Three questions on symmetric group algebras

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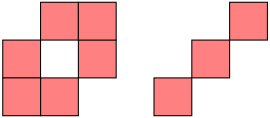
slides: http://www.cip.ifi.lmu.de/~grinberg/algebra/harvard2024.pdf

Are Specht modules pure?

- Let $\mathcal{A} = \mathbf{k}[S_n]$ be the group algebra of the symmetric group S_n (aka $\mathfrak{S}_n$) over a commutative ring $\mathbf{k}$.
- Let D be a diagram with n cells. For instance, for $n = 9$, we can have

$D_9 =$  .

- Let $\mathcal{S}^D$ be its Specht module. This is a left $\mathcal{A}$-module defined in any of the following ways:
  - as the span of polytabloids

$$\mathbf{e}_T = \sum_{\substack{w \in S_n \text{ preserves} \\ \text{the columns of } T}} (-1)^w \underbrace{\overline{wT}}_{\substack{\text{This means the} \\ \text{tabloid of } wT.}}$$

    within the Young module $\mathcal{M}^D$ (free $\mathbf{k}$-module on tabloids);

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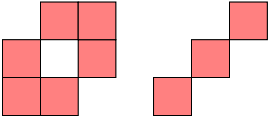
$D_9 =$  .

- Let $\mathcal{S}^D$ be its Specht module. This is a left $\mathcal{A}$-module defined in any of the following equivalent ways:
  - as the left ideal of $\mathcal{A}$ generated by $\mathbf{N}_T \mathbf{P}_T$, where T is any filling of D with the numbers $1, 2, \ldots, n$, and where

$$\mathbf{P}_T = \sum_{\substack{w \in S_n \text{ preserves} \\ \text{the rows of } T}} w \qquad \text{and} \qquad \mathbf{N}_T = \sum_{\substack{w \in S_n \text{ preserves} \\ \text{the columns of } T}} (-1)^w w;$$

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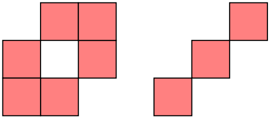
$D_9 =$  .

- Let $\mathcal{S}^D$ be its Specht module. This is a left $\mathcal{A}$-module defined in any of the following equivalent ways:
  - as the span of certain determinants in a polynomial ring (many options here).

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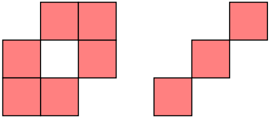
$D_9 =$  .

- Let $\mathcal{S}^D$ be its Specht module.
- **Question.** Is $\mathcal{S}^D$ a direct addend of the Young module $\mathcal{M}^D$ as a $\mathbf{k}$-module?
- Note that proving this for $\mathbf{k} = \mathbb{Z}$ would suffice.
- **Equivalent question.** If $\mathbf{k}$ is a finite field, is $\dim_{\mathbf{k}} \mathcal{S}^D$ independent on $\mathbf{k}$?
- **Better hope:** Does $\mathcal{S}^D$ have a combinatorially meaningful basis?

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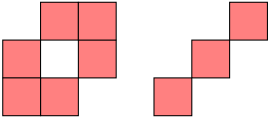
$D_9 =$  .

- Let $\mathcal{S}^D$ be its Specht module.
- **Question.** Is $\mathcal{S}^D$ a direct addend of the Young module $\mathcal{M}^D$ as a $\mathbf{k}$-module?
- Well-known positive answer when D is a skew Young diagram (Garnir's standard basis theorem).
- I think the answer is still positive when D is row-convex (Reiner/Shimozono 1993).
- Same questions exist for Schur and Weyl modules (over GL_n), but not sure if still equivalent.

The Gelfand–Tsetlin subalgebra

- Let $\mathbf{m}_k := t_{1,k} + t_{2,k} + \cdots + t_{k-1,k}$ be the k-th *Young–Jucys–Murphy element* for each $k \in [n]$ (where $t_{i,j}$ means the transposition $i \leftrightarrow j$).
- The $\mathbf{k}$-subalgebra of $\mathcal{A} = \mathbf{k}[S_n]$ generated by $\mathbf{m}_1, \mathbf{m}_2, \ldots, \mathbf{m}_n$ is commutative, and known as the *Gelfand–Tsetlin algebra*.
- **Question.** Is it free as a $\mathbf{k}$-module? (dimension $=$ # of involutions $=$ # of straight-shaped standard tableaux.)
- Again, proving it for $\mathbf{k} = \mathbb{Z}$ is enough.
- True for $n \leq 6$.
- Well-known positive answer for $\mathbf{k} = \mathbb{Q}$ (explicit basis: the diagonal vectors $e_{T,T}$ of the seminormal basis of $\mathbf{k}[S_n]$).
- Partial result: For all $i_1 < i_2 < \cdots < i_k$, we have

$$\mathbf{m}_{i_1}\mathbf{m}_{i_2}\cdots\mathbf{m}_{i_k} = \sum_{\substack{w \in S_n; \\ \operatorname{NoSt} w = \{i_1, i_2, \ldots, i_k\}}} w.$$

The simplest-looking open question you'll see today

- For any permutation $w \in S_n$, define

$$\operatorname{exc} w := (\text{\# of } i \in [n] \text{ such that } w(i) > i) \qquad \text{and}$$
$$\operatorname{anxc} w := (\text{\# of } i \in [n] \text{ such that } w(i) < i).$$

- For any $a, b \in \mathbb{N}$, define

$$\mathbf{X}_{a,b} := \sum_{\substack{w \in S_n; \\ \operatorname{exc} w = a; \\ \operatorname{anxc} w = b}} w \in \mathbf{k}[S_n].$$

- **Conjecture.** These elements $\mathbf{X}_{a,b}$ for all $a, b \in \mathbb{N}$ commute (for fixed n). In other words, $\mathbf{X}_{a,b}\mathbf{X}_{c,d} = \mathbf{X}_{c,d}\mathbf{X}_{a,b}$ for all $a, b, c, d \in \mathbb{N}$.
- Checked for all $n \leq 7$.
- This generalizes a limiting case of the Bethe subalgebra (Mukhin/Tarasov/Varchenko).