Higher Lie idempotents
Frédéric Patras and Christophe Reutenauer


Errata and questions - I (version 2)

• Page 1: Typo: ”caracteristic” should be ”characteristic”.

• Pages 1 and 2: Typo: ”envelopping” should be ”enveloping” (this typo appears several times).

• Page 2 and further: Typo: ”familly” should be ”family” (this typo appears several times).

• Page 2: Maybe ”Given a familly of Lie idempotents” should be ”Given an arbitrary Lie idempotent”? I think the constructions of the higher Lie idempotents depend only on one Lie idempotent $\iota$ and (in the case of higher Lie idempotents of the third kind) on a family of coefficients $a_{\mu}$.

• Page 3: Typo: ”reodering” should be ”reordering”.

• Page 4: Between Definition 2.2 and the Example, you write that ”the $\iota$-descent algebra decomposes as a direct sum

$$D_\iota = \bigoplus_{n=0}^{\infty} D_{\iota n}.$$"

It might be useful to notice here that this is a direct sum of vector spaces, not of algebras (under the convolution $*$).

• Page 5: In the proof of Lemma 3.1, you write: ”More generally, for any $l \geq 2$ and $k \geq 3$, let $\Delta_{l2}$ be [...]”. I don’t see any reason to require $l \geq 2$ and $k \geq 3$ here; everything is just as correct for any $l \geq 0$ and $m \geq 0$.

• Page 6: In the proof of Lemma 3.1, the $\sum_{\sigma \in S_n}$ should be $\sum_{\sigma \in S_k}$.

• Page 6: In the proof of Lemma 3.1, you write: ”If we apply $\Pi_k$ to the whole sum” (in the fourth line of page 6). I think you are applying $\Pi_k \otimes^k$ here, not $\Pi_k$.

• Page 6: In the proof of Lemma 3.1, you have a typo: ”Aplying” should be ”Applying”.

• Page 6: In the proof of Lemma 3.1, you write: ”Now, this sum is equal to $\sum (t_{\mu_1} \otimes \ldots \otimes t_{\mu_k}) \circ \sigma (x_1 \otimes \ldots \otimes x_k)$, where $\sigma$ denotes here the natural action of the symmetric group on $A^\otimes n$”. First, there should be a whitespace after ”Now,”. Second, the $\sigma (x_1 \otimes \ldots \otimes x_k)$ should be a $\sigma^{-1} (x_1 \otimes \ldots \otimes x_k)$, because $\sigma (x_1 \otimes \ldots \otimes x_k)$ is $x_{\sigma^{-1}(1)} \otimes \ldots \otimes x_{\sigma^{-1}(k)}$ rather than $x_{\sigma(1)} \otimes \ldots \otimes x_{\sigma(k)}$. Third, I think you mean $A^\otimes k$ instead of $A^\otimes n$ (unless you want to talk about general $n$).
• **Page 6:** In the proof of Lemma 3.1, you write: "Since the coproduct is cocommutative, we deduce that
\[(t_{\mu_1} \otimes \cdots \otimes t_{\mu_k}) \circ (\Pi^k_k) \circ \Delta_{kk} \circ (t_{\lambda_1} \otimes \cdots \otimes t_{\lambda_k}) = \sum (t_{\mu_1} \otimes \cdots \otimes t_{\mu_k}) \circ \sigma \circ \Delta_k = \sum (t_{\mu_1} \otimes \cdots \otimes t_{\mu_k}) \circ \Delta_k,\]
which implies (ii)." The \(\sigma\) here should be a \(\sigma^{-1}\). (Also, what somewhat confused me is that cocommutativity is used in the passage from \(\sum (t_{\mu_1} \otimes \cdots \otimes t_{\mu_k}) \circ \sigma \circ \Delta_k\) to \(\sum (t_{\mu_1} \otimes \cdots \otimes t_{\mu_k}) \circ \Delta_k\), not in the passage from \((t_{\mu_1} \otimes \cdots \otimes t_{\mu_k}) \circ (\Pi^k_k) \circ \Delta_{kk} \circ (t_{\lambda_1} \otimes \cdots \otimes t_{\lambda_k})\) to \((t_{\mu_1} \otimes \cdots \otimes t_{\mu_k}) \circ \sigma \circ \Delta_k\). It thus would probably better to mention cocommutativity after the long equation rather than before it.)

• **Page 7:** In the proof of Theorem 3.4, you write: "Thus \(f\) is idempotent if and only if \([\ldots]\)". But in general, only the "if" part of this is true (and fortunately, only the "if" part is needed), since nobody has told us that the \(t_{\alpha}\) are linearly independent.

• **Page 7:** In the proof of Theorem 3.4, it would be clearer if you replace \((n_1 + \ldots + n_k)!/n_1! \ldots n_k!\) by \((n_1 + \ldots + n_k)!/(n_1! \ldots n_k!)\). (I consider the notation \(a/b_1 b_2 b_k\) for \(a/(b_1 b_2 b_k)\) outdated and ambiguous, although it seems to be still in use.)

• **Page 7:** In Definition 4.1, I feel it would be good to point out three things explicitly:
  - The "1" in "\(F^v_\lambda := \left(1 - \sum_{l(\mu) < l(\lambda)} F^v_\mu\right) \circ E^v_\lambda\)" means the identity map \(\text{id}_{A_n} \in \text{End}(A_n)\), not the unity of the algebra \(\mathcal{L}(A)\).
  - For \(n = 0\), the element \(F^v_\lambda\) is defined as \(E^v_\lambda = \text{id}_{A_0} = \eta \circ \epsilon\) (here we are using the identification of \(\text{End}(A_0)\) with the space of all graded endomorphisms of \(A\) whose image is \(\subseteq A_0\)). (While this can be seen as a consequence of the formula \(F^v_\lambda := \left(1 - \sum_{l(\mu) < l(\lambda)} F^v_\mu\right) \circ E^v_\lambda\) applied to \(\lambda = ()\), it would be helpful to point this out explicitly).
  - The maps \(F^v_\lambda\) are called the "higher Lie idempotents of the second kind".

• **Page 7:** In Definition 4.1, it wouldn’t harm to say that the "induction base"
\[F^v_{(n):} = E^v_{(n)} = \iota_n\] is, itself, a particular case of the "induction step" \(F^v_\lambda := \left(1 - \sum_{l(\mu) < l(\lambda)} F^v_\mu\right) \circ E^v_\lambda\). In fact, if we substitute \(\lambda = (n)\) in \(F^v_\lambda := \left(1 - \sum_{l(\mu) < l(\lambda)} F^v_\mu\right) \circ E^v_\lambda\), then we get \(F^v_{(n)} = \left(1 - \sum_{l(\mu) < l((n))} F^v_\mu\right) \circ E^v_{(n)}\), but the sum \(\sum_{l(\mu) < l((n))} F^v_\mu\) is empty since \(l((n)) = 1\), and thus this becomes \(F^v_{(n)} = E^v_{(n)}\).

This fact allows us to use \(F^v_\lambda = \left(1 - \sum_{l(\mu) < l(\lambda)} F^v_\mu\right) \circ E^v_\lambda\) not only for \(\lambda \neq (n)\) but also for all \(\lambda\). This is used in several proofs in your paper.

• **Page 7:** In the Remark 1) at the end of page 7, you made a typo: "othogonal" should be "orthogonal".
• Page 9: On the first line of this page, you write: "$$F^t_\mu \circ F^t_\beta = \delta_{\mu\beta}$$". This should be $$F^t_\mu \circ F^t_\beta = \delta_{\mu\beta} F^t_\mu$$. (The only thing you actually use, though, is that $$F^t_\mu \circ F^t_\beta = 0$$ for $$\mu \neq \beta$$ when $$l(\mu)$$ and $$l(\beta)$$ are both $$< k$$.)

• Page 9: In the proof of Theorem 4.3, you write: "we have by Def.4.1 that $$E^t_\lambda (x) = F^t_\lambda (x)$$ plus a sum of $$E^t_{\lambda_1} \circ \ldots \circ E^t_{\lambda_k}$$". First, either you should replace the $$E^t_\lambda (x)$$ and $$F^t_\lambda (x)$$ here by $$E^t_i$$ and $$F^t_i$$, or you should replace the $$E^t_{\lambda_1} \circ \ldots \circ E^t_{\lambda_k}$$ by an $$(E^t_{\lambda_1} \circ \ldots \circ E^t_{\lambda_k}) (x)$$. Second, "sum" is slightly imprecise; you mean a linear combination rather than a sum (the coefficients in this combination can be both +1 and −1).

• Page 9: In the proof of Theorem 4.3, you write: "the elements $$(a_1, ..., a_k) = (1/k!) \sum_{k \in S_k} a_{\sigma(1)} \ldots a_{\sigma(k)}$$. Replace $$\sum_{k \in S_k}$$ by $$\sum_{\sigma \in S_k}$$ here.

• Page 9: In the proof of Theorem 4.3, you write: "Since $$A$$ is a graded cocommutative connected bialgebra of characteristic zero, it is by the Cartier-Milnor-Moore theorem isomorphic to the enveloping algebra of $$Prim (A)$$. Hence, by the Poincaré-Birkhoff-Witt theorem it is the direct sum of its subspaces $$A^\lambda$$, where for any partition $$\lambda$$, the latter subspace is spanned by the elements $$(a_1, ..., a_k) = (1/k!) \sum_{k \in S_k} a_{\sigma(1)} \ldots a_{\sigma(k)}$$, for any choice of homogeneous primitive elements $$a_i$$, with deg $$(a_i) = \lambda_i$$ and $$\lambda = (\lambda_1, ..., \lambda_k)$$."

This is a correct argument (up to the typos I mentioned above), but somewhat an overkill. In fact, you only need the easy part of the Cartier-Milnor-Moore theorem and only the easy part of the Poincaré-Birkhoff-Witt theorem to show that $$A$$ is the sum of its subspaces $$A^\lambda$$ (we don’t yet know that it is the direct sum), and this is already enough for your proof of Theorem 4.3. (I can detail this argument better if you wish, but I have a feeling that you already know this). Maybe you need something stronger (like the direct sum assertion) to prove Corollary 4.4 though (I don’t understand your proof at the moment), but I would always try to do without - maybe this will net us an explicit constructive proof of Poincaré-Birkhoff-Witt or Cartier-Milnor-Moore at the end...

• Page 9: In the proof of Theorem 4.3, you write: "It is equal to $$\sum_{\mu} \Pi_k \circ t_{\mu_1} \otimes \ldots \otimes t_{\mu_k} \circ \Delta_k (a_1 \ldots a_k)". I would put the $$t_{\mu_1} \otimes \ldots \otimes t_{\mu_k}$$ term in brackets here.

• Page 9: In the last absatz of page 9, you write: "the cofree cocommutative coalgebra on a vector space $$V$$". But I think it is more common to say “over a vector space $$V$$” rather than “on a vector space $$V$$”. (You yourself say “over” in Corollary 4.4.)

• Page 10: In Corollary 4.4, replace "$$\bigoplus \frac{1}{n!} t^\otimes n \circ \Delta_n$$" by "$$\bigoplus \frac{1}{n!} t^\otimes n \circ \Delta_n$$" (otherwise, this map would not be a coalgebra homomorphism).

• Page 10: In Corollary 4.4, replace the $$\mapsto$$ arrow by $$\rightarrow$$ arrow.

---

1By the "easy part", I mean the statement that a graded cocommutative connected bialgebra over a field of characteristic 0 is always generated as an algebra by its primitive elements.

2Here, the "easy part" is the statement that the symmetrization map $$S (g) \rightarrow U (g)$$ is surjective. (This only makes sense in characteristic 0.)
• **Page 10:** In Corollary 4.4, replace 

\[
\frac{1}{l(\lambda)!} \left( 1 - \sum_{l(\mu) < l(\lambda)} F^\mu \right) \circ \Pi_k \]

by 

\[
\frac{1}{l(\lambda)!} \left( 1 - \sum_{l(\mu) < l(\lambda)} F^\mu \right) \circ \Pi_k
\]

(\text{this change is needed to "balance out" the \( \frac{1}{n!} \) factor I added to \( \bigoplus_{n \in \mathbb{N}} l^\otimes_n \circ \Delta_n \)).

• **Page 10:** You write that "The corollary follows, once it is noted that Sym^\lambda (Prim (A)) is canonically isomorphic to A^\lambda, through the map \( \Pi_k \)." I do understand why Sym^\lambda (Prim A) is canonically isomorphic to A^\lambda through the map \( \Pi_k \). But I don’t understand how Corollary 4.4 follows from this! In particular, I don’t see how the \( \frac{1}{l(\lambda)!} \left( 1 - \sum_{l(\mu) < l(\lambda)} F^\mu \right) \) term appears.

• **Page 11:** In the proof of Theorem 5.1, you write: "We multiply this by \( e_n \) on the right in \( \mathcal{L} (A) \)." I think this is confusing: Multiplying something in \( \mathcal{L} (A) \)

\[
\sum_{\lambda \in \mathcal{N}} \sum_{S \subseteq \mathcal{V}} \lambda^S \text{ through the map } \Pi
\]

\[\text{for all } S \subseteq \mathcal{V}, \text{ through the map } \Pi \]

...
means convolution, but you want composition. Maybe you could just say "We compose this with $\epsilon_n$ on the right"?

- **Page 11:** In the proof of Theorem 5.1, you write:
  
  "Thus we obtain $e_n = \alpha e_n$, since $e_{\mu} \circ e_n = 0$ by Lemma 3.2. Thus, in case $e_n \neq 0$, $\alpha = 1$; and in case $e_n = 0$, we must have also $\iota_n = 0$, and we may take $\alpha = 1$ in (*).
  
  This argument is correct, but I think it can be simplified as follows:
  "Thus we obtain $e_n = \alpha e_n$, since $e_{\mu} \circ e_n = 0$ by Lemma 3.2. Thus, we can replace $\alpha e_n$ by $e_n$ in (*), and get $\iota_n = e_n + \sum_\mu \epsilon_n$.

Elements of $A$ satisfying $(\deg (a_i) = \lambda_i$ for all $i \in \{1, 2, ..., k\})$. In other words,

$$A^\lambda = \left\langle \left\{ \frac{1}{k!} \sum_{\sigma \in S_k} a_{\sigma(1)} \otimes a_{\sigma(2)} \otimes \ldots \otimes a_{\sigma(k)} \mid \text{all } a_i \text{ are primitive and homogeneous} \right\} \right\rangle$$

$$= \left\langle \left\{ \Pi_k \left( \frac{1}{k!} \sum_{\sigma \in S_k} a_{\sigma(1)} \otimes a_{\sigma(2)} \otimes \ldots \otimes a_{\sigma(k)} \right) \mid \text{all } a_i \text{ are primitive and homogeneous} \right\} \right\rangle$$

(since $\frac{1}{k!} \sum_{\sigma \in S_k} a_{\sigma(1)} a_{\sigma(2)} \ldots a_{\sigma(k)} = \Pi_k \left( \frac{1}{k!} \sum_{\sigma \in S_k} a_{\sigma(1)} \otimes a_{\sigma(2)} \otimes \ldots \otimes a_{\sigma(k)} \right)$ for any $(a_1, a_2, ..., a_k) \in A^k$)

$$= \Pi_k \left( \left\langle \left\{ \frac{1}{k!} \sum_{\sigma \in S_k} a_{\sigma(1)} \otimes a_{\sigma(2)} \otimes \ldots \otimes a_{\sigma(k)} \mid \text{all } a_i \text{ are primitive and homogeneous} \right\} \right\rangle \right)$$

$$= \Pi_k \left( \left\langle \left\{ \frac{1}{k!} \sum_{\sigma \in S_k} a_{\sigma(1)} \otimes a_{\sigma(2)} \otimes \ldots \otimes a_{\sigma(k)} \mid \text{all } a_i \text{ are primitive and homogeneous} \right\} \right\rangle \right)$$

(since $\Pi_k$ is $F$-linear)

$$= \Pi_k \left( \text{Sym}^\lambda (\text{Prim} A) \right) \quad \text{(by (A1))}.$$ 

Hence, $\Pi_k$ restricts to a surjective homomorphism $\text{Sym}^\lambda (\text{Prim} A) \rightarrow A^\lambda$.

Moreover, let $\tilde{\Pi}$ be the homomorphism $\bigoplus_{n \in \mathbb{N}} \Pi_n \left|_{((\text{Prim} A)^{\otimes n})^{S_n}} \right.: \bigoplus_{n \in \mathbb{N}} \left( (\text{Prim} A)^{\otimes n} \right)^{S_n} \rightarrow A$ (composed of the homomorphisms $\Pi_n \left|_{((\text{Prim} A)^{\otimes n})^{S_n}} \right.: (\text{Prim} A)^{\otimes n} \rightarrow A$ for all $n \in \mathbb{N}$). This homomorphism $\tilde{\Pi}$ sends $\frac{1}{n!} \sum_{\sigma \in S_n} a_{\sigma(1)} \otimes a_{\sigma(2)} \otimes \ldots \otimes a_{\sigma(n)}$ to $\frac{1}{n!} \sum_{\sigma \in S_n} a_{\sigma(1)} a_{\sigma(2)} \ldots a_{\sigma(n)}$ for every $n \in \mathbb{N}$ and $(a_1, a_2, ..., a_n) \in (\text{Prim} A)^n$. According to the Poincaré-Birkhoff-Witt theorem, this homomorphism $\tilde{\Pi}$ is an isomorphism (since the Cartier-Milnor-Moore theorem yields $A \cong U (\text{Prim} A)$, and under the identification of $A$ with $U (\text{Prim} A)$ the homomorphism $\tilde{\Pi}$ becomes the symmetrization map $S (\text{Prim} A) \rightarrow U (\text{Prim} A)$). Hence, $\tilde{\Pi}$ is injective.
This simplified argument has the additional advantage of being valid when $k$ is not necessarily a field.

- **Page 11**: In the proof of Theorem 5.1, you made a typo: "matrix fom" should be "matrix form".

- **Page 11**: In the proof of Theorem 5.1, you write: "It is clear that (i) implies (iv)". But is this really clear on its own, or is it clear using the fact that $D(A)$ is closed under convolution (a consequence of Theorem 9.2 in [R2], but [R2] only considers the case when $A$ is the tensor algebra of an alphabet)?

- **Page 12**: In the proof of Lemma 5.3, replace $\iota^{\mu_2}$ by $\iota^{\mu_2}$ (you forgot to make the 2 an index).

- **Pages 12 and 13**: In the proof of Theorem 5.4, you write: "Moreover:

\[
\left( \sum_{\mu < [n]} \mathcal{E}_\mu^t \right)^2 = (1 - \mathcal{E}_{[n]}^t)^2 = 1 - \mathcal{E}_{[n]}^t = \sum_{\mu < [n]} \mathcal{E}_\mu^t,
\]

and:

\[
\left( \sum_{\mu < [n]} \mathcal{E}_\mu^t \right) \circ \mathcal{E}_{[n]}^t = (1 - \mathcal{E}_{[n]}^t) \circ \mathcal{E}_{[n]}^t = 0.
\]

"}

These formulas are not literally true, because $\sum_{\mu < [n]} \mathcal{E}_\mu^t$ is $p_n - \mathcal{E}_{[n]}^t$ rather than

Now, since $\text{Sym}^\lambda(\text{Prim} A) \subseteq \left( (\text{Prim} A)^\otimes k \right)^{S_k}$, we have

\[
\Pi |_{\text{Sym}^\lambda(\text{Prim} A)} = \left( \Pi \big|_{\left( (\text{Prim} A)^\otimes k \right)^{S_k}} \right) |_{\text{Sym}^\lambda(\text{Prim} A)}
= \Pi_k \left|_{\left( (\text{Prim} A)^\otimes k \right)^{S_k}} \right|_{\text{Sym}^\lambda(\text{Prim} A)}
\]

(since $\Pi = \bigoplus_{n \in \mathbb{N}} \Pi_n \big|_{\left( (\text{Prim} A)^\otimes n \right)^{S_n}}$)

\[
= \left( \Pi_k \big|_{\left( (\text{Prim} A)^\otimes k \right)^{S_k}} \right) |_{\text{Sym}^\lambda(\text{Prim} A)} = \Pi_k |_{\text{Sym}^\lambda(\text{Prim} A)}.
\]

Since $\Pi |_{\text{Sym}^\lambda(\text{Prim} A)}$ is injective (because $\Pi$ is injective), this yields that $\Pi_k |_{\text{Sym}^\lambda(\text{Prim} A)}$ is injective. Now, consider the surjective homomorphism $\text{Sym}^\lambda(\text{Prim} A) \rightarrow A^\lambda$ to which $\Pi_k$ restricts. This homomorphism is also injective (since $\Pi_k |_{\text{Sym}^\lambda(\text{Prim} A)}$ is injective), and thus it is an isomorphism. Thus, $\Pi_k$ restricts to an isomorphism $\text{Sym}^\lambda(\text{Prim} A) \rightarrow A^\lambda$. Hence, $\text{Sym}^\lambda(\text{Prim} A)$ is isomorphic to $A^\lambda$ through the map $\Pi_k$, qed.
1 - \( \mathcal{E}_{[n]}^t \) (since

\[
\sum_{\mu < [n]} \mathcal{E}_{\mu}^t + \mathcal{E}_{[n]}^t = \sum_{\mu \leq [n]} \mathcal{E}_{\mu}^t = \sum_{\mu \text{ is a partition of } n} \mathcal{E}_{\mu}^t = \sum_{\lambda \text{ is a partition of } n} \mathcal{E}_{\mu}^t = \sum_{\mu \text{ is a composition of } n; p(\mu) = \lambda} a_{\mu}^t \cdot t_{\mu} = p_n
\]

). Only if you restrict all maps to the \( n \)-th graded component of \( A \), these equations become true. Alternatively, you could replace these equations by

\[
\left( \sum_{\mu < [n]} \mathcal{E}_{\mu}^t \right)^2 = (p_n - \mathcal{E}_{[n]}^t)^2 = \sum_{p_n = p_n} = \mathcal{E}_{[n]}^t - \mathcal{E}_{[n]}^t \circ p_n - p_n \circ \mathcal{E}_{[n]}^t + \mathcal{E}_{[n]}^t = p_n - \mathcal{E}_{[n]}^t = \sum_{\mu < [n]} \mathcal{E}_{\mu}^t,
\]

and:

\[
\left( \sum_{\mu < [n]} \mathcal{E}_{\mu}^t \right) \circ \mathcal{E}_{[n]}^t = (p_n - \mathcal{E}_{[n]}^t) \circ \mathcal{E}_{[n]}^t = p_n \circ \mathcal{E}_{[n]}^t - (\mathcal{E}_{[n]}^t)^2 = 0.
\]

A similar inaccuracy appears at the end of page 13: There you write

\[
h \circ 1 = h \circ (h + g + k) = bh + h \circ g.
\]

This is not wrong, but not exactly clear: Probably you want to say

\[
h = h \circ p_n = h \circ (h + g + k) = bh + h \circ g.
\]

- **Page 13:** You write: “In other words, \( \mathcal{E}_{[n]}^t \) and \( \sum_{\mu < [n]} \mathcal{E}_{\mu}^t \) are two orthogonal idempotents.” But in order to show this, you must not only prove that \( (\mathcal{E}_{[n]}^t)^2 = \mathcal{E}_{[n]}^t \),

\[
\left( \sum_{\mu < [n]} \mathcal{E}_{\mu}^t \right)^2 = \sum_{\mu < [n]} \mathcal{E}_{\mu}^t \text{ and } \left( \sum_{\mu < [n]} \mathcal{E}_{\mu}^t \right) \circ \mathcal{E}_{[n]}^t = 0 \text{ (this you have proven), but also prove that } \mathcal{E}_{[n]}^t \circ \left( \sum_{\mu < [n]} \mathcal{E}_{\mu}^t \right) = 0. \text{ This is easy, of course:}
\]

\[
\mathcal{E}_{[n]}^t \circ \left( \sum_{\mu < [n]} \mathcal{E}_{\mu}^t \right) = \mathcal{E}_{[n]}^t \circ (p_n - \mathcal{E}_{[n]}^t) = \mathcal{E}_{[n]}^t \circ p_n - (\mathcal{E}_{[n]}^t)^2 = 0.
\]

But it should be mentioned, I think.
Page 14: You write: "It follows that the coefficients $a_{\mu}^\epsilon$ of the higher Lie idempotents of the third kind depend polynomially of $\epsilon$.

First, I don’t understand how this follows from $p_n = \sum_{|\mu|=n} F_{\mu}^\epsilon$. While all $F_{\mu}^\epsilon$ are (by definition) linear combinations (with constant coefficients) of compositions of various $\epsilon_{\mu}$, it is not clear (to me) why they are linear combinations (with coefficients polynomial in $\epsilon$) of convolutions of various $\epsilon_{\mu}$. I do know that $\mathcal{D}_{\nu}$ is closed under convolution (by Theorem 5.1, since $\epsilon_{\nu} \in \langle \epsilon, e \rangle \subseteq \mathcal{D}(A)$), and this yields that they are linear combinations of convolutions of various $\epsilon_{\mu}$, but why with coefficients polynomial in $\epsilon$?

Second, even if we can show that we can write $p_n$ as a linear combination of $\epsilon_{\mu}$ with coefficients polynomial in $\epsilon$, then it is not clear to me why these coefficients, when specializing at $\epsilon = 1$, become our $a_{\mu}^1$ - in fact, the $a_{\mu}^1$ are not always uniquely determined by $p_n = \sum_{|\mu|=n} a_{\mu}^1 \epsilon_{\mu}$ (since the $\epsilon_{\mu}$ are not always linearly independent), so the $a_{\mu}^1$ you have started with might not be the same as the $a_{\mu}^1$ you get by writing $p_n$ as a linear combination of $\epsilon_{\mu}$ and specializing at $\epsilon = 1$ (although both families of $a_{\mu}^1$ satisfy $p_n = \sum_{|\mu|=n} a_{\mu}^1 \epsilon_{\mu}$).

I am interested in how you actually show that the $a_{\mu}^\epsilon$ depend polynomially of $\epsilon$ in such a way that specialization at $\epsilon = 1$ yields our initial $a_{\mu}^1$. I think I can show this (with some handwaving) under the additional condition that $a_{[n]}^1 = 1$ for every $n$. Here is how my proof (roughly) goes:

Start with the equations $p_n = \sum_{|\mu|=n} a_{\mu}^1 \epsilon_{\mu}$. By repeated convolution, these equations yield equations of the form $p_{\nu} = \sum_{|\mu|=|\nu|; \mu \geq \nu} a_{\mu,\nu}^1 \epsilon_{\mu}$ (with $a_{\mu,\nu}^1$ being scalars, and $a_{\mu,[n]}^1 = a_{\mu}^1$) for all partitions $\nu$, where $\mu \geq \nu$ means that the composition $\mu$ can be obtained by splitting some parts of $\nu$ into smaller parts (this defines a partial order $\geq$ on compositions). Since $a_{[n]}^1 = 1$ for every $n$, we find that $a_{\nu,\nu}^1 = 1$ for every composition $\nu$. Now, the equations $p_{\nu} = \sum_{|\mu|=|\nu|; \mu \geq \nu} a_{\mu,\nu}^1 \epsilon_{\mu}$ show us that $(a_{\mu,\nu}^1)_{|\mu|=|\nu|=n}$ is an upper triangular matrix, and the equations $a_{\nu,\nu}^1 = 1$ show that its diagonal entries are $= 1$. Hence, it has an inverse matrix $(b_{\mu,\nu}^1)_{|\mu|=|\nu|=n}$ which satisfies $\epsilon_{\nu} = \sum_{|\mu|=|\nu|; \mu \geq \nu} b_{\mu,\nu}^1 \epsilon_{\mu}$ for all compositions $\nu$, and again is upper triangular and has diagonal entries $= 1$. The same argument, done for $e$ instead of $\epsilon$, shows that there exists a matrix $(b_{\mu,\nu}^e)_{|\mu|=|\nu|=n}$ which satisfies $e_{\nu} = \sum_{|\mu|=|\nu|; \mu \geq \nu} b_{\mu,\nu}^e \epsilon_{\mu}$ for all compositions $\nu$, and again is upper triangular and has its diagonal entries $= 1$. Now,
the matrix \((\epsilon \cdot b_{\mu,\nu}^\epsilon + (1 - \epsilon) \cdot b_{\mu,\nu}^\epsilon)_{|\mu|=|\nu|=n}\) satisfies

\[
\nu^\epsilon = \epsilon \cdot \nu + (1 - \epsilon) \cdot e_\nu = \epsilon \cdot \sum_{|\mu|=|\nu|; \mu \geq \nu} b_{\mu,\nu}^\epsilon p_\mu + (1 - \epsilon) \cdot \sum_{|\mu|=|\nu|; \mu \geq \nu} b_{\mu,\nu}^\epsilon p_\mu
\]

\[
= \sum_{|\mu|=|\nu|; \mu \geq \nu} \left(\epsilon \cdot b_{\mu,\nu}^\epsilon + (1 - \epsilon) \cdot b_{\mu,\nu}^\epsilon\right) p_\mu
\]

for all compositions \(\nu\), and again is upper triangular and has its diagonal entries = 1. Hence, its inverse matrix \((a_{\mu,\nu}^\epsilon)_{|\mu|=|\nu|=n}\) satisfies \(p_n = \sum_{|\mu|=n} a_{\mu,n}^\epsilon \nu^\epsilon\), but its entries \(a_{\mu,\nu}^\epsilon\) are polynomials in the entries of \((\epsilon \cdot b_{\mu,\nu}^\epsilon + (1 - \epsilon) \cdot b_{\mu,\nu}^\epsilon)_{|\mu|=|\nu|=n}\) (because if \(C\) is an upper triangular matrix with diagonal entries = 1, then the entries of \(C^{-1}\) are polynomials in the entries of \(C\)), and thus polynomials in \(\epsilon\). This gives us what we want.

But I cannot get rid of the condition that \(a_{\nu,\nu}^\epsilon = 1\) for every \(n\) (not only for the one we are working with, but also for the smaller \(n\), because we need all \(a_{\nu,\nu}^\epsilon\) to be 1).

**HOWEVER**, I think that I can modify your proof of Theorem 5.4 in a different way to make it valid:

First of all, let us generalize the results of Section 3 from one Lie idempotent to two Lie idempotents:

**Lemma 5.6.** Let \(\iota\) and \(\rho\) be two Lie idempotents. Then, any two compositions \(\lambda\) and \(\mu\) such that \(|\lambda| \neq |\mu|\) satisfy \(\iota_\lambda \circ \rho_\mu = 0\).

This is a very obvious fact (it is obvious because the image of \(\rho_\mu\) lies in the \(|\mu|\)-th graded component of \(H\), whereas \(\iota_\lambda\) sends every graded component of \(H\) except of the \(|\lambda|\)-th one to 0), and it generalizes the property \(\iota_\lambda \circ \iota_\mu = 0\) for \(|\lambda| \neq |\mu|\).

Less trivially, we have:

**Lemma 5.7.** Let \(\iota\) and \(\rho\) be two Lie idempotents. Let \(\mu\) and \(\lambda\) be two compositions of the same weight and the same length \(k\).

(i) If \(p(\lambda) \neq p(\mu)\), then \(\iota_\mu \circ \rho_\lambda = 0\).

(ii) If \(p(\lambda) = p(\mu)\), then \(\iota_\mu \circ \rho_\lambda = N_\rho_\mu\), where \(N\) is the number of permutations of \(\{1, 2, \ldots, k\}\) which act trivially on the sequence \(p(\mu) = p(\lambda)\). (This number \(N\) only depends on \(p(\lambda) = p(\mu)\), and will often be denoted by \(N(p(\lambda))\) or by \(N(\lambda)\).)

For the proof of Lemma 5.7, proceed in the same way as in the proof of Lemma 3.1. You will need the identity \(\iota \circ \rho = \rho\), which follows from \(\iota \mid_{\text{Prim } A} = \text{id}_{\text{Prim } A}\) (because both \(\iota\) and \(\rho\) are Lie idempotents, i.e., projections on \(\text{Prim } A\)).

Similarly:

**Lemma 5.8.** Let \(\iota\) and \(\rho\) be two Lie idempotents. Let \(\mu\) and \(\lambda\) be two compositions of the same weight such that \(l(\mu) > l(\lambda)\). Then \(\iota_\mu \circ \rho_\lambda = 0\).

This is proven in the same way as Lemma 3.2.

Next, we need a kind of generalization of Lemma 5.3:

**Lemma 5.9.** Let \(\iota\) and \(\rho\) be two Lie idempotents. Let \(\lambda\) be a partition. For

---

4In the following Lemmas 5.6, 5.7, 5.8 and 5.9, we don’t assume that \(D(A) = D_1\).
every composition $\mu$ with $p(\mu) = \lambda$, let $b_\mu^i$ and $b_\mu^o$ be two scalars. Then,

$$\left( \sum_{p(\mu) = \lambda} b_\mu^i t_\mu \right) \circ \left( \sum_{p(\mu) = \lambda} b_\mu^o \rho_\mu \right) = \left( \sum_{p(\mu) = \lambda} b_\mu^i \right) N \left( \sum_{p(\mu) = \lambda} b_\mu^o \rho_\mu \right),$$

where $N$ is the number of permutations of $\{1, 2, ..., k\}$ which act trivially on the sequence $\lambda$.

The proof of this lemma proceeds in the same way as the identity \( \left( \sum_{p(\mu) = \lambda} b_\mu t_\mu \right)^2 = \left( \sum_{p(\mu) = \lambda} b_\mu \right) N \left( \sum_{p(\mu) = \lambda} b_\mu \rho_\mu \right) \) was proven in the proof of Lemma 5.3. Here are the details of the proof:

**Proof of Lemma 5.9.** For every composition $\mu$ satisfying $p(\mu) = \lambda$, we know that $N$ is the number of permutations of $\{1, 2, ..., k\}$ which act trivially on the sequence $p(\mu)$ (because $N$ is defined as the number of permutations of $\{1, 2, ..., k\}$ which act trivially on the sequence $\lambda$, but we have $\lambda = p(\mu)$). Hence, for every composition $\mu$ satisfying $p(\mu) = \lambda$, we have $i_\mu \circ \rho_\mu = N \rho_\mu$ (by Lemma 5.7 (ii), applied to $\mu$ instead of $\lambda$). Since composition of linear maps is bilinear, we have

$$\left( \sum_{p(\mu) = \lambda} b_\mu^i t_\mu \right) \circ \left( \sum_{p(\mu) = \lambda} b_\mu^o \rho_\mu \right) = \sum_{p(\mu) = \lambda} \sum_{\mu\rho_\mu = N \rho_\mu} b_\mu^i b_\mu^o \circ \rho_\mu = N \sum_{p(\mu) = \lambda} \sum_{\mu\rho_\mu = N \rho_\mu} b_\mu^i b_\mu^o \rho_\mu = \left( \sum_{p(\mu) = \lambda} b_\mu^i \right) N \left( \sum_{p(\mu) = \lambda} b_\mu^o \rho_\mu \right)$$

(since composition of linear maps is bilinear).

This proves Lemma 5.9.

Now to the proof of Theorem 5.4. We proceed in the same way as you do (with one exception: we don’t have to assume $h \neq 0$) until your Claim 5.5 (which we cannot make anymore, since we haven’t assumed that $h \neq 0$). Then, just as you, we prove $h \circ g = (1 - b) h$ and $k \circ g = (b - 1) h$. Now I am going to show that $h^2 = h$.

First of all, we have $p_n = \sum_{|\mu| = n} \frac{1}{n!} e_\mu$ \(^5\). Let us define a scalar $a_\mu^e$ by $a_\mu^e = \frac{1}{n!}$ for every partition $\mu$. Then, $p_n = \sum_{|\mu| = n} \frac{1}{n!} e_\mu = \sum_{|\mu| = n} a_\mu^e e_\mu$. Hence, in the same way as we defined an element $E^\lambda$ for every partition $\lambda$ in Definition 5.2, we can define

---

\(^5\)This is a known fact (I knew it in the form $p_n = \sum_{\ell=0}^n \frac{1}{\ell!} \sum_{(n_1, n_2, ..., n_{\ell}) \in \{1, 2, ..., n\}^\ell} (e_{a_1} \ast e_{a_2} \ast ... \ast e_{a_\ell})$). It can be easily derived from the fact that $e = \log_{a_\ast} (\text{id})$, so that $\text{id} = \exp_{a_\ast} e = \exp_{a_\ast} (e_1 + e_2 + e_3 + ...)$. 

---
an element $\mathcal{E}_\lambda^e$ for every partition $\lambda$ by the formula

$$\mathcal{E}_\lambda^e := \sum_{p(\mu) = \lambda} \frac{a_\mu^e \cdot e_\mu}{n!} = \sum_{p(\mu) = \lambda} \frac{1}{n!} e_\mu.$$ 

From Lemmas 5.7 and 5.8 (applied to $e$ and $\iota$ instead of $\iota$ and $\rho$), we conclude that $\mathcal{E}_\lambda^e \circ \mathcal{E}_\mu^e = 0$ for every partition $\mu < \lambda$. Hence,

$$\mathcal{E}_\lambda^e \circ \sum_{\mu < \lambda} \frac{k}{e_\mu} = \mathcal{E}_\lambda^e \circ \left( \sum_{\mu < \lambda} \mathcal{E}_\mu^e \right) = \sum_{\mu < \lambda} \mathcal{E}_\mu^e \circ \mathcal{E}_\mu^e = 0.$$

On the other hand, for every partition $\lambda$, let $N(\lambda)$ denote the number of permutations of $\{1, 2, ..., k\}$ which act trivially on the sequence $\lambda$. We have $\mathcal{E}_\lambda^e = \sum_{p(\mu) = \lambda} \frac{1}{n!} e_\mu$ and $h = \mathcal{E}_\lambda^\iota = \sum_{p(\mu) = \lambda} a_\mu^\iota \cdot e_\mu$, so that

$$\mathcal{E}_\lambda^e \circ h = \left( \sum_{p(\mu) = \lambda} \frac{1}{n!} e_\mu \right) \circ \left( \sum_{p(\mu) = \lambda} a_\mu^\iota \cdot e_\mu \right) = \left( \sum_{p(\mu) = \lambda} \frac{1}{n!} \right) N(\lambda) \cdot \left( \sum_{p(\mu) = \lambda} a_\mu^\iota \cdot e_\mu \right)$$

(by Lemma 5.9, applied to $N(\lambda), \frac{1}{n!}, a_\mu^\iota, e$ and $\iota$ instead of $N, b_\mu^\iota, b_\mu^\iota, \iota$ and $\rho$)

$$= \left( \sum_{p(\mu) = \lambda} \frac{1}{n!} \right) N(\lambda) h.$$

Now, compare

$$\mathcal{E}_\lambda^e \circ k \circ g = 0 \circ g = 0$$

with

$$\mathcal{E}_\lambda^e \circ \sum_{p(\mu) = \lambda} \frac{k}{n!} = (b - 1) \left( \sum_{p(\mu) = \lambda} \frac{1}{n!} \right) N(\lambda) h.$$

This yields

$$(b - 1) \left( \sum_{p(\mu) = \lambda} \frac{1}{n!} \right) N(\lambda) h = 0.$$

Since $\left( \sum_{p(\mu) = \lambda} \frac{1}{n!} \right) N(\lambda)$ is invertible in $k$ (in fact, $\left( \sum_{p(\mu) = \lambda} \frac{1}{n!} \right) N(\lambda) \neq 0$ obviously; we can even prove that $\left( \sum_{p(\mu) = \lambda} \frac{1}{n!} \right) N(\lambda) = 1$, but we don’t need this),
this becomes \((b - 1) h = 0\), so that \(h = bh\). Compared with \(h \circ h = bh\) (which follows from the proof of Lemma 5.3), this yields \(h \circ h = h\), so that \(h\) is an idempotent.

Since \(g^2 = g\) (because \(g = \sum_{\mu > \lambda} E_{\mu}\), and by the induction assumption the \(E_{\mu}\) are orthogonal idempotents), \(h \circ g = (1 - b) h = - (b - 1) h = 0\) and \(k \circ g = (b - 1) h = 0\), we can continue the proof as you do after you prove Claim 5.5. This proves Theorem 5.4.

- **Page 14:** There is a typo: \(b^i\) should be \(b^\iota\).
- **Page 15:** You write: ”and the proof of theorem 5.3 is complete”. The theorem is Theorem 5.4, not 5.3.
- **Page 16:** In reference [R1], typo: ”representations”.
