

Generalized cohomology quotients of the symmetric functions

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ICERM: Combinatorial Algebraic Geometry

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What is this about?

- From a modern point of view, **Schubert calculus** (a.k.a. classical enumerative geometry, or Hilbert's 15th problem) is about two cohomology rings:

$$H^* \left(\underbrace{\text{Gr}(k, n)}_{\text{Grassmannian}} \right) \text{ and } H^* \left(\underbrace{\text{Fl}(n)}_{\text{flag variety}} \right)$$

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- Classical result: as rings,

$$\begin{aligned} H^*(\text{Gr}(k, n)) \\ \cong (\text{symmetric polynomials in } x_1, x_2, \dots, x_k \text{ over } \mathbb{Z}) \\ / (h_{n-k+1}, h_{n-k+2}, \dots, h_n)_{\text{ideal}}, \end{aligned}$$

where the h_i are complete homogeneous symmetric polynomials (to be defined soon).

- (Small) **Quantum cohomology** is a deformation of cohomology from the 1980–90s. For the Grassmannian, it is

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- Many properties of classical cohomology still hold here. In particular: $\text{QH}^*(\text{Gr}(k, n))$ has a $\mathbb{Z}[q]$ -module basis $(\overline{s_\lambda})_{\lambda \in P_{k,n}}$ of (projected) Schur polynomials (to be defined soon), with λ ranging over all partitions with $\leq k$ parts and each part $\leq n - k$. The structure constants are the **Gromov–Witten invariants**. References:

- Aaron Bertram, Ionut Ciocan-Fontanine, William Fulton, *Quantum multiplication of Schur polynomials*, 1999.
- Alexander Postnikov, *Affine approach to quantum Schubert calculus*, 2005.

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- Let \mathbf{k} be a commutative ring.
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- Let \mathcal{S} denote the ring of *symmetric* polynomials in \mathcal{P} .
These are the polynomials $f \in \mathcal{P}$ satisfying

$$f(x_1, x_2, \dots, x_k) = f(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(k)})$$

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- **Theorem (Artin ≤ 1944):** The \mathcal{S} -module \mathcal{P} is free with basis

$$(x^\alpha)_{\alpha \in \mathbb{N}^k; \alpha_i < i \text{ for each } i} \quad (\text{or, informally: } \left(x_1^{<1} x_2^{<2} \cdots x_k^{<k} \right)).$$

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Example: For $k = 3$, this basis is $(1, x_3, x_3^2, x_2, x_2x_3, x_2x_3^2)$.

- The ring \mathcal{S} of symmetric polynomials in $\mathcal{P} = \mathbf{k}[x_1, x_2, \dots, x_k]$ has several bases, usually indexed by certain sets of (integer) partitions.

First, let us recall what partitions are:

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Examples: $(4, 2, 2, 0, 0, 0, \dots)$ and $(3, 2, 0, 0, 0, 0, \dots)$ and $(5, 0, 0, 0, 0, 0, \dots)$ are three partitions.
 $(2, 3, 2, 0, 0, 0, \dots)$ and $(2, 1, 1, 1, \dots)$ are not.

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- Thus there is a bijection

$$\{k\text{-partitions}\} \rightarrow \{\text{partitions with at most } k \text{ nonzero entries}\},$$
$$\lambda \mapsto (\lambda_1, \lambda_2, \dots, \lambda_k, 0, 0, 0, \dots).$$

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- If $\lambda \in \mathbb{N}^k$ is a k -partition, then its **Young diagram** $Y(\lambda)$ is defined as a table made out of k left-aligned rows, where the i -th row has λ_i boxes.

Example: If $k = 6$ and $\lambda = (5, 5, 3, 2, 0, 0)$, then

$$Y(\lambda) = \begin{array}{|c|c|c|c|c|} \hline \square & \square & \square & \square & \square \\ \hline \square & \square & \square & \square & \square \\ \hline \square & \square & \square & & \\ \hline \square & \square & & & \\ \hline \square & \square & & & \\ \hline \square & \square & & & \\ \hline \end{array} .$$

(Empty rows are invisible.)

- The same convention applies to partitions.

Symmetric polynomials: the e -basis

- For each $m \in \mathbb{Z}$, we let e_m denote the m -th *elementary symmetric polynomial*:

$$e_m = \sum_{1 \leq i_1 < i_2 < \dots < i_m \leq k} x_{i_1} x_{i_2} \cdots x_{i_m} = \sum_{\substack{\alpha \in \{0,1\}^k; \\ |\alpha| = m}} x^\alpha \in \mathcal{S}.$$

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- Note that $e_m = 0$ when $m > k$.

- For each $m \in \mathbb{Z}$, we let h_m denote the m -th *complete homogeneous symmetric polynomial*:

$$h_m = \sum_{1 \leq i_1 \leq i_2 \leq \dots \leq i_m \leq k} x_{i_1} x_{i_2} \cdots x_{i_m} = \sum_{\substack{\alpha \in \mathbb{N}^k; \\ |\alpha| = m}} x^\alpha \in \mathcal{S}.$$

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- **Theorem:** $(h_\lambda)_\lambda$ is a k -partition is a basis of the \mathbf{k} -module \mathcal{S} . (Another basis!)

Symmetric polynomials: the s -basis (Schur polynomials)

- For each k -partition λ , we let s_λ be the λ -th Schur polynomial:

$$s_\lambda = \frac{\det \left(\left(x_i^{\lambda_j + k - j} \right)_{1 \leq i \leq k, 1 \leq j \leq k} \right)}{\det \left(\left(x_i^{k - j} \right)_{1 \leq i \leq k, 1 \leq j \leq k} \right)} \quad (\text{alternant formula})$$

$$= \det \left((h_{\lambda_i - i + j})_{1 \leq i \leq k, 1 \leq j \leq k} \right) \quad (\text{Jacobi-Trudi}).$$

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- Theorem:** The equality above holds, and s_λ is a symmetric polynomial with nonnegative coefficients. Explicitly,

$$s_\lambda = \sum_{\substack{T \text{ is a semistandard } \lambda\text{-tableau} \\ \text{with entries } 1, 2, \dots, k}} \prod_{i=1}^k x_i^{(\text{number of } i\text{'s in } T)},$$

where a *semistandard λ -tableau with entries $1, 2, \dots, k$* is a way of putting an integer $i \in \{1, 2, \dots, k\}$ into each box of $Y(\lambda)$ such that the entries **weakly** increase along rows and **strictly** increase along columns.

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- **Theorem:** The equality above holds, and s_λ is a symmetric polynomial with nonnegative coefficients.
- **Theorem:** $(s_\lambda)_{\lambda \text{ is a } k\text{-partition}}$ is a basis of the \mathbf{k} -module \mathcal{S} .

- If λ and μ are two k -partitions, then the product $s_\lambda s_\mu$ can be again written as a \mathbf{k} -linear combination of Schur polynomials (since these form a basis):

$$s_\lambda s_\mu = \sum_{\nu \text{ is a } k\text{-partition}} c_{\lambda, \mu}^\nu s_\nu,$$

where the $c_{\lambda, \mu}^\nu$ lie in \mathbf{k} . These $c_{\lambda, \mu}^\nu$ are called the *Littlewood-Richardson coefficients*.

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where the $c_{\lambda, \mu}^\nu$ lie in \mathbf{k} . These $c_{\lambda, \mu}^\nu$ are called the *Littlewood-Richardson coefficients*.

- **Theorem:** These Littlewood-Richardson coefficients $c_{\lambda, \mu}^\nu$ are nonnegative integers (and count something).

- We have defined

$$s_{\lambda} = \det \left((h_{\lambda_i - i + j})_{1 \leq i \leq k, 1 \leq j \leq k} \right)$$

for k -partitions λ .

Apply the same definition to arbitrary $\lambda \in \mathbb{Z}^k$.

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- **Proposition:** If $\alpha \in \mathbb{Z}^k$, then s_α is either 0 or equals $\pm s_\lambda$ for some k -partition λ .
(So we get nothing really new.)

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More precisely: Let

$$\beta = (\alpha_1 + (k - 1), \alpha_2 + (k - 2), \dots, \alpha_k + (k - k)).$$

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$$s_\lambda = \det \left((h_{\lambda_i - i + j})_{1 \leq i \leq k, 1 \leq j \leq k} \right)$$

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- Also, the alternant formula still holds if all $\lambda_i + (k-i)$ are ≥ 0 .

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- **Theorem (G.):** The \mathbf{k} -module \mathcal{P}/J is free with basis

$$\begin{aligned} & (\overline{x^\alpha})_{\alpha \in \mathbb{N}^k; \alpha_i < n-k+i \text{ for each } i} \\ & \left(\text{informally: } \overline{\left(x_1^{<n-k+1} x_2^{<n-k+2} \dots x_n^{<n} \right)} \right) \end{aligned}$$

where the overline $\overline{\quad}$ means “projection” onto whatever quotient we need (here: from \mathcal{P} onto \mathcal{P}/J).

(This basis has $n(n-1)\cdots(n-k+1)$ elements.)

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A slightly less general setting: symmetric a_1, a_2, \dots, a_k and J

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- Let $\omega = \underbrace{(n-k, n-k, \dots, n-k)}_{k \text{ entries}}$ and

$$\begin{aligned} P_{k,n} &= \{ \lambda \text{ is a } k\text{-partition} \mid \lambda_1 \leq n-k \} \\ &= \{ k\text{-partitions } \lambda \subseteq \omega \}. \end{aligned}$$

- Here, for two k -partitions α and β , we say that $\alpha \subseteq \beta$ if and only if $\alpha_i \leq \beta_i$ for all i .
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 - **classical cohomology:** If $\mathbf{k} = \mathbb{Z}$ and $a_1 = a_2 = \dots = a_k = 0$, then \mathcal{S}/I becomes the cohomology ring $H^*(\mathrm{Gr}(k, n))$; the basis $(\overline{s_\lambda})_{\lambda \in P_{k,n}}$ corresponds to the Schubert classes.
 - **quantum cohomology:** If $\mathbf{k} = \mathbb{Z}[q]$ and $a_1 = a_2 = \dots = a_{k-1} = 0$ and $a_k = -(-1)^k q$, then \mathcal{S}/I becomes the quantum cohomology ring $\mathrm{QH}^*(\mathrm{Gr}(k, n))$.

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- The above theorem lets us work in these rings (and more generally) without relying on geometry.

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$$\nu^\vee := (n - k - \nu_k, n - k - \nu_{k-1}, \dots, n - k - \nu_1) \in P_{k,n}.$$

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- Equivalent restatement:** Each $\nu \in P_{k,n}$ and $f \in \mathcal{S}/I$ satisfy $\text{coeff}_\omega (\overline{s_\nu} f) = \text{coeff}_{\nu^\vee} (f)$.

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- **Proposition (G.):** Let m be a positive integer. Then,

$$\overline{h_{n+m}} = \sum_{j=0}^{k-1} (-1)^j a_{k-j} \overline{s_{(m,1^j)}},$$

where $(m, 1^j) := (m, \underbrace{1, 1, \dots, 1}_{j \text{ ones}}, 0, 0, 0, \dots)$ (a hook-shaped k -partition).

The Pieri rule for symmetric polynomials

- If α and β are two k -partitions, then we say that α/β is a *horizontal strip* if and only if the Young diagram $Y(\alpha)$ is obtained from $Y(\beta)$ by adding some (possibly none) extra boxes with no two of these new boxes lying in the same column.

Example: If $k = 4$ and $\alpha = (5, 3, 2, 1)$ and $\beta = (3, 2, 2, 0)$, then α/β is a horizontal strip, since

$$Y(\beta) = \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} \subseteq \begin{array}{|c|c|c|c|c|} \hline & & & X & X \\ \hline & & & X & \\ \hline & & & & \\ \hline & & & & \\ \hline & & & X & \\ \hline \end{array} = Y(\alpha)$$

with no two X 's in the same column.

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- Furthermore, given $j \in \mathbb{N}$, we say that α/β is a *horizontal j -strip* if α/β is a horizontal strip and $|\alpha| - |\beta| = j$.
- **Theorem (Pieri).** Let λ be a k -partition. Let $j \in \mathbb{N}$. Then,

$$s_\lambda h_j = \sum_{\substack{\mu \text{ is a } k\text{-partition;} \\ \mu/\lambda \text{ is a} \\ \text{horizontal } j\text{-strip}}} s_\mu.$$

- Theorem (G.):** Let $\lambda \in P_{k,n}$. Let $j \in \{0, 1, \dots, n - k\}$.
 Then,

$$\overline{s_\lambda h_j} = \sum_{\substack{\mu \in P_{k,n}; \\ \mu/\lambda \text{ is a} \\ \text{horizontal } j\text{-strip}}} \overline{s_\mu} - \sum_{i=1}^k (-1)^i a_i \sum_{\nu \subseteq \lambda} c_{(n-k-j+1, 1^{i-1}), \nu}^\lambda \overline{s_\nu}.$$

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- This generalizes the h-Pieri rule from Bertram, Ciocan-Fontanine and Fulton, but note that $c_{(n-k-j+1, 1^{i-1}), \nu}^\lambda$ may be > 1 .

- **Example:** For $n = 7$ and $k = 3$, we have

$$\begin{aligned} \overline{s_{(4,3,2)} h_2} &= \overline{s_{(4,4,3)}} + a_1 (\overline{s_{(4,2)}} + \overline{s_{(3,2,1)}} + \overline{s_{(3,3)}}) \\ &\quad - a_2 (\overline{s_{(4,1)}} + \overline{s_{(2,2,1)}} + \overline{s_{(3,1,1)}} + 2\overline{s_{(3,2)}}) \\ &\quad + a_3 (\overline{s_{(2,2)}} + \overline{s_{(2,1,1)}} + \overline{s_{(3,1)}}) . \end{aligned}$$

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- Multiplying by e_j appears harder: For $n = 5$ and $k = 3$, we have

$$\overline{s_{(2,2,1)}} e_2 = a_1 \overline{s_{(2,2)}} - 2a_2 \overline{s_{(2,1)}} + a_3 (\overline{s_{(2)}} + \overline{s_{(1,1)}}) + a_1^2 \overline{s_{(1)}} - 2a_1 a_2 \overline{s_{()}}.$$

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- Multiplying by e_j appears harder: For $n = 5$ and $k = 3$, we have

$$\overline{s_{(2,2,1)}} e_3 = -a_1 \overline{s_{(2,2)}} + a_2 \overline{s_{(2,1)}} + a_0^2 \overline{s_{(2)}} - 2a_0 a_1 \overline{s_{(1)}} + a_1^2 \overline{s_{()}} .$$

So, even multiplying by e_k can give a mess...

A “rim hook algorithm”

- For $QH^*(Gr(k, n))$, Bertram, Ciocan-Fontanine and Fulton give a “rim hook algorithm” that rewrites an arbitrary \overline{s}_μ as $(-1)^{\text{something}} q^{\text{something}} \overline{s}_\lambda$ with $\lambda \in P_{k,n}$.
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- **Theorem (G.):** Let μ be a k -partition with $\mu_1 > n - k$. Let

$$W = \left\{ \lambda = (\lambda_1, \lambda_2, \dots, \lambda_k) \in \mathbb{Z}^k \mid \lambda_1 = \mu_1 - n \right. \\ \left. \text{and } \lambda_i - \mu_i \in \{0, 1\} \text{ for all } i \in \{2, 3, \dots, k\} \right\}.$$

(Not all elements of W are k -partitions, but all belong to \mathbb{Z}^k , so we know how to define s_λ for them.)

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Then,

$$\overline{s_\mu} = \sum_{j=1}^k (-1)^{k-j} a_j \sum_{\substack{\lambda \in W; \\ |\lambda| = |\mu| - (n-k+j)}} \overline{s_\lambda}.$$

- **Conjecture:** Let $b_i = (-1)^{n-k-1} a_i$ for each $i \in \{1, 2, \dots, k\}$. Let $\lambda, \mu, \nu \in P_{k,n}$. Then, $(-1)^{|\lambda|+|\mu|-|\nu|} \text{coeff}_\nu(\overline{s_\lambda s_\mu})$ is a polynomial in b_1, b_2, \dots, b_k with coefficients in \mathbb{N} .
- Verified for all $n \leq 8$ using SageMath.
- This would generalize positivity of Gromov–Witten invariants.

- **Theorem (G.):** The \mathbf{k} -module \mathcal{S}/I is free with basis

$$(\overline{m_\lambda})_{\lambda \in P_{k,n}},$$

where

$$m_\lambda = \left(\begin{array}{l} \text{the sum of all distinct permutations of} \\ \text{the monomial } x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_k^{\lambda_k} \end{array} \right)$$

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- The family $(\overline{p_\lambda})_{\lambda \in P_{k,n}}$ built of the power-sum symmetric functions p_λ is not generally a basis (not even if $\mathbf{k} = \mathbb{Q}$ and $a_i = 0$).
- What about other bases? Forgotten symmetric functions?

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More questions

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- What is the S_k -module structure on \mathcal{P}/J ?
- **Almost-theorem (G., needs to be checked):** Assume that \mathbf{k} is a \mathbb{Q} -algebra. Then, as S_k -modules,

$$\mathcal{P}/J \cong (\mathcal{P}/\mathcal{PS}^+)^{\times \binom{n}{k}} \cong \left(\underbrace{\mathbf{k}S_k}_{\text{regular rep}} \right)^{\times \binom{n}{k}},$$

where \mathcal{PS}^+ is the ideal of \mathcal{P} generated by symmetric polynomials with constant term 0.

- Let us recall symmetric **functions** (not polynomials) now; we'll need them soon anyway.

$$\mathcal{S} := \{\text{symmetric polynomials in } x_1, x_2, \dots, x_k\};$$

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- So why not replace the \mathbf{e}_j by $\mathbf{e}_j - b_j$ too?

- **Theorem (G.):** Assume that $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k$ as well as $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \dots$ are elements of Λ such that

$$\deg \mathbf{a}_i < n - k + i \quad \text{and} \quad \deg \mathbf{b}_i < k + i.$$

Then,

$$\Lambda / (\mathbf{h}_{n-k+1} - \mathbf{a}_1, \mathbf{h}_{n-k+2} - \mathbf{a}_2, \dots, \mathbf{h}_n - \mathbf{a}_k, \\ \mathbf{e}_{k+1} - \mathbf{b}_1, \mathbf{e}_{k+2} - \mathbf{b}_2, \mathbf{e}_{k+3} - \mathbf{b}_3, \dots)_{\text{ideal}}$$

is a free \mathbf{k} -module with basis $(\overline{\mathbf{s}}_\lambda)_{\lambda \in P_{k,n}}$.

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 - Darij Grinberg, *A basis for a quotient of symmetric polynomials (draft)*, <http://www.cip.ifi.lmu.de/~grinberg/algebra/basisquot.pdf> ,
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- **Main ideas:**
 - Use Gröbner bases to show that \mathcal{P}/J is free with basis $(\overline{x^\alpha})_{\alpha \in \mathbb{N}^k; \alpha_i < n-k+i \text{ for each } i}$.
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 - Using that + Jacobi–Trudi, show that \mathcal{S}/I is free with basis $(\overline{s_\lambda})_{\lambda \in P_{k,n}}$.
 - As for the rest, compute in $\Lambda \dots$ a lot.

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- Gröbner bases are “particularly uncomplicated” generating sets for ideals in polynomial rings.
(But take the word “basis” with a grain of salt – they can have redundant elements, for example.)

- A *monomial order* is a total order on the monomials in \mathcal{P} with the properties that
 - $1 \leq m$ for each monomial m ;
 - $a \leq b$ implies $am \leq bm$ for any monomials a, b, m ;
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- The *degree-lexicographic order* is the monomial order defined as follows: Two monomials $a = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k}$ and $b = x_1^{\beta_1} x_2^{\beta_2} \cdots x_k^{\beta_k}$ satisfy $a > b$ if and only if
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 - each nonzero polynomial $f \in \mathcal{P}$ has a well-defined *leading monomial* (= the highest monomial appearing in f).
 - a polynomial f is called *quasi-monic* if the coefficient of its leading term in f is invertible.

- If \mathcal{I} is an ideal of \mathcal{P} , then a *Gröbner basis* of \mathcal{I} (for a fixed monomial order) means a family $(f_i)_{i \in G}$ of quasi-monic polynomials that
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 - The quadruple $(y^3 - z^3, x^2 - yz, xy - z^2, xz - y^2)$ is a Gröbner basis of \mathcal{I} . (Thanks SageMath, and whatever packages it uses for this.)

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- **Theorem (Macaulay's basis theorem).** Let \mathcal{I} be an ideal of \mathcal{P} that has a Gröbner basis $(f_i)_{i \in G}$. A monomial \mathfrak{m} will be called *reduced* if it is not divisible by the leading term of any f_i . Then, the projections of the reduced monomials form a basis of the \mathbf{k} -module \mathcal{P}/\mathcal{I} .

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Thus, $(\bar{x}) \cup (\overline{y^j z^\ell})_{j < 3}$ is a basis of \mathcal{P}/\mathcal{I} .

- It is easy to prove the identity

$$h_p(x_{i..k}) = \sum_{t=0}^{i-1} (-1)^t e_t(x_{1..i-1}) h_{p-t}(x_{1..k})$$

for all $i \in \{1, 2, \dots, k+1\}$ and $p \in \mathbb{N}$.

Here, $x_{a..b}$ means x_a, x_{a+1}, \dots, x_b .

- Use this to show that

$$\left(h_{n-k+i}(x_{i..k}) - \sum_{t=0}^{i-1} (-1)^t e_t(x_{1..i-1}) a_{i-t} \right)_{i \in \{1, 2, \dots, k\}}$$

is a Gröbner basis of the ideal J wrt the degree-lexicographic order.

- Thus, Macaulay's basis theorem shows that

$(\overline{x^\alpha})_{\alpha \in \mathbb{N}^k; \alpha_i < n-k+i \text{ for each } i}$ is a basis of the \mathbf{k} -module \mathcal{P}/J .

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- Combining these yields that $(\overline{s_\lambda x^\alpha})_{\lambda \in P_{k,n}; \alpha \in \mathbb{N}^k; \alpha_i < i \text{ for each } i}$ spans $\mathcal{P}/I\mathcal{P} = \mathcal{P}/J$.

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- But we also know that the family $(\overline{x^\alpha})_{\alpha \in \mathbb{N}^k; \alpha_i < n-k+i \text{ for each } i}$ is a basis of \mathcal{P}/J .

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- Jacobi–Trudi lets you recursively reduce each $\overline{s_\lambda}$ with $\lambda \notin P_{k,n}$ into smaller $\overline{s_\mu}$'s.
 $\implies (\overline{s_\lambda})_{\lambda \in P_{k,n}}$ spans \mathcal{S}/I .
- On the other hand, $(x^\alpha)_{\alpha \in \mathbb{N}^k; \alpha_i < i \text{ for each } i}$ spans \mathcal{P} as an \mathcal{S} -module (Artin).
- Combining these yields that $(\overline{s_\lambda x^\alpha})_{\lambda \in P_{k,n}; \alpha \in \mathbb{N}^k; \alpha_i < i \text{ for each } i}$ spans $\mathcal{P}/I\mathcal{P} = \mathcal{P}/J$.
- But we also know that the family $(\overline{x^\alpha})_{\alpha \in \mathbb{N}^k; \alpha_i < n-k+i \text{ for each } i}$ is a basis of \mathcal{P}/J .
- What can you say if a \mathbf{k} -module has a basis $(a_v)_{v \in V}$ and a spanning family $(b_u)_{u \in U}$ of the same finite size ($|U| = |V| < \infty$)?

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- Maybe the most important one:

Bernstein's identity: Let λ be a partition. Let $m \in \mathbb{Z}$ be such that $m \geq \lambda_1$. Then,

$$\sum_{i \in \mathbb{N}} (-1)^i \mathbf{h}_{m+i} (\mathbf{e}_i)^\perp \mathbf{s}_\lambda = \mathbf{s}_{(m, \lambda_1, \lambda_2, \lambda_3, \dots)}.$$

Here, $\mathbf{f}^\perp \mathbf{g}$ means “ \mathbf{g} skewed by \mathbf{f} ” (so that $(\mathbf{s}_\mu)^\perp \mathbf{s}_\lambda = \mathbf{s}_{\lambda/\mu}$).

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