v - u < i$, and therefore the (u, v)-th entry of the infinite sum $I - M + M^2 - M^3 + M^4 \pm \cdots$ is only affected by the first $v - u + 1$ addends of this sum.

Lemma B.1. Let $T = [p, q]$ be an integer interval of size $m = q - p + 1$ (so that $q = p + m - 1$). Then, the map

$$\begin{aligned} \mathrm{UT}_{\mathbb{Z}} &\to \mathrm{UT}_m, \\ A &\mapsto \operatorname{sub}_T^T A \end{aligned} \tag{1}$$

is a group morphism.

Proof of Lemma B.1. It is easy to see that this map is well-defined (i.e., that $\operatorname{sub}_T^T A \in \mathrm{UT}_m$ for each $A \in \mathrm{UT}_{\mathbb{Z}}$). (Indeed, more generally, any principal submatrix of an upper unitriangular matrix is again upper unitriangular.) It is clear that $\operatorname{sub}_T^T (I_{\mathbb{Z}}) = I_m$. It remains to show that $\operatorname{sub}_T^T (AB) = \operatorname{sub}_T^T A \cdot \operatorname{sub}_T^T B$ for all $A, B \in \mathrm{UT}_{\mathbb{Z}}$.

So let $A, B \in \mathrm{UT}_{\mathbb{Z}}$ be arbitrary. Write these matrices A and B as $A = (a_{i,j})_{i,j\in\mathbb{Z}}$ and $B = (b_{i,j})_{i,j\in\mathbb{Z}}$. Since $T = [p, q] = [p, p + m - 1]$ (because $q = p + m - 1$), we thus have

$$\operatorname{sub}_T^T A = (a_{p+i-1,p+j-1})_{i,j\in[m]} \qquad \text{and} \qquad \operatorname{sub}_T^T B = (b_{p+i-1,p+j-1})_{i,j\in[m]}$$

and therefore

$$\begin{aligned} \operatorname{sub}_T^T A \cdot \operatorname{sub}_T^T B &= (a_{p+i-1,p+j-1})_{i,j\in[m]} \cdot (b_{p+i-1,p+j-1})_{i,j\in[m]} \\ &= \left(\sum_{k=1}^{m} a_{p+i-1,p+k-1} b_{p+k-1,p+j-1} \right)_{i,j\in[m]} \end{aligned} \tag{2}$$

(by the definition of matrix multiplication).

The matrix A is upper-triangular (since $A \in \mathrm{UT}_{\mathbb{Z}}$), so we have

$$a_{i,k} = 0 \qquad \text{for all } i > k. \tag{3}$$

Likewise,

$$b_{k,j} = 0 \qquad \text{for all } k > j. \tag{4}$$

Thus, we can easily see that if $i, j \in T$, then

$$a_{i,k} b_{k,j} = 0 \qquad \text{for all integers } k \notin T. \tag{5}$$

(*Proof:* Let k be an integer such that $k \notin T$. Thus, $k \notin T = [p, q]$. Hence, either $k < p$ or $k > q$. In the former case, we have $k < p \leq i$ (since $i \in T = [p, q]$) and therefore $i > k$, so that $a_{i,k} = 0$ (by (3)), whence $\underbrace{a_{i,k}}_{=0} b_{k,j} = 0$. In the latter case, we have $k > q \geq j$ (since $j \in T = [p, q]$

entails $j \leq q$) and therefore $b_{k,j} = 0$ (by (4)), whence $a_{i,k} \underbrace{b_{k,j}}_{=0} = 0$. Hence, we have proved $a_{i,k}b_{k,j} = 0$ in both cases. Thus, (5) is proved.)
Hence, for any $i, j \in T$, the (i,j)-th entry of the matrix AB is

$$\begin{aligned} &\sum_{k\in\mathbb{Z}} a_{i,k}b_{k,j} \qquad \text{(by the definition of } AB\text{)} \\ &= \sum_{k\in T} a_{i,k}b_{k,j} \qquad \left(\begin{array}{c}\text{since (5) shows that} \\ \text{any addend } a_{i,k}b_{k,j} \text{ equals 0 unless } k \in T, \\ \text{and thus we can restrict the sum to} \\ \text{only range over the } k \in T\end{array}\right).\end{aligned}$$

Therefore, from $T = [p,q]$, we obtain

$$\operatorname{sub}_T^T(AB) = \left(\sum_{k\in T} a_{p+i-1,k}b_{k,p+j-1}\right)_{i,j\in[m]}. \tag{6}$$

However, each $i, j \in [m]$ satisfy

$$\begin{aligned} &\sum_{k=1}^{m} a_{p+i-1,p+k-1}b_{p+k-1,p+j-1} \\ &= \underbrace{\sum_{k=p}^{p+m-1}}_{\substack{=\sum\limits_{k\in T} \\ \text{(since } T=[p,p+m-1])}} a_{p+i-1,k}b_{k,p+j-1} \\ &\qquad \text{(here, we have substituted } k \text{ for } p+k-1 \text{ in the sum)} \\ &= \sum_{k\in T} a_{p+i-1,k}b_{k,p+j-1}.\end{aligned}$$

Thus, the right hand sides of the equalities (2) and (6) are equal. Hence, their left hand sides are also equal. In other words, $\operatorname{sub}_T^T(AB) = \operatorname{sub}_T^T A \cdot \operatorname{sub}_T^T B$. This completes the proof of Lemma B.1. ■

Lemma B.2. Let A be an invertible $m \times m$-matrix. Let P and Q be two subsets of $[m]$ such that $|P| = |Q|$. Let $\widetilde{P} := [m] \setminus P$ and $\widetilde{Q} := [m] \setminus Q$ be their complements. Let $\sum P$ be the sum of all elements of P, and let $\sum Q$ be the sum of all elements of Q. Then,

$$\det\left(\operatorname{sub}_P^Q A\right) = (-1)^{\sum P + \sum Q} \det A \cdot \det\left(\operatorname{sub}_{\widetilde{Q}}^{\widetilde{P}}\left(A^{-1}\right)\right).$$

Lemma B.2 is [Grinbe20, Exercise 6.56] (with slightly different notations: $\operatorname{sub}_P^Q A$ is denoted $\operatorname{sub}_{w(P)}^{w(Q)} A$ there). Alternatively, it can be easily derived from [LLPT95, (APP.1.5.2)] (since the adjugate matrix $\operatorname{adj} A$ of A, which is denoted by ${}^{+}A$ in [LLPT95], is known to equal $(\det A) \cdot A^{-1}$). ■

Lemma B.3. Let B be an infinite matrix in $\mathrm{UT}_{\mathbb{Z}}$. Let $T = [p, q]$ be an integer interval. Let U and V be two subsets of T such that $|U| = |V|$. Then,

$$\det\left(\operatorname{sub}_U^V B\right) = (-1)^{\sum U + \sum V} \det\left(\operatorname{sub}_{T\setminus V}^{T\setminus U}\left(B^{-1}\right)\right).$$

(Note that B^{-1} exists: Indeed, the matrix B belongs to the group $\mathrm{UT}_{\mathbb{Z}}$ and thus has an inverse.)

Proof of Lemma B.3. Let m be the size $q - p + 1$ of the interval $T = [p, q]$. Then, $q = m + p - 1$. Furthermore, Lemma B.1 says that the map (1) is a group morphism. Hence, $\left(\operatorname{sub}_T^T B\right)^{-1} = \operatorname{sub}_T^T\left(B^{-1}\right)$. Moreover, $\operatorname{sub}_T^T B \in \mathrm{UT}_m$ (since the map (1) has target UT_m), so that the matrix $\operatorname{sub}_T^T B$ is upper unitriangular. Thus, $\det\left(\operatorname{sub}_T^T B\right) = 1$ (since the determinant of a triangular matrix is the product of its diagonal entries, and therefore the determinant of a unitriangular matrix is 1).

Let A be the submatrix $\operatorname{sub}_T^T B$ of B. Then,

$$A^{-1} = \left(\operatorname{sub}_T^T B\right)^{-1} = \operatorname{sub}_T^T\left(B^{-1}\right) \qquad \text{and} \tag{7}$$

$$\det A = \det\left(\operatorname{sub}_T^T B\right) = 1. \tag{8}$$

For any subset S of $\mathbb{Z}$ and any integer z, we let $S + z$ denote the set $\{s + z \mid s \in S\}$. Visually speaking, this is simply the set S shifted by z units to the right along the number line. Clearly, $|S + z| = |S|$ and $(S + z) + (-z) = S$. Moreover, any two subsets S and T of $\mathbb{Z}$ and any integer z satisfy

$$(S \setminus T) + z = (S + z) \setminus (T + z), \tag{9}$$

since the operation of adding z to each integer is a bijection.

We have

$$\begin{aligned} T &= [p, q] \\ &= [1 + p - 1,\ m + p - 1] \qquad (\text{since } p = 1 + p - 1 \text{ and } q = m + p - 1) \\ &= [1, m] + (p - 1) \\ &= [m] + (p - 1) \qquad (\text{since } [1, m] = m). \end{aligned} \tag{10}$$

Therefore, the map $i \mapsto i + (p - 1)$ is a bijection from $[m]$ to T. This bijection induces a bijection $J \mapsto J + (p - 1)$ from the set of all subsets of $[m]$

to the set of all subsets of T. Hence, any subset S of T has the form $S = S' + (p-1)$ for a unique subset $S' \subseteq [m]$. In particular, the subsets U and V of T thus have the forms $U = P + (p-1)$ and $V = Q + (p-1)$ for unique subsets $P, Q \subseteq [m]$. Consider these P, Q. Clearly, $|U| = |P + (p-1)| = |P|$ and $|V| = |Q + (p-1)| = |Q|$, so that $|P| = |U| = |V| = |Q|$.

Let $\widetilde{P} := [m] \setminus P$ and $\widetilde{Q} := [m] \setminus Q$ be the complements of P and Q within $[m]$. Let $\sum P$ be the sum of all elements of P, and let $\sum Q$ be the sum of all elements of Q. From $U = P + (p-1)$, we obtain

$$\sum U = \sum (P + (p-1)) = \sum P + |P| \cdot (p-1)$$

(since the elements of $P + (p-1)$ are simply the $|P|$ elements of P with $p-1$ added to each). Likewise, $\sum V = \sum Q + |Q| \cdot (p-1)$. Adding these two equalities together, we find

$$\begin{aligned} \sum U + \sum V &= \left(\sum P + |P| \cdot (p-1)\right) + \left(\sum Q + |Q| \cdot (p-1)\right) \\ &= \sum P + \sum Q + \underbrace{(|P| + |Q|)}_{\substack{=|Q|+|Q| \\ (\text{since } |P|=|Q|)}} \cdot (p-1) \\ &= \sum P + \sum Q + \underbrace{(|Q| + |Q|)}_{=2|Q| \equiv 0 \bmod 2} \cdot (p-1) \\ &\equiv \sum P + \sum Q \bmod 2. \end{aligned}$$

Hence,

$$(-1)^{\sum U + \sum V} = (-1)^{\sum P + \sum Q}. \tag{11}$$

Next, we recall that $A = \operatorname{sub}_T^T B = \operatorname{sub}_{[p,q]}^{[p,q]} B$ (since $T = [p,q]$). Hence, for all $i, j \in [m]$, the (i,j)-th entry of A is the $(i+p-1, j+p-1)$-th entry of B. Consequently, for any $X, Y \subseteq [m]$, we have

$$\operatorname{sub}_X^Y A = \operatorname{sub}_{X+(p-1)}^{Y+(p-1)} B.$$

Applying this to $X = P$ and $Y = Q$, we obtain

$$\operatorname{sub}_P^Q A = \operatorname{sub}_{P+(p-1)}^{Q+(p-1)} B = \operatorname{sub}_U^V B \tag{12}$$

(since $P + (p-1) = U$ and $Q + (p-1) = V$).

Furthermore, from $\widetilde{P} = [m] \setminus P$, we obtain

$$\begin{aligned} \widetilde{P} + (p-1) &= ([m] \setminus P) + (p-1) \\ &= \underbrace{([m] + (p-1))}_{=T} \setminus \underbrace{(P + (p-1))}_{=U} \qquad \text{(by (9))} \\ &= T \setminus U. \end{aligned}$$

Similarly, $\widetilde{Q} + (p-1) = T \setminus V$.

However, (7) says that $A^{-1} = \operatorname{sub}_T^T \left(B^{-1}\right) = \operatorname{sub}_{[p,q]}^{[p,q]} \left(B^{-1}\right)$ (since $T = [p,q]$). Thus, for all $i, j \in [m]$, the (i,j)-th entry of A^{-1} is the $(i+p-1, j+p-1)$-th entry of B^{-1}. Consequently, for any $X, Y \subseteq [m]$, we have

$$\operatorname{sub}_X^Y \left(A^{-1}\right) = \operatorname{sub}_{X+(p-1)}^{Y+(p-1)} \left(B^{-1}\right).$$

Applying this to $X = \widetilde{Q}$ and $Y = \widetilde{P}$, we obtain

$$\operatorname{sub}_{\widetilde{Q}}^{\widetilde{P}} \left(A^{-1}\right) = \operatorname{sub}_{\widetilde{Q}+(p-1)}^{\widetilde{P}+(p-1)} \left(B^{-1}\right) = \operatorname{sub}_{T \setminus V}^{T \setminus U} \left(B^{-1}\right) \tag{13}$$

(since $\widetilde{P} + (p-1) = T \setminus U$ and $\widetilde{Q} + (p-1) = T \setminus V$).

Now, Lemma B.2 yields

$$\begin{aligned} \det\left(\operatorname{sub}_P^Q A\right) &= \underbrace{(-1)^{\sum P + \sum Q}}_{\substack{=(-1)^{\sum U + \sum V} \\ \text{(by (11))}}} \underbrace{\det A}_{\substack{=1 \\ \text{(by (8))}}} \cdot \det\left(\operatorname{sub}_{\widetilde{Q}}^{\widetilde{P}} \left(A^{-1}\right)\right) \\ &= (-1)^{\sum U + \sum V} \cdot \det\left(\operatorname{sub}_{\widetilde{Q}}^{\widetilde{P}} \left(A^{-1}\right)\right). \end{aligned}$$

In view of (12) and (13), we can rewrite this as

$$\det\left(\operatorname{sub}_U^V B\right) = (-1)^{\sum U + \sum V} \det\left(\operatorname{sub}_{T \setminus V}^{T \setminus U} \left(B^{-1}\right)\right).$$

Thus, Lemma B.3 is proved. ∎

Lemma B.4. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_q)$ be a partition, and let $\lambda' = \left(\lambda'_1, \lambda'_2, \ldots, \lambda'_p\right)$ be its conjugate partition. (The entries λ_i and λ'_j are allowed to be 0.) For each $i \in [q]$, let us set $\alpha_i := \lambda_i - i$. For each $j \in [p]$, let us set $\beta_j := \lambda'_j - j$ and $\eta_j := -1 - \beta_j$. Then, the two sets $\{\alpha_1, \alpha_2, \ldots, \alpha_q\}$ and $\{\eta_1, \eta_2, \ldots, \eta_p\}$ are disjoint, and their union is the integer interval $[-q, p-1]$.

Lemma B.4 is Proposition 3.18 **(f)** in the detailed version of the paper [Grinbe19] (this detailed version is downloadable from the arXiv as an ancillary file). (Note that the μ in the paper corresponds to our λ', and that the conditions $p \geq \lambda_1$ and $q \geq \mu_1$ in the paper follow from our assumptions $\lambda' = \left(\lambda'_1, \lambda'_2, \ldots, \lambda'_p\right)$ and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_q)$, respectively.) ∎

Lemma B.5. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_q)$ be a partition, and let $\lambda' = \left(\lambda'_1, \lambda'_2, \ldots, \lambda'_p\right)$ be its conjugate partition. (The entries λ_i and λ'_j are allowed to be 0.)

Let $\mu = (\mu_1, \mu_2, \ldots, \mu_q)$ be a partition, and let $\mu' = \left(\mu'_1, \mu'_2, \ldots, \mu'_p\right)$ be its conjugate partition. (The entries μ_i and μ'_j are allowed to be 0.)

Let $B = (b_{i,j})_{i,j\in\mathbb{Z}} \in \mathrm{UT}_{\mathbb{Z}}$ be an upper unitriangular matrix. Let $(c_{i,j})_{i,j\in\mathbb{Z}} = B^{-1}$ be its inverse matrix. Then,

$$\begin{aligned}&\det\left(b_{\mu_i - i,\ \lambda_j - j}\right)_{i,j\in[q]}\\ &= (-1)^{(\lambda_1+\lambda_2+\cdots+\lambda_q)+(\mu_1+\mu_2+\cdots+\mu_q)} \det\left(c_{i-\lambda'_i-1,\ j-\mu'_j-1}\right)_{i,j\in[p]}.\end{aligned}$$

Proof of Lemma B.5. Let T be the integer interval $[-q, p-1]$. For each $i \in [q]$, let us set $\alpha_i := \lambda_i - i$ and $\gamma_i := \mu_i - i$. For each $j \in [p]$, let us set $\beta_j := \lambda'_j - j$ and $\eta_j := -1 - \beta_j$ and $\delta_j := \mu'_j - j$ and $\omega_j := -1 - \delta_j$.

Then, Lemma B.4 says that the two sets $\{\alpha_1, \alpha_2, \ldots, \alpha_q\}$ and $\{\eta_1, \eta_2, \ldots, \eta_p\}$ are disjoint, and their union is the integer interval $[-q, p-1]$. Therefore, the set $\{\eta_1, \eta_2, \ldots, \eta_p\}$ is the complement of the set $\{\alpha_1, \alpha_2, \ldots, \alpha_q\}$ in the interval $[-q, p-1]$. In other words,

$$\begin{aligned}\{\eta_1, \eta_2, \ldots, \eta_p\} &= \underbrace{[-q, p-1]}_{=T} \setminus \{\alpha_1, \alpha_2, \ldots, \alpha_q\}\\ &= T \setminus \{\alpha_1, \alpha_2, \ldots, \alpha_q\}. \qquad (14)\end{aligned}$$

The same argument (applied to μ, μ', γ_i, δ_j and ω_j instead of λ, λ', α_i, β_j and η_j) yields

$$\{\omega_1, \omega_2, \ldots, \omega_p\} = T \setminus \{\gamma_1, \gamma_2, \ldots, \gamma_q\}. \qquad (15)$$

Since λ is a partition, we have $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_q$ and thus $\lambda_1 - 1 > \lambda_2 - 2 > \cdots > \lambda_q - q$. In other words, $\alpha_1 > \alpha_2 > \cdots > \alpha_q$ (since $\alpha_i = \lambda_i - i$ for each i). Hence,

$$\{\alpha_1, \alpha_2, \ldots, \alpha_q\} = \{\alpha_q < \alpha_{q-1} < \cdots < \alpha_1\}. \qquad (16)$$

Similarly,

$$\{\gamma_1, \gamma_2, \ldots, \gamma_q\} = \{\gamma_q < \gamma_{q-1} < \cdots < \gamma_1\}. \qquad (17)$$

These two equalities show that both sets $\{\gamma_1, \gamma_2, \ldots, \gamma_q\}$ and $\{\alpha_1, \alpha_2, \ldots, \alpha_q\}$ have size q, so that they have the same size. In other words, $|\{\gamma_1, \gamma_2, \ldots, \gamma_q\}| = |\{\alpha_1, \alpha_2, \ldots, \alpha_q\}|$.

Furthermore, recall that the two sets $\left\{\alpha_1, \alpha_2, \ldots, \alpha_q\right\}$ and $\left\{\eta_1, \eta_2, \ldots, \eta_p\right\}$ are disjoint, and their union is the integer interval $[-q, p-1]$. Hence, in particular, $\left\{\alpha_1, \alpha_2, \ldots, \alpha_q\right\}$ is a subset of $[-q, p-1] = T$. Similarly, $\left\{\gamma_1, \gamma_2, \ldots, \gamma_q\right\}$ is a subset of T as well.

Thus, Lemma B.3 (applied to $-q$, $p-1$, $\left\{\gamma_1, \gamma_2, \ldots, \gamma_q\right\}$ and $\left\{\alpha_1, \alpha_2, \ldots, \alpha_q\right\}$ instead of p, q, U and V) yields

$$\det\left(\operatorname{sub}_{\left\{\gamma_1,\gamma_2,\ldots,\gamma_q\right\}}^{\left\{\alpha_1,\alpha_2,\ldots,\alpha_q\right\}} B\right) = (-1)^{\Sigma\left\{\gamma_1,\gamma_2,\ldots,\gamma_q\right\}+\Sigma\left\{\alpha_1,\alpha_2,\ldots,\alpha_q\right\}} \det\left(\operatorname{sub}_{T\setminus\left\{\alpha_1,\alpha_2,\ldots,\alpha_q\right\}}^{T\setminus\left\{\gamma_1,\gamma_2,\ldots,\gamma_q\right\}} \left(B^{-1}\right)\right). \tag{18}$$

In view of (16) and (17), we have

$$\operatorname{sub}_{\left\{\gamma_1,\gamma_2,\ldots,\gamma_q\right\}}^{\left\{\alpha_1,\alpha_2,\ldots,\alpha_q\right\}} B = \operatorname{sub}_{\left\{\gamma_q<\gamma_{q-1}<\cdots<\gamma_1\right\}}^{\left\{\alpha_q<\alpha_{q-1}<\cdots<\alpha_1\right\}} B = \left(b_{\gamma_{q+1-i},\alpha_{q+1-j}}\right)_{i,j\in[q]}$$

(since $B = \left(b_{i,j}\right)_{i,j\in\mathbb{Z}}$). Hence,

$$\det\left(\operatorname{sub}_{\left\{\gamma_1,\gamma_2,\ldots,\gamma_q\right\}}^{\left\{\alpha_1,\alpha_2,\ldots,\alpha_q\right\}} B\right) = \det\left(b_{\gamma_{q+1-i},\alpha_{q+1-j}}\right)_{i,j\in[q]} = \det\left(b_{\gamma_i,\alpha_j}\right)_{i,j\in[q]} \tag{19}$$

(here, we have substituted $q+1-i$ and $q+1-j$ for i and j in the matrix, which effectively rotates the matrix by 180°; this rotation does not change the determinant, because it is a composition of a row permutation and a column permutation with the same sign).

On the other hand, let us recall that $\left\{\eta_1, \eta_2, \ldots, \eta_p\right\} = T \setminus \left\{\alpha_1, \alpha_2, \ldots, \alpha_q\right\}$. But λ' is a partition; thus, $\lambda'_1 \geq \lambda'_2 \geq \cdots \geq \lambda'_p$ and therefore $\lambda'_1 - 1 > \lambda'_2 - 2 > \cdots > \lambda'_p - p$. In other words, $\beta_1 > \beta_2 > \cdots > \beta_p$ (since $\beta_j = \lambda'_j - j$ for each j). Hence, $-1-\beta_1 < -1-\beta_2 < \cdots < -1-\beta_p$. In other words, $\eta_1 < \eta_2 < \cdots < \eta_p$ (since $\eta_j = -1-\beta_j$ for each j). Thus, $\left\{\eta_1, \eta_2, \ldots, \eta_p\right\} = \left\{\eta_1 < \eta_2 < \cdots < \eta_p\right\}$. Comparing this with (14), we obtain

$$T \setminus \left\{\alpha_1, \alpha_2, \ldots, \alpha_q\right\} = \left\{\eta_1 < \eta_2 < \cdots < \eta_p\right\}.$$

The same argument (applied to μ, μ', γ_i, δ_j and ω_j instead of λ, λ', α_i, β_j and η_j) yields

$$T \setminus \left\{\gamma_1, \gamma_2, \ldots, \gamma_q\right\} = \left\{\omega_1 < \omega_2 < \cdots < \omega_p\right\}.$$

In view of these two equalities, we have

$$\operatorname{sub}_{T\setminus\left\{\alpha_1,\alpha_2,\ldots,\alpha_q\right\}}^{T\setminus\left\{\gamma_1,\gamma_2,\ldots,\gamma_q\right\}} \left(B^{-1}\right) = \operatorname{sub}_{\left\{\eta_1<\eta_2<\cdots<\eta_p\right\}}^{\left\{\omega_1<\omega_2<\cdots<\omega_p\right\}} \left(B^{-1}\right) = \left(c_{\eta_i,\omega_j}\right)_{i,j\in[p]}$$

(since $B^{-1} = \left(c_{i,j}\right)_{i,j\in\mathbb{Z}}$). Thus,

$$\det\left(\operatorname{sub}_{T\setminus\left\{\alpha_1,\alpha_2,\ldots,\alpha_q\right\}}^{T\setminus\left\{\gamma_1,\gamma_2,\ldots,\gamma_q\right\}}\left(B^{-1}\right)\right) = \det\left(c_{\eta_i,\omega_j}\right)_{i,j\in[p]}. \qquad (20)$$

Furthermore, $\alpha_1 > \alpha_2 > \cdots > \alpha_q$ shows that the numbers $\alpha_1, \alpha_2, \ldots, \alpha_q$ are distinct; hence,

$$\begin{aligned}\sum\left\{\alpha_1,\alpha_2,\ldots,\alpha_q\right\} &= \alpha_1+\alpha_2+\cdots+\alpha_q\\ &= \left(\lambda_1-1\right)+\left(\lambda_2-2\right)+\cdots+\left(\lambda_q-q\right)\\ &\qquad \left(\text{since } \alpha_i=\lambda_i-i \text{ for all } i\right)\\ &= \left(\lambda_1+\lambda_2+\cdots+\lambda_q\right)-\left(1+2+\cdots+q\right).\end{aligned}$$

Similarly,

$$\sum\left\{\gamma_1,\gamma_2,\ldots,\gamma_q\right\} = \left(\mu_1+\mu_2+\cdots+\mu_q\right)-\left(1+2+\cdots+q\right).$$

Adding these two equalities together, we find

$$\begin{aligned}&\sum\left\{\alpha_1,\alpha_2,\ldots,\alpha_q\right\}+\sum\left\{\gamma_1,\gamma_2,\ldots,\gamma_q\right\}\\ &= \left(\lambda_1+\lambda_2+\cdots+\lambda_q\right)-\left(1+2+\cdots+q\right)\\ &\qquad + \left(\mu_1+\mu_2+\cdots+\mu_q\right)-\left(1+2+\cdots+q\right)\\ &= \left(\lambda_1+\lambda_2+\cdots+\lambda_q\right)+\left(\mu_1+\mu_2+\cdots+\mu_q\right)-\underbrace{2\left(1+2+\cdots+q\right)}_{\equiv 0 \bmod 2}\\ &\equiv \left(\lambda_1+\lambda_2+\cdots+\lambda_q\right)+\left(\mu_1+\mu_2+\cdots+\mu_q\right) \bmod 2.\end{aligned}$$

Hence,

$$\begin{aligned}&(-1)^{\sum\left\{\alpha_1,\alpha_2,\ldots,\alpha_q\right\}+\sum\left\{\gamma_1,\gamma_2,\ldots,\gamma_q\right\}}\\ &= (-1)^{\left(\lambda_1+\lambda_2+\cdots+\lambda_q\right)+\left(\mu_1+\mu_2+\cdots+\mu_q\right)}. \qquad (21)\end{aligned}$$

Finally, using (19), we can rewrite (18) as

$$\begin{aligned}&\det\left(b_{\gamma_i,\alpha_j}\right)_{i,j\in[q]}\\ &= \underbrace{(-1)^{\sum\left\{\gamma_1,\gamma_2,\ldots,\gamma_q\right\}+\sum\left\{\alpha_1,\alpha_2,\ldots,\alpha_q\right\}}}_{\substack{=(-1)^{\sum\left\{\alpha_1,\alpha_2,\ldots,\alpha_q\right\}+\sum\left\{\gamma_1,\gamma_2,\ldots,\gamma_q\right\}}\\ =(-1)^{\left(\lambda_1+\lambda_2+\cdots+\lambda_q\right)+\left(\mu_1+\mu_2+\cdots+\mu_q\right)}\\ \text{(by (21))}}} \underbrace{\det\left(\operatorname{sub}_{T\setminus\left\{\alpha_1,\alpha_2,\ldots,\alpha_q\right\}}^{T\setminus\left\{\gamma_1,\gamma_2,\ldots,\gamma_q\right\}}\left(B^{-1}\right)\right)}_{\substack{=\det\left(c_{\eta_i,\omega_j}\right)_{i,j\in[p]}\\ \text{(by (20))}}}\\ &= (-1)^{\left(\lambda_1+\lambda_2+\cdots+\lambda_q\right)+\left(\mu_1+\mu_2+\cdots+\mu_q\right)}\det\left(c_{\eta_i,\omega_j}\right)_{i,j\in[p]}.\end{aligned}$$

In view of $\gamma_i = \mu_i - i$ and $\alpha_j = \lambda_j - j$ and $\eta_i = -1 - \underbrace{\beta_i}_{=\lambda'_i - i} = -1 - \left(\lambda'_i - i\right) = i - \lambda'_i - 1$ and $\omega_j = -1 - \underbrace{\delta_j}_{=\mu'_j - j} = -1 - \left(\mu'_j - j\right) = j - \mu'_j - 1$,

we can rewrite this as

$$\begin{aligned} &\det\left(b_{\mu_i - i,\ \lambda_j - j}\right)_{i,j\in[q]} \\ &= (-1)^{(\lambda_1+\lambda_2+\cdots+\lambda_q)+(\mu_1+\mu_2+\cdots+\mu_q)} \det\left(c_{i-\lambda'_i-1,\ j-\mu'_j-1}\right)_{i,j\in[p]}. \end{aligned}$$

This proves Lemma B.5. ■

Lemma B.6. Let $\left(u_{i,j}\right)_{i,j\in[p]}$ be a $p\times p$-matrix. Furthermore, let $\alpha_1, \alpha_2, \ldots, \alpha_p, \beta_1, \beta_2, \ldots, \beta_p$ be any $2p$ scalars. Then,

$$\det\left(\alpha_i\beta_j u_{i,j}\right)_{i,j\in[p]} = \left(\prod_{i=1}^{p}\alpha_i\right)\left(\prod_{j=1}^{p}\beta_j\right)\cdot\det\left(u_{i,j}\right)_{i,j\in[p]}.$$

Proof of Lemma B.6. The matrix $\left(\alpha_i\beta_j u_{i,j}\right)_{i,j\in[p]}$ is obtained from the matrix $\left(u_{i,j}\right)_{i,j\in[p]}$ by

- multiplying the i-th row by α_i for each $i\in[p]$, and then
- multiplying the j-th column by β_j for each $j\in[p]$.

Each of these operations multiplies the determinant of the matrix by the corresponding factor α_i or β_j. Thus, in total, the determinant gets multiplied by $\left(\prod_{i=1}^{p}\alpha_i\right)\left(\prod_{j=1}^{p}\beta_j\right)$. This proves Lemma B.6. ■

Now, we can prove (6.9):

Proof of (6.9): Write the partitions λ, λ', μ and μ' in the forms

$$\begin{aligned} \lambda &= \left(\lambda_1, \lambda_2, \ldots, \lambda_q\right), & \lambda' &= \left(\lambda'_1, \lambda'_2, \ldots, \lambda'_p\right), \\ \mu &= \left(\mu_1, \mu_2, \ldots, \mu_q\right), & \mu' &= \left(\mu'_1, \mu'_2, \ldots, \mu'_p\right) \end{aligned}$$

for some $p, q\in\mathbb{N}$ (where, of course, the entries λ_i, λ'_j, μ_i and μ'_j are allowed to be 0).

Recall that

$$\mathbf{H}\left(x\mid a\right) = \left(h_{j-i}\left(x\mid\tau^{i+1}a\right)\right)_{i,j\in\mathbb{Z}} \qquad \text{and}$$

$$\mathbf{E}\left(x\mid a\right) = \left((-1)^{j-i}e_{j-i}\left(x\mid\tau^{j}a\right)\right)_{i,j\in\mathbb{Z}}$$

are two upper unitriangular matrices in $\operatorname{UT}_{\mathbb{Z}}$ (since $e_0(x \mid a) = 0$ and $e_k(x \mid a) = 0$ whenever $k < 0$). From (6.6), we know that the matrix $\mathbf{E}(x \mid a)$ is the inverse matrix of $\mathbf{H}(x \mid a)$. Hence, Lemma B.5 (applied to $B = \mathbf{H}(x \mid a)$ and $b_{i,j} = h_{j-i}\left(x \mid \tau^{i+1}a\right)$ and $c_{i,j} = (-1)^{j-i} e_{j-i}\left(x \mid \tau^j a\right)$) yields

$$\begin{aligned}
&\det\left(h_{(\lambda_j - j)-(\mu_i - i)}\left(x \mid \tau^{\mu_i - i+1}a\right)\right)_{i,j\in[q]} \\
&= (-1)^{(\lambda_1+\lambda_2+\cdots+\lambda_q)+(\mu_1+\mu_2+\cdots+\mu_q)} \\
&\qquad \det\left((-1)^{\left(j-\mu'_j-1\right)-\left(i-\lambda'_i-1\right)} e_{\left(j-\mu'_j-1\right)-\left(i-\lambda'_i-1\right)}\left(x \mid \tau^{j-\mu'_j-1}a\right)\right)_{i,j\in[p]}.
\end{aligned}$$

In view of $(\lambda_j - j) - (\mu_i - i) = \lambda_j - \mu_i - j + i$ and $\left(j - \mu'_j - 1\right) - \left(i - \lambda'_i - 1\right) = \lambda'_i - \mu'_j - i + j$, we can rewrite this as

$$\begin{aligned}
&\det\left(h_{\lambda_j-\mu_i-j+i}\left(x \mid \tau^{\mu_i-i+1}a\right)\right)_{i,j\in[q]} \\
&= (-1)^{(\lambda_1+\lambda_2+\cdots+\lambda_q)+(\mu_1+\mu_2+\cdots+\mu_q)} \\
&\qquad \det\left((-1)^{\lambda'_i-\mu'_j-i+j} e_{\lambda'_i-\mu'_j-i+j}\left(x \mid \tau^{j-\mu'_j-1}a\right)\right)_{i,j\in[p]}.
\end{aligned}$$

In view of

$$\det\left(h_{\lambda_j-\mu_i-j+i}\left(x \mid \tau^{\mu_i-i+1}a\right)\right)_{i,j\in[q]} = \det\left(h_{\lambda_i-\mu_j-i+j}\left(x \mid \tau^{\mu_j-j+1}a\right)\right)_{i,j\in[q]}$$

(since the determinant of a matrix does not change when we transpose it) and

$$\begin{aligned}
&\det\left(\underbrace{(-1)^{\lambda'_i-\mu'_j-i+j}}_{=(-1)^{\lambda'_i-i}(-1)^{\mu'_j-j}} e_{\lambda'_i-\mu'_j-i+j}\left(x \mid \tau^{j-\mu'_j-1}a\right)\right)_{i,j\in[p]} \\
&= \det\left((-1)^{\lambda'_i-i}(-1)^{\mu'_j-j} e_{\lambda'_i-\mu'_j-i+j}\left(x \mid \tau^{j-\mu'_j-1}a\right)\right)_{i,j\in[p]} \\
&= \left(\prod_{i=1}^{p}(-1)^{\lambda'_i-i}\right)\left(\prod_{j=1}^{p}(-1)^{\mu'_j-j}\right)\cdot\det\left(e_{\lambda'_i-\mu'_j-i+j}\left(x \mid \tau^{j-\mu'_j-1}a\right)\right)_{i,j\in[p]}
\end{aligned}$$

(by Lemma B.6, applied to $\alpha_i = (-1)^{\lambda'_i-i}$ and $\beta_j = (-1)^{\mu'_j-j}$ and $u_{i,j} =$

$e_{\lambda'_i-\mu'_j-i+j}\left(x \mid \tau^{j-\mu'_j-1}a\right)$), we can rewrite this as

$$\begin{aligned}
&\det\left(h_{\lambda_i-\mu_j-i+j}\left(x \mid \tau^{\mu_j-j+1}a\right)\right)_{i,j\in[q]} \\
&= (-1)^{(\lambda_1+\lambda_2+\cdots+\lambda_q)+(\mu_1+\mu_2+\cdots+\mu_q)} \\
&\qquad \left(\prod_{i=1}^{p}(-1)^{\lambda'_i-i}\right)\left(\prod_{j=1}^{p}(-1)^{\mu'_j-j}\right)\cdot\det\left(e_{\lambda'_i-\mu'_j-i+j}\left(x \mid \tau^{j-\mu'_j-1}a\right)\right)_{i,j\in[p]} \\
&= (-1)^{(\lambda_1+\lambda_2+\cdots+\lambda_q)+(\mu_1+\mu_2+\cdots+\mu_q)}\left(\prod_{i=1}^{p}(-1)^{\lambda'_i-i}\right)\left(\prod_{j=1}^{p}(-1)^{\mu'_j-j}\right) \\
&\qquad \cdot\det\left(e_{\lambda'_i-\mu'_j-i+j}\left(x \mid \tau^{j-\mu'_j-1}a\right)\right)_{i,j\in[p]}. \qquad (22)
\end{aligned}$$

However,

$$\begin{aligned}
&(-1)^{(\lambda_1+\lambda_2+\cdots+\lambda_q)+(\mu_1+\mu_2+\cdots+\mu_q)}\left(\prod_{i=1}^{p}(-1)^{\lambda'_i-i}\right)\left(\prod_{j=1}^{p}(-1)^{\mu'_j-j}\right) \\
&= (-1)^{(\lambda_1+\lambda_2+\cdots+\lambda_q)+(\mu_1+\mu_2+\cdots+\mu_q)+\sum_{i=1}^{p}(\lambda'_i-i)+\sum_{j=1}^{p}(\mu'_j-j)} \\
&= 1,
\end{aligned}$$

since

$$\begin{aligned}
&\underbrace{(\lambda_1+\lambda_2+\cdots+\lambda_q)}_{=|\lambda|}+\underbrace{(\mu_1+\mu_2+\cdots+\mu_q)}_{=|\mu|}+\underbrace{\sum_{i=1}^{p}(\lambda'_i-i)}_{=\sum_{i=1}^{p}\lambda'_i-\sum_{i=1}^{p}i}+\underbrace{\sum_{j=1}^{p}\left(\mu'_j-j\right)}_{=\sum_{j=1}^{p}\mu'_j-\sum_{j=1}^{p}j} \\
&= |\lambda|+|\mu|+\underbrace{\sum_{i=1}^{p}\lambda'_i}_{=|\lambda'|=|\lambda|}-\underbrace{\sum_{i=1}^{p}i}_{=1+2+\cdots+p}+\underbrace{\sum_{j=1}^{p}\mu'_j}_{=|\mu'|=|\mu|}-\underbrace{\sum_{j=1}^{p}j}_{=1+2+\cdots+p} \\
&= |\lambda|+|\mu|+|\lambda|+(1+2+\cdots+p)+|\mu|+(1+2+\cdots+p) \\
&= 2\left(|\lambda|+|\mu|+(1+2+\cdots+p)\right) \qquad \text{is even.}
\end{aligned}$$

Thus, we can rewrite (22) as

$$\begin{aligned}
&\det\left(h_{\lambda_i-\mu_j-i+j}\left(x \mid \tau^{\mu_j-j+1}a\right)\right)_{i,j\in[q]} \\
&= 1\cdot\det\left(e_{\lambda'_i-\mu'_j-i+j}\left(x \mid \tau^{j-\mu'_j-1}a\right)\right)_{i,j\in[p]} \\
&= \det\left(e_{\lambda'_i-\mu'_j-i+j}\left(x \mid \tau^{j-\mu'_j-1}a\right)\right)_{i,j\in[p]} \\
&= \det\left(e_{\lambda'_i-\mu'_j-i+j}\left(x \mid \tau^{-\mu'_j+j-1}a\right)\right)_{i,j\in[p]} \qquad \left(\text{since } j-\mu'_j-1=-\mu'_j+j-1\right).
\end{aligned}$$

In view of (6.8), this can be rewritten as

$$s_{\lambda/\mu}(x \mid a) = \det\left(e_{\lambda'_i-\mu'_j-i+j}\left(x \mid \tau^{-\mu'_j+j-1}a\right)\right)_{i,j\in[p]}.$$

This proves (6.9). ■

Applying (6.9) to $\mu = \varnothing$, we obtain the second equality in (6.7).

6. **page 18, (6.9):** See the previous bullet point for a proof of (6.9).

7. **page 18, (6.10):** Let me prove (6.10) here. We will need a simple lemma about determinants.

If p and q are two integers such that $p \leq q+1$, then $[p,q]$ shall denote the set of all integers m such that $p \leq m \leq q$. We call this set an *integer interval*. It has size $q-p+1$ (so it is empty if $p = q+1$).

For a given $r \in \mathbb{N}$, we shall denote the integer interval $[1,r] = \{1,2,\ldots,r\}$ by $[r]$.

Our lemma says the following:

Lemma B.7. Let $(u_{i,j})_{i,j\in[p]}$ be a $p\times p$-matrix for some $p \in \mathbb{N}$. Let U and V be two subsets of $[p]$ satisfying $|U|+|V| > p$. Assume that

$$u_{i,j} = 0 \qquad \text{for all } i \in U \text{ and } j \in V. \tag{23}$$

Then, $\det (u_{i,j})_{i,j\in[p]} = 0$.

Proof of Lemma B.7. By the definition of a determinant, we have

$$\det (u_{i,j})_{i,j\in[p]} = \sum_{\sigma\in S_p} \operatorname{sign}\sigma \cdot \prod_{i=1}^{p} u_{i,\sigma(i)} \tag{24}$$

(where S_p is the symmetric group of all permutations of $[p]$). Now we shall show that each $\sigma \in S_p$ satisfies

$$\prod_{i=1}^{p} u_{i,\sigma(i)} = 0. \tag{25}$$

(*Proof:* Let $\sigma \in S_p$. Then, $|\sigma(U)| = |U| > p - |V|$ (since $|U|+|V| > p$), so that $\sigma(U) \not\subseteq [p]\setminus V$ (since $\sigma(U) \subseteq [p]\setminus V$ would entail $|\sigma(U)| \leq |[p]\setminus V| = p-|V|$, contradicting $|\sigma(U)| > p-|V|$). In other words, there exists some $j \in \sigma(U)$ such that $j \notin [p]\setminus V$. Consider this j. From $j \in \sigma(U)$, we obtain $j = \sigma(k)$ for some $k \in U$. Consider this k. Now, $k \in U$ and $j \in V$ (since $j \in [p]$ but $j \notin [p]\setminus V$). Hence, (23) (applied to $i = k$) yields $u_{k,j} = 0$.

In other words, $u_{k,\sigma(k)} = 0$ (since $j = \sigma(k)$). Thus, one of the factors of the product $\prod_{i=1}^{p} u_{i,\sigma(i)}$ is 0 (namely, the k-th factor). Hence, this whole product is 0. This proves (25).)

Now, (24) becomes

$$\det\left(u_{i,j}\right)_{i,j\in[p]} = \sum_{\sigma\in S_p} \operatorname{sign}\sigma \cdot \underbrace{\prod_{i=1}^{p} u_{i,\sigma(i)}}_{\substack{=0\\ \text{(by (24))}}} = 0.$$

This proves Lemma B.7. ■

We can now prove (6.10):

Proof of (6.10). Assume that we don't have $0 \leq \lambda_i' - \mu_i' \leq n$ for all i. We must prove that $s_{\lambda/\mu}(x \mid a) = 0$.

Write the partitions λ' and μ' as $\lambda' = \left(\lambda_1', \lambda_2', \ldots, \lambda_p'\right)$ and $\mu' = \left(\mu_1', \mu_2', \ldots, \mu_p'\right)$. Then, all $i > p$ satisfy $\lambda_i' = 0$ and $\mu_i' = 0$ and therefore $\lambda_i' - \mu_i' = 0 - 0 = 0$. Note also that $\lambda_1' \geq \lambda_2' \geq \lambda_3' \geq \cdots$ (since λ' is a partition) and $\mu_1' \geq \mu_2' \geq \mu_3' \geq \cdots$ (similarly). From (6.9), we obtain

$$s_{\lambda/\mu}(x \mid a) = \det\left(e_{\lambda_i'-\mu_j'-i+j}\left(x \mid \tau^{-\mu_j'+j-1}a\right)\right)_{i,j\in[p]}. \tag{26}$$

We have assumed that we don't have $0 \leq \lambda_i' - \mu_i' \leq n$ for all i. In other words, there exists some $i \geq 1$ such that we don't have $0 \leq \lambda_i' - \mu_i' \leq n$. Consider this i, and denote it by k. Thus, we don't have $0 \leq \lambda_k' - \mu_k' \leq n$. Hence, we have either $\lambda_k' - \mu_k' < 0$ or $\lambda_k' - \mu_k' > n$. We are thus in one of the following two cases:

Case 1: We have $\lambda_k' - \mu_k' < 0$.

Case 2: We have $\lambda_k' - \mu_k' > n$.

Consider Case 1 first. In this case, we have $\lambda_k' - \mu_k' < 0$. Hence, we cannot have $k > p$ (since all $i > p$ satisfy $\lambda_i' - \mu_i' = 0$, which would yield $\lambda_k' - \mu_k' = 0$ if we had $k > p$). Thus, $k \leq p$, so that $k \in [p]$.

Now, let $i \in [k, p]$ and $j \in [k]$. Then, $i \geq k$ (since $i \in [k, p]$), so that $k \leq i$ and thus $\lambda_k' \geq \lambda_i'$ (since $\lambda_1' \geq \lambda_2' \geq \lambda_3' \geq \cdots$). Thus, $\lambda_i' \leq \lambda_k'$. Furthermore, $j \leq k$ (since $j \in [k]$), so that $\mu_j' \geq \mu_k'$. Thus,

$$\underbrace{\lambda_i'}_{\leq \lambda_k'} - \underbrace{\mu_j'}_{\geq \mu_k'} - \underbrace{i}_{\geq k} + \underbrace{j}_{\leq k} \leq \lambda_k' - \mu_k' - k + k = \lambda_k' - \mu_k' < 0,$$

and therefore $e_{\lambda_i'-\mu_j'-i+j}\left(x \mid \tau^{-\mu_j'+j-1}a\right) = 0$ (since $e_\ell\left(x \mid \tau^{-\mu_j'+j-1}a\right) = 0$ for any $\ell < 0$).

Forget that we fixed i and j. We thus have shown that

$$e_{\lambda'_i-\mu'_j-i+j}\left(x \mid \tau^{-\mu'_j+j-1}a\right) = 0 \qquad \text{for all } i \in [k,p] \text{ and } j \in [k].$$

Hence, Lemma B.7 (applied to $u_{i,j} = e_{\lambda'_i-\mu'_j-i+j}\left(x \mid \tau^{-\mu'_j+j-1}a\right)$) yields that

$$\det\left(e_{\lambda'_i-\mu'_j-i+j}\left(x \mid \tau^{-\mu'_j+j-1}a\right)\right)_{i,j\in[p]} = 0$$

(since $\underbrace{|[k,p]|}_{=p-k+1} + \underbrace{|[k]|}_{=k} = (p-k+1)+k = p+1 > p$). Hence, (26) rewrites as $s_{\lambda/\mu}(x \mid a) = 0$. This proves (6.10) in Case 1.

Let us now consider Case 2. In this case, we have $\lambda'_k - \mu'_k > n$. Hence, we cannot have $k > p$ (since all $i > p$ satisfy $\lambda'_i - \mu'_i = 0$, which would yield $\lambda'_k - \mu'_k = 0 \leq n$ if we had $k > p$). Thus, $k \leq p$, so that $k \in [p]$.

Now, let $i \in [k]$ and $j \in [k,p]$. Then, $j \geq k$ (since $j \in [k,p]$), so that $k \leq j$ and thus $\mu'_k \geq \mu'_j$ (since $\mu'_1 \geq \mu'_2 \geq \mu'_3 \geq \cdots$). Hence, $\mu'_j \leq \mu'_k$. Furthermore, $i \leq k$ (since $i \in [k]$), so that $\lambda'_i \geq \lambda'_k$. Thus,

$$\underbrace{\lambda'_i}_{\geq\lambda'_k} - \underbrace{\mu'_j}_{\leq\mu'_k} - \underbrace{i}_{\leq k} + \underbrace{j}_{\geq k} \geq \lambda'_k - \mu'_k - k + k = \lambda'_k - \mu'_k > n,$$

and therefore $e_{\lambda'_i-\mu'_j-i+j}\left(x \mid \tau^{-\mu'_j+j-1}a\right) = 0$ (since $e_\ell\left(x \mid \tau^{-\mu'_j+j-1}a\right) = 0$ for any $\ell > n$).

Forget that we fixed i and j. We thus have shown that

$$e_{\lambda'_i-\mu'_j-i+j}\left(x \mid \tau^{-\mu'_j+j-1}a\right) = 0 \qquad \text{for all } i \in [k] \text{ and } j \in [k,p].$$

Hence, Lemma B.7 (applied to $u_{i,j} = e_{\lambda'_i-\mu'_j-i+j}\left(x \mid \tau^{-\mu'_j+j-1}a\right)$) yields that

$$\det\left(e_{\lambda'_i-\mu'_j-i+j}\left(x \mid \tau^{-\mu'_j+j-1}a\right)\right)_{i,j\in[p]} = 0$$

(since $\underbrace{|[k]|}_{=k} + \underbrace{|[k,p]|}_{=p-k+1} = k+(p-k+1) = p+1 > p$). Hence, (26) rewrites as $s_{\lambda/\mu}(x \mid a) = 0$. This proves (6.10) in Case 2.

We have now proved (6.10) in both Cases 1 and 2; this completes the proof of (6.10). ■

8. **page 19, proof of (6.12):** In the last displayed equation of the proof, replace "$e_{j-k}\left(y \mid \tau^{n+j}a\right)$" by "$e_{j-i}\left(y \mid \tau^{n+j}a\right)$".

9. **page 19, proof of (6.13):** Let me explain in some more detail how the equality
$$\bigwedge^r \mathbf{H}(x, y \mid a) = \bigwedge^r \mathbf{H}(x \mid a) \cdot \bigwedge^r \mathbf{H}(y \mid \tau^n a)$$
yields the claim (6.13).

Indeed, a well-known corollary of the Cauchy–Binet theorem (specifically, [Grinbe20, Corollary 7.182], or rather its version for infinite matrices[2]) yields
$$\begin{aligned}
&\det\left(\operatorname{sub}^{\{\lambda_1-1,\lambda_2-2,\ldots,\lambda_r-r\}}_{\{\mu_1-1,\mu_2-2,\ldots,\mu_r-r\}}\left(\mathbf{H}(x \mid a)\cdot\mathbf{H}(y \mid \tau^n a)\right)\right)\\
&= \sum_{g_1<g_2<\cdots<g_r} \det\left(\operatorname{sub}^{\{g_1,g_2,\ldots,g_r\}}_{\{\mu_1-1,\mu_2-2,\ldots,\mu_r-r\}}\left(\mathbf{H}(x \mid a)\right)\right)\\
&\qquad\qquad \cdot \det\left(\operatorname{sub}^{\{\lambda_1-1,\lambda_2-2,\ldots,\lambda_r-r\}}_{\{g_1,g_2,\ldots,g_r\}}\left(\mathbf{H}(y \mid \tau^n a)\right)\right),
\end{aligned}$$
where the sum ranges over all strictly increasing r-tuples $(g_1 < g_2 < \cdots < g_r)$ of integers. Using (6.12)(ii), we can rewrite this as
$$\begin{aligned}
&\det\left(\operatorname{sub}^{\{\lambda_1-1,\lambda_2-2,\ldots,\lambda_r-r\}}_{\{\mu_1-1,\mu_2-2,\ldots,\mu_r-r\}}\left(\mathbf{H}(xy \mid a)\right)\right)\\
&= \sum_{g_1<g_2<\cdots<g_r} \det\left(\operatorname{sub}^{\{g_1,g_2,\ldots,g_r\}}_{\{\mu_1-1,\mu_2-2,\ldots,\mu_r-r\}}\left(\mathbf{H}(x \mid a)\right)\right)\\
&\qquad\qquad \cdot \det\left(\operatorname{sub}^{\{\lambda_1-1,\lambda_2-2,\ldots,\lambda_r-r\}}_{\{g_1,g_2,\ldots,g_r\}}\left(\mathbf{H}(y \mid \tau^n a)\right)\right)\\
&= \sum_{\substack{S \text{ is an } r\text{-element}\\ \text{set of integers}}} \det\left(\operatorname{sub}^{S}_{\{\mu_1-1,\mu_2-2,\ldots,\mu_r-r\}}\left(\mathbf{H}(x \mid a)\right)\right)\\
&\qquad\qquad \cdot \det\left(\operatorname{sub}^{\{\lambda_1-1,\lambda_2-2,\ldots,\lambda_r-r\}}_{S}\left(\mathbf{H}(y \mid \tau^n a)\right)\right)\\
&\qquad\qquad \left(\begin{array}{c}\text{here, we have substituted } S\\ \text{for the set } \{g_1, g_2, \ldots, g_r\}\end{array}\right)\\
&= \sum_{\substack{\nu \text{ is a partition}\\ \text{of length } \leq r}} \det\left(\operatorname{sub}^{\{\nu_1-1,\nu_2-2,\ldots,\nu_r-r\}}_{\{\mu_1-1,\mu_2-2,\ldots,\mu_r-r\}}\left(\mathbf{H}(x \mid a)\right)\right)\\
&\qquad\qquad \cdot \det\left(\operatorname{sub}^{\{\lambda_1-1,\lambda_2-2,\ldots,\lambda_r-r\}}_{\{\nu_1-1,\nu_2-2,\ldots,\nu_r-r\}}\left(\mathbf{H}(y \mid \tau^n a)\right)\right) \qquad (27)
\end{aligned}$$
(here, we have reindexed the sum, since each r-element set S of integers can be written uniquely in the form $\{\nu_1 - 1, \nu_2 - 2, \ldots, \nu_r - r\}$ for a partition ν of length $\leq r$, and conversely, any set of the latter form is an r-element set of integers).

[2]The version for infinite matrices is proved in the same way as the version for finite matrices, as long as (formal) convergence is taken care of (and that is easy when the matrices in question are upper-triangular).

However, in the second sentence of the proof of (6.13), it was said that

$$s_{\lambda/\mu}(x,y \mid a) = \det\left(\operatorname{sub}^{\{\lambda_1-1,\lambda_2-2,\ldots,\lambda_r-r\}}_{\{\mu_1-1,\mu_2-2,\ldots,\mu_r-r\}}(\mathbf{H}(xy \mid a))\right).$$

Similarly, for any partition ν of length $\leq n$, we have

$$s_{\nu/\mu}(x \mid a) = \det\left(\operatorname{sub}^{\{\nu_1-1,\nu_2-2,\ldots,\nu_r-r\}}_{\{\mu_1-1,\mu_2-2,\ldots,\mu_r-r\}}(\mathbf{H}(x \mid a))\right)$$

and

$$s_{\lambda/\nu}(y \mid \tau^n a) = \det\left(\operatorname{sub}^{\{\lambda_1-1,\lambda_2-2,\ldots,\lambda_r-r\}}_{\{\nu_1-1,\nu_2-2,\ldots,\nu_r-r\}}(\mathbf{H}(y \mid \tau^n a))\right).$$

In view of these three equalities, we can rewrite (27) as

$$\begin{aligned} s_{\lambda/\mu}(x,y \mid a) &= \sum_{\substack{\nu \text{ is a partition} \\ \text{of length } \leq r}} s_{\nu/\mu}(x \mid a)\, s_{\lambda/\nu}(y \mid \tau^n a) \\ &= \sum_{\nu \text{ is a partition}} s_{\nu/\mu}(x \mid a)\, s_{\lambda/\nu}(y \mid \tau^n a) \end{aligned}$$

(here, we have removed the condition "of length $\leq r$" from the sum; this does not change the sum, since all newly introduced addends are zero[3]). This proves (6.13).

10. **page 19, the paragraph containing (6.14):** Replace "Let $x^{(i)}, \ldots, x^{(n)}$ be" by "Let $x^{(1)}, \ldots, x^{(n)}$ be".

11. **page 19, the paragraph containing (6.14):** Replace "where $x^{(i)} = \left(x_1^{(1)}, \ldots, x_{r_i}^{(i)}\right)$" by "where $x^{(i)} = \left(x_1^{(i)}, \ldots, x_{r_i}^{(i)}\right)$".

12. **page 19, (6.14):** On the left hand side of (6.14), replace "$x^{(i)}, \ldots, x^{(n)}$" by "$x^{(1)}, \ldots, x^{(n)}$".

13. **page 19, last paragraph:** Let me explain why

$$s_{\lambda/\mu}(x \mid a) = \prod_{i \geq 1} h_{\lambda_i - \mu_i}\left(x \mid \tau^{\mu_i - i + 1} a\right) \qquad (28)$$

when $\lambda - \mu$ is a horizontal strip:

[3] *Proof.* Let ν is a partition of length $> r$. We must show that $s_{\nu/\mu}(x \mid a)\, s_{\lambda/\nu}(y \mid \tau^n a) = 0$.

Since ν has length $> r$, we have $l(\nu) > r$. Thus, $\nu'_1 = l(\nu) > r \geq \max(l(\lambda), l(\mu)) \geq l(\lambda) = \lambda'_1$ and therefore $0 > \lambda'_1 - \nu'_1$. Hence, we don't have $0 \leq \lambda'_1 - \nu'_1 \leq n$. Thus, we don't have $0 \leq \lambda'_i - \nu'_i \leq n$ for all i (since this inequality fails for $i = 1$). Thus, (6.10) (applied to ν, $\tau^n a$ and y instead of μ, a and x) yields $s_{\lambda/\nu}(y \mid \tau^n a) = 0$. Therefore, $s_{\nu/\mu}(x \mid a) \underbrace{s_{\lambda/\nu}(y \mid \tau^n a)}_{=0} = 0$ as well.

Assume that $\lambda - \mu$ is a horizontal strip. Write the partitions λ and μ as $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_q)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_q)$, where $q \in \mathbb{N}$ is sufficiently large (namely, $q \geq \max(\ell(\lambda), \ell(\mu))$). Then, for any $i, j \in [q]$ satisfying $i > j$, we have

$$h_{\lambda_i - \mu_j - i + j}\left(x \mid \tau^{\mu_j - j + 1} a\right) = 0. \tag{29}$$

(*Proof:* Let $i, j \in [q]$ satisfy $i > j$. Thus, $j < i$, so that $\lambda_j \geq \lambda_i$ (since $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots$) and $\mu_j \geq \mu_i$ (similarly). In other words, $\lambda_i \leq \lambda_j$ and $\mu_i \leq \mu_j$.

The skew partition $\lambda - \mu$ is a horizontal strip, i.e., contains no two cells in the same column. However, if we had $\mu_j < \lambda_i$, then the two distinct cells (i, λ_i) and (j, λ_i) would both belong to $\lambda - \mu$ (indeed, we would have $(i, \lambda_i) \in \lambda - \mu$ because of $\mu_i \leq \mu_j < \lambda_i \leq \lambda_i$, and we would have $(j, \lambda_i) \in \lambda - \mu$ because of $\mu_j < \lambda_i \leq \lambda_j$), which would contradict the preceding sentence (since these two cells clearly lie in the same column). Thus, we cannot have $\mu_j < \lambda_i$. In other words, we have $\mu_j \geq \lambda_i$. Hence, $\lambda_i - \underbrace{\mu_j}_{\geq \lambda_i} - \underbrace{i}_{>j} + j < \lambda_i - \lambda_i - j + j = 0$, and therefore $h_{\lambda_i - \mu_j - i + j}\left(x \mid \tau^{\mu_j - j + 1} a\right) = 0$ (since $h_k\left(x \mid \tau^{\mu_j - j + 1} a\right) = 0$ for any $k < 0$). This proves (29).)

Now, (29) shows that the matrix $\left(h_{\lambda_i - \mu_j - i + j}\left(x \mid \tau^{\mu_j - j + 1} a\right)\right)_{i, j \in [q]}$ is upper-triangular. Hence, its determinant is the product of its diagonal entries. In other words,

$$\begin{aligned}
&\det\left(h_{\lambda_i - \mu_j - i + j}\left(x \mid \tau^{\mu_j - j + 1} a\right)\right)_{i, j \in [q]} \\
&= \prod_{i=1}^{q} \underbrace{h_{\lambda_i - \mu_i - i + i}\left(x \mid \tau^{\mu_i - i + 1} a\right)}_{\substack{= h_{\lambda_i - \mu_i}\left(x \mid \tau^{\mu_i - i + 1} a\right) \\ (\text{since } \lambda_i - \mu_i - i + i = \lambda_i - \mu_i)}} \\
&= \prod_{i=1}^{q} h_{\lambda_i - \mu_i}\left(x \mid \tau^{\mu_i - i + 1} a\right).
\end{aligned}$$

In view of (6.8), we can rewrite this as

$$s_{\lambda/\mu}(x \mid a) = \prod_{i=1}^{q} h_{\lambda_i - \mu_i}\left(x \mid \tau^{\mu_i - i + 1} a\right). \tag{30}$$

This is a finite product, but we can extend it to an infinite product over all $i \geq 1$; this will not change the value of the product, since all the newly inserted factors $h_{\lambda_i - \mu_i}\left(x \mid \tau^{\mu_i - i + 1} a\right)$ for $i > q$ will equal 1 (because if $i > q$, then $\lambda_i = 0$ and $\mu_i = 0$ and therefore $\lambda_i - \mu_i = 0 - 0 = 0$, so that

$h_{\lambda_i-\mu_i}\left(x \mid \tau^{\mu_i-i+1}a\right) = h_0\left(x \mid \tau^{\mu_i-i+1}a\right) = 1$). Hence, (30) can be rewritten as

$$s_{\lambda/\mu}\left(x \mid a\right) = \prod_{i\geq 1} h_{\lambda_i-\mu_i}\left(x \mid \tau^{\mu_i-i+1}a\right).$$

This proves (28). ■

14. **page 19, last paragraph:** At the end of the last display on this page, there is a period. This period should be a comma.

15. **page 20:** "*column strict*" should be "*column-strict*".

16. **page 20, proof of (6.17):** Replace "$A_{\delta_{m+n}}\left(x,y\right)$" by "$A_{\delta_{m+n}}\left(x,y \mid a\right)$". Likewise, replace "$A_{\delta_n}\left(x\right)$" by "$A_{\delta_n}\left(x \mid a\right)$". Likewise, replace "$A_{\delta_m}\left(y\right)$" by "$A_{\delta_m}\left(y \mid a\right)$".

17. **page 20, proof of (6.17):** The equality

$$A_{\delta_{m+n}}\left(x,y\right) = \sum_{\lambda\subset(m^n)} \left(-1\right)^{\left|\widehat{\lambda}\right|} A_{\lambda+\delta_n}\left(x\right) A_{\widehat{\lambda}'+\delta_m}\left(y\right)$$

should be

$$A_{\delta_{m+n}}\left(x,y \mid a\right) = \sum_{\lambda\subset(m^n)} \left(-1\right)^{\left|\widehat{\lambda}\right|} A_{\lambda+\delta_n}\left(x \mid a\right) A_{\widehat{\lambda}'+\delta_m}\left(y \mid a\right) \qquad (31)$$

instead.

18. **page 20, proof of (6.17):** Let me explain how the equality (31) is proved. Some lemmas will be needed for the proof. The first lemma is known as the Laplace expansion formula along multiple rows:

Lemma B.8. Let $k \in \mathbb{N}$. Let A be any $k \times k$-matrix. For any subset I of $[k]$, we let $\sum I$ denote the sum of all elements of I, and we let $\widetilde{I}$ denote the complement $[k] \setminus I$ of I. (For instance, if $k = 4$ and $I = \{1,4\}$, then $\sum I = 1 + 4 = 5$ and $\widetilde{I} = \{2,3\}$.)

Let P be a subset of $[k]$. Then,

$$\det A = \sum_{\substack{Q\subseteq[k];\\ |Q|=|P|}} \left(-1\right)^{\sum P+\sum Q} \det\left(\operatorname{sub}_P^Q A\right) \det\left(\operatorname{sub}_{\widetilde{P}}^{\widetilde{Q}} A\right).$$

Lemma B.8 is [Grinbe20, Theorem 6.156], applied to $n = k$. (Note that our $\operatorname{sub}_P^Q A$ is denoted $\operatorname{sub}_{w(P)}^{w(Q)} A$ in [Grinbe20].) ■

Next, we state two combinatorial lemmas. Recall that the partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ that satisfy $\lambda_1 \leq m$ are precisely the partitions λ that satisfy

$\lambda \subseteq (m^n)$ (where "(m^n) is to be read in exponential notation – i.e., it means an n-tuple of m's). For each such partition λ, we can define the set

$$Q_\lambda^{(n,m)} := \{m - \lambda_i + i \mid i \in [n]\} = \{m - \lambda_1 + 1,\ m - \lambda_2 + 2,\ \ldots,\ m - \lambda_n + n\}$$

of integers. We first claim the following:

Lemma B.9. Let $n, m \in \mathbb{N}$. Then:

(a) For any partition $\lambda \subseteq (m^n)$ (that is, for any partition λ that satisfies $\lambda \subseteq (m^n)$), we have

$$Q_\lambda^{(n,m)} = \{m - \lambda_1 + 1 < m - \lambda_2 + 2 < \cdots < m - \lambda_n + n\}.$$

(b) For any partition $\lambda \subseteq (m^n)$, the set $Q_\lambda^{(n,m)}$ is an n-element subset of $[n+m]$.

(c) The map

$$\{\text{partitions } \lambda \subseteq (m^n)\} \to \{n\text{-element subsets of } [n+m]\},$$
$$\lambda \mapsto Q_\lambda^{(n,m)}$$

is a bijection.

Proof. **(a)** Let $\lambda \subseteq (m^n)$ be a partition. We must show that

$$Q_\lambda^{(n,m)} = \{m - \lambda_1 + 1 < m - \lambda_2 + 2 < \cdots < m - \lambda_n + n\}.$$

By its definition, the set $Q_\lambda^{(n,m)}$ is defined as $\{m - \lambda_1 + 1,\ m - \lambda_2 + 2,\ \ldots,\ m - \lambda_n + n\}$. Thus, it remains to prove that

$$m - \lambda_1 + 1 < m - \lambda_2 + 2 < \cdots < m - \lambda_n + n.$$

In other words, it remains to prove that $m - \lambda_i + i < m - \lambda_{i+1} + (i+1)$ for each $i \in [n-1]$.

So let us prove this. Let $i \in [n-1]$. Since λ is a partition, we have $\lambda_i \geq \lambda_{i+1}$. Hence,

$$m - \underbrace{\lambda_i}_{\geq \lambda_{i+1}} + \underbrace{i}_{<i+1} < m - \lambda_{i+1} + (i+1).$$

This is exactly what we needed to prove. Thus, Lemma B.9 **(a)** is proved.

(b) Let $\lambda \subseteq (m^n)$ be a partition. Lemma B.9 **(a)** shows that

$$Q_\lambda^{(n,m)} = \{m - \lambda_1 + 1 < m - \lambda_2 + 2 < \cdots < m - \lambda_n + n\}.$$

Hence, $Q_\lambda^{(n,m)}$ is an n-element set.

Let $i \in [n]$. Then, $\lambda_i \leq m$ (since $\lambda \subseteq (m^n)$) and $\lambda_i \geq 0$ (obviously) and $i \geq 1$ and $i \leq n$ (since $i \in [n]$). Now, $m - \underbrace{\lambda_i}_{\geq 0} + \underbrace{i}_{\leq n} \leq m + n$ and $m - \underbrace{\lambda_i}_{\leq m} + \underbrace{i}_{\geq 1} \geq m - m + 1 = 1$. Combining these two inequalities, we obtain $m - \lambda_i + i \in [m+n]$.

Forget that we fixed i. We thus have shown that $m - \lambda_i + i \in [m+n]$ for each $i \in [n]$. In other words, $\{m - \lambda_i + i \mid i \in [n]\} \subseteq [m+n]$. Thus,

$$Q_\lambda^{(n,m)} = \{m - \lambda_i + i \mid i \in [n]\} \subseteq [m+n] = [n+m].$$

Since we know that $Q_\lambda^{(n,m)}$ is an n-element set, we thus conclude that $Q_\lambda^{(n,m)}$ is an n-element subset of $[n+m]$. This proves Lemma B.9 **(b)**.

(c) Lemma B.9 **(b)** shows that for any partition $\lambda \subseteq (m^n)$, the set $Q_\lambda^{(n,m)}$ is an n-element subset of $[n+m]$. Hence, the map

$$\begin{aligned} \{\text{partitions } \lambda \subseteq (m^n)\} &\to \{n\text{-element subsets of } [n+m]\}, \\ \lambda &\mapsto Q_\lambda^{(n,m)} \end{aligned} \tag{32}$$

is well-defined. It remains to show that this map is bijective. We will prove its injectivity and its surjectivity separately:

- *Injectivity:* Let us show that the map (32) is injective. For this purpose, we must show that any partition $\lambda \subseteq (m^n)$ can be uniquely reconstructed from the set $Q_\lambda^{(n,m)}$. But this is easy: Lemma B.9 **(a)** shows that

  $$Q_\lambda^{(n,m)} = \{m - \lambda_1 + 1 < m - \lambda_2 + 2 < \cdots < m - \lambda_n + n\}.$$

  Hence, the n numbers $m - \lambda_1 + 1,\ m - \lambda_2 + 2,\ \ldots,\ m - \lambda_n + n$ are the elements of the set $Q_\lambda^{(n,m)}$ in increasing order. Thus, knowing the set $Q_\lambda^{(n,m)}$, we can reconstruct these n numbers, and therefore also the entries $\lambda_1, \lambda_2, \ldots, \lambda_n$ of λ; hence, the whole partition λ can be reconstructed from $Q_\lambda^{(n,m)}$. This proves that the map (32) is injective.

- *Surjectivity:* Let us now show that the map (32) is surjective. For this purpose, we fix some n-element subset R of $[n+m]$. We must show that there is a partition $\lambda \subseteq (m^n)$ such that $Q_\lambda^{(n,m)} = R$.

  We know that R is an n-element subset of $[n+m]$; thus, it can be written as $R = \{r_1 < r_2 < \cdots < r_n\}$ for some elements $r_1 < r_2 <$

$\cdots < r_n$ of $[n+m]$. Consider these elements. Define an n-tuple $\lambda := (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{Z}^n$ by setting

$$\lambda_i := m - r_i + i \qquad \text{for each } i \in [n].$$

Then, we have $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ [4] and $m \geq \lambda_1$ [5] and $\lambda_n \geq 0$ [6]. Combining these inequalities, we obtain $m \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$. Hence, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is a partition satisfying $\lambda \subseteq (m^n)$. Furthermore, each $i \in [n]$ satisfies $m - \lambda_i + i = r_i$ (since the definition of λ yields $\lambda_i = m - r_i + i$). But the definition of $Q_\lambda^{(n,m)}$ says that

$$\begin{aligned} Q_\lambda^{(n,m)} &= \{m - \lambda_i + i \mid i \in [n]\} \\ &= \{r_i \mid i \in [n]\} \qquad (\text{since each } i \in [n] \text{ satisfies } m - \lambda_i + i = r_i) \\ &= R \qquad (\text{since } R = \{r_1 < r_2 < \cdots < r_n\} = \{r_i \mid i \in [n]\}). \end{aligned}$$

Thus, we have found a partition $\lambda \subseteq (m^n)$ such that $Q_\lambda^{(n,m)} = R$. This completes our proof that the map (32) is surjective.

We have now shown that the map (32) is both injective and surjective. Hence, this map is bijective, i.e., is a bijection. This proves Lemma B.9 **(c)**. □

Next, we shall connect the $Q_\lambda^{(n,m)}$ construction to the partition $\widehat{\lambda}'$ that appears on the right hand sides of (0.11′) and of (6.17). First, we recall some notations:

- If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \subseteq (m^n)$ is a partition, then its complementary partition $\widehat{\lambda} = \left(\widehat{\lambda}_1, \widehat{\lambda}_2, \ldots, \widehat{\lambda}_n\right)$ is defined by $\widehat{\lambda}_i = m - \lambda_{n+1-i}$ for all $i \in [n]$. Note that this depends on m and n. The complementary partition $\widehat{\lambda}$ again satisfies $\widehat{\lambda} \subseteq (m^n)$, and can be described visually as

[4] *Proof.* We must show that $\lambda_i \geq \lambda_{i+1}$ for each $i \in [n-1]$.

So let $i \in [n-1]$. Then, $r_i < r_{i+1}$ (since $r_1 < r_2 < \cdots < r_n$) and thus $r_i \leq r_{i+1} - 1$ (since r_i and r_{i+1} are integers). But the definition of λ_i shows that $\lambda_i = m - r_i + i$, whereas the definition of λ_{i+1} shows that $\lambda_{i+1} = m - r_{i+1} + (i+1)$. Hence,

$$\lambda_i = m - \underbrace{r_i}_{\leq r_{i+1}-1} + i \geq m - (r_{i+1} - 1) + i = m - r_{i+1} + (i+1) = \lambda_{i+1}.$$

Thus, we have proved that $\lambda_i \geq \lambda_{i+1}$ for each $i \in [n-1]$, qed.

[5] *Proof.* By the definition of λ_1, we have $\lambda_1 = m - r_1 + 1$. But $r_1 \geq 1$, since $r_1 \in \{r_1 < r_2 < \cdots < r_n\} = R \subseteq [n+m]$. Thus, $\lambda_1 = m - \underbrace{r_1}_{\geq 1} + 1 \leq m - 1 + 1 = m$, so that $m \geq \lambda_1$.

[6] *Proof.* By the definition of λ_n, we have $\lambda_n = m - r_n + n$. But $r_n \leq n + m$, since $r_n \in \{r_1 < r_2 < \cdots < r_n\} = R \subseteq [n+m]$. Thus, $\lambda_n = m - \underbrace{r_n}_{\leq n+m} + n \geq m - (n+m) + n = 0$.

follows: Its Young diagram is the complement of the Young diagram of λ in the rectangle (m^n), rotated by 180°. It is furthermore easy to see that $\widehat{\widehat{\lambda}} = \lambda$ and

$$\left|\widehat{\lambda}\right| = nm - |\lambda| . \tag{33}$$

- If λ is any partition, then λ' is the conjugate partition of λ, obtained by reflecting the Young diagram of λ across the main diagonal (so that rows become columns and vice versa). Clearly, if $\lambda \subseteq (m^n)$, then $\lambda' \subseteq (n^m)$.

Now our next lemma is the following:

Lemma B.10. Let $\lambda \subseteq (m^n)$ be a partition. Then, $\lambda' \subseteq (n^m)$ and

$$Q_{\lambda'}^{(m,n)} = [n+m] \setminus Q_{\widehat{\lambda}}^{(n,m)}.$$

Proof. From $\lambda \subseteq (m^n)$, we obtain $\lambda' \subseteq (n^m)$. Thus, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ (since $\lambda \subseteq (m^n)$) and $\lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_m)$ (since $\lambda' \subseteq (n^m)$).

For each $i \in [n]$, let us set $\alpha_i := \lambda_i - i$. For each $j \in [m]$, let us set $\beta_j := \lambda'_j - j$ and $\eta_j := -1 - \beta_j$. Then, Lemma B.4 (applied to $q = n$ and $p = m$) yields that the two sets $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ and $\{\eta_1, \eta_2, \ldots, \eta_m\}$ are disjoint, and their union is the integer interval $[-n, m-1]$. Hence,

$$\{\eta_1, \eta_2, \ldots, \eta_m\} = [-n, m-1] \setminus \{\alpha_1, \alpha_2, \ldots, \alpha_n\} . \tag{34}$$

For any subset S of $\mathbb{Z}$ and any integer z, we let $S + z$ denote the set $\{s + z \mid s \in S\}$. Visually speaking, this is simply the set S shifted by z units to the right along the number line. Clearly, any two subsets S and T of $\mathbb{Z}$ and any integer z satisfy

$$(S \setminus T) + z = (S + z) \setminus (T + z) , \tag{35}$$

since the operation of adding z to each integer is a bijection.

Now, from (34), we obtain

$$\begin{aligned}
&\{\eta_1, \eta_2, \ldots, \eta_m\} + (n+1) \\
&= ([-n, m-1] \setminus \{\alpha_1, \alpha_2, \ldots, \alpha_n\}) + (n+1) \\
&= \underbrace{([-n, m-1] + (n+1))}_{\substack{=[1,(m-1)+(n+1)] \\ =[1,n+m]=[n+m]}} \setminus \underbrace{(\{\alpha_1, \alpha_2, \ldots, \alpha_n\} + (n+1))}_{\substack{=\{\alpha_1+(n+1),\alpha_2+(n+1),\ldots,\alpha_n+(n+1)\} \\ =\{\alpha_i+(n+1) \mid i \in [n]\} \\ =\{\alpha_{n+1-i}+(n+1) \mid i \in [n]\} \\ \text{(here, we have substituted } n+1-i \text{ for } i \text{ in the set,} \\ \text{since the map } [n]\to[n],\ i \mapsto n+1-i \text{ is a bijection)}}} \qquad \text{(by (35))} \\
&= [n+m] \setminus \{\alpha_{n+1-i} + (n+1) \mid i \in [n]\} .
\end{aligned} \tag{36}$$

However, each $i \in [n]$ satisfies $\alpha_{n+1-i} = \lambda_{n+1-i} - (n+1-i)$ (by the definition of α_{n+1-i}) and thus

$$\alpha_{n+1-i} + (n+1) = \lambda_{n+1-i} - (n+1-i) + (n+1) = \underbrace{\lambda_{n+1-i}}_{\substack{=m-\widehat{\lambda}_i \\ \text{(since the definition of } \widehat{\lambda} \\ \text{yields } \widehat{\lambda}_i = m - \lambda_{n+1-i})}} + i = m - \widehat{\lambda}_i + i.$$

Hence,

$$\left\{ \underbrace{\alpha_{n+1-i} + (n+1)}_{=m-\widehat{\lambda}_i+i} \ \mid \ i \in [n] \right\} = \left\{ m - \widehat{\lambda}_i + i \ \mid \ i \in [n] \right\} = Q_{\widehat{\lambda}}^{(n,m)}$$

(since the definition of $Q_{\widehat{\lambda}}^{(n,m)}$ yields $Q_{\widehat{\lambda}}^{(n,m)} = \left\{ m - \widehat{\lambda}_i + i \ \mid \ i \in [n] \right\}$). Thus, we can rewrite (36) as

$$\{\eta_1, \eta_2, \ldots, \eta_m\} + (n+1) = [n+m] \setminus Q_{\widehat{\lambda}}^{(n,m)}. \tag{37}$$

On the other hand, each $i \in [m]$ satisfies

$$\underbrace{\eta_i}_{\substack{=-1-\beta_i \\ \text{(by the definition of } \eta_i)}} + (n+1) = -1 - \beta_i + (n+1) = n - \underbrace{\beta_i}_{\substack{=\lambda_i'-i \\ \text{(by the definition of } \beta_i)}} = n - \left(\lambda_i' - i\right) = n - \lambda_i' + i. \tag{38}$$

Now,

$$\begin{aligned} & \{\eta_1, \eta_2, \ldots, \eta_m\} + (n+1) \\ & = \{\eta_1 + (n+1),\ \eta_2 + (n+1),\ \ldots,\ \eta_m + (n+1)\} \\ & = \{\eta_i + (n+1) \ \mid \ i \in [m]\} \\ & = \left\{ n - \lambda_i' + i \ \mid \ i \in [m] \right\} \qquad \text{(by (38))} \\ & = Q_{\lambda'}^{(m,n)} \end{aligned}$$

(since the definition of $Q_{\lambda'}^{(m,n)}$ yields $Q_{\lambda'}^{(m,n)} = \left\{ n - \lambda_i' + i \ \mid \ i \in [m] \right\}$). Comparing this with (37), we obtain $Q_{\lambda'}^{(m,n)} = [n+m] \setminus Q_{\widehat{\lambda}}^{(n,m)}$. This proves Lemma B.10. □

We can easily restate Lemma B.10 as follows:

Lemma B.11. Let $\lambda \subseteq (m^n)$ be a partition. Then, $\widehat{\lambda}' \subseteq (n^m)$ and

$$Q^{(m,n)}_{\widehat{\lambda}'} = [n+m] \setminus Q^{(n,m)}_{\lambda}.$$

Proof. From $\lambda \subseteq (m^n)$, we obtain $\widehat{\lambda} \subseteq (m^n)$ and $\widehat{\widehat{\lambda}} = \lambda$ (by the basic properties of $\widehat{\lambda}$). Thus, Lemma B.10 (applied to $\widehat{\lambda}$ instead of λ) yields $\widehat{\lambda}' \subseteq (n^m)$ and

$$Q^{(m,n)}_{\widehat{\lambda}'} = [n+m] \setminus Q^{(n,m)}_{\widehat{\widehat{\lambda}}} = [n+m] \setminus Q^{(n,m)}_{\lambda}$$

(since $\widehat{\widehat{\lambda}} = \lambda$). This proves Lemma B.11. $\square$

We are now ready to prove (31):

Proof of (31). From the definition (6.2) of $A_{\delta_{m+n}}(x, y \mid a)$, we know that

$$A_{\delta_{m+n}}(x, y \mid a) = \det B, \tag{39}$$

where B is the $(n+m) \times (n+m)$-matrix whose (i,j)-th entry (for all $i, j \in [n+m]$) is

$$\begin{cases} (x_i \mid a)^{m+n-j}, & \text{if } i \leq n; \\ (y_{i-n} \mid a)^{m+n-j}, & \text{if } i > n. \end{cases}$$

Consider this matrix B.

Now, set $k := n+m$, so that B is a $k \times k$-matrix. Let furthermore $P := [n]$. This is a subset of $[k]$, since $n \leq n+m = k$. Moreover, $|P| = |[n]| = n$.

We shall use the notations of Lemma B.8, so that in particular we have

$$\begin{aligned} \widetilde{P} &= \Big[\underbrace{k}_{=n+m}\Big] \setminus \underbrace{P}_{=[n]} = [n+m] \setminus [n] = \{n+1, n+2, \ldots, n+m\} \\ &= \{n+1 < n+2 < \cdots < n+m\}. \end{aligned}$$

Now, Lemma B.8 (applied to $A = B$) yields

$$\begin{aligned} \det B &= \sum_{\substack{Q \subseteq [k]; \\ |Q| = |P|}} (-1)^{\sum P + \sum Q} \det\left(\operatorname{sub}_P^Q B\right) \det\left(\operatorname{sub}_{\widetilde{P}}^{\widetilde{Q}} B\right) \\ &= \sum_{\substack{Q \subseteq [n+m]; \\ |Q| = n}} (-1)^{\sum P + \sum Q} \det\left(\operatorname{sub}_P^Q B\right) \det\left(\operatorname{sub}_{\widetilde{P}}^{\widetilde{Q}} B\right) \\ &\qquad (\text{since } k = n+m \text{ and } |P| = n) \\ &= \sum_{\lambda \subseteq (m^n)} (-1)^{\sum P + \sum Q^{(n,m)}_{\lambda}} \det\left(\operatorname{sub}_P^{Q^{(n,m)}_{\lambda}} B\right) \det\left(\operatorname{sub}_{\widetilde{P}}^{\widetilde{Q^{(n,m)}_{\lambda}}} B\right) \end{aligned}$$

(here, we have substituted $Q_\lambda^{(n,m)}$ for Q, because Lemma B.9 **(c)** says that the map

$$\{\text{partitions } \lambda \subseteq (m^n)\} \to \{n\text{-element subsets of } [n+m]\},$$
$$\lambda \mapsto Q_\lambda^{(n,m)}$$

is a bijection).

Now, we shall simplify the right hand side of this equality. Fix a partition $\lambda \subseteq (m^n)$. Then, by the definition of $\widetilde{Q_\lambda^{(n,m)}}$, we have

$$\begin{aligned}\widetilde{Q_\lambda^{(n,m)}} &= \Big[\underbrace{k}_{=n+m}\Big] \setminus Q_\lambda^{(n,m)} = [n+m] \setminus Q_\lambda^{(n,m)} \\ &= Q_{\widehat{\lambda}'}^{(m,n)} \end{aligned} \tag{40}$$

(by Lemma B.11). Note that Lemma B.11 also yields $\widehat{\lambda}' \subseteq (n^m)$.

Furthermore, recall that $P = [n] = \{1 < 2 < \cdots < n\}$ and $Q_\lambda^{(n,m)} = \{m - \lambda_1 + 1 < m - \lambda_2 + 2 < \cdots < m - \lambda_n + n\}$ (by Lemma B.9 **(a)**). Hence,

$$\begin{aligned}\operatorname{sub}_P^{Q_\lambda^{(n,m)}} B &= \operatorname{sub}_{\{1<2<\cdots<n\}}^{\{m-\lambda_1+1<m-\lambda_2+2<\cdots<m-\lambda_n+n\}} B \\ &= \left((x_i \mid a)^{m+n-\left(m-\lambda_j+j\right)}\right)_{1\leq i,j\leq n} \\ &\qquad \left(\begin{array}{c}\text{by the definition of } B \text{, since} \\ \text{all } i \in \{1<2<\cdots<n\} \text{ satisfy } i \leq n\end{array}\right) \\ &= \left((x_i \mid a)^{(\lambda+\delta_n)_j}\right)_{1\leq i,j\leq n}\end{aligned}$$

(since $m+n-\left(m-\lambda_j+j\right) = \lambda_j + n - j = (\lambda+\delta_n)_j$ for all $j \in [n]$) and therefore

$$\begin{aligned}\det\left(\operatorname{sub}_P^{Q_\lambda^{(n,m)}} B\right) &= \det\left((x_i \mid a)^{(\lambda+\delta_n)_j}\right)_{1\leq i,j\leq n} \\ &= A_{\lambda+\delta_n}(x \mid a)\end{aligned} \tag{41}$$

(by the definition of $A_{\lambda+\delta_n}(x \mid a)$). Furthermore, we have $\widetilde{P} = \{n+1 < n+2 < \cdots < n+m\}$ and

$$\begin{aligned}\widetilde{Q_\lambda^{(n,m)}} &= Q_{\widehat{\lambda}'}^{(m,n)} \qquad \text{(by (40))} \\ &= \left\{n - \widehat{\lambda}'_1 + 1 < n - \widehat{\lambda}'_2 + 2 < \cdots < n - \widehat{\lambda}'_m + m\right\}\end{aligned}$$

(by Lemma B.9 **(a)**, applied to m, n and $\widehat{\lambda}'$ instead of n, m and λ). Hence,

$$\begin{aligned}
\operatorname{sub}_{\widetilde{P}}^{\widetilde{Q_\lambda^{(n,m)}}} B &= \operatorname{sub}_{\{n+1<n+2<\cdots<n+m\}}^{\{n-\widehat{\lambda}'_1+1<n-\widehat{\lambda}'_2+2<\cdots<n-\widehat{\lambda}'_m+m\}} B \\
&= \left((y_i \mid a)^{m+n-\left(n-\widehat{\lambda}'_j+j\right)} \right)_{1\leq i,j\leq m} \\
&\qquad \left(\begin{array}{c} \text{by the definition of } B \text{, since} \\ \text{all } i \in \{n+1<n+2<\cdots<n+m\} \text{ satisfy } i>n \end{array} \right) \\
&= \left((y_i \mid a)^{\left(\widehat{\lambda}'+\delta_m\right)_j} \right)_{1\leq i,j\leq m}
\end{aligned}$$

(since $m+n-\left(n-\widehat{\lambda}'_j+j\right) = \widehat{\lambda}'_j + m - j = \left(\widehat{\lambda}'+\delta_m\right)_j$ for all $j\in[m]$) and therefore

$$\begin{aligned}
\det\left(\operatorname{sub}_{\widetilde{P}}^{\widetilde{Q_\lambda^{(n,m)}}} B\right) &= \det\left((y_i \mid a)^{\left(\widehat{\lambda}'+\delta_m\right)_j} \right)_{1\leq i,j\leq m} \\
&= A_{\widehat{\lambda}'+\delta_m}(y \mid a) \qquad (42)
\end{aligned}$$

(by the definition of $A_{\widehat{\lambda}'+\delta_m}(y \mid a)$). Finally, we have

$$\begin{aligned}
&\sum \underbrace{P}_{=[n]} + \sum \underbrace{Q_\lambda^{(n,m)}}_{\substack{=\{m-\lambda_1+1<m-\lambda_2+2<\cdots<m-\lambda_n+n\} \\ \text{(by Lemma B.9 (a))}}} \\
&= \underbrace{\sum [n]}_{=1+2+\cdots+n} + \underbrace{\sum \{m-\lambda_1+1<m-\lambda_2+2<\cdots<m-\lambda_n+n\}}_{=(m-\lambda_1+1)+(m-\lambda_2+2)+\cdots+(m-\lambda_n+n)} \\
&= (1+2+\cdots+n) + \underbrace{(m-\lambda_1+1)+(m-\lambda_2+2)+\cdots+(m-\lambda_n+n)}_{=\underbrace{(m+m+\cdots+m)}_{n\text{ times}} - (\lambda_1+\lambda_2+\cdots+\lambda_n)+(1+2+\cdots+n)} \\
&= (1+2+\cdots+n) + \underbrace{\underbrace{(m+m+\cdots+m)}_{n\text{ times}}}_{=nm} - \underbrace{(\lambda_1+\lambda_2+\cdots+\lambda_n)}_{\substack{=|\lambda| \\ (\text{since } \lambda=(\lambda_1,\lambda_2,\ldots,\lambda_n))}} + (1+2+\cdots+n) \\
&= (1+2+\cdots+n) + nm - |\lambda| + (1+2+\cdots+n) \\
&= \underbrace{2\cdot(1+2+\cdots+n)}_{\equiv 0 \bmod 2} + nm - |\lambda| \equiv nm - |\lambda| = \left|\widehat{\lambda}\right| \bmod 2 \qquad (\text{by } (33))
\end{aligned}$$

and thus

$$(-1)^{\sum P + \sum Q_\lambda^{(n,m)}} = (-1)^{\left|\widehat{\lambda}\right|}. \qquad (43)$$

Forget that we fixed λ. We thus have proved (40), (41), (42) and (43) for each partition $\lambda \subseteq (m^n)$.

Hence, our above computation of $\det B$ can be continued as follows:

$$\det B = \sum_{\lambda \subseteq (m^n)} \underbrace{(-1)^{\sum P + \sum Q_\lambda^{(n,m)}}}_{\substack{=(-1)^{|\widehat{\lambda}|} \\ \text{(by (43))}}} \underbrace{\det\left(\operatorname{sub}_P^{Q_\lambda^{(n,m)}} B\right)}_{\substack{=A_{\lambda+\delta_n}(x|a) \\ \text{(by (41))}}} \underbrace{\det\left(\operatorname{sub}_{\widetilde{P}}^{\widetilde{Q_\lambda^{(n,m)}}} B\right)}_{\substack{=A_{\widehat{\lambda}'+\delta_m}(y|a) \\ \text{(by (42))}}}$$

$$= \sum_{\lambda \subseteq (m^n)} (-1)^{|\widehat{\lambda}|} A_{\lambda+\delta_n}(x \mid a) A_{\widehat{\lambda}'+\delta_m}(y \mid a).$$

In view of (39), we can rewrite this as

$$A_{\delta_{m+n}}(x, y \mid a) = \sum_{\lambda \subseteq (m^n)} (-1)^{|\widehat{\lambda}|} A_{\lambda+\delta_n}(x \mid a) A_{\widehat{\lambda}'+\delta_m}(y \mid a).$$

This proves (31). □

19. **page 21, Remark:** The chain of equalities

$$A_\alpha(x \mid a) = \det\left(\sum_{\beta_k \geq 0} x_i^{\beta_k} e_{\beta_k - \alpha_j}\left(a^{(\alpha_j)}\right)\right)$$
$$= \sum_\beta \det\left(x_i^{\beta_k}\right) \det\left(e_{\beta_k - \alpha_j}\left(a^{(\alpha_j)}\right)\right)$$

should be replaced by

$$A_\alpha(x \mid a) = \det\left(\sum_{k \geq 0} x_i^k e_{\alpha_j - k}\left(a^{(\alpha_j)}\right)\right)_{1 \leq i,k \leq n}$$
$$= \sum_\beta \det\left(x_i^{\beta_k}\right)_{1 \leq i,k \leq n} \det\left(e_{\alpha_j - \beta_k}\left(a^{(\alpha_j)}\right)\right)_{1 \leq k,j \leq n}. \tag{44}$$

Let me also explain in more detail how the last equality sign here is proved (the first one is clear from the definition of $A_\alpha(x \mid a)$ and the formula for $(x \mid a)^r$ stated at the beginning of the Remark). We shall use the Cauchy–Binet formula for infinite matrices, but this time the infinite matrices are infinite leftwards and upwards, respectively. Namely, we let $-\mathbb{N}$ be the set $\{0, -1, -2, \ldots\}$ of all nonpositive integers, and we define the $[n] \times (-\mathbb{N})$-matrix

$$U := \left(x_i^{-j}\right)_{i \in [n],\ j \in -\mathbb{N}} = \begin{pmatrix} \cdots & x_1^2 & x_1^1 & x_1^0 \\ \cdots & x_2^2 & x_2^1 & x_2^0 \\ \cdots & \vdots & \vdots & \vdots \\ \cdots & x_n^2 & x_n^1 & x_n^0 \end{pmatrix}$$

and the $(-\mathbb{N}) \times [n]$-matrix

$$V := \left(e_{\alpha_j+i}\left(a^{(\alpha_j)}\right)\right)_{i\in-\mathbb{N},\ j\in[n]} = \begin{pmatrix} \vdots & \vdots & \vdots & \vdots \\ e_{\alpha_2-2}\left(a^{(\alpha_2)}\right) & e_{\alpha_2-2}\left(a^{(\alpha_2)}\right) & \cdots & e_{\alpha_n-2}\left(a^{(\alpha_n)}\right) \\ e_{\alpha_2-1}\left(a^{(\alpha_2)}\right) & e_{\alpha_2-1}\left(a^{(\alpha_2)}\right) & \cdots & e_{\alpha_n-1}\left(a^{(\alpha_n)}\right) \\ e_{\alpha_1-0}\left(a^{(\alpha_1)}\right) & e_{\alpha_2-0}\left(a^{(\alpha_2)}\right) & \cdots & e_{\alpha_n-0}\left(a^{(\alpha_n)}\right) \end{pmatrix}.$$

Then, UV is the $[n] \times [n]$-matrix whose (i,j)-th entry (for all $i, j \in [n]$) is

$$\sum_{k\in-\mathbb{N}} x_i^{-k} e_{\alpha_j+k}\left(a^{(\alpha_j)}\right) = \sum_{k\in\mathbb{N}} x_i^k e_{\alpha_j-k}\left(a^{(\alpha_j)}\right)$$
$$\left(\begin{array}{c}\text{here, we have substituted } -k \text{ for } k, \\ \text{since the map } \mathbb{N} \to -\mathbb{N},\ k \mapsto -k \\ \text{is a bijection}\end{array}\right)$$
$$= \sum_{k\geq 0} x_i^k e_{\alpha_j-k}\left(a^{(\alpha_j)}\right).$$

In other words,

$$UV = \left(\sum_{k\geq 0} x_i^k e_{\alpha_j-k}\left(a^{(\alpha_j)}\right)\right)_{1\leq i,k\leq n}. \tag{45}$$

On the other hand, the Cauchy–Binet formula (applied to the matrices U and V) yields

$$\det(UV) = \sum_{b_1<b_2<\cdots<b_n\leq 0} \det\underbrace{\left(\operatorname{sub}^{\{b_1,b_2,\ldots,b_n\}}_{\{1,2,\ldots,n\}} U\right)}_{\substack{=\left(x_i^{-b_j}\right)_{i,j\in[n]} \\ \text{(by the definition of } U\text{)}}} \cdot \det\underbrace{\left(\operatorname{sub}^{\{1,2,\ldots,n\}}_{\{b_1,b_2,\ldots,b_n\}} V\right)}_{\substack{=\left(e_{\alpha_j+b_i}\left(a^{(\alpha_j)}\right)\right)_{i,j\in[n]} \\ \text{(by the definition of } V\text{)}}}$$
$$= \sum_{b_1<b_2<\cdots<b_n\leq 0} \det\left(x_i^{-b_j}\right)_{i,j\in[n]} \cdot \det\left(e_{\alpha_j+b_i}\left(a^{(\alpha_j)}\right)\right)_{i,j\in[n]}$$
$$= \sum_{\beta_1>\beta_2>\cdots>\beta_n\geq 0} \det\left(x_i^{\beta_j}\right)_{i,j\in[n]} \cdot \det\left(e_{\alpha_j-\beta_i}\left(a^{(\alpha_j)}\right)\right)_{i,j\in[n]}$$

(here, we have substituted $-\beta_1, -\beta_2, \ldots, -\beta_n$ for $b_1, b_2, \ldots, b_n$ in the sum). Thus,

$$\det(UV) = \sum_{\beta_1>\beta_2>\cdots>\beta_n\geq 0} \det\underbrace{\left(x_i^{\beta_j}\right)_{i,j\in[n]}}_{\substack{=\left(x_i^{\beta_k}\right)_{i,k\in[n]} \\ =\left(x_i^{\beta_k}\right)_{1\leq i,k\leq n}}} \cdot \det\underbrace{\left(e_{\alpha_j-\beta_i}\left(a^{(\alpha_j)}\right)\right)_{i,j\in[n]}}_{\substack{=\left(e_{\alpha_j-\beta_k}\left(a^{(\alpha_j)}\right)\right)_{k,j\in[n]} \\ =\left(e_{\alpha_j-\beta_k}\left(a^{(\alpha_j)}\right)\right)_{1\leq k,j\leq n}}}$$
$$= \sum_{\beta_1>\beta_2>\cdots>\beta_n\geq 0} \det\left(x_i^{\beta_k}\right)_{1\leq i,k\leq n} \det\left(e_{\alpha_j-\beta_k}\left(a^{(\alpha_j)}\right)\right)_{1\leq k,j\leq n}.$$

In view of (45), we can rewrite this as

$$\det\left(\sum_{k\geq 0} x_i^k e_{\alpha_j - k}\left(a^{(\alpha_j)}\right)\right)_{1\leq i,k\leq n}$$
$$= \sum_{\beta_1>\beta_2>\cdots>\beta_n\geq 0} \det\left(x_i^{\beta_k}\right)_{1\leq i,k\leq n} \det\left(e_{\alpha_j-\beta_k}\left(a^{(\alpha_j)}\right)\right)_{1\leq k,j\leq n}.$$

This proves the second equality sign in (44).

20. **page 21, (6.18):** Let me briefly explain why the sum here is over the partitions $\mu \subset \lambda$ rather than over all partitions $\mu \in \mathbb{N}^n$.

    Indeed, of course, the sum that is initially obtained from (44) by doing what is said here (i.e., dividing both sides by $\Delta(x)$ and replacing α, β by $\lambda + \delta, \mu + \delta$) is a sum ranging over all partitions $\mu \in \mathbb{N}^n$. However, the partitions μ that don't satisfy $\mu \subset \lambda$ do not actually contribute anything to the sum, since the corresponding addends are all 0 (because an argument similar to the proof of (6.10) shows that $\det\left(e_{\lambda_i-\mu_j-i+j}\left(a^{(\lambda_j+n-j)}\right)\right)_{i,j\in[n]} = 0$ for all such μ). Thus, the sum can be restricted (without changing its value) to range only over the partitions $\mu \subset \lambda$.

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