Shuffle-compatible permutation statistics II: the exterior peak set

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April 1, 2018

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This draft is a continuation of the preprint “Shuffle-compatible permutation statistics” by Ira M. Gessel and Yan Zhuang ([GesZhu17]) (but can be read independently from the latter).

All of the following is a rough draft, and proofs are merely outlined.

## Acknowledgments

We thank Yan Zhuang and Ira Gessel for helpful conversations and a few corrections. The SageMath computer algebra system [SageMath] has been used in finding some of the results below.

## 1. Notations and definitions

Let us first introduce the definitions and notations that we will use in the rest of this paper. Many of these definitions appear in [GesZhu17] already; we have tried to deviate from the notations of [GesZhu17] as little as possible.

### 1.1. Permutations and other basic concepts

**Definition 1.1.** We let $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ and $\mathbb{P} = \{1, 2, 3, \ldots\}$. Both of these sets are understood to be equipped with their standard total order.

**Definition 1.2.** Let $n \in \mathbb{Z}$. We shall use the notation $[n]$ for the totally ordered set $\{1, 2, \ldots, n\}$ (with the usual order relation inherited from $\mathbb{Z}$). Note that $[n] = \emptyset$ when $n \leq 0$.

**Definition 1.3.** Let $n \in \mathbb{N}$. An $n$-permutation shall mean a word with $n$ letters, which are distinct and belong to $\mathbb{P}$. Equivalently, an $n$-permutation shall be regarded as an injective map $[n] \to \mathbb{P}$ (the image of $i$ under this map being the $i$-th letter of the word).

For example, $(3, 6, 4)$ and $(9, 1, 2)$ are 3-permutations, but $(2, 1, 2)$ is not.
**Definition 1.4.** A permutation is defined to be an \( n \)-permutation for some \( n \in \mathbb{N} \). If \( \pi \) is an \( n \)-permutation for some \( n \in \mathbb{N} \), then the number \( n \) is called the size (or length) of the permutation \( \pi \) and is denoted by \( |\pi| \). A permutation is said to be nonempty if it is nonempty as a word (i.e., if its size is \( > 0 \)).

Note that the meaning of “permutation” we have just defined is unusual (most authors define a permutation to be a bijection from a set to itself); we are following \([GesZhu17]\) in defining permutations this way.

**Definition 1.5.** Let \( n \in \mathbb{N} \). Two \( n \)-permutations \( \alpha \) and \( \beta \) are said to be order-equivalent if they have the following property: For every two integers \( i, j \in [n] \), we have \( \alpha(i) < \alpha(j) \) if and only if \( \beta(i) < \beta(j) \).

**Definition 1.6. (a)** A permutation statistic is a map \( st \) from the set of all permutations to an arbitrary set that has the following property: Whenever \( \alpha \) and \( \beta \) are two order-equivalent permutations, we have \( st(\alpha) = st(\beta) \).

(b) Let \( st \) be a permutation statistic. Two permutations \( \alpha \) and \( \beta \) are said to be \( st \)-equivalent if they satisfy \( |\alpha| = |\beta| \) and \( st(\alpha) = st(\beta) \). The relation “\( st \)-equivalent” is an equivalence relation; its equivalence classes are called \( st \)-equivalence classes.

**Remark 1.7.** Let \( n \in \mathbb{N} \). Let us call an \( n \)-permutation \( \pi \) standard if its letters are \( 1, 2, \ldots, n \) (in some order). The standard \( n \)-permutations are in bijection with the \( n! \) permutations of the set \( \{1, 2, \ldots, n\} \) in the usual sense of this word (i.e., the bijections from this set to itself).

It is easy to see that for each \( n \)-permutation \( \sigma \), there exists a unique standard \( n \)-permutation \( \pi \) order-equivalent to \( \sigma \). Thus, a permutation statistic is uniquely determined by its values on standard permutations. Consequently, we can view permutation statistics as statistics defined on standard permutations, i.e., on permutations in the usual sense of the word.

### 1.2. Compositions

**Definition 1.8.** A composition is a finite list of positive integers. If \( I = (i_1, i_2, \ldots, i_n) \) is a composition, then the nonnegative integer \( i_1 + i_2 + \cdots + i_n \) is called the size of \( I \) and is denoted by \( |I| \); we furthermore say that \( I \) is a composition of \( |I| \).

**Definition 1.9.** Let \( n \in \mathbb{N} \). For each composition \( I = (i_1, i_2, \ldots, i_k) \) of \( n \), we define a subset \( \text{Des} I \) of \([n-1]\) by
\[
\text{Des} I = \{i_1, i_1 + i_2, i_1 + i_2 + i_3, \ldots, i_1 + i_2 + \cdots + i_{k-1}\} = \{i_1 + i_2 + \cdots + i_s \mid s \in [k-1]\}.
\]
On the other hand, for each subset $A = \{a_1 < a_2 < \cdots < a_k\}$ of $[n - 1]$, we define a composition $\text{Comp} A$ of $n$ by

$$\text{Comp} A = (a_1, a_2 - a_1, a_3 - a_2, \ldots, a_k - a_{k-1}, n - a_k).$$

(The definition of $\text{Comp} A$ should be understood to give $\text{Comp} A = (n)$ if $A = \emptyset$. Note that $\text{Comp} A$ depends not only on the set $A$ itself, but also on $n$. We hope that $n$ will always be clear from the context when we use this notation.)

We thus have defined a map $\text{Des}$ (from the set of all compositions of $n$ to the set of all subsets of $[n - 1]$) and a map $\text{Comp}$ (in the opposite direction). These two maps are mutually inverse bijections.

**Definition 1.10.** Let $n \in \mathbb{N}$. Let $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$ be an $n$-permutation.

- **(a)** The *descents* of $\pi$ are the elements $i \in [n - 1]$ satisfying $\pi_i > \pi_{i+1}$.
- **(b)** The *descent set* of $\pi$ is defined to be the set of all descents of $\pi$. This set is denoted by $\text{Des} \pi$, and is always a subset of $[n - 1]$.
- **(c)** The *descent composition* of $\pi$ is defined to be the composition $\text{Comp} (\text{Des} \pi)$ of $n$. This composition is denoted by $\text{Comp} \pi$.
- **(d)** The *peaks* of $\pi$ are the elements $i \in \{2, 3, \ldots, n - 1\}$ satisfying $\pi_{i-1} < \pi_i > \pi_{i+1}$.
- **(e)** The *peak set* of $\pi$ is defined to be the set of all peaks of $\pi$. This set is denoted by $\text{Pk} \pi$, and is always a subset of $\{2, 3, \ldots, n - 1\}$.
- **(f)** The *left peaks* of $\pi$ are the elements $i \in [n - 1]$ satisfying $\pi_{i-1} > \pi_i > \pi_{i+1}$, where we set $\pi_0 = 0$.
- **(g)** The *left peak set* of $\pi$ is defined to be the set of all left peaks of $\pi$. This set is denoted by $\text{Lpk} \pi$, and is always a subset of $[n - 1]$. It is easy to see that (for $n \geq 2$) we have

$$\text{Lpk} \pi = \text{Pk} \pi \cup \{1 \mid \pi_1 > \pi_2\}.$$  

(Of course, $\{1 \mid \pi_1 > \pi_2\}$ is the 1-element set $\{1\}$ if $\pi_1 > \pi_2$, and the empty set $\emptyset$ otherwise.)
- **(h)** The *right peaks* of $\pi$ are the elements $i \in \{2, 3, \ldots, n\}$ satisfying $\pi_{i-1} < \pi_i > \pi_{i+1}$, where we set $\pi_n = 0$.
- **(i)** The *right peak set* of $\pi$ is defined to be the set of all right peaks of $\pi$. This set is denoted by $\text{Rpk} \pi$, and is always a subset of $\{2, 3, \ldots, n\}$. It is easy to see that (for $n \geq 2$) we have

$$\text{Rpk} \pi = \text{Pk} \pi \cup \{n \mid \pi_{n-1} < \pi_n\}.$$  

- **(j)** The *exterior peaks* of $\pi$ are the elements $i \in [n]$ satisfying $\pi_{i-1} < \pi_i > \pi_{i+1}$, where we set $\pi_0 = 0$ and $\pi_{n+1} = 0$.
- **(k)** The *exterior peak set* of $\pi$ is defined to be the set of all exterior peaks of $\pi$. This set is denoted by $\text{Epk} \pi$, and is always a subset of $[n]$. It is easy to see
that (for \( n \geq 2 \)) we have
\[
\text{Epk } \pi = \text{Pk } \pi \cup \{ 1 \mid \pi_1 > \pi_2 \} \cup \{ n \mid \pi_{n-1} < \pi_n \}
= \text{Lpk } \pi \cup \text{Rpk } \pi.
\]
(For \( n = 1 \), we have \( \text{Epk } \pi = \{ 1 \} \).)

For example, the 6-permutation \( \pi = (4, 1, 3, 9, 6, 8) \) has
\[
\text{Des } \pi = \{ 1, 4 \}, \quad \text{Comp } \pi = (1, 3, 2), \quad \text{Pk } \pi = \{ 4 \},
\]
\[
\text{Lpk } \pi = \{ 1, 4 \}, \quad \text{Rpk } \pi = \{ 4, 6 \}, \quad \text{Epk } \pi = \{ 1, 4, 6 \}.
\]

For another example, the 6-permutation \( \pi = (1, 4, 3, 2, 9, 8) \) has
\[
\text{Des } \pi = \{ 2, 3, 5 \}, \quad \text{Comp } \pi = (2, 1, 2, 1), \quad \text{Pk } \pi = \{ 2, 5 \},
\]
\[
\text{Lpk } \pi = \{ 2, 5 \}, \quad \text{Rpk } \pi = \{ 2, 5 \}, \quad \text{Epk } \pi = \{ 2, 5 \}.
\]

Notice that Definition 1.10 actually defines several permutation statistics. For example, Definition 1.10(b) defines the permutation statistic Des, whose codomain is the set of all subsets of \( \mathbb{P} \). Also, Definition 1.10(c) defines the permutation statistic Comp, whose codomain is the set of all compositions. The main permutation statistic that we will study in this paper is Epk, which is defined in Definition 1.10(k); its codomain is the set of all subsets of \( \mathbb{P} \).

The following simple fact expresses the set Epk \( \pi \) corresponding to an \( n \)-permutation \( \pi \) in terms of Des \( \pi \):

**Proposition 1.11.** Let \( n \) be a positive integer. Let \( \pi \) be an \( n \)-permutation. Then,
\[
\text{Epk } \pi = (\text{Des } \pi \cup \{ n \}) \setminus (\text{Des } \pi + 1),
\]
where \( \text{Des } \pi + 1 \) denotes the set \( \{ i + 1 \mid i \in \text{Des } \pi \} \).

**Proof of Proposition 1.11.** Write \( \pi \) in the form \( \pi = (\pi_1, \pi_2, \ldots, \pi_n) \). Set \( \pi_0 = 0 \) and \( \pi_{n+1} = 0 \). Recall that Des \( \pi \) is defined as the set of all descents of \( \pi \). In other words,
\[
\text{Des } \pi = (\text{the set of all descents of } \pi) = \{ i \in [n-1] \mid \pi_i > \pi_{i+1} \}
\]
(because the descents of \( \pi \) are defined to be the \( i \in [n-1] \) satisfying \( \pi_i > \pi_{i+1} \)).

But \((\pi_1, \pi_2, \ldots, \pi_n) = \pi \) is an \( n \)-permutation, and thus has no equal entries. Hence, for each \( i \in [n-1] \), we have \( \pi_i \neq \pi_{i+1} \). Thus, for each \( i \in [n-1] \), we have the equivalence \((\pi_i \geq \pi_{i+1}) \iff (\pi_i > \pi_{i+1})\). Therefore,
\[
\{ i \in [n-1] \mid \pi_i \geq \pi_{i+1} \} = \{ i \in [n-1] \mid \pi_i > \pi_{i+1} \} = \text{Des } \pi.
\]

On the other hand, \( \pi_n \in [n] \), so that \( \pi_n > 0 = \pi_{n+1} \). Hence, \( n \) is an element of the set \( \{ i \in \{ n \} \mid \pi_i > \pi_{i+1} \} \). Clearly, this set cannot have any other element (since it is a subset of \( \{ n \} \)); thus, \( \{ i \in \{ n \} \mid \pi_i > \pi_{i+1} \} = \{ n \} \).
But \([n] = [n - 1] \cup \{n\}\), so that
\[
\begin{align*}
\{i \in [n] \mid \pi_i &> \pi_{i+1}\} \\
= \{i \in [n - 1] \cup \{n\} \mid \pi_i &> \pi_{i+1}\} \\
= \{i \in [n - 1] \mid \pi_i > \pi_{i+1}\} \cup \{i \in \{n\} \mid \pi_i > \pi_{i+1}\} = \text{Des} \pi \cup \{n\}. \quad (1)
\end{align*}
\]

On the other hand, \(\pi_1 \in [n]\), so that \(\pi_1 > 0 = \pi_0\). Hence, we do not have \(\pi_0 \geq \pi_1\). Thus, 0 is not an element of the set \(\{i \in \{0\} \mid \pi_i \geq \pi_{i+1}\}\). Clearly, this set cannot have any other element (since it a subset of \(\{0\}\)); thus, \(\{i \in \{0\} \mid \pi_i \geq \pi_{i+1}\} = \emptyset\).

But \(\{0,1,\ldots,n-1\} = \{0\} \cup [n-1]\), so that
\[
\begin{align*}
\{i \in \{0,1,\ldots,n-1\} \mid \pi_i &\geq \pi_{i+1}\} \\
= \{i \in \{0\} \cup [n-1] \mid \pi_i &\geq \pi_{i+1}\} \\
= \{i \in \{0\} \mid \pi_i \geq \pi_{i+1}\} \cup \{i \in [n-1] \mid \pi_i \geq \pi_{i+1}\} = \emptyset \cup \text{Des} \pi = \text{Des} \pi.
\end{align*}
\]

Hence,
\[
\text{Des} \pi = \{i \in \{0,1,\ldots,n-1\} \mid \pi_i \geq \pi_{i+1}\},
\]
so that
\[
\text{Des} \pi + 1 = \{i + 1 \mid i \in \{0,1,\ldots,n-1\} \text{ satisfies } \pi_i \geq \pi_{i+1}\}
\]
\[
= \{j \in [n] \mid \pi_{j-1} \geq \pi_j\}
\]
\[
= \{i \in [n] \mid \pi_{i-1} \geq \pi_i\} \quad (2)
\]
(here, we have renamed the index \(j\) as \(i\)).

But \(\text{Ep}_{\pi} \pi\) is the set of all exterior peaks of \(\pi\) (by the definition of \(\text{Ep}_{\pi} \pi\)). Thus,
\[
\text{Ep}_{\pi} \pi = (\text{the set of all exterior peaks of } \pi)
\]
\[
= \left\{ i \in [n] \mid \pi_{i-1} < \pi_i > \pi_{i+1} \right\}
\]
(by the definition of an “exterior peak” of \(\pi\))
\[
\begin{align*}
= \left\{ i \in [n] \mid \pi_i > \pi_{i+1} \text{ and } \pi_{i-1} < \pi_i \right\}
\end{align*}
\]
\[
\begin{align*}
\iff (\pi_i > \pi_{i+1} \text{ and } \pi_{i-1} < \pi_i)
\end{align*}
\]
\[
\begin{align*}
\iff (\pi_i > \pi_{i+1} \text{ and not } \pi_{i-1} \geq \pi_i)
\end{align*}
\]
\[
\begin{align*}
\iff (\pi_i > \pi_{i+1} \text{ and not } \pi_{i-1} \geq \pi_i)
\end{align*}
\]
\[
\begin{align*}
= \left\{ i \in [n] \mid \pi_i > \pi_{i+1} \text{ and not } \pi_{i-1} \geq \pi_i \right\}
\end{align*}
\]
\[
\begin{align*}
= \left\{ i \in [n] \mid \pi_{i-1} > \pi_i \right\}
\end{align*}
\]
\[
\begin{align*}
= \left\{ i \in [n] \mid \pi_{i-1} > \pi_i \text{ and } \pi_{i-1} \geq \pi_i \right\}
\end{align*}
\]
\[
\begin{align*}
= \left\{ i \in [n] \mid \pi_{i-1} > \pi_i \text{ and } \pi_{i-1} \geq \pi_i \right\}
\end{align*}
\]
\[
\begin{align*}
\iff (\text{Des} \pi \cup \{n\}) \quad (\text{by } (1))
\end{align*}
\]
\[
\begin{align*}
\iff (\text{Des} \pi + 1) \quad (\text{by } (2))
\end{align*}
\]
\[
\begin{align*}
= (\text{Des} \pi \cup \{n\}) \setminus (\text{Des} \pi + 1).
\end{align*}
\]
This proves Proposition 1.11

1.3. Shuffles and shuffle-compatibility

**Definition 1.12.** Let \( \pi \) and \( \sigma \) be two permutations.

(a) We say that \( \pi \) and \( \sigma \) are **disjoint** if no letter appears in both \( \pi \) and \( \sigma \).

(b) Assume that \( \pi \) and \( \sigma \) are disjoint. Set \( m = |\pi| \) and \( n = |\sigma| \). Let \( \tau \) be an \((m+n)\)-permutation. Then, we say that \( \tau \) is a **shuffle** of \( \pi \) and \( \sigma \) if both \( \pi \) and \( \sigma \) appear as subsequences of \( \tau \).

(c) We let \( S(\pi, \sigma) \) be the set of all shuffles of \( \pi \) and \( \sigma \).

For example, the permutations \((3,1)\) and \((6,2,9)\) are disjoint, whereas the permutations \((3,2)\) and \((6,2,9)\) are not. The shuffles of the two disjoint permutations \((3,1)\) and \((2,6)\) are

\[
(3,1,2,6), \quad (3,2,1,6), \quad (3,2,6,1), \\
(2,3,1,6), \quad (2,3,6,1), \quad (2,6,3,1).
\]

If \( \pi \) and \( \sigma \) are two disjoint permutations, then \( S(\pi, \sigma) = S(\sigma, \pi) \) is an \( \binom{m+n}{m} \)-element set, where \( m = |\pi| \) and \( n = |\sigma| \).

**Definition 1.12 (b)** is not new; it is used, e.g., in [Greene88]. From the point of view of combinatorics on words, it is somewhat naive, as it fails to properly generalize to the case when the words \( \pi \) and \( \sigma \) are no longer disjoint. But we will not be considering this general case, since our results do not seem to straightforwardly extend to it (although we might have to look more closely); thus, Definition 1.12 will suffice for us.

**Definition 1.13.** Let \( st \) be a permutation statistic. We say that \( st \) is **shuffle-compatible** if and only if it has the following property: For any two disjoint permutations \( \pi \) and \( \sigma \), the multiset

\[
\{st(\tau) \mid \tau \in S(\pi, \sigma)\}
\]

depends only on \( st(\pi), st(\sigma), |\pi| \) and \( |\sigma| \).

---

1In this general case, it is best to define a shuffle of \( \pi \) and \( \sigma \) as a word of the form \((\gamma_{\eta(1)}, \gamma_{\eta(2)}, \ldots, \gamma_{\eta(m+n)})\), where \((\gamma_1, \gamma_2, \ldots, \gamma_{m+n})\) is the word \((\pi_1, \pi_2, \ldots, \pi_m, \sigma_1, \sigma_2, \ldots, \sigma_n)\), and where \( \eta \) is some permutation of the set \( \{1, 2, \ldots, m+n\} \) (that is, a bijection from this set to itself) satisfying \( \eta^{-1}(1) < \eta^{-1}(2) < \cdots < \eta^{-1}(m) \) (this causes the letters \( \pi_1, \pi_2, \ldots, \pi_m \) to appear in the word \((\gamma_{\eta(1)}, \gamma_{\eta(2)}, \ldots, \gamma_{\eta(m+n)})\) in this order) and \( \eta^{-1}(m+1) < \eta^{-1}(m+2) < \cdots < \eta^{-1}(m+n) \) (this causes the letters \( \sigma_1, \sigma_2, \ldots, \sigma_n \) to appear in the word \((\gamma_{\eta(1)}, \gamma_{\eta(2)}, \ldots, \gamma_{\eta(m+n)})\) in this order). Furthermore, the proper generalization of \( S(\pi, \sigma) \) to this case would be a multiset, not a mere set.
2. Extending enriched $P$-partitions and the exterior peak set

We are going to define $\mathcal{Z}$-enriched $P$-partitions, which are a straightforward generalization of the notions of “$P$-partitions”, “enriched $P$-partitions” and “left enriched $P$-partitions”. We will then consider a new particular case of this notion, which leads to a proof of the shuffle-compatibility of $\text{Epk}$ conjectured in [GesZhu17].

2.1. $\mathcal{Z}$-enriched $(P, \gamma)$-partitions

Convention 2.1. By abuse of notation, we will often use the same notation for a poset $P = (X, \leq)$ and its ground set $X$ when there is no danger of confusion. In particular, if $x$ is some object, then “$x \in P$” shall mean “$x \in X$”.

Definition 2.2. A labeled poset means a pair $(P, \gamma)$ consisting of a finite poset $P = (X, \leq)$ and an injective map $\gamma : X \to A$ for some totally ordered set $A$. The injective map $\gamma$ is called the labeling of the labeled poset $(P, \gamma)$. The poset $P$ is called the ground poset of the labeled poset $(P, \gamma)$.

Convention 2.3. Let $\mathcal{N}$ be a totally ordered set, whose (strict) order relation will be denoted by $\prec$. Let $+$ and $-$ be two distinct symbols. Let $\mathcal{Z}$ be a subset of the set $\mathcal{N} \times \{+, -\}$. For each $q = (n, s) \in \mathcal{Z}$, we denote the element $n \in \mathcal{N}$ by $|q|$, and we call the element $s \in \{+, -\}$ the sign of $q$. If $n \in \mathcal{N}$, then we will denote the two elements $(n, +)$ and $(n, -)$ of $\mathcal{N} \times \{+, -\}$ by $+n$ and $-n$, respectively.

We equip the set $\mathcal{Z}$ with a total order, whose (strict) order relation $\prec$ is defined by

$$(n, s) \prec (n', s') \text{ if and only if either } n \prec n' \text{ or } (n = n' \text{ and } s = - \text{ and } s' = +).$$

Let $\text{Pow} \mathcal{N}$ be the ring of all power series over $\mathbb{Q}$ in the indeterminates $x_n$ for $n \in \mathcal{N}$.

We fix $\mathcal{N}$ and $\mathcal{Z}$ throughout Subsection 2.1. That is, any result in this subsection is tacitly understood to begin with “Let $\mathcal{N}$ be a totally ordered set, whose (strict) order relation will be denoted by $\prec$, and let $\mathcal{Z}$ be a subset of the set $\mathcal{N} \times \{+, -\}$”.

Whenever $\prec$ denotes some strict order, the corresponding weak order will be denoted by $\preceq$. (Thus, $a \preceq b$ means “$a \prec b$ or $a = b$”.)

Definition 2.4. Let $(P, \gamma)$ be a labeled poset. A $\mathcal{Z}$-enriched $(P, \gamma)$-partition means a map $f : P \to \mathcal{Z}$ such that for all $x < y$ in $P$, the following conditions hold:
(i) We have \( f(x) \preceq f(y) \).

(ii) If \( f(x) = f(y) = +n \) for some \( n \in \mathbb{N} \), then \( \gamma(x) < \gamma(y) \).

(iii) If \( f(x) = f(y) = -n \) for some \( n \in \mathbb{N} \), then \( \gamma(x) > \gamma(y) \).

(Of course, this concept depends on \( \mathbb{N} \) and \( \mathbb{Z} \), but these will always be clear from the context.)

Example 2.5. Let \( P \) be the poset with the following Hasse diagram:

```
      b
     / \    \
    c   d
   / \  /  \
  a   
```

and let \( \gamma : P \to \mathbb{Z} \) be a labeling that satisfies \( \gamma(a) < \gamma(b) < \gamma(c) < \gamma(d) \) (for example, \( \gamma \) could be the map that sends \( a, b, c, d \) to \( 2, 3, 5, 7 \), respectively). Then, a \( \mathbb{Z} \)-enriched \((P, \gamma)\)-partition is a map \( f : P \to \mathbb{Z} \) satisfying the following conditions:

(i) We have \( f(a) \preceq f(c) \preceq f(b) \) and \( f(a) \preceq f(d) \preceq f(b) \).

(ii) We cannot have \( f(c) = f(b) = +n \) with \( n \in \mathbb{N} \). Also, we cannot have \( f(d) = f(b) = +n \) with \( n \in \mathbb{N} \).

(iii) We cannot have \( f(a) = f(c) = -n \) with \( n \in \mathbb{N} \). Also, we cannot have \( f(a) = f(d) = -n \) with \( n \in \mathbb{N} \).

For example, if \( \mathbb{N} = \mathbb{P} \) (the totally ordered set of positive integers, with its usual ordering) and \( \mathbb{Z} = \mathbb{N} \times \{+, -\} \), then the map \( f : P \to \mathbb{Z} \) sending \( a, b, c, d \) to \( +2, -3, +2, -3 \) (respectively) is a \( \mathbb{Z} \)-enriched \((P, \gamma)\)-partition. Notice that the total ordering on \( \mathbb{Z} \) in this case is given by

\[
-1 < +1 < -2 < +2 < -3 < +3 < \cdots,
\]

rather than by the familiar total order on \( \mathbb{Z} \).

The concept of a “\( \mathbb{Z} \)-enriched \((P, \gamma)\)-partition” generalizes three notions in existing literature: that of a “\((P, \gamma)\)-partition”, that of an “enriched \((P, \gamma)\)-partition”, and that of a “left enriched \((P, \gamma)\)-partition”\(^2\).

\(^2\) The ideas behind these three concepts are due to Stanley [Stanle72], Stembridge [Stembr97, §2] and Petersen [Peters05], respectively, but the precise definitions are not standardized through
Example 2.6. (a) If $\mathcal{N} = \mathbb{P}$ (the totally ordered set of positive integers) and $\mathcal{Z} = \mathcal{N} \times \{+\} = \{+n \mid n \in \mathcal{N}\}$, then the $\mathcal{Z}$-enriched $(P, \gamma)$-partitions are simply the $(P, \gamma)$-partitions into $\mathcal{N}$, composed with the canonical bijection $\mathcal{N} \to \mathcal{Z}$, $n \mapsto (+n)$.

(b) If $\mathcal{N} = \mathbb{P}$ (the totally ordered set of positive integers) and $\mathcal{Z} = \mathcal{N} \times \{+\} \cup \{-\} = \{+n \mid n \in \mathcal{N}\}$, then the $\mathcal{Z}$-enriched $(P, \gamma)$-partitions are the enriched $(P, \gamma)$-partitions.

(c) If $\mathcal{N} = \mathbb{N}$ (the totally ordered set of nonnegative integers) and $\mathcal{Z} = (\mathcal{N} \times \{+\} \cup \{-\}) \setminus \{0\}$, then the $\mathcal{Z}$-enriched $(P, \gamma)$-partitions are the left enriched $(P, \gamma)$-partitions. Note that $+0$ and $-0$ here stand for the pairs $(0, +)$ and $(0, -)$; thus, they are not the same thing.

Definition 2.7. If $(P, \gamma)$ is a labeled poset, then $\mathcal{E}(P, \gamma)$ shall denote the set of all $\mathcal{Z}$-enriched $(P, \gamma)$-partitions.

Definition 2.8. Let $P$ be any finite poset. Then, $\mathcal{L}(P)$ shall denote the set of all linear extensions of $P$. A linear extension of $P$ shall be understood simultaneously as a totally ordered set extending $P$ and as a list $(w_1, w_2, \ldots, w_n)$ of all elements of $P$ such that no two integers $i < j$ satisfy $w_i \geq w_j$ in $P$.

Let us prove some basic facts about $\mathcal{Z}$-enriched $(P, \gamma)$-partitions, straightforwardly generalizing classical results proven by Stanley and Gessel (for the case of “plain” $(P, \gamma)$-partitions), Stembridge (for enriched $(P, \gamma)$-partitions) and Petersen (for left enriched $(P, \gamma)$-partitions):

Proposition 2.9. For any labeled poset $(P, \gamma)$, we have

$$\mathcal{E}(P, \gamma) = \bigsqcup_{w \in \mathcal{L}(P)} \mathcal{E}(w, \gamma).$$

Proof of Proposition 2.9. Imitate, e.g., the proof of [Stembr97, Lemma 2.1].

Definition 2.10. Let $(P, \gamma)$ be a labeled poset. We define a power series $\Gamma_{\mathcal{Z}}(P, \gamma) \in \text{Pow} \mathcal{N}$ by

$$\Gamma_{\mathcal{Z}}(P, \gamma) = \sum_{f \in \mathcal{E}(P, \gamma)} \prod_{p \in P} x_{|f(p)|}.$$ 

This is easily seen to be convergent in the usual topology on $\text{Pow} \mathcal{N}$. We define a “$(P, \gamma)$-partition” as in [Stembr97, §1.1]; this definition differs noticeably from Stanley’s (in particular, Stanley requires $f(x) \geq f(y)$ instead of $f(x) \leq f(y)$, but the differences do not end here). We define an “enriched $(P, \gamma)$-partition” as in [Stembr97, §2]. Finally, we define a “left enriched $(P, \gamma)$-partition” to be a $\mathcal{Z}$-enriched $(P, \gamma)$-partition where $\mathcal{N} = \mathbb{N}$ and $\mathcal{Z} = (\mathcal{N} \times \{+\} \cup \{-\}) \setminus \{0\}$; it does not literally appear in [Peters05, Definition 4.1] in this form.
Corollary 2.11. For any labeled poset \((P, \gamma)\), we have
\[
\Gamma_Z(P, \gamma) = \sum_{w \in L(P)} \Gamma_Z(w, \gamma).
\]

Proof of Corollary 2.11. Follows straight from Proposition 2.9.

Definition 2.12. Let \(P\) be any set. Let \(A\) be a totally ordered set. Let \(\gamma : P \to A\) and \(\delta : P \to A\) be two maps. We say that \(\gamma\) and \(\delta\) are order-equivalent if the following holds: For every pair \((p, q) \in P \times P\), we have \(\gamma(p) \leq \gamma(q)\) if and only if \(\delta(p) \leq \delta(q)\).

Proposition 2.13. Let \((P, \gamma)\) and \((Q, \delta)\) be two labeled posets. Let \((P \sqcup Q, \varepsilon)\) be a labeled poset whose ground poset \(P \sqcup Q\) is the disjoint union of \(P\) and \(Q\), and whose labeling \(\varepsilon\) is such that the restriction of \(\varepsilon\) to \(P\) is order-equivalent to \(\gamma\) and such that the restriction of \(\varepsilon\) to \(Q\) is order-equivalent to \(\delta\). Then,
\[
\Gamma_Z(P, \gamma) \Gamma_Z(Q, \delta) = \Gamma_Z(P \sqcup Q, \varepsilon).
\]

Proof of Proposition 2.13. We WLOG assume that the ground sets \(P\) and \(Q\) are disjoint; thus, we can identify \(P \sqcup Q\) with the union \(P \cup Q\). The map
\[
\mathcal{E}(P \sqcup Q, \varepsilon) \to \mathcal{E}(P, \gamma) \times \mathcal{E}(Q, \delta),
\]
\[
f \mapsto (f \mid_P, f \mid_Q)
\]
(3)
is a bijection (this is easy to see). Now,
\[
\Gamma_Z(P \sqcup Q, \varepsilon) = \sum_{f \in \mathcal{E}(P \sqcup Q, \varepsilon)} \prod_{p \in P \cup Q} x_{|f(p)|} = \sum_{f \in \mathcal{E}(P \sqcup Q, \varepsilon)} \left( \prod_{p \in P} x_{|f(p)|} \right) \left( \prod_{p \in Q} x_{|f(p)|} \right)
\]
\[
= \left( \prod_{p \in P} x_{|f(p)|} \right) \left( \prod_{p \in Q} x_{|f(p)|} \right)
\]
\[
= \left( \sum_{g \in \mathcal{E}(P, \gamma)} \prod_{p \in P} x_{|g(p)|} \right) \left( \sum_{h \in \mathcal{E}(Q, \delta)} \prod_{p \in Q} x_{|h(p)|} \right)
\]
(3)
\[
= \Gamma_Z(P, \gamma) \Gamma_Z(Q, \delta).
\]

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Definition 2.14. Let \( n \in \mathbb{N} \). Let \( \pi \) be any \( n \)-permutation. (Recall that we have defined the concept of an “\( n \)-permutation” in Definition 1.3.) Then, \( ([n], \pi) \) is a labeled poset (in fact, \( \pi \) is an injective map \( [n] \to \{1,2,3,\ldots\} \), and thus can be considered a labeling). We define \( \Gamma_Z(\pi) \) to be the power series \( \Gamma_Z([n], \pi) \).

We shall now prove two simple facts of auxiliary use:

**Proposition 2.15.** Let \( w \) be a finite totally ordered set with ground set \( W \). Let \( n = |W| \). Let \( w \) be the unique poset isomorphism \( w \to [n] \). Let \( \gamma : W \to \{1,2,3,\ldots\} \) be any injective map. Then, \( \Gamma_Z(\gamma \circ \overline{w}^{-1}) = \Gamma_Z([n], \gamma \circ \overline{w}^{-1}) \).

**Proof of Proposition 2.15.** The map \( \gamma \circ \overline{w}^{-1} : [n] \to \{1,2,3,\ldots\} \) is an injective map, thus an \( n \)-permutation. Hence, \( \Gamma_Z(\gamma \circ \overline{w}^{-1}) \) is well-defined, and in fact we have \( \Gamma_Z(\gamma \circ \overline{w}^{-1}) = \Gamma_Z([n], \gamma \circ \overline{w}^{-1}) \).

**Corollary 2.16.** Let \( (P, \gamma) \) be a labeled poset. Let \( n = |P| \). Then,

\[
\Gamma_Z(P, \gamma) = \sum_{x: P \to [n] \text{ bijective poset homomorphism}} \Gamma_Z(\gamma \circ x^{-1}).
\]

**Proof of Corollary 2.16.** For each totally ordered set \( w \) with ground set \( P \), we let \( \overline{w} \) be the unique poset isomorphism \( w \to [n] \). Corollary 2.11 yields

\[
\Gamma_Z(P, \gamma) = \sum_{w \in \mathcal{L}(P)} \underbrace{\Gamma_Z(w, \gamma)}_{=\Gamma_Z(\gamma \circ \overline{w}^{-1})} = \sum_{w \in \mathcal{L}(P)} \Gamma_Z(\gamma \circ \overline{w}^{-1}).
\]

But the linear extensions of \( P \) are in bijection with the bijective poset homomorphisms \( x : P \to [n] \); the bijection sends a linear extension \( w \) of \( P \) to the bijective poset homomorphism \( \overline{w} : P \to [n] \). Thus, we can substitute \( x \) for \( w \) in the sum

\[
\sum_{w \in \mathcal{L}(P)} \Gamma_Z(\gamma \circ \overline{w}^{-1}),
\]

obtaining

\[
\sum_{w \in \mathcal{L}(P)} \Gamma_Z(\gamma \circ \overline{w}^{-1}) = \sum_{x: P \to [n] \text{ bijective poset homomorphism}} \Gamma_Z(\gamma \circ x^{-1}).
\]
Combining this with (4), we end up with the claim of Corollary 2.16.

**Corollary 2.17.** Let \( n \in \mathbb{N} \) and \( m \in \mathbb{N} \). Let \( \pi \) be an \( n \)-permutation and let \( \sigma \) be an \( m \)-permutation such that \( \pi \) and \( \sigma \) are disjoint. Then,

\[
\Gamma_Z(\pi) \Gamma_Z(\sigma) = \sum_{\tau \in S(\pi, \sigma)} \Gamma_Z(\tau).
\]

**Proof of Corollary 2.17.** Let \( \epsilon \) be the map \([n] \cup [m] \to \{1, 2, 3, \ldots\}\) which restricts to \( \pi \) on the \([n]\) part and restricts to \( \sigma \) on the \([m]\) part. This map \( \epsilon \) is an \((n + m)\)-permutation, since \( \pi \) and \( \sigma \) are disjoint.

Let \( \rho : [n] \cup [m] \to [n + m] \) be the strictly order-preserving bijection which sends the elements of \([n]\) to \(1, 2, \ldots, n\) and sends the elements of \([m]\) to \(n + 1, n + 2, \ldots, n + m\).

The definitions of \( \Gamma_Z(\pi) \) and \( \Gamma_Z(\sigma) \) yield

\[
\Gamma_Z(\pi) \Gamma_Z(\sigma) = \Gamma_Z([n], \pi) \Gamma_Z([m], \sigma)
= \Gamma_Z([n] \cup [m], \epsilon) \quad \text{(by Proposition 2.13)}
= \sum_{x : [n] \cup [m] \to [n + m] \text{ bijective poset homomorphism}} \Gamma_Z(\epsilon \circ x^{-1}) \quad \text{(by Corollary 2.16)}
= \sum_{\tau \in S(\sigma, \pi)} \Gamma_Z(\tau).
\]

Here, the last equality sign makes use of the (easy) fact that the map

\[
\{\text{bijective poset homomorphisms } x : [n] \cup [m] \to [n + m] \to S(\sigma, \pi), \quad x \mapsto \epsilon \circ x^{-1}\}
\]

is a well-defined bijection.

2.2. Exterior peaks

So far we have been doing general nonsense. Let us now specialize to a situation that is connected to exterior peaks.

**Convention 2.18.** From now on, we set \( \mathcal{N} = \{0, 1, 2, \ldots\} \cup \{\infty\} \), with total order given by \( 0 < 1 < 2 < \cdots < \infty \), and we set

\[
\mathcal{Z} = (\mathcal{N} \times \{+, -\}) \setminus \{-0, +\infty\}
= \{+0\} \cup \{+n \mid n \in \{1, 2, 3, \ldots\}\} \cup \{-n \mid n \in \{1, 2, 3, \ldots\}\} \cup \{-\infty\}.
\]

Recall that the total order on \( \mathcal{Z} \) has

\( +0 < -1 < +1 < -2 < +2 < \cdots < -\infty \).
Definition 2.19. A map $\chi$ from a subset $S$ of $\mathbb{Z}$ to a totally ordered set $K$ is said to be V-shaped if there exists some $t \in S$ such that the map $\chi |_{\{s \in S \mid s \leq t\}}$ is strictly decreasing while the map $\chi |_{\{s \in S \mid s \geq t\}}$ is strictly increasing. Notice that this $t \in S$ is uniquely determined in this case; namely, it is the unique $k \in S$ that minimizes $\chi(k)$.

Thus, roughly speaking, a map from a totally ordered set is V-shaped if and only if it is strictly decreasing up until a certain value of its argument, and then strictly increasing afterwards.

Definition 2.20. Let $n \in \mathbb{N}$.

(a) Let $f : [n] \rightarrow \mathbb{Z}$ be any map. Then, $|f|$ shall denote the map $[n] \rightarrow \mathcal{N}$, $i \mapsto |f(i)|$.

(b) Let $g : [n] \rightarrow \mathcal{N}$ be any map. Then, we define a monomial $x_g$ in $\text{Pow } \mathcal{N}$ by $x_g = \prod_{i=1}^{n} x_{g(i)}$. Note that this allows us to rewrite the definition of $\Gamma_Z(\pi)$ as follows: If $\pi$ is any $n$-permutation, then

$$\Gamma_Z(\pi) = \sum_{f \in E([n], \pi)} \prod_{p \in [n]} x_{|f(p)|} = \sum_{f \in E([n], \pi)} x_{|f|}. \quad (5)$$

(c) Let $g : [n] \rightarrow \mathcal{N}$ be any map. Let $\pi$ be an $n$-permutation. We shall say that $g$ is $\pi$-amenable if it has the following properties:

(i') The map $\pi |_{g^{-1}(0)}$ is strictly increasing. (This allows the case when $g^{-1}(0) = \emptyset$.)

(ii') For each $h \in g([n]) \cap \{1, 2, 3, \ldots\}$, the map $\pi |_{g^{-1}(h)}$ is V-shaped.

(iii') The map $\pi |_{g^{-1}(\infty)}$ is strictly decreasing. (This allows the case when $g^{-1}(\infty) = \emptyset$.)

(iv') The map $g$ is weakly increasing.

Proposition 2.21. Let $n \in \mathbb{N}$. Let $\pi$ be any $n$-permutation. Then,

$$\Gamma_Z(\pi) = \sum_{g : [n] \rightarrow \mathcal{N} \text{ is } \pi\text{-amenable}} 2^{|g([n]) \cap \{1, 2, 3, \ldots\}|} x_g.$$  

Proof of Proposition 2.21. The claim will immediately follow from (5) once we have shown the following two observations:

Observation 1: If $f \in \mathcal{E}([n], \pi)$, then the map $|f| : [n] \rightarrow \mathcal{N}$ is $\pi$-amenable.
Observation 2: If \( g : [n] \to \mathcal{N} \) is a \( \pi \)-amenable map, then there exist precisely \( 2|g([n])\cap\{1,2,3,\ldots\}| \) maps \( f \in \mathcal{E}([n],\pi) \) satisfying \( |f| = g \).

But both of these observations are easy:

[Proof of Observation 1: This is a simple consequence of the definition of a \( \mathcal{Z} \)-enriched \(([n],\pi)\)-partition. Let me spell it out: Let \( f \in \mathcal{E}([n],\pi) \). Thus, \( f \) is an \( \mathcal{Z} \)-enriched \(([n],\pi)\)-partition. In other words, \( f \) is a map \( [n] \to \mathcal{Z} \) such that for all \( x < y \) in \([n] \), the following conditions hold:

(i) We have \( f(x) \approx f(y) \).

(ii) If \( f(x) = f(y) = +h \) for some \( h \in \mathcal{N} \), then \( \pi(x) < \pi(y) \).

(iii) If \( f(x) = f(y) = -h \) for some \( h \in \mathcal{N} \), then \( \pi(x) > \pi(y) \).

Condition (i) shows that the map \( f \) is weakly increasing. Condition (ii) shows that for each \( h \in \mathcal{N} \), the map \( \pi|_{f^{-1}(+h)} \) is strictly increasing. Condition (iii) shows that for each \( h \in \mathcal{N} \), the map \( \pi|_{f^{-1}(-h)} \) is strictly decreasing.

Now, set \( g = |f| \). Then, \( g^{-1}(0) = f^{-1}(+0) \) (since \(-0 \notin \mathcal{Z}\)). But the map \( \pi|_{f^{-1}(+h)} \) is strictly increasing\(^3\) Thus, the map \( \pi|_{g^{-1}(0)} \) is strictly increasing (since \( g^{-1}(0) = f^{-1}(+0) \)). Hence, Condition (i') in the definition of “\( \pi \)-amenable” holds. Similarly, Condition (iii') in that definition also holds.

Now, fix \( h \in g([n]) \cap \{1,2,3,\ldots\} \). Then, the set \( g^{-1}(h) \) is nonempty (since \( h \in g([n]) \)), and can be written as the union of its two disjoint subsets \( f^{-1}(+h) \) and \( f^{-1}(-h) \). Furthermore, each element of \( f^{-1}(-h) \) is smaller than each element of \( f^{-1}(+h) \) (since \( f \) is weakly increasing), and we know that the map \( \pi|_{f^{-1}(-h)} \) is strictly decreasing while the map \( \pi|_{f^{-1}(+h)} \) is strictly increasing. Hence, the map \( \pi|_{g^{-1}(h)} \) is strictly decreasing up until some value of its argument, and then strictly increasing afterwards. In other words, the map \( \pi|_{g^{-1}(h)} \) is V-shaped. Thus, Condition (ii') in the definition of “\( \pi \)-amenable” holds. Finally, Condition (iv') in the definition of “\( \pi \)-amenable” holds because \( f \) is weakly increasing. We have hence checked all four conditions; thus, \( g \) is \( \pi \)-amenable. This proves Observation 1.\]

[Proof of Observation 2: Let \( g : [n] \to \mathcal{N} \) be a \( \pi \)-amenable map. Consider a map \( f \in \mathcal{E}([n],\pi) \) satisfying \( |f| = g \). We are wondering to what extent the map \( f \) is determined by \( g \) and \( \pi \).

Everything that we said in the proof of Observation 1 is true.

In order to determine the map \( f \), it clearly suffices to determine the sets \( f^{-1}(q) \) for all \( q \in \mathcal{Z} \). In other words, it suffices to determine the set \( f^{-1}(+0) \), the set \( f^{-1}(-\infty) \) and the sets \( f^{-1}(+h) \) and \( f^{-1}(-h) \) for all \( h \in \{1,2,3,\ldots\} \).

Recall from the proof of Observation 1 that \( g^{-1}(0) = f^{-1}(+0) \). Thus, \( f^{-1}(+0) \) is uniquely determined by \( g \). Similarly, \( f^{-1}(-\infty) \) is uniquely determined by \( g \).]

\(^3\)because for each \( h \in \mathcal{N} \), the map \( \pi|_{f^{-1}(+h)} \) is strictly increasing
g. Thus, we can focus on the remaining sets $f^{-1}(+h)$ and $f^{-1}(-h)$ for $h \in \{1,2,3,\ldots\}$.

Fix $h \in \{1,2,3,\ldots\}$. Recall that the set $g^{-1}(h)$ is the union of its two disjoint subsets $f^{-1}(+h)$ and $f^{-1}(-h)$. Thus, $f^{-1}(+h)$ and $f^{-1}(-h)$ are complementary subsets of $g^{-1}(h)$. If $g^{-1}(h) = \emptyset$, then this uniquely determines $f^{-1}(+h)$ and $f^{-1}(-h)$. Thus, we focus only on the case when $g^{-1}(h) \neq \emptyset$.

So assume that $g^{-1}(h) \neq \emptyset$. Hence, $h \in g([n])$, so that $h \in g([n]) \cap \{1,2,3,\ldots\}$. Since the map $g$ is $\pi$-amenable, we thus conclude that the map $\pi \mid_{g^{-1}(h)}$ is V-shaped (by Condition (ii') in the definition of “$\pi$-amenable”).

The map $g$ is weakly increasing (by Condition (iv') in the definition of “$\pi$-amenable”). Hence, $g^{-1}(h)$ is an interval of $[n]$. Let $\alpha \in \mathbb{Z}$ and $\gamma \in \mathbb{Z}$ be such that $g^{-1}(h) = [\alpha, \gamma]$ (where $[\alpha, \gamma]$ means the interval $\{\alpha, \alpha+1, \ldots, \gamma\}$).

As in the proof of Observation 1, we can see that each element of $f^{-1}(-h)$ is smaller than each element of $f^{-1}(+h)$. Since the union of $f^{-1}(-h)$ and $f^{-1}(+h)$ is $g^{-1}(h) = [\alpha, \gamma]$, we thus conclude that there exists some $\beta \in [\alpha - 1, \gamma]$ such that $f^{-1}(-h) = [\alpha, \beta]$ and $f^{-1}(+h) = [\beta + 1, \gamma]$. Consider this $\beta$. Clearly, $f^{-1}(-h)$ and $f^{-1}(+h)$ are uniquely determined by $\beta$; we just need to find out which values $\beta$ can take.

As in the proof of Observation 1, we can see that the map $\pi \mid_{f^{-1}(-h)}$ is strictly decreasing while the map $\pi \mid_{f^{-1}(+h)}$ is strictly increasing. Let $k$ be the element of $g^{-1}(h)$ minimizing $\pi(k)$. Then, the map $\pi$ is strictly decreasing on the set $\{u \in g^{-1}(h) \mid u \leq k\}$ and strictly increasing on the set $\{u \in g^{-1}(h) \mid u \geq k\}$ (since the map $\pi \mid_{g^{-1}(h)}$ is V-shaped).

The map $\pi \mid_{f^{-1}(-h)}$ is strictly decreasing. In other words, the map $\pi$ is strictly decreasing on the set $f^{-1}(-h) = [\alpha, \beta]$. On the other hand, the map $\pi$ is strictly increasing on the set $\{u \in g^{-1}(h) \mid u \geq k\}$. Hence, the two sets $[\alpha, \beta]$ and $\{u \in g^{-1}(h) \mid u \geq k\}$ cannot have more than one point in common (since $\pi$ is strictly decreasing on one and strictly increasing on the other). Thus, $k \geq \beta$. A similar argument shows that $k \leq \beta + 1$. Combining these inequalities, we obtain $k \in \{\beta, \beta + 1\}$, so that $\beta \in \{k, k - 1\}$. This shows that $\beta$ can take only two values: $k$ and $k - 1$.

Now, let us take a bird’s eye view. We have shown that for each $h \in g([n]) \cap \{1,2,3,\ldots\}$, the sets $f^{-1}(+h)$ and $f^{-1}(-h)$ are uniquely determined once the integer $\beta$ is chosen, and that this integer $\beta$ can be chosen in two ways. (As we have seen, all other values of $h$ do not matter.) Thus, in total, the map $f$ is uniquely determined up to $|g([n]) \cap \{1,2,3,\ldots\}|$ decisions, where each decision allows choosing from two values. Thus, there are at most $2^{|g([n]) \cap \{1,2,3,\ldots\}|}$ maps $f \in \mathcal{E}([n], \pi)$ satisfying $|f| = g$. Working the above argument backwards, we see that each possible decision actually leads to a map $f \in \mathcal{E}([n], \pi)$ satisfying $|f| = g$; thus, there are exactly $2^{|g([n]) \cap \{1,2,3,\ldots\}|}$ maps $f \in \mathcal{E}([n], \pi)$ satisfying $|f| = g$. This proves Observation 2. \qed

Now, let us observe that if $g : [n] \to \mathcal{N}$ is a weakly increasing map (for some
Then the fibers of \( g \) (that is, the subsets \( g^{-1}(h) \) of \([n]\) for various \( h \in \mathcal{N} \)) are intervals of \([n]\) (possibly empty). Of course, when these fibers are nonempty, they have smallest elements and largest elements. We shall next study these elements more closely.

**Definition 2.22.** Let \( n \in \mathbb{N} \). Let \( g : [n] \to \mathcal{N} \) be any map. We define a subset \( \text{FE}(g) \) of \([n]\) as follows:

\[
\text{FE}(g) = \left\{ \min \left( g^{-1}(h) \right) \mid h \in \{1, 2, 3, \ldots, \infty\} \text{ with } g^{-1}(h) \neq \emptyset \right\} 
\cup \left\{ \max \left( g^{-1}(h) \right) \mid h \in \{0, 1, 2, 3, \ldots\} \text{ with } g^{-1}(h) \neq \emptyset \right\}.
\]

In other words, \( \text{FE}(g) \) is the set comprising the smallest elements of all nonempty fibers of \( g \) except for \( g^{-1}(0) \) as well as the largest elements of all nonempty fibers of \( g \) except for \( g^{-1}(\infty) \). We shall refer to the elements of \( \text{FE}(g) \) as the fiber-ends of \( g \).

We can rewrite Proposition 2.21 as follows, exhibiting its analogy with [Stembr97, Proposition 2.2]:

**Proposition 2.23.** Let \( n \in \mathbb{N} \). Let \( \pi \) be any \( n \)-permutation. Then,

\[
\Gamma_Z (\pi) = \sum_{g : [n] \to \mathcal{N} \text{ is weakly increasing; } \text{Epk } \pi \subseteq \text{FE}(g)} 2|g([n]) \cap \{1, 2, 3, \ldots\}|x_g.
\]

**Proof of Proposition 2.23** This will follow from Proposition 2.21 once we know that the \( \pi \)-amenable maps \( g : [n] \to \mathcal{N} \) are precisely the weakly increasing maps \( g : [n] \to \mathcal{N} \) satisfying \( \text{Epk } \pi \subseteq \text{FE}(g) \). But this is easy to check. (The main idea: If \( g : [n] \to \mathcal{N} \) is a weakly increasing map, then all nonempty fibers \( g^{-1}(h) \) of \( g \) are intervals, and Conditions (i'), (ii') and (iii') in the definition of “\( \pi \)-amenable” say precisely that no peak of \( \pi \) can appear in the interior of any such fiber; i.e., all peaks must appear at fiber-ends. Of course, this needs some exceptions for \( g^{-1}(0) \) and \( g^{-1}(\infty) \), but this is all straightforward.)}

**Definition 2.24.** Let \( n \in \mathbb{N} \). If \( \Lambda \) is any subset of \([n]\), then we define a power series \( K_{n,\Lambda}^Z \in \text{Pow}_N \) by

\[
K_{n,\Lambda}^Z = \sum_{g : [n] \to \mathcal{N} \text{ is weakly increasing; } \Lambda \subseteq \text{FE}(g)} 2|g([n]) \cap \{1, 2, 3, \ldots\}|x_g.
\]

Thus, if \( \Lambda = \text{Epk } \pi \) for some \( n \)-permutation \( \pi \), then Proposition 2.23 shows that

\[
\Gamma_Z (\pi) = K_{n,\Lambda}^Z.
\]
Corollary 2.17 now leads directly to the following multiplication rule:

**Corollary 2.25.** Let \( n \in \mathbb{N} \) and \( m \in \mathbb{N} \). Let \( \pi \) be an \( n \)-permutation. Let \( \sigma \) be an \( m \)-permutation. Then,

\[
K_{n,\text{Epk}\pi} \cdot K_{m,\text{Epk}\sigma} = \sum_{\tau \in S(\pi,\sigma)} K_{n+m,\text{Epk}\tau}.
\]

Proof of Corollary 2.25. From (7), we obtain \( \Gamma_Z(\pi) = K_{n,\Lambda}^Z \). Similarly, \( \Gamma_Z(\sigma) = K_{m,\text{Epk}\sigma}^Z \). Multiplying these two equalities, we obtain \( \Gamma_Z(\pi) \cdot \Gamma_Z(\sigma) = K_{n,\text{Epk}\pi}^Z \cdot K_{m,\text{Epk}\sigma}^Z \). Hence,

\[
K_{n,\text{Epk}\pi} \cdot K_{m,\text{Epk}\sigma} = \Gamma_Z(\pi) \cdot \Gamma_Z(\sigma) = \sum_{\tau \in S(\pi,\sigma)} \Gamma_Z(\tau) \quad \text{(by Corollary 2.17)}
\]

\[
= \sum_{\tau \in S(\pi,\sigma)} K_{n+m,\text{Epk}\tau} \quad \text{(by (7))}
\]

This proves Corollary 2.25. \( \square \)

**Definition 2.26.** A set \( S \) of integers is said to be lacunar if each \( s \in S \) satisfies \( s+1 \notin S \).

The following observation is easy:

**Proposition 2.27.** Let \( n \) be a positive integer. Let \( \pi \) be an \( n \)-permutation. Then, \( \text{Epk}\pi \) is a lacunar and nonempty subset of \([n]\).

Proof of Proposition 2.27. The set \( \text{Epk}\pi \) is lacunar (since two consecutive integers cannot both be exterior peaks of \( \pi \)), and is also nonempty (since \( \pi^{-1}(n) \) is an exterior peak of \( \pi \)).

This proves Proposition 2.27. \( \square \)

Proposition 2.27 actually has a sort of converse:

**Proposition 2.28.** Let \( n \) be a positive integer. Let \( \Lambda \) be a subset of \([n]\). Then, there exists an \( n \)-permutation \( \pi \) satisfying \( \Lambda = \text{Epk}\pi \) if and only if \( \Lambda \) is lacunar and nonempty.

Proof of Proposition 2.28. \( \Longrightarrow \): We need to prove that for any \( n \)-permutation \( \pi \), the set \( \text{Epk}\pi \) is lacunar and nonempty. But this follows immediately from Proposition 2.27. This proves the \( \Longrightarrow \) direction of Proposition 2.28.

\( \Longleftarrow \): Assume that \( \Lambda \) is lacunar and nonempty. We must prove that there exists an \( n \)-permutation \( \pi \) satisfying \( \Lambda = \text{Epk}\pi \). Such an \( n \)-permutation \( \pi \) can be constructed as follows:
• Write the set \( \Lambda \) in the form \( \Lambda = \{ u_1 < u_2 < \cdots < u_\ell \} \) (where \( \ell = |\Lambda| \)). Thus, \( \ell \geq 1 \) (since \( \Lambda \) is nonempty), and we can represent the set \([n] \setminus \Lambda\) as a union of disjoint intervals as follows:

\[
[n] \setminus \Lambda = [1, u_1 - 1] \cup [u_1 + 1, u_2 - 1] \cup [u_2 + 1, u_3 - 1] \cup \cdots \cup [u_{\ell-1} + 1, u_\ell - 1] \cup [u_\ell + 1, n].
\]

• Let \( \pi \) take the values \( n, n - 1, \ldots, n - \ell + 1 \) on the elements of \( \Lambda \). (For example, this can be achieved by setting \( \pi(u_i) = n + 1 - i \) for each \( i \in [\ell] \).

• Let \( \pi \) take the values \( 1, 2, \ldots, n - \ell \) on the elements of \([n] \setminus \Lambda\) in such a way that:

(A) on each of the intervals \([1, u_1 - 1], [u_1 + 1, u_2 - 1], [u_2 + 1, u_3 - 1], \ldots, [u_{\ell-1} + 1, u_\ell - 1], [u_\ell + 1, n]\), the map \( \pi \) is either strictly increasing or strictly decreasing;

(B) if the interval \([1, u_1 - 1]\) is nonempty, then the map \( \pi \) is strictly increasing on this interval;

(C) if the interval \([u_\ell + 1, n]\) is nonempty, then the map \( \pi \) is strictly decreasing on this interval.

(This is indeed possible, because if the two intervals \([1, u_1 - 1]\) and \([u_\ell + 1, n]\) are both nonempty, then they are distinct (since \( \ell \geq 1 \)).)

Any \( n \)-permutation \( \pi \) constructed in this way will satisfy \( \Lambda = \text{Epk} \, \pi \). Indeed, it is clear that \( \pi \) satisfies

\[ \pi(u) > \pi(v) \quad \text{for all } u \in \Lambda \text{ and } v \in [n] \setminus \Lambda. \]

Hence, any element of \( \Lambda \) is an exterior peak of \( \pi \). Conversely, an element of \([n] \setminus \Lambda\) cannot be an exterior peak of \( \pi \) (because our construction of \( \pi \) guarantees that any \( s \in [n] \setminus \Lambda \) satisfies either \( s - 1 \in [n] \) and \( \pi(s - 1) > \pi(s) \) or \( s + 1 \in [n] \) and \( \pi(s + 1) > \pi(s) \)). Thus, the exterior peaks of \( \pi \) are precisely the elements of \( \Lambda \); in other words, we have \( \Lambda = \text{Epk} \, \pi \). This proves the \( \iff \) direction of Proposition 2.28. \( \Box \)

**Proposition 2.29.** Let \( n \in \mathbb{N} \). Then, the family

\[ (K_{\mathbb{Z}} \mathbb{Z}^n, \Lambda)_{\Lambda \subseteq [n]} \]

is \( \mathbb{Q} \)-linearly independent.

**Proof of Proposition 2.29** Let \( \Omega \) be the subset \( \{1, 3, 5, \ldots \} \cap [n] = \{ i \in [n] \mid i \text{ is odd} \} \) of \([n]\). This is clearly a lacunar subset of \([n]\). We are going to prove the following claim:
Claim 1: (a) If \( n \) is odd, then the only syzygy\(^4\) of the family \( (K_{n,\Lambda}^Z)_{\Lambda \subseteq [n]} \) is lacunar is \( \sum_{\Lambda \subseteq \Omega} (-1)^{|\Lambda|} K_{n,\Lambda}^Z = 0 \).

(b) If \( n \) is even, then the family \( (K_{n,\Lambda}^Z)_{\Lambda \subseteq [n]} \) is lacunar is \( \mathbb{Q} \)-linearly independent.

For each subset \( \Lambda \) of \( [n] \), define a power series \( L_{n,\Lambda}^Z \in \text{Pow} \mathcal{N} \) by

\[
L_{n,\Lambda}^Z = \sum_{\substack{g : [n] \to \mathcal{N} \\
\text{weakly increasing;} \\
\Lambda \cap \text{FE}(g) = \emptyset}} 2^{|g([n]) \cap \{1,2,3,...\}|} x_g.
\]

Then, for each lacunar subset \( \Lambda \) of \( [n] \), the equality \( \ref{eq:K_n,Lambda}^6 \) becomes

\[
K_{n,\Lambda}^Z = \sum_{\substack{g : [n] \to \mathcal{N} \\
\text{weakly increasing;} \\
\Lambda \subseteq \text{FE}(g)}} 2^{|g([n]) \cap \{1,2,3,...\}|} x_g = \sum_{\substack{Q \subseteq \Lambda}} (-1)^{|Q|} L_{n,Q}^Z
\]

(by the principle of inclusion and exclusion). Thus, (again by inclusion and exclusion) each lacunar subset \( \Lambda \) of \( [n] \) satisfies

\[
L_{n,\Lambda}^Z = \sum_{\substack{Q \subseteq \Lambda}} (-1)^{|Q|} K_{n,Q}^Z.
\]

Hence, the two families \( (K_{n,\Lambda}^Z)_{\Lambda \subseteq [n]} \) and \( (L_{n,\Lambda}^Z)_{\Lambda \subseteq [n]} \) can be obtained from each other by a unitriangular transition matrix (unitriangular with respect to inclusion\(^5\)). Thus, the syzygies of these two families are in bijection with each other. Hence, in order to prove Claim 1, it suffices to prove the following claim:

Claim 2: (a) If \( n \) is odd, then the only syzygy of the family \( (L_{n,\Lambda}^Z)_{\Lambda \subseteq [n]} \) is lacunar is \( L_{n,\Omega}^Z = 0 \).

(b) If \( n \) is even, then the family \( (L_{n,\Lambda}^Z)_{\Lambda \subseteq [n]} \) is lacunar is \( \mathbb{Q} \)-linearly independent.

---

\( ^4 \)If \((v_h)_{h \in \mathcal{H}} \) is a family of vectors in a vector space over a field \( \mathbb{F} \), then a syzygy of this family \((v_h)_{h \in \mathcal{H}} \) means a family \((\lambda_h)_{h \in \mathcal{H}} \in \mathbb{F}^\mathcal{H}\) of scalars in \( \mathbb{F} \) satisfying \( \sum_{h \in \mathcal{H}} \lambda_h v_h = 0 \).

Thus, a syzygy is what is commonly called a “linear dependence relation” (at least when the scalars \( \lambda_h \) are not all 0). By abuse of notation, we shall speak of the “syzygy \( \sum_{h \in \mathcal{H}} \lambda_h v_h = 0 \)” meaning not the equality \( \sum_{h \in \mathcal{H}} \lambda_h v_h = 0 \) but the family of coefficients \((\lambda_h)_{h \in \mathcal{H}}\).

When we say “the only syzygy”, we mean “the only nonzero syzygy up to scalar multiples”.

\( ^5 \)We are using the fact that a subset of a lacunar subset is lacunar.
Let $G$ be the set of all weakly increasing maps $g : [n] \to \mathcal{N}$. Let $R$ be the free $\mathbb{Q}$-vector space with basis $G$; its standard basis will be denoted by $(\{g\})_{g \in G}$. We define a $\mathbb{Q}$-linear map

$$\Phi : R \to \text{Pow } \mathcal{N},$$

$$[g] \mapsto 2^{|g([n]) \cap \{1, 2, 3, \ldots\}|} x_g.$$

This map $\Phi$ is easily seen to be injective (since the maps $g \in G$ are weakly increasing, and thus can be uniquely recovered from the monomials $x_g$).

For each subset $\Lambda$ of $[n]$, we define an element $\tilde{L}_\Lambda$ of $R$ by

$$\tilde{L}_\Lambda = \sum_{g \in G ; \Lambda \cap \text{FE}(g) = \emptyset} [g].$$

Then, each subset $\Lambda$ of $[n]$ satisfies

$$L^{Z}_{n, \Lambda} = \sum_{g : [n] \to \mathcal{N} \text{ is weakly increasing; } \Lambda \cap \text{FE}(g) = \emptyset} 2^{|g([n]) \cap \{1, 2, 3, \ldots\}|} x_g = \sum_{g \in G ; \Lambda \cap \text{FE}(g) = \emptyset} \Phi ([g])$$

(by the definition of $\Phi$)

$$= \sum_{g \in G ; \Lambda \cap \text{FE}(g) = \emptyset} [g] = \Phi \left( \tilde{L}_\Lambda \right).$$

Hence, the family $(L^{Z}_{n, \Lambda})_{\Lambda \subseteq [n]}$ is lacunar is the image of the family $(\tilde{L}_\Lambda)_{\Lambda \subseteq [n]}$ is lacunar under the map $\Phi$. Thus, the syzygies of the two families are in bijection (since the map $\Phi$ is injective\textsuperscript{6}). Hence, in order to prove Claim 2, it suffices to prove the following claim:

Claim 3: (a) If $n$ is odd, then the only syzygy of the family $(\tilde{L}_\Lambda)_{\Lambda \subseteq [n]}$ is lacunar is $\tilde{L}_\Omega = 0$.

\textsuperscript{6}We are here using the following obvious fact:

Let $V$ and $W$ be two vector spaces over a field $\mathbb{F}$. Let $(v_h)_{h \in H} \in V^H$ be a family of vectors in $V$. Let $\phi : V \to W$ be an injective $\mathbb{F}$-linear map. Then, the syzygies of the families $(v_h)_{h \in H} \in V^H$ and $(f(v_h))_{h \in H} \in W^H$ are in bijection. (Actually, these syzygies, when regarded as families of scalars, are literally the same.) In particular, the family $(v_h)_{h \in H}$ is $\mathbb{F}$-linearly independent if and only if the family $(f(v_h))_{h \in H}$ is $\mathbb{F}$-linearly independent.
(b) If \( n \) is even, then the family \( \left( \tilde{L}_\Lambda \right)_{\Lambda \subseteq [n]} \) is \( \mathbb{Q} \)-linearly independent.

Let us agree that if \( g \in G \), then we will set \( g(0) = 0 \) and \( g(n+1) = \infty \). Hence, \( g(i) \) will be a well-defined value of \( N \) for each \( i \in \{0,1,\ldots,n+1\} \).

If \( g \in G \), then we let \( \text{Stag}(g) \) be the subset \( \{i \in [n+1] \mid g(i) = g(i-1)\} \) of \([n+1]\). It is easy to see that the family \( \left( \sum_{g \in G; \text{Stag}(g) = T} [g] \right)_{T \subseteq [n+1]; T \neq [n+1]} \) of elements of \( \mathbb{R} \) is \( \mathbb{Q} \)-linearly independent. Hence, the family \( \left( \sum_{g \in G; \text{Stag}(g) \supseteq T} [g] \right)_{T \subseteq [n+1]; T \neq [n+1]} \) of elements of \( \mathbb{R} \) is \( \mathbb{Q} \)-linearly independent, too (because this family is obtained from the previous family \( \left( \sum_{g \in G; \text{Stag}(g) = T} [g] \right)_{T \subseteq [n+1]; T \neq [n+1]} \) via a unitriangular change-of-basis matrix. Therefore, the only syzygy of the family

\[
\left( \sum_{g \in G; \text{Stag}(g) \supseteq T} [g] \right)_{T \subseteq [n+1]; T \neq [n+1]} \] is \( \sum_{g \in G; \text{Stag}(g) \supseteq [n+1]} [g] = 0 \) (since it is easy to see that no \( g \in G \) satisfies \( \text{Stag}(g) \supseteq [n+1] \), which is why \( \sum_{g \in G; \text{Stag}(g) \supseteq [n+1]} [g] \) is indeed 0).

But if \( \Lambda \) is a lacunar subset of \([n]\), and if \( g \in G \), then we have the following

---

7Proof. Clearly, any two elements of this family are supported on different basis elements (i.e., any \([g]\) appearing in one of them cannot appear in any other). It thus remains to show that these elements are \( \neq 0 \). In other words, it remains to show that for any proper subset \( T \) of \([n+1]\), we have \( \sum_{g \in G; \text{Stag}(g) = T} [g] \neq 0 \). But this is easy: Just construct some \( g \in G \) satisfying \( \text{Stag}(g) = T \).

8Unitriangular with respect to the reverse inclusion order (notice that \( \sum_{g \in G; \text{Stag}(g) = T} [g] = 0 \) for \( T = [n+1] \), so the exclusion of \([n+1]\) makes sense and does not mess up our computations)
logical equivalence:

\[
\Lambda \cap \text{FE}(g) = \emptyset
\]

\[\iff \text{(no } i \in \Lambda \text{ satisfies } i \in \text{FE}(g))\]

\[\iff \text{(each } i \in \Lambda \text{ satisfies } i \notin \text{FE}(g))\]

\[\iff \left\{ \begin{array}{l}
\text{each } i \in \Lambda \text{ satisfies } i \neq \min \left( g^{-1}(h) \right) \text{ for all } h \in \{1, 2, 3, \ldots, \infty\} \\
\iff (g(i) = g(i-1)) \\
\text{(since } g \text{ is weakly increasing, and } g(0) = 0)
\end{array} \right.\]

\[\text{and } i \neq \max \left( g^{-1}(h) \right) \text{ for all } h \in \{0, 1, 2, 3, \ldots\} \]

\[\iff (g(i) = g(i+1)) \]

\[\text{(since } g \text{ is weakly increasing, and } g(n+1) = \infty)\]

\[\iff \left\{ \begin{array}{l}
\text{each } i \in \Lambda \text{ satisfies } g(i) = g(i-1) \text{ and } g(i) = g(i+1) \\
\iff (i \in \text{Stag}(g)) \\
\iff (i+1 \in \text{Stag}(g))
\end{array} \right.\]

\[\iff \text{(each } i \in \Lambda \text{ satisfies } i \in \text{Stag}(g) \text{ and } i+1 \in \text{Stag}(g))\]

\[\iff (\Lambda \cup (\Lambda + 1) \subseteq \text{Stag}(g)) \iff (\text{Stag}(g) \supseteq \Lambda \cup (\Lambda + 1)).\]

Hence, if \( \Lambda \) is a lacunar subset of \([n]\), then the condition \( \Lambda \cap \text{FE}(g) = \emptyset \) is equivalent to the condition \( \text{Stag}(g) \supseteq \Lambda \cup (\Lambda + 1) \). Thus, for each lacunar subset \( \Lambda \) of \([n]\), the definition of \( \tilde{L}_\Lambda \) becomes

\[
\tilde{L}_\Lambda = \sum_{g \in G; \Lambda \cap \text{FE}(g) = \emptyset} [g] = \sum_{g \in G; \text{Stag}(g) \supseteq \Lambda \cup (\Lambda + 1)} [g].
\]

Hence, the family \( (\tilde{L}_\Lambda)_{\Lambda \subseteq [n]} \) is lacunar

is a subfamily of the family \( \sum_{g \in G; \text{Stag}(g) \supseteq T} [g] \) \( T \subseteq [n+1] \)

(because if \( \Lambda \) is a lacunar subset of \([n]\), then \( \Lambda \cup (\Lambda + 1) \) is a well-defined subset of \([n+1]\), and moreover \( \Lambda \) can be uniquely recovered from \( \Lambda \cup (\Lambda + 1) \)). The further argument depends on the parity of \( n \):

\[\text{This takes a bit of thought to check. You need to show that if } \Lambda_1 \text{ and } \Lambda_2 \text{ are two lacunar subsets of } [n] \text{ satisfying } \Lambda_1 \cup (\Lambda_1 + 1) = \Lambda_2 \cup (\Lambda_2 + 1), \text{ then } \Lambda_1 = \Lambda_2. \text{ In order to prove this, assume the contrary, and conclude that there is a smallest element } h \text{ of the symmetric difference } \Lambda_1 \Delta \Lambda_2. \text{ WLOG assume that } h \in \Lambda_2 \setminus \Lambda_1, \text{ and argue that } h \text{ belongs to } \Lambda_1 \cup (\Lambda_1 + 1) \text{ but not to } \Lambda_2 \cup (\Lambda_2 + 1), \text{ which contradicts } \Lambda_1 \cup (\Lambda_1 + 1) = \Lambda_2 \cup (\Lambda_2 + 1).]
If $n$ is odd, then the vanishing element $\sum_{g \in G; \text{Stag}(g) \supseteq [n+1]} [g]$ does appear in the family $\left( \tilde{L}_\Lambda \right)_{\Lambda \subseteq [n]}$ is lacunar, because there exists a lacunar subset $\Lambda$ of $[n]$ satisfying $\Lambda \cup (\Lambda + 1) = [n+1]$: Namely, this $\Lambda$ is $\Omega$. Thus, the only syzygy of the family $\left( \tilde{L}_\Lambda \right)_{\Lambda \subseteq [n]}$ is lacunar is $\tilde{L}_\Omega = 0$ (since the only syzygy of the family $\left( \tilde{L}_\Lambda \right)_{\Lambda \subseteq [n]}$ is lacunar $\tilde{L}_\Omega = 0$).

If $n$ is even, then the vanishing element $\sum_{g \in G; \text{Stag}(g) \supseteq [n+1]} [g]$ does not appear in the family $\left( \tilde{L}_\Lambda \right)_{\Lambda \subseteq [n]}$ is lacunar, since no lacunar subset $\Lambda$ of $[n]$ satisfies $\Lambda \cup (\Lambda + 1) = [n+1]$. Hence, the syzygy $\sum_{g \in G; \text{Stag}(g) \supseteq [n+1]} [g] = 0$ of the family $\left( \tilde{L}_\Lambda \right)_{\Lambda \subseteq [n]}$ is lacunar disappears when we pass to the subfamily $\left( \tilde{L}_\Lambda \right)_{\Lambda \subseteq [n]}$ is lacunar. Consequently, the subfamily $\left( \tilde{L}_\Lambda \right)_{\Lambda \subseteq [n]}$ is lacunar is Q-linearly independent.

This proves Claim 3. As explained above, this yields Claim 2, hence also Claim 1, and thus completes the proof of Proposition 2.29.

**Corollary 2.30.** The family

$$\left( K_{n,\Lambda}^Z \right)_{n>0; \Lambda \subseteq [n]} \text{ is lacunar and nonempty} \cup \left( K_{0,\emptyset}^Z \right)$$

(where $\left( K_{0,\emptyset}^Z \right)$ is a 1-element family) is Q-linearly independent.

**Proof of Corollary 2.30.** For each $n \in \mathbb{N}$, the family $\left( K_{n,\Lambda}^Z \right)_{\Lambda \subseteq [n]}$ is lacunar and nonempty is Q-linearly independent (by Proposition 2.29). Furthermore, these families for varying $n > 0$ live in linearly disjoint subspaces of $\text{Pow} \, \mathcal{N}$ (because for each $n > 0$ and $\Lambda \subseteq [n]$, the power series $K_{n,\Lambda}^Z$ is homogeneous of degree $n$). Thus, the union $\left( K_{n,\Lambda}^Z \right)_{n>0; \Lambda \subseteq [n]}$ of all these families must also be Q-linearly independent. Adding the 1-element family $\left( K_{0,\emptyset}^Z \right)$ to this union preserves the Q-linear independence, since it lives in yet another linearly disjoint
subspace of \( \text{Pow} \mathcal{N} \) (namely, in its 0-th graded component). This proves Corollary 2.30.

**Definition 2.31.** Let \( \Pi_\mathcal{Z} \) be the \( \mathbb{Q} \)-vector subspace of \( \text{Pow} \mathcal{N} \) spanned by the family \( \left( K^\mathcal{Z}_{n, \Lambda} \right)_{n > 0; \; \Lambda \subseteq [n]} \) is lacunar and nonempty \( \cup \left( K^\mathcal{Z}_{0, \emptyset} \right) \). Then, \( \Pi_\mathcal{Z} \) is also the \( \mathbb{Q} \)-vector subspace of \( \text{Pow} \mathcal{N} \) spanned by the family \( \left( K^\mathcal{Z}_{n, \text{Ep}k \pi} \right)_{n \in \mathbb{N}} ; \; \pi \) is an \( n \)-permutation (by Proposition 2.28). In other words, \( \Pi_\mathcal{Z} \) is also the \( \mathbb{Q} \)-vector subspace of \( \text{Pow} \mathcal{N} \) spanned by the family \( \left( \Gamma_\mathcal{Z} (\pi) \right)_{n \in \mathbb{N}} ; \; \pi \) is an \( n \)-permutation (because of (7)). Hence, Corollary 2.17 shows that \( \Pi_\mathcal{Z} \) is closed under multiplication. Since furthermore \( \Gamma_\mathcal{Z} (\text{id}_0) = 1 \) (for the empty 0-permutation \( \text{id}_0 \)), we can thus conclude that \( \Pi_\mathcal{Z} \) is a \( \mathbb{Q} \)-subalgebra of \( \text{Pow} \mathcal{N} \).

We can now finally prove what we came here for:

**Theorem 2.32.** The permutation statistic \( \text{Ep}k \) is shuffle-compatible.

**Proof of Theorem 2.32** We must prove that \( \text{Ep}k \) is shuffle-compatible. In other words, we must prove that for any two disjoint permutations \( \pi \) and \( \sigma \), the multiset \( \{ \text{Ep}k (\tau) \mid \tau \in S (\pi, \sigma) \} \) depends only on \( \text{Ep}k (\pi) \), \( \text{Ep}k (\sigma) \), \(|\pi|\) and \(|\sigma|\). In other words, we must prove that if \( \pi \) and \( \sigma \) are two disjoint permutations, and if \( \pi' \) and \( \sigma' \) are two disjoint permutations satisfying \( \text{Ep}k (\pi) = \text{Ep}k (\pi') \), \( \text{Ep}k (\sigma) = \text{Ep}k (\sigma') \), \(|\pi| = |\pi'|\) and \(|\sigma| = |\sigma'|\), then the multiset \( \{ \text{Ep}k (\tau) \mid \tau \in S (\pi, \sigma) \} \) equals the multiset \( \{ \text{Ep}k (\tau) \mid \tau \in S (\pi', \sigma') \} \).

So let \( \pi \) and \( \sigma \) be two disjoint permutations, and let \( \pi' \) and \( \sigma' \) be two disjoint permutations satisfying \( \text{Ep}k (\pi) = \text{Ep}k (\pi') \), \( \text{Ep}k (\sigma) = \text{Ep}k (\sigma') \), \(|\pi| = |\pi'|\) and \(|\sigma| = |\sigma'|\).

Define \( n \in \mathbb{N} \) by \( n = |\pi| = |\pi'| \) (this is well-defined, since \(|\pi| = |\pi'|\)). Likewise, define \( m \in \mathbb{N} \) by \( m = |\sigma| = |\sigma'| \). Thus, \( \pi \) is an \( n \)-permutation, while \( \sigma \) is an \( m \)-permutation.

WLOG assume that \( n + m > 0 \) (since otherwise, both \( n \) and \( m \) are 0, which causes all four permutations \( \pi, \pi', \sigma \) and \( \sigma' \) to be empty; but then our goal is
obviously satisfied). Corollary 2.25 yields
\[
K_{n,Epk(\pi)}^Z \cdot K_{m,Epk(\sigma)}^Z = \sum_{\tau \in S(\pi,\sigma)} K_{n+m,Epk(\tau)}^Z = \sum_{\Lambda \subseteq [n+m]} K_{n+m,\Lambda}^Z \sum_{\tau \in S(\pi,\sigma); \ Epk(\tau) = \Lambda} \tau \in S(\pi,\sigma) \mid Epk(\tau) = \Lambda\mid K_{n+m,\Lambda}^Z
\]
because if \( \tau \in S(\pi,\sigma) \), then Epk(\tau) is a lacunar and nonempty subset of \([n+m]\) (by Proposition 2.27 applied to \( n+m \) and \( \tau \))
\[
= \sum_{\Lambda \subseteq [n+m]} \mid \{ \tau \in S(\pi,\sigma) \mid Epk(\tau) = \Lambda\mid K_{n+m,\Lambda}^Z
\]
Similarly,
\[
K_{n,Epk(\pi')}^Z \cdot K_{m,Epk(\sigma')}^Z = \sum_{\Lambda \subseteq [n+m]} \mid \{ \tau \in S(\pi',\sigma') \mid Epk(\tau) = \Lambda\mid K_{n+m,\Lambda}^Z
\]
The left-hand sides of these two equalities are identical (since Epk(\pi) = Epk(\pi') and Epk(\sigma) = Epk(\sigma')). Thus, their right-hand sides must also be identical. In other words, we have
\[
\sum_{\Lambda \subseteq [n+m]} \mid \{ \tau \in S(\pi,\sigma) \mid Epk(\tau) = \Lambda\mid K_{n+m,\Lambda}^Z
\]
Since the family \((K_{n+m,\Lambda}^Z)_{\Lambda \subseteq [n+m]}\) is lacunar and nonempty (by Proposition 2.29), this shows that
\[
\mid \{ \tau \in S(\pi,\sigma) \mid Epk(\tau) = \Lambda\mid = \mid \{ \tau \in S(\pi',\sigma') \mid Epk(\tau) = \Lambda\mid
\]
for each lacunar nonempty subset \( \Lambda \) of \([n+m]\). In other words, the multiset \{Epk(\tau) \mid \tau \in S(\pi,\sigma)\} equals the multiset \{Epk(\tau) \mid \tau \in S(\pi',\sigma')\} (because both of these multisets consist of lacunar nonempty subsets \( \Lambda \) of \([n+m]\), and the previous sentence shows that these subsets appear with equal multiplicities in them). This completes our proof of Theorem 2.32.

The permutation statistic \((Lpk, val)\) (see [GesZhu17] for its definition) is equivalent to Epk 10. Thus, an analogue of Theorem 2.32 holds for the statistic \((Lpk, val)\).

10Indeed, val is equivalent to epk by [GesZhu17] Lemma 2.1 (e); but knowing epk allows you to compute Epk from Lpk and vice versa (since Epk differs from Lpk only in the possible element \( n \)).
**Question 2.33.** Our concept of a “$Z$-enriched $(P, \gamma)$-partition” generalizes the concept of an “enriched $(P, \gamma)$-partition” by restricting ourselves to a subset $Z$ of $\mathcal{N} \times \{+, -\}$. (This does not sound like much of a generalization when stated like this, but as we have seen the behavior of the power series $\Gamma_Z(P, \gamma)$ depends strongly on what $Z$ is, and is not all anticipated by the $Z = \mathcal{N} \times \{+, -\}$ case.) A different generalization of enriched $(P, \gamma)$-partitions (introduced by Hsiao and Petersen in [HsiPet10]) are the colored $(P, \gamma)$-partitions, where the two-element set $\{+, -\}$ is replaced by the set $\{1, \omega, \ldots, \omega^{m-1}\}$ of all $m$-th roots of unity (where $m$ is a chosen positive integer, and $\omega$ is a fixed primitive $m$-th root of unity). We can play various games with this concept. The most natural thing to do seems to be to consider arbitrary total orders $<_0, <_1, \ldots, <_{m-1}$ on the codomain $A$ of the labeling $\gamma$ (perhaps with some nice properties such as all intervals being finite) and an arbitrary subset $Z$ of $\mathcal{N} \times \{1, \omega, \ldots, \omega^{m-1}\}$, and define a $Z$-enriched colored $(P, \gamma)$-partition to be a map $f : P \to Z$ such that every $x < y$ in $P$ satisfy the following conditions:

1. We have $f(x) \preceq f(y)$. (Here, the total order on $\mathcal{N} \times \{1, \omega, \ldots, \omega^{m-1}\}$ is defined by $(n, \omega^i) \preceq (n', \omega^{i'})$ if and only if either $n < n'$ or ($n = n'$ and $i < i'$)

   (for $i, i' \in \{0, 1, \ldots, m-1\}$).

2. If $f(x) = f(y) = (n, \omega^i)$ for some $n \in \mathcal{N}$ and $i \in \{0, 1, \ldots, m-1\}$, then $\gamma(x) <_i \gamma(y)$.

Is this a useful concept, and can it be used to study permutation statistics?

3. Descent statistics and quasisymmetric functions

In this section, we shall recall the concepts of descent statistics and their shuffle algebras (introduced in [GesZhu17]), and apply them to Epk.

3.1. Descent statistics

**Definition 3.1.** Let $st$ be a permutation statistic. We say that $st$ is a descent statistic if and only if $st\pi$ (for $\pi$ a permutation) depends only on the descent composition $\text{Comp}\pi$ of $\pi$. In other words, $st$ is a descent statistic if and only if every two permutations $\pi$ and $\sigma$ satisfying $\text{Comp}\pi = \text{Comp}\sigma$ satisfy $st\pi = st\sigma$.

Equivalently, a permutation statistic $st$ is a descent statistic if and only if every two permutations $\pi$ and $\sigma$ satisfying $|\pi| = |\sigma|$ and $\text{Des}\pi = \text{Des}\sigma$ satisfy $st\pi = st\sigma$. 

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st \sigma. (This is indeed equivalent, because for two permutations \( \pi \) and \( \sigma \), the condition \(|\pi| = |\sigma| \) and Des \( \pi = \text{Des} \sigma \) is equivalent to (Comp \( \pi = \text{Comp} \sigma \)).

For example, the permutation statistic Des is a descent statistic, because each permutation \( \pi \) satisfies Des \( \pi = \text{Des} (\text{Comp} \pi) \). Also, Pk is a descent statistic, since each permutation \( \pi \) satisfies

\[
P k \pi = (\text{Des} \pi) \setminus (\{1\} \cup (\text{Des} \pi + 1)),
\]

where Des \( \pi + 1 \) denotes the set \( \{i + 1 \mid i \in \text{Des} \pi\} \) (and, as we have just said, Des \( \pi \) can be recovered from Comp \( \pi \)). Furthermore, Epk is a descent statistic, since each \( n \)-permutation \( \pi \) (for a positive integer \( n \)) satisfies

\[
E pk \pi = (\text{Des} \pi \cup \{n\}) \setminus (\text{Des} \pi + 1)
\]

(and both Des \( \pi \) and \( n \) can be recovered from Comp \( \pi \)). The permutation statistics Lpk and Rpk (and, of course, Comp) are descent statistics as well, as one can easily check.

**Definition 3.2.** Let \( st \) be a descent statistic. Then, we can regard \( st \) as a map from the set of all compositions (rather than from the set of all permutations). Namely, for any composition \( I \), we define \( st I \) (an element of the codomain of \( st \)) by setting

\[
st I = st \pi \quad \text{for any permutation} \ \pi \ \text{satisfying} \ \text{Comp} \ \pi = I.
\]

This is well-defined (because for every composition \( I \), there exists at least one permutation \( \pi \) satisfying Comp \( \pi = I \), and all such permutations \( \pi \) have the same value of \( st \pi \)). In the following, we shall regard every descent statistic \( st \) simultaneously as a map from the set of all permutations and as a map from the set of all compositions.

Note that this definition leads to a new interpretation of Des \( I \) for a composition \( I \): It is now defined as Des \( \pi \) for any permutation \( \pi \) satisfying Comp \( \pi = I \). This could clash with the old meaning of Des \( I \) introduced in Definition 1.9. Fortunately, these two meanings of Des \( I \) are exactly the same, so there is no conflict of notation.

However, Definition 3.2 causes an ambiguity for expressions like “Des \((i_1, i_2, \ldots, i_n)\)”:

Here, the “\((i_1, i_2, \ldots, i_n)\)” might be understood either as a permutation, or as a composition, and the resulting descent sets Des \((i_1, i_2, \ldots, i_n)\) are not the same. A similar ambiguity occurs for any descent statistic \( st \) instead of Des. We hope that this ambiguity will not arise in this paper due to our explicit typecasting of permutations and compositions; but the reader should be warned that it can arise if one takes the notation too literally.
Definition 3.3. Let $st$ be a descent statistic.

(a) Two compositions $J$ and $K$ are said to be $st$-equivalent if and only if they have the same size and satisfy $st J = st K$. Equivalently, two compositions $J$ and $K$ are $st$-equivalent if and only if there exist two $st$-equivalent permutations $\pi$ and $\sigma$ satisfying $J = \text{Comp} \pi$ and $K = \text{Comp} \sigma$.

(b) The relation "$st$-equivalent" is an equivalence relation on compositions; its equivalence classes are called $st$-equivalence classes of compositions.

3.2. Quasisymmetric functions

We now recall the definition of quasisymmetric functions; see [GriRei17, Chapter 5] (and various other modern textbooks) for more details about this:

Definition 3.4. Consider the ring of power series $Q[[x_1, x_2, x_3, \ldots]]$ in infinitely many commuting indeterminates over $Q$. A power series $f \in Q[[x_1, x_2, x_3, \ldots]]$ is said to be quasisymmetric if it has the following property: For any positive integers $a_1, a_2, \ldots, a_k$ and any two strictly increasing sequences $(i_1 < i_2 < \cdots < i_k)$ and $(j_1 < j_2 < \cdots < j_k)$ of positive integers, the coefficient of $x_1^{i_1} x_2^{i_2} \cdots x_k^{i_k}$ in $f$ equals the coefficient of $x_1^{j_1} x_2^{j_2} \cdots x_k^{j_k}$ in $f$. A quasisymmetric function is a quasisymmetric power series $f \in Q[[x_1, x_2, x_3, \ldots]]$ that has bounded degree (i.e., there exists an $N \in \mathbb{N}$ such that each monomial appearing in $f$ has degree $\leq N$). The quasisymmetric functions form a $Q$-subalgebra of $Q[[x_1, x_2, x_3, \ldots]]$; this $Q$-subalgebra is denoted by $QSym$ and called the ring of quasisymmetric functions over $Q$.

The $Q$-algebra $QSym$ has much interesting structure (e.g., it is a Hopf algebra), but we will need very little of it (and we will introduce it whenever it becomes important). One simple yet crucial feature of $QSym$ that we will immediately use is the fundamental basis of $QSym$:

Definition 3.5. For any composition $\alpha$, we define the fundamental quasisymmetric function $F_\alpha$ to be the power series

$$\sum_{i_1 \leq i_2 \leq \cdots \leq i_n, i_j < i_{j+1} \text{ for each } j \in \text{Des } \alpha} x_{i_1} x_{i_2} \cdots x_{i_n} \in QSym,$$

where $n = |\alpha|$ is the size of $\alpha$. The family $(F_\alpha)_\alpha$ is a composition is a basis of the $Q$-vector space $QSym$; it is known as the fundamental basis of $QSym$.

We notice that the fundamental quasisymmetric function $F_\alpha$ is denoted by $L_\alpha$ in [GriRei17 §5.2].

The multiplication of fundamental quasisymmetric functions is intimately related to shuffles of permutations:
**Theorem 3.6.** Let $\pi$ and $\sigma$ be two disjoint permutations. Let $J = \text{Comp} \pi$ and $K = \text{Comp} \sigma$. For any composition $L$, let $c_{J,K}^L$ be the number of permutations with descent composition $L$ among the shuffles of $\pi$ and $\sigma$. Then,

$$F_J F_K = \sum_L c_{J,K}^L F_L$$

(where the sum is over all compositions $L$).

Theorem 3.6 is [GesZhu17, Theorem 4.1]; it can also be written in the following form:

**Proposition 3.7.** Let $\pi$ and $\sigma$ be two disjoint permutations. Then,

$$F_{\text{Comp} \pi} F_{\text{Comp} \sigma} = \sum_{\chi \in S(\pi, \sigma)} F_{\text{Comp} \chi}.$$

For a proof of Proposition 3.7 (and therefore also of the equivalent Theorem 3.6), we refer to [GriRei17, (5.2.6)] (which makes the additional requirement that the letters of $\pi$ are $1, 2, \ldots, |\pi|$ and the letters of $\sigma$ are $|\pi| + 1, |\pi| + 2, \ldots, |\pi| + |\sigma|$; but this requirement is not used in the proof and thus can be dropped).

### 3.3. Shuffle algebras

Any shuffle-compatible permutation statistic $s_t$ gives rise to a shuffle algebra $A_{s_t}$, defined as follows:

**Definition 3.8.** Let $s_t$ be a shuffle-compatible permutation statistic. For each permutation $\pi$, let $[\pi]_{s_t}$ denote the $s_t$-equivalence class of $\pi$.

Let $A_{s_t}$ be the free $\mathbb{Q}$-vector space whose basis is the set of all $s_t$-equivalence classes of permutations. We define a multiplication on $A_{s_t}$ by setting

$$[\pi]_{s_t} [\sigma]_{s_t} = \sum_{\tau \in S(\pi, \sigma)} [\tau]_{s_t}$$

for any two disjoint permutations $\pi$ and $\sigma$. It is easy to see that this multiplication is well-defined and associative, and turns $A_{s_t}$ into a $\mathbb{Q}$-algebra whose unity is the $s_t$-equivalence class of the empty permutation. (In the particular case when $s_t$ is a descent statistic, this shall be proven again in Proposition 3.9 (a) below.) This $\mathbb{Q}$-algebra is denoted by $A_{s_t}$, and is called the shuffle algebra of $s_t$. It is a graded $\mathbb{Q}$-algebra; its $n$-th graded component (for each $n \in \mathbb{N}$) is spanned by the $s_t$-equivalence classes of all $n$-permutations.
**Proposition 3.9.** Let $s^t$ be a shuffle-compatible descent statistic.

(a) The multiplication on $A_{s^t}$ defined in Definition 3.8 is well-defined and associative, and turns $A_{s^t}$ into a $\mathbb{Q}$-algebra whose unity is the $s^t$-equivalence class of the empty permutation.

(b) There is a surjective $\mathbb{Q}$-algebra homomorphism $p_{s^t}: \mathcal{QSym} \rightarrow A_{s^t}$ that satisfies

$$p_{s^t}(F_{\text{Comp}\,\pi}) = [\pi]_{s^t}$$

for every permutation $\pi$.

A central property of the shuffle algebra $A_{s^t}$ of a shuffle-compatible descent statistic is its relation to $\mathcal{QSym}$. This relation is given by [GesZhu17, Theorem 4.3], which we restate as follows:

**Theorem 3.10.** Let $s^t$ be a descent statistic.

(a) The descent statistic $s^t$ is shuffle-compatible if and only if there exist a $\mathbb{Q}$-algebra $A$ with basis $(u_\alpha)$ (indexed by $s^t$-equivalence classes $\alpha$ of compositions) and a $\mathbb{Q}$-algebra homomorphism $\phi_{s^t}: \mathcal{QSym} \rightarrow A$ with the property that whenever $\alpha$ is an $s^t$-equivalence class of compositions, we have

$$\phi_{s^t}(F_L) = u_\alpha$$

for each $L \in \alpha$.

(b) In this case, the $\mathbb{Q}$-linear map

$$A_{s^t} \rightarrow A, \quad [\pi]_{s^t} \mapsto u_{\alpha},$$

where $\alpha$ is the $s^t$-equivalence class of the composition $\text{Comp}\,\pi$, is a $\mathbb{Q}$-algebra isomorphism $A_{s^t} \rightarrow A$.

For the sake of completeness, we shall give proofs of Proposition 3.9 and Theorem 3.10 (independent of [GesZhu17]) in Subsection 4.4. We shall perhaps also use some other notations that were introduced in [GesZhu17]. The reader should thus consult [GesZhu17] for unexplained notations.

### 3.4. The shuffle algebra of $E_{pk}$

Theorem 2.32 yields that the permutation statistic $E_{pk}$ is shuffle-compatible. Hence, the shuffle algebra $A_{E_{pk}}$ is well-defined. We have not much to say about the shuffle algebra $A_{E_{pk}}$; we know that it is a graded quotient algebra of $\mathcal{QSym}$ (due to the homomorphism $p_{s^t}$ from Proposition 3.9 (b)), and that for each positive integer $n$, its $n$-th graded component has dimension $f_{n+2} - 1$, where $(f_0, f_1, f_2, \ldots)$ is the Fibonacci sequence (defined by $f_0 = 0$ and $f_1 = 1$ and the recursive relation $f_m = f_{m-1} + f_{m-2}$ for all $m \geq 2$). Moreover, the following holds:
**Theorem 3.11.** The map given by

\[ [\pi]_{\text{Epk}} \mapsto K_{n,\text{Epk}} \]

is a Q-algebra isomorphism from \( A_{\text{Epk}} \) to \( \Pi_{\mathcal{Z}} \).

In the process of proving Theorem 3.11, we will also prove Theorem 2.32 again.

**Proof of Theorem 3.11.** For each positive integer \( n \) and each \( n \)-permutation \( \pi \), we have

\[ \text{Epk} \pi = (\text{Des} \pi \cup \{n\}) \setminus (\text{Des} \pi + 1) \]

(by Proposition 1.11). Thus, \( \text{Epk} \pi \) is uniquely determined by \( \text{Des} \pi \) and \( n \). (Of course, this holds for \( n = 0 \) as well, because in this case only one \( \pi \) exists.) Hence, \( \text{Epk} \) is a descent statistic. Thus, for every composition \( L \), a set \( \text{Epk} L \) is defined (according to Definition 3.2); explicitly, \( \text{Epk} L = \text{Epk} \pi \) whenever \( \pi \) is a permutation satisfying \( \text{Comp} \pi = L \).

Recall that \( (F_L)_L \) is a composition is a basis of the Q-vector space \( \text{QSym} \). Let \( \phi_{\text{Epk}} : \text{QSym} \rightarrow \Pi_{\mathcal{Z}} \) be the Q-linear map that sends each \( F_L \) (for each \( n \in \mathbb{N} \) and each composition \( L \) of \( n \)) to \( K_{n,\text{Epk}}^L \in \Pi_{\mathcal{Z}} \). This Q-linear map \( \phi_{\text{Epk}} \) respects multiplication \(^{11}\) and sends \( 1 \in \text{QSym} \) to \( 1 \in \Pi_{\mathcal{Z}} \).\(^ {12}\) Thus, \( \phi_{\text{Epk}} \) is a Q-algebra homomorphism.

\(^{11}\) **Proof.** Let \( n \in \mathbb{N} \) and \( m \in \mathbb{N} \). Let \( J \) be a composition of \( n \), and let \( K \) be a composition of \( m \). Fix an \( n \)-permutation \( \pi \) satisfying \( \text{Comp} \pi = J \), and fix an \( m \)-permutation \( \sigma \) satisfying \( \text{Comp} \sigma = K \). Now,

\[ \phi_{\text{Epk}} (F_J) \cdot \phi_{\text{Epk}} (F_K) = K_{n,\text{Epk}}^J \cdot K_{m,\text{Epk}}^K = K_{n+m,\text{Epk}}^{J,K} \]

(by Corollary 2.25).

Comparing this with

\[ \phi_{\text{Epk}} \left( \sum_{\tau \in S(\pi,\sigma)} F_{\text{Comp} \tau} \right) = \sum_{\tau \in S(\pi,\sigma)} \phi_{\text{Epk}} (F_{\text{Comp} \tau}) = \sum_{\tau \in S(\pi,\sigma)} K_{n+m,\text{Epk}}^{\text{Comp} \tau} \]

we obtain \( \phi_{\text{Epk}} (F_J) \cdot \phi_{\text{Epk}} (F_K) = \phi_{\text{Epk}} (F_J F_K) \). Since the map \( \phi_{\text{Epk}} \) is Q-linear, this yields that \( \phi_{\text{Epk}} \) respects multiplication (since \( (F_L)_L \) is a composition is a basis of the Q-vector space \( \text{QSym} \)).

\(^{12}\) This is easy, since \( 1 = F() \).
The family \( K_{n,\Lambda}^Z \) for \( n > 0; \Lambda \subseteq [n] \) is lacunar and nonempty \( \bigcup K_{0,\varnothing}^Z \) spans \( \Pi_Z \) (by the definition of \( \Pi_Z \)) and is \( \mathbb{Q} \)-linearly independent (by Corollary 2.30). Thus, it is a basis of \( \Pi_Z \).

For each positive integer \( n \), there is a canonical bijection between the Epk-equivalence classes of \( n \)-permutations and the nonempty lacunar subsets \( \Lambda \) of \( [n] \) (indeed, the bijection sends any equivalence class \([\pi]\) to \( \text{Epk}\pi \)).

Hence, Theorem 3.10 (a) (applied to \( A = \Pi_Z \), \( \text{st} = \text{Epk} \) and \( u_\alpha = K_{n,\Lambda}^Z \), where \( \alpha \) is an Epk-equivalence class of \( n \)-permutations and where \( \Lambda \) is the corresponding nonempty lacunar subset) shows that the descent statistic Epk is shuffle-compatible. This proves Theorem 2.32 again. Theorem 3.10 (b) then yields that the \( \mathbb{Q} \)-linear map

\[ A_{\text{Epk}} \to \Pi_Z, \quad [\pi]_{\text{Epk}} \mapsto K_{n,\text{Epk}\pi}^Z \]

is a \( \mathbb{Q} \)-algebra isomorphism from \( A_{\text{Epk}} \) to \( \Pi_Z \). Hence, Theorem 3.11 is proven.

4. The kernel of the map \( \text{QSym} \to A_{\text{Epk}} \)

4.1. The kernel of a descent statistic

Now, we shall focus on a feature of shuffle-compatible descent statistics that seems to have been overlooked so far: their kernels.

**Definition 4.1.** Let \( \text{st} \) be a descent statistic. Then, \( K_{\text{st}} \) shall mean the \( \mathbb{Q} \)-vector subspace of \( \text{QSym} \) spanned by all elements of the form \( F_J - F_K \), where \( J \) and \( K \) are two \( \text{st} \)-equivalent compositions. (See Definition 3.3 for the definition of “\( \text{st} \)-equivalent compositions”.) We shall refer to \( K_{\text{st}} \) as the *kernel* of \( \text{st} \).

The following basic linear-algebraic lemma will be useful:

**Lemma 4.2.** Let \( \text{st} \) be a descent statistic. Let \( A \) be a \( \mathbb{Q} \)-vector space with basis \( (u_\alpha) \) indexed by \( \text{st} \)-equivalence classes \( \alpha \) of compositions. Let \( \phi_{\text{st}} : \text{QSym} \to A \) be a \( \mathbb{Q} \)-linear map with the property that whenever \( \alpha \) is an \( \text{st} \)-equivalence class of compositions, we have

\[ \phi_{\text{st}}(F_L) = u_\alpha \quad \text{for each} \quad L \in \alpha. \]  

Then, \( \text{Ker} (\phi_{\text{st}}) = K_{\text{st}} \).

**Proof of Lemma 4.2.** Let us first show that \( \text{Ker} (\phi_{\text{st}}) \subseteq K_{\text{st}} \).

---

13 Proposition 2.28 shows that this is indeed a bijection.
14 or empty, if \( n = 0 \)
Indeed, let $x \in \text{Ker} (\phi_{st})$ be arbitrary. Write $x \in \text{QSym}$ in the form $x = \sum_{L} x_{L} F_{L}$, where the sum ranges over all compositions $L$, and where the $x_{L}$ are elements of $Q$ (all but finitely many of which are zero). Now, $x \in \text{Ker} (\phi_{st})$, so that $\phi_{st} (x) = 0$. Thus,

$$0 = \phi_{st} (x) = \sum_{L} x_{L} \phi_{st} (F_{L}) \left( \text{since } x = \sum_{L} x_{L} F_{L} \right)$$

$$= \sum_{\alpha} \sum_{L \in \alpha} x_{L} \phi_{st} (F_{L}) \left( \text{where the first sum is over all st-equivalence classes } \alpha \text{ of compositions} \right)$$

$$= \sum_{\alpha} \sum_{L \in \alpha} x_{L} u_{\alpha} = \sum_{\alpha} \left( \sum_{L \in \alpha} x_{L} \right) u_{\alpha}.$$

Since the family $(u_{\alpha})$ is linearly independent (because it is a basis of $A$), we thus conclude that

$$\sum_{L \in \alpha} x_{L} = 0 \quad (9)$$

for each st-equivalence class $\alpha$ of compositions.

Now, for each st-equivalence class $\alpha$ of compositions, we fix an element $L_{\alpha}$ of $\alpha$. Then, for each st-equivalence class $\alpha$ of compositions and each composition $L \in \alpha$, we have

$$F_{L} = F_{L_{\alpha}} \in \mathcal{K}_{st} \quad (10)$$

(since the compositions $L$ and $L_{\alpha}$ are st-equivalent$^{15}$).

Now,

$$x = \sum_{L} x_{L} F_{L}$$

$$= \sum_{\alpha} \sum_{L \in \alpha} x_{L} (F_{L} - F_{L_{\alpha}}) + \sum_{\alpha} x_{L} F_{L_{\alpha}}$$

$$\in \sum_{\alpha} \sum_{L \in \alpha} x_{L} \mathcal{K}_{st} + \sum_{\alpha} 0 F_{L_{\alpha}} \subseteq \mathcal{K}_{st}.$$

Now, forget that we fixed $x$. We thus have proven that $x \in \mathcal{K}_{st}$ for each $x \in \text{Ker} (\phi_{st})$. In other words, $\text{Ker} (\phi_{st}) \subseteq \mathcal{K}_{st}$.

$^{15}$since both compositions $L$ and $L_{\alpha}$ lie in the same st-equivalence class $\alpha$
Conversely, it is easy to see that $\mathcal{K}_{st} \subseteq \text{Ker}(\phi_{st})$. Combining this with $\text{Ker}(\phi_{st}) \subseteq \mathcal{K}_{st}$, we obtain $\mathcal{K}_{st} = \text{Ker}(\phi_{st})$. This proves Lemma 4.2.

Theorem 3.10 easily yields the following fact:

**Proposition 4.3.** Let $st$ be a descent statistic. Then, $st$ is shuffle-compatible if and only if $\mathcal{K}_{st}$ is an ideal of $\text{QSym}$. Furthermore, in this case, $\mathcal{A}_{st} \cong \text{QSym} / \mathcal{K}_{st}$.

**Proof of Proposition 4.3.** $\implies$: Assume that $st$ is shuffle-compatible. Thus, Theorem 3.10 (a) shows that there exist a $\mathbb{Q}$-algebra $A$ with basis $(u_\alpha)$ indexed by st-equivalence classes $\alpha$ of compositions, and a $\mathbb{Q}$-algebra homomorphism $\phi_{st} : \text{QSym} \to A$ with the property that whenever $\alpha$ is an st-equivalence class of compositions, we have

$$\phi_{st}(F_L) = u_\alpha \quad \text{for each } L \in \alpha. \quad (11)$$

Consider this $A$ and this $\phi_{st}$.

Lemma 4.2 yields that $\mathcal{K}_{st} = \text{Ker}(\phi_{st})$. But the map $\phi_{st}$ is a $\mathbb{Q}$-algebra homomorphism. Thus, its kernel $\text{Ker}(\phi_{st})$ is an ideal of $\text{QSym}$. In other words, $\mathcal{K}_{st}$ is an ideal of $\text{QSym}$ (since $\mathcal{K}_{st} = \text{Ker}(\phi_{st})$).

It remains to show that $\mathcal{A}_{st} \cong \text{QSym} / \mathcal{K}_{st}$. This is easy: Each element of the basis $(u_\alpha)$ of the $\mathbb{Q}$-vector space $A$ is contained in the image of $\phi_{st}$ (because of (11)). Therefore, the homomorphism $\phi_{st}$ is surjective. Thus, $\phi_{st}(\text{QSym}) = A$. Hence, $A = \phi_{st}(\text{QSym}) \cong \text{QSym} / \text{Ker}(\phi_{st})$ (by the homomorphism theorem). But Theorem 3.10 (b) shows that $\mathcal{A}_{st} \cong A$. Thus, $\mathcal{A}_{st} \cong A \cong \text{QSym} / \text{Ker}(\phi_{st}) \cong \text{QSym} / \mathcal{K}_{st}$. This finishes the proof of the $\implies$ direction of Proposition 4.3.

$\Longleftarrow$: Assume that $\mathcal{K}_{st}$ is an ideal of $\text{QSym}$. We must prove that $st$ is shuffle-compatible.

We shall not use this direction of Proposition 4.3 so let us merely sketch the proof. Let $A$ be the $\mathbb{Q}$-algebra $\text{QSym} / \mathcal{K}_{st}$. Let $\phi_{st}$ be the canonical projection $\text{QSym} \to A$; this is clearly a $\mathbb{Q}$-algebra homomorphism.

For each st-equivalence class $\alpha$ of compositions, we define an element $u_\alpha$ of $A$ by requiring that

$$u_\alpha = \phi_{st}(F_L) \quad \text{whenever } L \in \alpha.$$  

**Proof.** Recall that $\mathcal{K}_{st}$ is the $\mathbb{Q}$-vector subspace of $\text{QSym}$ spanned by all elements of the form $F_J - F_K$, where $J$ and $K$ are two st-equivalent compositions. Hence, in order to prove that $\mathcal{K}_{st} \subseteq \text{Ker}(\phi_{st})$, it suffices to show that $F_J - F_K \in \text{Ker}(\phi_{st})$, whenever $J$ and $K$ are two st-equivalent compositions. So let $J$ and $K$ be two st-equivalent compositions. We must show that $F_J - F_K \in \text{Ker}(\phi_{st})$.

The two compositions $J$ and $K$ are st-equivalent. Hence, they lie in one and the same st-equivalence class. Let $\alpha$ be this st-equivalence class. Then, $J \in \alpha$ and therefore $\phi_{st}(F_J) = u_\alpha$ (by (8), applied to $L = J$). Similarly, $\phi_{st}(F_K) = u_\alpha$. Now, $\phi_{st}(F_J - F_K) = \phi_{st}(F_J) - \phi_{st}(F_K) = u_\alpha - u_\alpha = 0$. In other words, $F_J - F_K \in \text{Ker}(\phi_{st})$. This completes our proof.
This is easily seen to be well-defined, because the image $\phi_{st}(F_L)$ depends only on $a$ but not on $L$ (indeed, if $J$ and $K$ are two elements of $a$, then $J$ and $K$ are st-equivalent, whence $F_J - F_K \in K_{st}$, whence $F_J \equiv F_K \mod K_{st}$ and therefore $\phi_{st}(F_J) = \phi_{st}(F_K)$).

It is not hard to see that the family $(u_a)$ (where $a$ ranges over all st-equivalence classes of compositions) is a basis of the Q-algebra $A$. Hence, Theorem 3.10(a) yields that $st$ is shuffle-compatible. This proves the $\Leftarrow$ direction of Proposition 4.3. \qed

Corollary 4.4. The kernel $K_{Epk}$ of the descent statistic $Epk$ is an ideal of $QSym$.

Proof of Corollary 4.4. This follows from Proposition 4.3 (applied to $st = Epk$), because of Theorem 2.32. \qed

We can study the kernel of any descent statistic; in particular, the case of shuffle-compatible descent statistics appears interesting. Since $QSym$ is isomorphic to a polynomial ring (as an algebra), it has many ideals, which are rather hopeless to classify or tame; but the ones obtained as kernels of shuffle-compatible descent statistics might be worth discussing.

4.2. The $F$-generating set of $K_{Epk}$

Let us now focus on $K_{Epk}$, the kernel of $Epk$.

Proposition 4.5. If $J = (j_1, j_2, \ldots, j_m)$ and $K$ are two compositions, then we shall write $J \rightarrow K$ if there exists an $\ell \in \{2, 3, \ldots, m\}$ such that $j_\ell > 2$ and $K = (j_1, j_2, \ldots, j_{\ell-1}, j_\ell - 1, j_\ell + 1, j_{\ell+1}, j_{\ell+2}, \ldots, j_m)$. (In other words, we write $J \rightarrow K$ if $K$ can be obtained from $J$ by “splitting” some entry $j_\ell > 2$ into two consecutive entries $1$ and $j_\ell - 1$, provided that this entry was not the first entry – i.e., we had $\ell > 1$ – and that this entry was greater than 2.)

The ideal $K_{Epk}$ of $QSym$ is spanned (as a Q-vector space) by all differences of the form $F_J - F_K$, where $J$ and $K$ are two compositions satisfying $J \rightarrow K$.

Example 4.6. We have $(2, 1, 4, 4) \rightarrow (2, 1, 1, 3, 4)$, since the composition $(2, 1, 1, 3, 4)$ is obtained from $(2, 1, 4, 4)$ by splitting the third entry (which is $4 > 2$) into two consecutive entries $1$ and $3$.

Similarly, $(2, 1, 4, 4) \rightarrow (2, 1, 4, 1, 3)$.

But we do not have $(3, 1) \rightarrow (1, 2, 1)$, because splitting the first entry of the composition is not allowed in the definition of the relation $\rightarrow$. Also, we do not have $(1, 2, 1) \rightarrow (1, 1, 1, 1)$, because the entry we are splitting must be $> 2$.

The word “consecutive” here means “in consecutive positions of $J$", not “consecutive integers”. So two consecutive entries of $J$ are two entries of the form $j_p$ and $j_{p+1}$ for some $p \in \{1, 2, \ldots, m-1\}$.

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Two compositions $J$ and $K$ satisfying $J \rightarrow K$ must necessarily satisfy $|J| = |K|$.

Here are all relations $\rightarrow$ between compositions of size 4:

$$(1, 3) \rightarrow (1, 1, 2).$$

Here are all relations $\rightarrow$ between compositions of size 5:

$$(1, 4) \rightarrow (1, 1, 3),$$
$$(1, 3, 1) \rightarrow (1, 1, 2, 1),$$
$$(1, 1, 3) \rightarrow (1, 1, 1, 2),$$
$$(2, 3) \rightarrow (2, 1, 2).$$

There are no relations $\rightarrow$ between compositions of size $\leq 3$.

**Proof of Proposition 4.5** We begin by proving some simple claims.

**Claim 1:** Let $n \in \mathbb{N}$. Let $J$ and $K$ be two compositions of size $n$. Then, $J \rightarrow K$ if and only if there exists some $k \in [n-1]$ such that

$$\Des K = \Des J \cup \{k\}, \quad \text{and} \quad k \notin \Des J, \quad k - 1 \in \Des J \quad \text{and} \quad k + 1 \notin \Des J \cup \{n\}.$$  

*Proof of Claim 1:* This is straightforward to check: “Splitting” an entry of a composition $C$ into two consecutive entries (summing up to the original entry) is always tantamount to adding a new element to $\Des C$. The rest is translating conditions.

If $n$ is a positive integer, and $L$ is any composition of $n$, then

$$\Ep k L = (\Des L \cup \{n\}) \setminus (\Des L + 1).$$  

(12) 

(This is a consequence of Proposition 1.11 applied to any $n$-permutation $\pi$ satisfying $L = \Comp \pi$.)

**Claim 2:** Let $J$ and $K$ be two compositions satisfying $J \rightarrow K$. Then, $\Ep k J = \Ep k K$.

*Proof of Claim 2:* Easy consequence of Claim 1 and (12).

For any two integers $a$ and $b$, we set $[a, b] = \{a, a + 1, \ldots, b\}$. (This is an empty set if $a > b$.)

It is easy to see that every composition $J$ of size $n > 0$ satisfies

$$[\max (\Ep k J), n - 1] \subseteq \Des J$$  

(13)
For each \( n \in \mathbb{N} \) and each subset \( S \) of \([n - 1]\), we define a subset \( S^c \) of \([n - 1]\) as follows:

\[
S^c_n = \{ s \in S \mid s - 1 \notin S \text{ or } [s, n - 1] \subseteq S \}.
\]

Also, for each \( n \in \mathbb{N} \) and each nonempty subset \( T \) of \([n]\), we define a subset \( \rho_n(T) \) of \([n - 1]\) as follows:

\[
\rho_n(T) = \begin{cases} 
T \setminus \{n\}, & \text{if } n \in T; \\
T \cup [\max T, n - 1], & \text{if } n \notin T.
\end{cases}
\]

**Claim 3:** Let \( n \in \mathbb{N} \). Let \( J \) be a composition of size \( n \). Then, \((\text{Des } J)_n^c = \rho_n(\text{Epk } J)\).

**Proof of Claim 3:** Let \( g \in (\text{Des } J)_n^c \). We shall show that \( g \in \rho_n(\text{Epk } J)\).

We have \( g \in (\text{Des } J)_n^c \subseteq \text{Des } J \) (since \( S^c_n \subseteq S \) for each subset \( S \) of \([n - 1]\)) and therefore \( \text{Des } J \neq \emptyset \). Hence, \( J \) is not the empty composition. In other words, \( n > 0 \).

From [12], we obtain \( \text{Epk } J = (\text{Des } J \cup \{n\}) \setminus (\text{Des } J + 1) \). Thus, the set \( \text{Epk } J \) is disjoint from \( \text{Des } J + 1 \). Furthermore, the set \( \text{Epk } J \) is nonempty.

We have \( g \in (\text{Des } J)_n^c \). Thus, \( g \) is an element of \( \text{Des } J \) satisfying \( g - 1 \notin \text{Des } J \) or \([g, n - 1] \subseteq \text{Des } J \) (by the definition of \((\text{Des } J)_n^c\)). We are thus in one of the following two cases:

**Case 1:** We have \( g - 1 \notin \text{Des } J \).

**Case 2:** We have \([g, n - 1] \subseteq \text{Des } J \).

Let us first consider Case 1. In this case, we have \( g - 1 \notin \text{Des } J \). In other words, \( g \notin \text{Des } J + 1 \). Combined with \( g \in \text{Des } J \subseteq \text{Des } J \cup \{n\} \), this yields \( g \in (\text{Des } J \cup \{n\}) \setminus (\text{Des } J + 1) \). Indeed, assume the contrary. Thus, \([\max (\text{Epk } J), n - 1] \notin \text{Des } J \). Hence, there exists some \( q \in [\max (\text{Epk } J), n - 1] \) satisfying \( q \notin \text{Des } J \). Let \( r \) be the largest such \( q \).

Thus, \( r \in [\max (\text{Epk } J), n - 1] \) but \( r \notin \text{Des } J \). From \( r \in [\max (\text{Epk } J), n - 1] \subseteq [n - 1] \), we obtain \( r + 1 \in [n] \). Also, from \( r \notin \text{Des } J \), we obtain \( r + 1 \notin \text{Des } J + 1 \).

From \( r \in [\max (\text{Epk } J), n - 1] \), we obtain \( r \geq \max (\text{Epk } J) \), so that \( r + 1 > r \geq \max (\text{Epk } J) \) and therefore \( r + 1 \notin \text{Epk } J \) (since a number that is higher than \( \max (\text{Epk } J) \) cannot belong to \( \text{Epk } J \)).

From [12], we obtain \( \text{Epk } J = (\text{Des } J \cup \{n\}) \setminus (\text{Des } J + 1) \).

If we had \( r + 1 \in \text{Des } J \cup \{n\} \), then we would have \( r + 1 \in (\text{Des } J \cup \{n\}) \setminus (\text{Des } J + 1) \) (since \( r + 1 \notin \text{Des } J + 1 \)). This would contradict \( r + 1 \notin \text{Epk } J = (\text{Des } J \cup \{n\}) \setminus (\text{Des } J + 1) \). Thus, we cannot have \( r + 1 \in \text{Des } J \cup \{n\} \). Therefore, \( r + 1 \notin \text{Des } J \cup \{n\} \).

Hence, \( r + 1 \neq n \) (since \( r + 1 \notin \text{Des } J \cup \{n\} \) but \( n \notin \{n\} \subseteq \text{Des } J \cup \{n\} \)). Combined with \( r + 1 \in [n] \), this yields \( r + 1 \in [n] \setminus \{n\} = [n - 1] \). Combined with \( r + 1 > \max (\text{Epk } J) \), this yields \( r + 1 \in [\max (\text{Epk } J), n - 1] \). Also, \( r + 1 \notin \text{Des } J \) (since \( r + 1 \notin \text{Des } J \cup \{n\} \)).

Thus, \( r + 1 \) is a \( q \in [\max (\text{Epk } J), n - 1] \) satisfying \( q \notin \text{Des } J \). This contradicts the fact that \( r \) is the largest such \( q \) (since \( r + 1 \) is clearly larger than \( r \)). This contradiction proves that our assumption was wrong; thus, [13] is proven.

Indeed, it contains at least the smallest element of the set \( \text{Des } J \cup \{n\} \) (since \( \text{Epk } J = (\text{Des } J \cup \{n\}) \setminus (\text{Des } J + 1) \)).
In other words, ρ definition of Thus, h Des J. Hence, J g [n] + Des J. From g g ∈ (Epk J) \ {n} (since g ∈ Epk J). But each nonempty subset T of [n] satisfies T \ {n} ⊆ ρ T (by the definition of ρ T). Applying this to T = Epk J, we obtain (Epk J) \ {n} ⊆ ρ (Epk J). Hence, g ∈ (Epk J) \ {n} ⊆ ρ (Epk J). Thus, g ∈ ρ (Epk J) is proven in Case 1.

Let us now consider Case 2. In this case, we have [g, n − 1] ⊆ Des J. Hence, each of the elements g, g + 1, . . . , n − 1 belongs to Des J. In other words, each of the elements g + 1, g + 2, . . . , n belongs to Des J + 1. Hence, none of the elements g + 1, g + 2, . . . , n belongs to Epk J (since the set Epk J is disjoint from Des J + 1). Thus, max (Epk J) ≤ g. Therefore, g ∈ [max (Epk J), n − 1] (since g ∈ Des J ⊆ [n − 1]).

Also, n ̸∈ Epk J. Hence, the definition of ρ (Epk J) yields ρ (Epk J) = Epk J ∪ [max (Epk J), n − 1]. Now,

\[ g \in [\max (\text{Epk J}), n - 1] \subseteq \text{Epk J} \cup [\max (\text{Epk J}), n - 1] = \rho (\text{Epk J}). \]

Hence, g ∈ ρ (Epk J) is proven in Case 2.

Thus, g ∈ ρ (Epk J) is proven in both Cases 1 and 2. This shows that g ∈ ρ (Epk J) always holds.

Forget that we fixed g. We thus have proven that g ∈ ρ (Epk J) for each g ∈ (Des J)o. In other words, (Des J)o ⊆ ρ (Epk J).

Now, let h ∈ ρ (Epk J) be arbitrary. We shall prove that h ∈ (Des J)o.

We are in one of the following two cases:

Case 1: We have n ∈ Epk J.

Case 2: We have n ̸∈ Epk J.

Let us first consider Case 1. In this case, we have n ∈ Epk J, and thus ρ (Epk J) = Epk J \ {n} (by the definition of ρ (Epk J)). Hence,

\[ h \in \rho (\text{Epk J}) = \text{Epk J} \setminus \{n\} \subseteq \text{Epk J} = (\text{Des J} \cup \{n\}) \setminus \{\text{Des J} + 1\}. \]

In other words, h ∈ Des J \ {n} and h ̸∈ Des J + 1. Since h ∈ Des J \ {n} and h ̸= n (because h ∈ ρ (Epk J) ⊆ [n − 1]), we obtain h ∈ (Des J \ {n}) \ {n} ⊆ Des J. From h ̸∈ Des J + 1, we obtain h − 1 ̸∈ Des J. Thus, h is an element of Des J satisfying h − 1 ̸∈ Des J or [h, n − 1] ⊆ Des J (in fact, h − 1 ̸∈ Des J holds). Thus, h ∈ (Des J)o (by the definition of (Des J)o). Thus, h ∈ (Des J)o is proven in Case 1.

Let us now consider Case 2. In this case, we have n ̸∈ Epk J. Hence, the definition of ρ (Epk J) yields ρ (Epk J) = (Epk J) ∪ [max (Epk J), n − 1]. Thus, h ∈ ρ (Epk J) = (Epk J) ∪ [max (Epk J), n − 1].

If h ∈ Epk J, then we can prove h ∈ (Des J)o just as in Case 1. Hence, let us WLOG assume that we don’t have h ∈ Epk J. Thus, h ̸∈ Epk J. Combined with

\footnote{Proof. Assume the contrary. Thus, n ∈ Epk J. But none of the elements g + 1, g + 2, . . . , n belongs to Epk J. Hence, n is not among the elements g + 1, g + 2, . . . , n. Therefore, g ≥ n, so that g = n. This contradicts g ∈ Des J ⊆ [n − 1]. This contradiction shows that our assumption was wrong, qed.}
h ∈ (Epk J) ∪ max (Epk J), n − 1, this yields

\[ h ∈ ((Epk J) ∪ [\text{max} (Epk J), n − 1]) \setminus (Epk J) = [\text{max} (Epk J), n − 1] \setminus (Epk J) \]

\[ \subseteq [\text{max} (Epk J), n − 1] \subseteq \text{Des} J \quad \text{(by (13)).} \]

Moreover, from \( h ∈ [\text{max} (Epk J), n − 1] \), we obtain \( h ≥ \text{max} (Epk J) \), so that

\[ [h, n − 1] \subseteq [\text{max} (Epk J), n − 1] \subseteq \text{Des} J \quad \text{(by (13)).} \]

Hence, \( h \) is an element of \( \text{Des} J \) satisfying \( h − 1 \notin \text{Des} J \) or \( [h, n − 1] \subseteq \text{Des} J \) (namely, \( [h, n − 1] \subseteq \text{Des} J \)). In other words, \( h ∈ (\text{Des} J)_n^\circ \) (by the definition of \( (\text{Des} J)_n^\circ \)). Thus, \( h ∈ (\text{Des} J)_n^\circ \) is proven in Case 2.

We have now proven \( h ∈ (\text{Des} J)_n^\circ \) in both Cases 1 and 2. Hence, \( h ∈ (\text{Des} J)_n^\circ \) always holds.

Forget that we fixed \( h \). We thus have shown that \( h ∈ (\text{Des} J)_n^\circ \) for each \( h ∈ \rho_n (\text{Epk} J) \). In other words, \( \rho_n (\text{Epk} J) \subseteq (\text{Des} J)_n^\circ \). Combining this with \( (\text{Des} J)_n^\circ \subseteq \rho_n (\text{Epk} J) \), we obtain \( (\text{Des} J)_n^\circ = \rho_n (\text{Epk} J) \). This proves Claim 3.

Claim 4: Let \( n ∈ \mathbb{N} \). Let \( J \) and \( K \) be two compositions of size \( n \) satisfying \( \text{Epk} J = \text{Epk} K \). Then, \( (\text{Des} J)_n^\circ = (\text{Des} K)_n^\circ \).

[Proof of Claim 4: Claim 3 yields \( (\text{Des} J)_n^\circ = \rho_n (\text{Epk} J) \) and similarly \( (\text{Des} K)_n^\circ = \rho_n (\text{Epk} K) \). Hence,

\[ (\text{Des} J)_n^\circ = \rho_n (\text{Epk} J) = (\text{Des} K)_n^\circ \].

This proves Claim 4.]

We let \( \rightarrow \) be the transitive-and-reflexive closure of the relation \( \rightarrow \). Thus, two compositions \( J \) and \( K \) satisfy \( J \rightarrow K \) if and only if there exists a sequence \( (L_0, L_1, \ldots, L_\ell) \) of compositions satisfying \( L_0 = J \) and \( L_\ell = K \) and \( L_0 \rightarrow L_1 \rightarrow \cdots \rightarrow L_\ell \).

Claim 5: Let \( n ∈ \mathbb{N} \). Let \( K \) be a composition of size \( n \). Then,

\[ \text{Comp} ((\text{Des} K)_n^\circ) \rightarrow K. \]

[Proof of Claim 5: We shall prove Claim 5 by strong induction on \( |(\text{Des} K) \setminus (\text{Des} K)_n^\circ| \). Thus, we fix an \( n ∈ \mathbb{N} \) and a composition \( K \) of size \( n \), and we assume (as the induction hypothesis) that each composition \( J \) of size \( n \) satisfying \( |(\text{Des} J) \setminus (\text{Des} J)_n^\circ| < |(\text{Des} K) \setminus (\text{Des} K)_n^\circ| \) satisfies \( \text{Comp} ((\text{Des} J)_n^\circ) \rightarrow J \). Our goal is to prove that \( \text{Comp} ((\text{Des} K)_n^\circ) \rightarrow K \).

Let \( A = \text{Des} K \). Thus, \( K = \text{Comp} A \) and \( A \subseteq [n − 1] \).

Applying (12) to \( L = K \), we obtain \( \text{Epk} K = (\text{Des} K \cup \{n\}) \setminus (\text{Des} K + 1) = (A \cup \{n\}) \setminus (A + 1) \) (since \( \text{Des} K = A \)).

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Also, \( A_n^0 \subseteq A \) (since \( S_n^0 \subseteq S \) for any subset \( S \) of \([n-1]\)). If \( A_n^0 = A \), then we are done (because if \( A_n^0 = A \), then \( \text{Comp} \left( \left( \text{Des} K \right)^{n}_{\rightarrow A} \right) = \text{Comp} \left( A_n^0 \right) \)).

\( \text{Comp} A = K \), and therefore the reflexivity of \( \rightarrow \) shows that \( \text{Comp} \left( \left( \text{Des} K \right)^{n}_{\rightarrow A} \right) \rightarrow K \). Hence, we WLOG assume that \( A_n^0 \neq A \). Thus, \( A_n^0 \) is a proper subset of \( A \) (since \( A_n^0 \subseteq A \)). Therefore, there exists a \( q \in A \) satisfying \( q \notin A_n^0 \). Let \( k \) be the \text{largest} such \( q \). Thus, \( k \in A \) and \( k \notin A_n^0 \). Hence, \( k \in A \setminus A_n^0 \). Also, \( k \in A \subseteq [n-1] \).

Let \( J = \text{Comp} \left( A \setminus \{k\} \right) \). Thus, \( \text{Des} J = A \setminus \{k\} \), so that \( A = \text{Des} J \cup \{k\} \) (since \( k \in A \)). Hence, \( \text{Des} K = A = \text{Des} J \cup \{k\} \). Also, \( k \notin A \setminus \{k\} = \text{Des} J \).

Furthermore, \( k - 1 \in \text{Des} J \) \(^{21}\) and \( k + 1 \notin \text{Des} J \cup \{n\} \). \(^{22}\) Hence, we have found a \( k \in [n-1] \) satisfying

\[
\text{Des} K = \text{Des} J \cup \{k\} , \quad k \notin \text{Des} J , \quad k - 1 \in \text{Des} J \quad \text{and} \quad k + 1 \notin \text{Des} J \cup \{n\} .
\]

Therefore, Claim 1 yields \( J \rightarrow K \). Thus, Claim 2 yields \( \text{EpK} J = \text{EpK} K \). Claim 4

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\(^{21}\)\textit{Proof}. Assume the contrary. Thus, \( k - 1 \notin \text{Des} J \). Hence, \( k - 1 \notin \text{Des} J \cup \{k\} \) as well (since \( k - 1 \neq k \)). In other words, \( k - 1 \notin A \) (since \( \text{Des} J \cup \{k\} = A \)). Therefore, the element \( k \) of \( A \) satisfies \( k - 1 \notin A \) or \([k, n-1] \subseteq A \). Thus, the definition of \( A_n^0 \) yields \( k \in A_n^0 \). This contradicts \( k \notin A_n^0 \). This contradiction shows that our assumption was false; qed.

\(^{22}\)\textit{Proof}. Assume the contrary. Thus, \( k + 1 \in \text{Des} J \cup \{n\} \). In other words, we have \( k + 1 \in \text{Des} J \) or \( k + 1 = n \). In other words, we are in one of the following two cases:

Case 1: We have \( k + 1 \in \text{Des} J \).

Case 2: We have \( k + 1 = n \).

Let us first consider Case 1. In this case, we have \( k + 1 \in \text{Des} J \). Hence, \( k + 1 \in \text{Des} J \subseteq \text{Des} J \cup \{k\} = A \). If we had \( k + 1 \notin A_n^0 \), then \( k + 1 \) would be a \( q \in A \) satisfying \( q \notin A_n^0 \), this would contradict the fact that \( k \) is the \text{largest} \( q \) (since \( k + 1 \) is larger than \( k \)). Hence, we cannot have \( k + 1 \notin A_n^0 \). Thus, we must have \( k + 1 \in A_n^0 \). In other words, \( k + 1 \) is an element of \( A \) satisfying \( (k + 1) - 1 \notin A \) or \([k + 1, n - 1] \subseteq A \) (by the definition of \( A_n^0 \)). Since \( (k + 1) - 1 \notin A \) is impossible (because \( (k + 1) - 1 = k \in A \)), we thus have \([k + 1, n - 1] \subseteq A \).

Now, \( [k, n - 1] = \{k\} \cup [k + 1, n - 1] \subseteq A \cup A = A \). Thus, the element \( k \) of \( A \) satisfies \( k - 1 \notin A \) or \([k, n - 1] \subseteq A \). In other words, \( k \in A_n^0 \) (by the definition of \( A_n^0 \)). This contradicts \( k \notin A_n^0 \). Thus, we have found a contradiction in Case 1.

Let us now consider Case 2. In this case, we have \( k + 1 = n \). Hence, \( k = n - 1 \), so that \([k, n - 1] = \{k\} \subseteq A \) (since \( k \in A \)). Thus, the element \( k \) of \( A \) satisfies \( k - 1 \notin A \) or \([k, n - 1] \subseteq A \). In other words, \( k \in A_n^0 \) (by the definition of \( A_n^0 \)). This contradicts \( k \notin A_n^0 \). Thus, we have found a contradiction in Case 2.

We have therefore found a contradiction in each of the two Cases 1 and 2. Thus, we always get a contradiction, so our assumption must have been wrong. Qed.
therefore yields \((\text{Des } J)^{\circ}_{n} = (\text{Des } K)^{\circ}_{n} = A_{n}^{\circ}\) (since \(\text{Des } K = A\)). Thus,

\[
\left|\frac{(\text{Des } J) \setminus (\text{Des } J)^{\circ}_{n}}{A \setminus \{k\}} \right| = \left|\frac{A \setminus \{k\} \setminus A_{n}^{\circ}}{(A \setminus A_{n}) \setminus \{k\}}\right| = \left|A \setminus A_{n}^{\circ}\right| - 1 \quad \text{(since } k \in A \setminus A_{n}^{\circ})
\]

< \left|A \setminus A_{n}^{\circ}\right| = \left|(\text{Des } K) \setminus (\text{Des } K)^{\circ}_{n}\right|

(since \(A = \text{Des } K\)). Thus, the induction hypothesis shows that \(\text{Comp } (\text{Des } J)^{\circ}_{n} \rightarrow J\). Combining this with \(J \rightarrow K\), we obtain \(\text{Comp } (\text{Des } J)^{\circ}_{n} \rightarrow K\) (since \(\rightarrow\) is the transitive-and-reflexive closure of the relation \(\rightarrow\)). In light of \((\text{Des } J)^{\circ}_{n} = (\text{Des } K)^{\circ}_{n}\), this rewrites as \(\text{Comp } (\text{Des } K)^{\circ}_{n} \rightarrow K\). Thus, Claim 5 is proven by induction.

Now, let \(K'\) be the \(\mathbb{Q}\)-vector subspace of \(\text{QSym}\) spanned by all differences of the form \(f_{J} - f_{K}\), where \(J\) and \(K\) are two compositions satisfying \(J \rightarrow K\).

**Claim 6:** Let \(J\) and \(K\) be two compositions such that \(J \rightarrow K\). Then, \(f_{J} - f_{K} \in K'\).

**Proof of Claim 6:** We have \(J \rightarrow K\). By the definition of the relation \(\rightarrow\), this means that there exists a sequence \((L_{0}, L_{1}, \ldots, L_{\ell})\) of compositions satisfying \(L_{0} = J\) and \(L_{\ell} = K\) and \(L_{0} \rightarrow L_{1} \rightarrow \cdots \rightarrow L_{\ell}\). Consider this sequence. For each \(i \in \{0, 1, \ldots, \ell - 1\}\), we have \(L_{i} \rightarrow L_{i+1}\) and thus \(f_{L_{i}} - f_{L_{i+1}} \in K'\) (by the definition of \(K'\)). Therefore, \(\sum_{i=0}^{\ell-1} (f_{L_{i}} - f_{L_{i+1}}) \in K'\). In light of

\[
\sum_{i=0}^{\ell-1} (f_{L_{i}} - f_{L_{i+1}}) = f_{L_{0}} - f_{L_{\ell}} \quad \text{(by the telescope principle)}
\]

\[= f_{J} - f_{K} \quad \text{(since } L_{0} = J\text{ and } L_{\ell} = K),
\]

this rewrites as \(f_{J} - f_{K} \in K'\). This proves Claim 6.]

**Claim 7:** We have \(K_{\text{Epk}} \subseteq K'\).

**Proof of Claim 7:** Recall that \(K_{\text{Epk}}\) is the \(\mathbb{Q}\)-vector subspace of \(\text{QSym}\) spanned by all elements of the form \(f_{J} - f_{K}\), where \(J\) and \(K\) are two Epk-equivalent compositions. Thus, it suffices to show that if \(J\) and \(K\) are two Epk-equivalent compositions, then \(f_{J} - f_{K} \in K'\).

So let \(J\) and \(K\) be two Epk-equivalent compositions. We must prove that \(f_{J} - f_{K} \in K'\).

The compositions \(J\) and \(K\) are Epk-equivalent; in other words, they have the same size and satisfy \(\text{Epk } J = \text{Epk } K\). Let \(n = |J| = |K|\). (This is well-defined, since the compositions \(J\) and \(K\) have the same size.)
Claim 4 yields \((\text{Des } J)\circ_n = (\text{Des } K)\circ_n\). But Claim 5 yields Comp \( ((\text{Des } K)\circ_n) \overset{\rightarrow}{\to} K\). Hence, Claim 6 (applied to Comp \( ((\text{Des } K)\circ_n) \)) instead of \( J \) shows that \( F_{\text{Comp}}( (\text{Des } K)\circ_n ) - F_K \in \mathcal{K}' \). The same argument (applied to \( J \) instead of \( K \)) shows that \( F_{\text{Comp}}( (\text{Des } J)\circ_n ) - F_J \in \mathcal{K}' \). Now,

\[
\left( F_{\text{Comp}}( (\text{Des } K)\circ_n ) - F_K \right) - \left( F_{\text{Comp}}( (\text{Des } J)\circ_n ) - F_J \right) = \underbrace{F_{\text{Comp}}( (\text{Des } K)\circ_n ) - F_{\text{Comp}}( (\text{Des } J)\circ_n )}_{= F_{\text{Comp}}( (\text{Des } K)\circ_n ) - F_{\text{Comp}}( (\text{Des } J)\circ_n ) \text{ (since } (\text{Des } J)\circ_n = (\text{Des } K)\circ_n \)} + F_J - F_K
\]

so that

\[
F_J - F_K = \left( F_{\text{Comp}}( (\text{Des } K)\circ_n ) - F_K \right) - \left( F_{\text{Comp}}( (\text{Des } J)\circ_n ) - F_J \right) \in \mathcal{K}' - \mathcal{K}' \subseteq \mathcal{K}'.
\]

This proves Claim 7.

Claim 8: We have \( \mathcal{K}' \subseteq \mathcal{K}_{\text{Epk}} \).

[Proof of Claim 8: Recall that \( \mathcal{K}' \) is the \( \mathbb{Q} \)-vector subspace of \( \text{QSym} \) spanned by all differences of the form \( F_J - F_K \), where \( J \) and \( K \) are two compositions satisfying \( J \to K \). Thus, it suffices to show that if \( J \) and \( K \) are two compositions satisfying \( J \to K \), then \( F_J - F_K \in \mathcal{K}_{\text{Epk}} \).

So let \( J \) and \( K \) be two compositions satisfying \( J \to K \). We must prove that \( F_J - F_K \in \mathcal{K}_{\text{Epk}} \).

We have \( J \to K \) and thus \( J \overset{\rightarrow}{\to} K \) (since \( \overset{\rightarrow}{\to} \) is the transitive-and-reflexive closure of the relation \( \to \)). Hence, Claim 6 shows that \( F_J - F_K \in \mathcal{K}' \). This proves Claim 8.]

Combining Claim 7 and Claim 8, we obtain \( \mathcal{K}_{\text{Epk}} = \mathcal{K}' \). Recalling the definition of \( \mathcal{K}' \), we can rewrite this as follows: \( \mathcal{K}_{\text{Epk}} \) is the \( \mathbb{Q} \)-vector subspace of \( \text{QSym} \) spanned by all differences of the form \( F_J - F_K \), where \( J \) and \( K \) are two compositions satisfying \( J \to K \). This proves Proposition 4.5.

4.3. The M-generating set of \( \mathcal{K}_{\text{Epk}} \)

Another characterization of the ideal \( \mathcal{K}_{\text{Epk}} \) of \( \text{QSym} \) can be obtained using the monomial basis of \( \text{QSym} \). Let us first recall how said basis is defined:
For any composition \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell) \), we let

\[
M_\alpha = \sum_{i_1 < i_2 < \cdots < i_\ell} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}
\]

(where the sum is over all strictly increasing \( \ell \)-tuples \( (i_1, i_2, \ldots, i_\ell) \) of positive integers). This power series \( M_\alpha \) belongs to \( \text{QSym} \). The family \( (M_\alpha)_\alpha \) is a composition is a basis of the \( \text{Q} \)-vector space \( \text{QSym} \); it is called the monomial basis of \( \text{QSym} \).

**Proposition 4.7.** If \( J = (j_1, j_2, \ldots, j_m) \) and \( K \) are two compositions, then we shall write \( J \rightarrow M_K \) if there exists an \( \ell \in \{2, 3, \ldots, m\} \) such that \( j_\ell > 2 \) and \( K = (j_1, j_2, \ldots, j_{\ell-1}, 2, j_\ell - 2, j_{\ell+1}, j_{\ell+2}, \ldots, j_m) \). (In other words, we write \( J \rightarrow M_K \) if \( K \) can be obtained from \( J \) by “splitting” some entry \( j_\ell > 2 \) into two consecutive entries \( 2 \) and \( j_\ell - 2 \), provided that this entry was not the first entry – i.e., we had \( \ell > 1 \) – and that this entry was greater than \( 2 \).)

The ideal \( K_{\text{Epk}} \) of \( \text{QSym} \) is spanned (as a \( \text{Q} \)-vector space) by all sums of the form \( M_J + M_K \), where \( J \) and \( K \) are two compositions satisfying \( J \rightarrow M_K \).

**Example 4.8.** We have \( (2,1,4,4) \rightarrow M_{(2,1,2,2,4)} \), since the composition \( (2,1,2,2,4) \) is obtained from \( (2,1,4,4) \) by splitting the third entry (which is \( 4 > 2 \)) into two consecutive entries \( 2 \) and \( 2 \).

Similarly, \( (2,1,4,4) \rightarrow M_{(2,1,4,2,2)} \) and \( (2,1,5,4) \rightarrow M_{(2,1,2,3,4)} \).

But we do not have \( (3,1) \rightarrow M_{(2,1,1)} \), because splitting the first entry of the composition is not allowed in the definition of the relation \( \rightarrow \).

Two compositions \( J \) and \( K \) satisfying \( J \rightarrow M_K \) must necessarily satisfy \( |J| = |K| \).

Here are all relations \( \rightarrow \) between compositions of size 4:

\[
(1,3) \rightarrow M_{(1,2,1)}
\]

Here are all relations \( \rightarrow \) between compositions of size 5:

\[
(1,4) \rightarrow M_{(1,2,2)},
(1,3,1) \rightarrow M_{(1,2,1,1)},
(1,1,3) \rightarrow M_{(1,1,2,1)},
(2,3) \rightarrow M_{(2,2,1)}.
\]

There are no relations \( \rightarrow \) between compositions of size \( \leq 3 \).
Before we start proving Proposition 4.7, let us recall a basic formula that connects the monomial quasisymmetric functions with the fundamental quasisymmetric functions:

**Proposition 4.9.** Let \( n \in \mathbb{N} \). Let \( \alpha \) be any composition of \( n \). Then,

\[
M_\alpha = \sum_{\beta \text{ is a composition of } n \text{ that refines } \alpha} (-1)^{\ell(\beta) - \ell(\alpha)} F_\beta.
\]

**Proof of Proposition 4.9.** This precisely [GriRei17, (5.2.2)]. (If the numbering shifts: This is the formula in Proposition 5.2.8.)

**Proposition 4.10.** Let \( n \) be a positive integer. Let \( C \) be a subset of \([n - 1]\).

(a) Then,

\[
M_{\text{Comp } C} = \sum_{B \supseteq C} (-1)^{|B \setminus C|} F_{\text{Comp } B}.
\]

(The bound variable \( B \) in this sum and any similar sums is supposed to be a subset of \([n - 1]\); thus, the above sum ranges over all subsets \( B \) of \([n - 1]\) satisfying \( B \supseteq C \).)

(b) Let \( k \in [n - 1] \) be such that \( k \notin C \). Then,

\[
M_{\text{Comp } C} + M_{\text{Comp } (C \cup \{k\})} = \sum_{\substack{B \supseteq C; \\
B \cup \{k\} \notin B \cup \{0\}}} (-1)^{|B \setminus C|} (F_{\text{Comp } B} - F_{\text{Comp } (B \cup \{k - 1\})}).
\]

(c) Let \( k \in [n - 1] \) be such that \( k \notin C \) and \( k - 1 \notin C \cup \{0\} \). Then,

\[
M_{\text{Comp } C} + M_{\text{Comp } (C \cup \{k\})} = \sum_{\substack{B \supseteq C; \\
B \cup \{k\} \notin B \cup \{0\}}} (-1)^{|B \setminus C|} \left(F_{\text{Comp } B} - F_{\text{Comp } (B \cup \{k - 1\})}\right).
\]

**Proof of Proposition 4.10.** (a) Proposition 4.10(a) is the result of applying Proposition 4.9 to \( \alpha = \text{Comp } C \) and using the standard dictionary between compositions of \( n \) and subsets of \([n - 1]\).

(b) Proposition 4.10(a) (applied to \( C \cup \{k\} \) instead of \( C \)) yields

\[
M_{\text{Comp } (C \cup \{k\})} = \sum_{B \supseteq C \cup \{k\}} (-1)^{|B \setminus (C \cup \{k\})|} F_{\text{Comp } B} = \sum_{\substack{B \supseteq C; \\
k \notin B}} (-1)^{|B \setminus C|} F_{\text{Comp } B}
\]

\[
= \sum_{\substack{B \supseteq C; \\
k \notin B}} (-1)^{|B \setminus C|} (\text{since } |B \setminus (C \cup \{k\})| = |B \setminus C| - 1 \text{ (since } k \notin C \text{))}
\]

\[
= \sum_{\substack{B \supseteq C; \\
k \notin B}} (-1)^{|B \setminus C|} F_{\text{Comp } B}.
\]
But Proposition 4.10(a) yields

\[ M_{\text{Comp } C} = \sum_{B \supseteq C} (-1)^{|B \setminus C|} F_{\text{Comp } B} \]

\[ = \sum_{B \supseteq C; \ k \in B} (-1)^{|B \setminus C|} F_{\text{Comp } B} + \sum_{B \supseteq C; \ k \notin B} (-1)^{|B \setminus C|} F_{\text{Comp } B}. \]

Adding (14) to this equality, we obtain

\[ M_{\text{Comp } C} + M_{\text{Comp } (C \cup \{k\})} \]

\[ = \sum_{B \supseteq C; \ k \in B} (-1)^{|B \setminus C|} F_{\text{Comp } B} + \sum_{B \supseteq C; \ k \notin B} (-1)^{|B \setminus C|} F_{\text{Comp } B} + \left( - \sum_{B \supseteq C; \ k \in B} (-1)^{|B \setminus C|} F_{\text{Comp } B} \right) \]

\[ = \sum_{B \supseteq C; \ k \notin B} (-1)^{|B \setminus C|} F_{\text{Comp } B}. \]

This proves Proposition 4.10(b).

(c) We have \( k - 1 \notin C \cup \{0\} \). Thus, \( k - 1 \notin C \) and \( k - 1 \neq 0 \). From \( k - 1 \neq 0 \), we obtain \( k - 1 \in [n - 1] \).

The map

\[ \{ B \subseteq [n - 1] \ | \ B \supseteq C \text{ and } k \notin B \} \]

\[ \rightarrow \{ B \subseteq [n - 1] \ | \ B \supseteq C \text{ and } k \notin B \text{ and } k - 1 \in B \} \]

sending each \( B \) to \( B \cup \{k - 1\} \) is a bijection (this is easy to check using the facts that \( k - 1 \notin C \) and \( k - 1 \in [n - 1] \)). We shall denote this map by \( \Phi \).
Proposition 4.10(b) yields

\[ \sum_{B \supseteq C; \ k \notin B; \ k-1 \notin B} (-1)^{|B \setminus C|} F_{\text{Comp} B} + \sum_{B \supseteq C; \ k \notin B; \ k-1 \notin B} (-1)^{|B \setminus C|} F_{\text{Comp} B} = \sum_{B \supseteq C; \ k \notin B; \ k-1 \notin B} (-1)^{|(B \cup \{k-1\}) \setminus C|} F_{\text{Comp} (B \cup \{k-1\})}\]

(here, we have substituted \( B \cup \{k-1\} \) for \( B \) in the sum (since the map \( \Phi \) is a bijection))

\[ = \sum_{B \supseteq C; \ k \notin B; \ k-1 \notin B} (-1)^{|B \setminus C|} F_{\text{Comp} B} + \sum_{B \supseteq C; \ k \notin B; \ k-1 \notin B} (-1)^{|B \setminus C|+1} F_{\text{Comp} (B \cup \{k-1\})} = (-1)^{|B \setminus C|+1} F_{\text{Comp} (B \cup \{k-1\})} \]

\[ = \sum_{B \supseteq C; \ k \notin B; \ k-1 \notin B} (-1)^{|B \setminus C|} F_{\text{Comp} B} - \sum_{B \supseteq C; \ k \notin B; \ k-1 \notin B} (-1)^{|B \setminus C|} F_{\text{Comp} (B \cup \{k-1\})} \]

\[ = \sum_{B \supseteq C; \ k \notin B; \ k-1 \notin B} (-1)^{|B \setminus C|} \left( F_{\text{Comp} B} - F_{\text{Comp} (B \cup \{k-1\})} \right). \]

This proves Proposition 4.10(c).  

**Proof of Proposition 4.7** We shall use the notation \( \langle f_i \mid i \in I \rangle \) for the \( \mathbb{Q} \)-linear span of a family \( \langle f_i \rangle_{i \in I} \) of elements of a \( \mathbb{Q} \)-vector space.

Define a \( \mathbb{Q} \)-vector subspace \( \mathcal{M} \) of \( \text{QSym} \) by

\[ \mathcal{M} = \left\langle M_J + M_K \mid J \text{ and } K \text{ are compositions satisfying } J \rightarrow M \right\rangle. \]

Then, our goal is to prove that \( \mathcal{K}^{\text{Epk}} = \mathcal{M} \).

We have

\[ \mathcal{M} = \left\langle M_J + M_K \mid J \text{ and } K \text{ are compositions satisfying } J \rightarrow M \right\rangle \]

\[ = \sum_{n \in \mathbb{N}} \left\langle M_J + M_K \mid J \text{ and } K \text{ are compositions of } n \text{ satisfying } J \rightarrow K \right\rangle \]

\[ \text{47} \]
(because if \( J \) and \( K \) are two compositions satisfying \( J \to M K \), then \( J \) and \( K \) have the same size).

Consider the binary relation \( \to \) defined in Proposition 4.5. Then, Proposition 4.5 yields
\[
K_{Epk} = \langle F_J - F_K \mid J \text{ and } K \text{ are compositions satisfying } J \to K \rangle
= \sum_{n \in \mathbb{N}} \langle F_J - F_K \mid J \text{ and } K \text{ are compositions of } n \text{ satisfying } J \to K \rangle
\]
(because if \( J \) and \( K \) are two compositions satisfying \( J \to K \), then \( J \) and \( K \) have the same size).

Now, fix \( n \in \mathbb{N} \). Let \( \Omega \) be the set of all pairs \((C, k)\) in which \( C \) is a subset of \([n - 1]\) and \( k \) is an element of \([n - 1]\) satisfying \( k \notin C, k - 1 \in C \) and \( k + 1 \notin C \cup \{n\}\).

For every \((C, k)\) \(\in \Omega\), we define two elements \(m_{C, k}\) and \(f_{C, k}\) of \(\mathbb{Q} \text{Sym}\) by
\[
\begin{align*}
m_{C, k} &= M_{\text{Comp} C} + M_{\text{Comp}(C \cup \{k + 1\})} \quad \text{and} \\
f_{C, k} &= F_{\text{Comp} C} - F_{\text{Comp}(C \cup \{k\})}.
\end{align*}
\]

We have the following:

**Claim 1:** We have
\[
\langle M_J + M_K \mid J \text{ and } K \text{ are compositions of } n \text{ satisfying } J \to M K \rangle
= \langle m_{C, k} \mid (C, k) \in \Omega \rangle.
\]

**Proof of Claim 1:** It is easy to see that two subsets \( C \) and \( D \) of \([n - 1]\) satisfy \( \text{Comp} C \to M \text{Comp} D \) if and only if there exists some \( k \in [n - 1] \) satisfying \( D = \]

\(\text{These two elements are well-defined, because both } C \cup \{k\} \text{ and } C \cup \{k + 1\} \text{ are subsets of } [n - 1] \text{ (since } k + 1 \notin C \cup \{n\} \text{ shows that } k + 1 \notin n).\)
\[ C \cup \{ k + 1 \}, k \notin C, k - 1 \in C \text{ and } k + 1 \notin C \cup \{ n \} . \]

Thus,

\[
\langle M \text{Comp}_C + M \text{Comp}_D \mid C \text{ and } D \text{ are subsets of } [n - 1] \text{ satisfying } \text{Comp}_C \rightarrow M \text{Comp}_D \rangle
= \langle M \text{Comp}_C + M \text{Comp}_D \mid C \text{ and } D \text{ are subsets of } [n - 1] \text{ such that there exists some } k \in [n - 1] \text{ satisfying } D = C \cup \{ k + 1 \}, k \notin C, k - 1 \in C \text{ and } k + 1 \notin C \cup \{ n \} \rangle
= \langle M \text{Comp}_C + M \text{Comp}(C \cup \{ k + 1 \}) \mid C \subseteq [n - 1] \text{ and } k \in [n - 1] \text{ are such that } k \notin C, k - 1 \in C \text{ and } k + 1 \notin C \cup \{ n \} \rangle
= \langle M \text{Comp}_C + M \text{Comp}(C \cup \{ k + 1 \}) \mid (C, k) \in \Omega \rangle
= \langle m_{C, k} \mid (C, k) \in \Omega \rangle .
\]

Now, recall that \( \text{Comp} \) is a bijection between the subsets of \([n - 1]\) and the compositions of \( n \). Hence,

\[
\langle M_J + M_K \mid J \text{ and } K \text{ are compositions of } n \text{ satisfying } J \rightarrow M K \rangle
= \langle M \text{Comp}_C + M \text{Comp}_D \mid C \text{ and } D \text{ are subsets of } [n - 1] \text{ satisfying } \text{Comp}_C \rightarrow M \text{Comp}_D \rangle
= \langle m_{C, k} \mid (C, k) \in \Omega \rangle .
\]

This proves Claim 1.]

**Claim 2:** We have

\[
\langle F_J - F_K \mid J \text{ and } K \text{ are compositions of } n \text{ satisfying } J \rightarrow K \rangle
= \langle f_{C, k} \mid (C, k) \in \Omega \rangle .
\]

[Proof of Claim 2: It is easy to see that two subsets \( C \) and \( D \) of \([n - 1]\) satisfy \( \text{Comp}_C \rightarrow \text{Comp}_D \) if and only if there exists some \( k \in [n - 1] \) satisfying \( D =

\[\text{To prove this, recall that “splitting” an entry of a composition } J \text{ into two consecutive entries (summing up to the original entry) is always tantamount to adding a new element to } \text{Des } J. \text{ It suffices to show that the conditions under which an entry of a composition } J \text{ can be split in the definition of the relation } \rightarrow_M \text{ are precisely the conditions } k \notin C, k - 1 \in C \text{ and } k + 1 \notin C \cup \{ n \} \text{ on } C = \text{Des } J. \text{ This is straightforward.}\]
Thus,
\[
\langle F_{\text{Comp} C} - F_{\text{Comp} D} \mid C \text{ and } D \text{ are subsets of } [n-1] \text{ satisfying } \text{Comp } C \rightarrow \text{Comp } D \rangle
\]
\[
= \langle F_{\text{Comp} C} - F_{\text{Comp} D} \mid C \text{ and } D \text{ are subsets of } [n-1] \text{ such that there exists some } k \in [n-1] \text{ satisfying } D = C \cup \{k\}, k \notin C, k-1 \in C \text{ and } k+1 \notin C \cup \{n\} \rangle
\]
\[
= \langle F_{\text{Comp} C} - F_{\text{Comp}(C \cup \{k\})} \mid C \subseteq [n-1] \text{ and } k \in [n-1] \text{ are such that } k \notin C, k-1 \in C \text{ and } k+1 \notin C \cup \{n\} \rangle
\]
\[
= \left\langle F_{\text{Comp} C} - F_{\text{Comp}(C \cup \{k\})} \mid (C,k) \in \Omega \right\rangle
\]
(by the definition of \(\Omega\))
\[
= \{f_{C,k} \mid (C,k) \in \Omega \}.
\]

Now, recall that Comp is a bijection between the subsets of \([n-1]\) and the compositions of \(n\). Hence,
\[
\langle F_J - F_K \mid J \text{ and } K \text{ are compositions of } n \text{ satisfying } J \rightarrow K \rangle
\]
\[
= \langle F_{\text{Comp} C} - F_{\text{Comp} D} \mid C \text{ and } D \text{ are subsets of } [n-1] \text{ satisfying } \text{Comp } C \rightarrow \text{Comp } D \rangle
\]
\[
= \{f_{C,k} \mid (C,k) \in \Omega \}.
\]

This proves Claim 2.

We define a partial order on the set \(\Omega\) by setting
\[
(B,k) \geq (C,\ell) \quad \text{ if and only if } \quad (k = \ell \text{ and } B \supseteq C).
\]
Thus, \(\Omega\) is a finite poset.

Claim 3: For every \((C,\ell) \in \Omega\), we have
\[
m_{C,\ell} = \sum_{(B,k) \in \Omega; (B,k) \geq (C,\ell)} (-1)^{|B \setminus C|} f_{B,k}.
\]

25To prove this, recall that “splitting” an entry of a composition \(J\) into two consecutive entries (summing up to the original entry) is always tantamount to adding a new element to Des \(J\). It suffices to show that the conditions under which an entry of a composition \(J\) can be split in the definition of the relation \(\rightarrow\) are precisely the conditions \(k \notin C, k-1 \in C \text{ and } k+1 \notin C \cup \{n\}\) on \(C = \text{Des } J\). This is straightforward.
Proof of Claim 3: Let \((C, \ell) \in \Omega\). Thus, \(C\) is a subset of \([n - 1]\) and \(\ell\) is an element of \([n - 1]\) satisfying \(\ell \not\in C\), \(\ell - 1 \in C\) and \(\ell + 1 \not\in C \cup \{n\}\). From \(\ell + 1 \not\in C \cup \{n\}\), we obtain \(\ell + 1 \not\in C\) and \(\ell + 1 \neq n\). From \(\ell + 1 \neq n\), we obtain \(\ell + 1 \in [n - 1]\). Also, \((\ell + 1) - 1 = \ell \not\in C \cup \{0\}\) (since \(\ell \not\in C\) and \(\ell \neq 0\)). Thus, Proposition 4.10\((c)\) (applied to \(k = \ell + 1\)) yields \(^{26}\)

\[
M_{\text{Comp}} C + M_{\text{Comp}} (C \cup (\ell + 1)) = \sum_{B \supseteq C; \ell + 1 \not\in B; \ell \not\in B} (-1)^{|B| - |C|} \left( F_{\text{Comp}} B - F_{\text{Comp}} (B \cup \{\ell\}) \right).
\]

But every \(B \subseteq [n - 1]\) satisfying \(B \supseteq C\) must satisfy \(\ell - 1 \in B\) (since \(\ell - 1 \in C \subseteq B\)). Hence, we can manipulate summation signs as follows:

\[
\sum_{B \supseteq C; \ell + 1 \not\in B; \ell \not\in B} = \sum_{B \supseteq C; \ell + 1 \not\in B; \ell \not\in B; \ell - 1 \in B} = \sum_{B \supseteq C; \ell \not\in B; \ell - 1 \in B; \ell + 1 \not\in B} = \sum_{B \supseteq C; \ell \not\in B; \ell + 1 \not\in B \cup \{n\}; \ell - 1 \in B}
\]

(since \(\ell + 1 \not\in B\) is equivalent to \(\ell + 1 \not\in B \cup \{n\}\))

(because \(\ell + 1 \neq n\))

(since the condition \((\ell \not\in B, \ell - 1 \in B\) and \(\ell + 1 \not\in B \cup \{n\}\)) on a subset \(B\) of \([n - 1]\) is equivalent to \((B, \ell) \in \Omega\))

(by the definition of \(\Omega\))

\[(17)\]

(since the condition \((k = \ell\) and \(B \supseteq C\)) on a \((B, k) \in \Omega\) is equivalent to \((B, k) \geq (C, \ell)\) (by the definition of the partial order on \(\Omega\))).

---

\(^{26}\)Here and in the following, the bound variable \(B\) in a sum always is understood to be a subset of \([n - 1]\).
Now, the definition of \( m_{C,\ell} \) yields
\[
m_{C,\ell} = M_{\text{Comp}} + M_{\text{Comp}(C \cup \{\ell+1\})} = \sum_{B \supseteq C; \ell+1 \notin B; \ell \notin B} (-1)^{|B \setminus C|} \left( F_{\text{Comp}} B - F_{\text{Comp}} (B \cup \{\ell\}) \right)
\]
\[
= \sum_{(B,k) \in \Omega; (B,k) \geq (C,\ell)} (-1)^{|B \setminus C|} \left( F_{\text{Comp}} B - \frac{F_{\text{Comp}} (B \cup \{\ell\})}{F_{\text{Comp}} (B \cup \{k\})} \right)
\]
\[
= \sum_{(B,k) \in \Omega; (B,k) \geq (C,\ell)} (-1)^{|B \setminus C|} f_{B,k}.
\]

This proves Claim 3.

Now, Claim 3 shows that the family \((m_{C,k})_{(C,k) \in \Omega}\) expands triangularly with respect to the family \((f_{C,k})_{(C,k) \in \Omega}\) with respect to the poset structure on \( \Omega \). Moreover, the expansion is unitorientable (because if \((B,k) = (C,\ell)\), then \( B = C \) and thus \((-1)^{|B \setminus C|} = (-1)^{|C \setminus C|} = (-1)^0 = 1\) and thus invertibly triangular (this means that the diagonal entries are invertible). Therefore, by a standard fact from linear algebra (see, e.g., [GriRei17, Corollary 11.1.19 (b)]), we conclude that the span of the family \((m_{C,k})_{(C,k) \in \Omega}\) equals the span of the family \((f_{C,k})_{(C,k) \in \Omega}\).

In other words,
\[
\langle m_{C,k} \mid (C,k) \in \Omega \rangle = \langle f_{C,k} \mid (C,k) \in \Omega \rangle.
\]

Now, Claim 1 yields
\[
\left\langle M_J + M_K \mid J \text{ and } K \text{ are compositions of } n \text{ satisfying } J \to K \right\rangle
\]= \langle m_{C,k} \mid (C,k) \in \Omega \rangle = \langle f_{C,k} \mid (C,k) \in \Omega \rangle
\]= \langle F_J - F_K \mid J \text{ and } K \text{ are compositions of } n \text{ satisfying } J \to K \rangle
(by Claim 2).

Now, forget that we fixed \( n \). We thus have proven that

\[
\left\langle M_J + M_K \mid J \text{ and } K \text{ are compositions of } n \text{ satisfying } J \looparrowright_M K \right\rangle = \left\langle F_J - F_K \mid J \text{ and } K \text{ are compositions of } n \text{ satisfying } J \looparrowright_M K \right\rangle
\]

for each \( n \in \mathbb{N} \). Thus,

\[
\sum_{n \in \mathbb{N}} \left\langle M_J + M_K \mid J \text{ and } K \text{ are compositions of } n \text{ satisfying } J \looparrowright_M K \right\rangle = \sum_{n \in \mathbb{N}} \left\langle F_J - F_K \mid J \text{ and } K \text{ are compositions of } n \text{ satisfying } J \looparrowright_M K \right\rangle.
\]

In light of

\[
\mathcal{K}_{\text{Epk}} = \sum_{n \in \mathbb{N}} \left\langle F_J - F_K \mid J \text{ and } K \text{ are compositions of } n \text{ satisfying } J \looparrowright_M K \right\rangle
\]

and

\[
\mathcal{M} = \sum_{n \in \mathbb{N}} \left\langle M_J + M_K \mid J \text{ and } K \text{ are compositions of } n \text{ satisfying } J \looparrowright_M K \right\rangle,
\]

this rewrites as \( \mathcal{M} = \mathcal{K}_{\text{Epk}} \). In other words, \( \mathcal{K}_{\text{Epk}} = \mathcal{M} \). This proves Proposition 4.7.

---

**Question 4.11.** It is worth analyzing the kernels of other known descent statistics (shuffle-compatible or not). We have seen above that \( \mathcal{K}_{\text{Epk}} \) is a “binomial subspace” of \( \text{QSym} \) in the fundamental basis (i.e., it can be spanned by elements of the form \( \lambda F_J + \mu F_K \) for \( J, K \in \text{Comp} \)) and a “binomial subspace” of \( \text{QSym} \) in the monomial basis (i.e., it can be spanned by elements of the form \( \lambda M_J + \mu M_K \) for \( J, K \in \text{Comp} \)). What other descent statistics have this property?

---

**4.4. Appendix: Proof of Proposition 3.9 and Theorem 3.10**

Let us now give proofs of Proposition 3.9 and Theorem 3.10 which we have promised above. We will mostly rely on Lemma 4.2 and on Proposition 3.7.

For the rest of Subsection 4.4, we shall make the following conventions:

**Convention 4.12.** Let \( \text{st} \) be a permutation statistic. For each permutation \( \pi \), let \([\pi]_{\text{st}}\) denote the st-equivalence class of \( \pi \). Let \( \mathcal{A}_{\text{st}} \) be the free \( \mathbb{Q} \)-vector space whose basis is the set of all st-equivalence classes of permutations. (This is well-defined whether or not \( \text{st} \) is shuffle-compatible.)
A magmatic algebra shall mean a $\mathbb{Q}$-vector space equipped with a binary operation which is written as multiplication (i.e., we write $ab$ for the image of a pair $(a,b)$ under this operation), but is not required to be associative (or have a unity). An (actual, i.e., associative unital) algebra is thus a magmatic algebra whose multiplication is associative and has a unity. In particular, any actual algebra is a magmatic algebra. A magmatic algebra homomorphism is a $\mathbb{Q}$-linear map between two magmatic algebras that preserves the multiplication.

We make $A_{st}$ into a magmatic algebra by setting
\[
[\pi]_{st} [\sigma]_{st} = \sum_{\tau \in S(\pi, \sigma)} [\tau]_{st}
\]
for any two disjoint permutations $\pi$ and $\sigma$. This is well-defined, because the right-hand side of (18) depends only on the st-equivalence classes $[\pi]_{st}$ and $[\sigma]_{st}$ rather than on the permutations $\pi$ and $\sigma$ themselves (this is because st is shuffle-compatible).

Define a $\mathbb{Q}$-linear map $p : \text{QSym} \to A_{st}$ by requiring that
\[
p(F_L) = [\pi]_{st} \quad \text{for every composition } L \text{ and every permutation } \pi \text{ with } \text{Comp } \pi = L.
\]
This is well-defined, because for any given composition $L$, any two permutations $\pi$ with $\text{Comp } \pi = L$ will have the same st-equivalence class $[\pi]_{st}$ (since st is a descent statistic).

Thus, each permutation $\pi$ satisfies
\[
p(F_{\text{Comp } \pi}) = [\pi]_{st}
\]
and therefore $[\pi]_{st} = p(F_{\text{Comp } \pi}) \in p(\text{QSym})$. Hence, $A_{st} \subseteq p(\text{QSym})$ (since the st-equivalence classes $[\pi]_{st}$ form a basis of $A_{st}$). Consequently, the map $p$ is surjective.

Moreover, we have
\[
p(ab) = p(a) p(b) \quad \text{for all } a, b \in \text{QSym}.
\]

[Proof of (20): Let $a, b \in \text{QSym}$. We must prove the equality (20). Since this equality is $\mathbb{Q}$-linear in each of $a$ and $b$, we WLOG assume that $a$ and $b$ belong to the fundamental basis of QSym. That is, $a = F_J$ and $b = F_K$ for two compositions $J$ and $K$. Consider these $J$ and $K$. Fix any two disjoint permutations $\pi$ and $\sigma$ such that $\text{Comp } \pi = J$ and $\text{Comp } \sigma = K$. (Such $\pi$ and $\sigma$ are easy to find.) The
definition of $p$ thus yields $p(F_J) = [\pi]_{st}$ and $p(F_K) = [\sigma]_{st}$. Hence,

$$p\left(\begin{array}{c} a \\ \rightarrow F_J \end{array}\right)p\left(\begin{array}{c} b \\ \rightarrow F_K \end{array}\right) = p(F_J)p(F_K) = [\pi]_{st}[\sigma]_{st}$$

\[= \sum_{\tau \in S(\pi, \sigma)} [\tau]_{st} \quad \text{(by (18))}\]

\[= \sum_{\chi \in S(\pi, \sigma)} [\chi]_{st} \quad \text{(21)}\]

(here, we have renamed the summation index $\tau$ as $\chi$). On the other hand, $a = F_J = F_{\text{Comp}} \pi$ (since $J = \text{Comp} \pi$) and $b = F_{\text{Comp}} \sigma$ (similarly); multiplying these equalities, we get

$$ab = F_{\text{Comp}} \pi F_{\text{Comp}} \sigma = \sum_{\chi \in S(\pi, \sigma)} F_{\text{Comp}} \chi$$

(by Proposition 3.7). Applying the map $p$ to this equality, we find

$$p(ab) = p\left(\sum_{\chi \in S(\pi, \sigma)} F_{\text{Comp}} \chi\right) = \sum_{\chi \in S(\pi, \sigma)} p(F_{\text{Comp}} \chi) = \sum_{\chi \in S(\pi, \sigma)} [\chi]_{st}$$

\[= p(a)p(b) \quad \text{(by (21))} .\]

This proves (20).

The equality (20) shows that $p$ is a magmatic algebra homomorphism (since $p$ is $\mathbb{Q}$-linear). Thus, using the surjectivity of $p$, we can easily see that the magmatic algebra $A_{st}$ is associative\footnote{\textit{Proof.} Let $u, v, w \in A_{st}$. We must show that $(uv)w = u(vw)$.

There exist $a, b, c \in \mathbb{Q} \text{Sym}$ such that $u = p(a)$, $v = p(b)$ and $w = p(c)$ (since $p$ is surjective).

Fix such $a, b, c$. Since $\mathbb{Q} \text{Sym}$ is an actual (i.e., associative unital) algebra, we have

$$p(abc) = p((ab)c) = p(ab)p(c) \quad \text{(since $p$ is a magmatic algebra homomorphism)}$$

\[= p(a)p(b)p(c) \quad \text{(since $p$ is a magmatic algebra homomorphism)} \]

\[= (p(a)p(b))p(c) = (uv)v \quad \text{(by (19))} \]

\[= (uv)w \quad \text{(by (18))} \]

\[= u(vw) \quad \text{(by (18))} \].\]

Thus, $u(vw) = (uv)w$. This proves (20).

A similar argument shows that $p(abc) = u(vw)$. Thus, $(uv)w = p(abc) = u(vw)$, qed.
is well-defined and associative, and turns \( \mathcal{A}_{\text{st}} \) into a \( \mathbb{Q} \)-algebra whose unity is the \( \text{st} \)-equivalence class of the empty permutation. This proves Proposition 3.9 (a).

(b) The map \( p : \text{QSym} \to \mathcal{A}_{\text{st}} \) is \( \mathbb{Q} \)-linear and respects multiplication (by (20)). Moreover, it sends the unity of \( \text{QSym} \) to the unity of the algebra \( \mathcal{A}_{\text{st}} \) 28 Thus, \( p \) is a \( \mathbb{Q} \)-algebra homomorphism. Moreover, recall that \( p \) is surjective and satisfies \( p \left( F_{\text{Comp} \pi} \right) = \left[ \pi \right]_{\text{st}} \) for every permutation \( \pi \). Hence, there is a surjective \( \mathbb{Q} \)-algebra homomorphism \( p_{\text{st}} : \text{QSym} \to \mathcal{A}_{\text{st}} \) that satisfies

\[
p_{\text{st}} \left( F_{\text{Comp} \pi} \right) = \left[ \pi \right]_{\text{st}} \quad \text{for every permutation } \pi
\]

(namely, \( p_{\text{st}} = p \)). This proves Proposition 3.9 (b).

\( \square \)

Proof of Theorem 3.10 (a) \( \implies \): Assume that \( \text{st} \) is shuffle-compatible. Proposition 3.9 (b) shows that there is a surjective \( \mathbb{Q} \)-algebra homomorphism \( p_{\text{st}} : \text{QSym} \to \mathcal{A}_{\text{st}} \) that satisfies

\[
p_{\text{st}} \left( F_{\text{Comp} \pi} \right) = \left[ \pi \right]_{\text{st}} \quad \text{for every permutation } \pi.
\]

(22)

Consider this \( p_{\text{st}} \).

If \( \alpha \) is an \( \text{st} \)-equivalence class of compositions, then we let \( u_\alpha \) denote the \( \text{st} \)-equivalence class \( \left[ \pi \right]_{\text{st}} \) of all permutations \( \pi \) whose descent composition \( \text{Comp} \pi \) belongs to \( \alpha \). (This is indeed a well-defined \( \text{st} \)-equivalence class, because \( \text{st} \) is a descent statistic.) This establishes a bijection between the \( \text{st} \)-equivalence classes of compositions and the \( \text{st} \)-equivalence classes of compositions. Thus, the family \( (u_\alpha) \) (indexed by \( \text{st} \)-equivalence classes \( \alpha \) of compositions) is just a reindexing of the basis of \( \mathcal{A}_{\text{st}} \) consisting of the \( \text{st} \)-equivalence classes \( \left[ \pi \right]_{\text{st}} \) of permutations. Consequently, this family is a basis of the \( \mathbb{Q} \)-vector space \( \mathcal{A}_{\text{st}} \). Moreover, \( p_{\text{st}} \) is a \( \mathbb{Q} \)-algebra homomorphism \( \text{QSym} \to \mathcal{A}_{\text{st}} \) with the property that whenever \( \alpha \) is an \( \text{st} \)-equivalence class of compositions, we have

\[
p_{\text{st}} \left( F_L \right) = u_\alpha \quad \text{for each } L \in \alpha.
\]

(Indeed, this follows from applying (22) to any permutation \( \pi \) satisfying \( \text{Comp} \pi = L \).)

Thus, there exist a \( \mathbb{Q} \)-algebra \( A \) (namely, \( A = \mathcal{A}_{\text{st}} \)) with basis \( (u_\alpha) \) (indexed by \( \text{st} \)-equivalence classes \( \alpha \) of compositions) and a \( \mathbb{Q} \)-algebra homomorphism \( \phi_{\text{st}} : \text{QSym} \to A \) (namely, \( \phi_{\text{st}} = p_{\text{st}} \)) with the property that whenever \( \alpha \) is an \( \text{st} \)-equivalence class of compositions, we have

\[
\phi_{\text{st}} \left( F_L \right) = u_\alpha \quad \text{for each } L \in \alpha.
\]

\( \text{Proof.} \) The unity of \( \text{QSym} \) is \( 1 = F_{()} \), where \( () \) denotes the empty composition. Now, let \( \emptyset \) denote the empty permutation. Then, the \( \text{st} \)-equivalence class \( \left[ \emptyset \right]_{\text{st}} \) is the unity of the algebra \( \mathcal{A}_{\text{st}} \). But the empty permutation \( \emptyset \) has descent composition \( \text{Comp} \emptyset = () \). Hence, the definition of \( p \) yields \( p \left( F_{()} \right) = \left[ \emptyset \right]_{\text{st}} \). In view of what we just said, this equality says that \( p \) sends the unity of \( \text{QSym} \) to the unity of the algebra \( \mathcal{A}_{\text{st}} \).
This proves the \(\implies\) direction of Theorem 3.10 (a).

\(\iff\): Assume that there exist a \(\mathbb{Q}\)-algebra \(A\) with basis \((u_\alpha)\) (indexed by st-equivalence classes \(\alpha\) of compositions) and a \(\mathbb{Q}\)-algebra homomorphism \(\phi_{st}: \text{QSym} \to A\) with the property that whenever \(\alpha\) is an st-equivalence class of compositions, we have

\[
\phi_{st}(F_L) = u_\alpha \quad \text{for each } L \in \alpha.
\]

Consider this \(A\), this \((u_\alpha)\) and this \(\phi_{st}\). Lemma 4.2 shows that \(\text{Ker}(\phi_{st}) = \mathcal{K}_{st}\). But \(\text{Ker}(\phi_{st})\) is an ideal of QSym (since \(\phi_{st}\) is a \(\mathbb{Q}\)-algebra homomorphism). In other words, \(\mathcal{K}_{st}\) is an ideal of QSym (since \(\text{Ker}(\phi_{st}) = \mathcal{K}_{st}\)).

Now, consider any two disjoint permutations \(\pi\) and \(\sigma\). Also, consider two further disjoint permutations \(\pi'\) and \(\sigma'\) satisfying \(\text{st}(\pi) = \text{st}(\pi')\), \(\text{st}(\sigma) = \text{st}(\sigma')\), \(|\pi| = |\pi'|\) and \(|\sigma| = |\sigma'|\). We shall show that \(\{\text{st}(\tau) \mid \tau \in S(\pi, \sigma)\} = \{\text{st}(\tau) \mid \tau \in S(\pi', \sigma')\}\) as multisets. This will show that the multiset \(\{\text{st}(\tau) \mid \tau \in S(\pi, \sigma)\}\) depends only on \(\text{st}(\pi), \text{st}(\sigma), |\pi|\) and \(|\sigma|\).

From \(\text{st}(\pi) = \text{st}(\pi')\) and \(|\pi| = |\pi'|\), we conclude that \(\pi\) and \(\pi'\) are st-equivalent. In other words, \(\text{Comp}_\pi\) and \(\text{Comp}_\pi\) are st-equivalent. Hence, \(F_{\text{Comp}_\pi} - F_{\text{Comp}_\pi'} \in \mathcal{K}_{st}\) (by the definition of \(\mathcal{K}_{st}\)), so that \(F_{\text{Comp}_\pi} \equiv F_{\text{Comp}_\pi'} \mod \mathcal{K}_{st}\).

Similarly, \(F_{\text{Comp}_\sigma} \equiv F_{\text{Comp}_\sigma'} \mod \mathcal{K}_{st}\). These two congruences, combined, yield \(F_{\text{Comp}_\pi} F_{\text{Comp}_\sigma} \equiv F_{\text{Comp}_\pi'} F_{\text{Comp}_\sigma'} \mod \mathcal{K}_{st}\), because \(\mathcal{K}_{st}\) is an ideal of QSym.

Let \(X\) be the codomain of the map \(\text{st}\). Let \(\mathbb{Q}[X]\) be the free \(\mathbb{Q}\)-vector space with basis \((|x|)_{x \in X}\). Then, we can define a \(\mathbb{Q}\)-linear map \(\text{st}: \text{QSym} \to \mathbb{Q}[X], F_j \mapsto [\text{st} f]\). This map \(\text{st}\) sends each of the generators of \(\mathcal{K}_{st}\) to 0 (by the definition of \(\mathcal{K}_{st}\)), and therefore sends the whole \(\mathcal{K}_{st}\) to 0. In other words, \(\text{st}(\mathcal{K}_{st}) = 0\).

We have \(F_{\text{Comp}_\pi} F_{\text{Comp}_\sigma} \equiv F_{\text{Comp}_\pi'} F_{\text{Comp}_\sigma'} \mod \mathcal{K}_{st}\) and thus

\[
\text{st} \left( F_{\text{Comp}_\pi} F_{\text{Comp}_\sigma} \right) = \text{st} \left( F_{\text{Comp}_\pi'} F_{\text{Comp}_\sigma'} \right) \quad (23)
\]

(since \(\text{st}(\mathcal{K}_{st}) = 0\)). But Proposition 3.7 yields

\[
F_{\text{Comp}_\pi} F_{\text{Comp}_\sigma} = \sum_{\chi \in S(\pi, \sigma)} F_{\text{Comp}_\chi}.
\]

Applying the map \(\text{st}\) to both sides of this equality, we find

\[
\text{st} \left( F_{\text{Comp}_\pi} F_{\text{Comp}_\sigma} \right) = \text{st} \left( \sum_{\chi \in S(\pi, \sigma)} F_{\text{Comp}_\chi} \right) = \sum_{\chi \in S(\pi, \sigma)} \text{st} \left( F_{\text{Comp}_\chi} \right) = \sum_{\chi \in S(\pi, \sigma)} [\text{st} \chi].
\]

Similarly,

\[
\text{st} \left( F_{\text{Comp}_\pi'} F_{\text{Comp}_\sigma'} \right) = \sum_{\chi \in S(\pi', \sigma')} [\text{st} \chi].
\]
But the left-hand sides of the last two equalities are equal (because of (23)); therefore, the right-hand sides must be equal as well. In other words,

$$\sum_{\chi \in S(\pi, \sigma')} \left[ st \chi \right] = \sum_{\chi \in S(\pi', \sigma')} \left[ st \chi \right].$$

This shows exactly that \( \{ st (\chi) \mid \chi \in S(\pi, \sigma) \} = \{ st (\chi) \mid \chi \in S(\pi', \sigma') \} \) as multisets. In other words, \( \{ st (\tau) \mid \tau \in S(\pi, \sigma) \} = \{ st (\tau) \mid \tau \in S(\pi', \sigma') \} \) as multisets. Thus, we have proven that the multiset \( \{ st (\chi) \mid \chi \in S(\pi, \sigma) \} \) depends only on \( st (\pi), st (\sigma), |\pi| \) and \(|\sigma|\). Hence, the statistic \( st \) is shuffle-compatible. This proves the \( \iff \) direction of Theorem 3.10 (a).

(b) Proposition 3.9 (b) shows that there is a surjective \( \mathbb{Q} \)-algebra homomorphism \( p_{st} : \text{QSym} \to A_{st} \) that satisfies

$$p_{st} (F_{\text{Comp}} \pi) = [\pi]_{st} \quad \text{for every permutation } \pi. \quad (24)$$

Consider this \( p_{st} \).

Let \( \gamma \) be the \( \mathbb{Q} \)-linear map

$$A_{st} \to A, \quad [\pi]_{st} \mapsto u_{\alpha},$$

where \( \alpha \) is the \( st \)-equivalence class of the composition \( \text{Comp} \pi \). This map \( \gamma \) is clearly well-defined (since the \( st \)-equivalence classes \( [\pi]_{st} \) form a basis of \( A_{st} \), and since the \( st \)-equivalence class of the composition \( \text{Comp} \pi \) depends only on the \( st \)-equivalence class \( [\pi]_{st} \) and not on the permutation \( \pi \) itself). Moreover, \( \gamma \) sends a basis of \( A_{st} \) (the basis formed by the \( st \)-equivalence classes \( [\pi]_{st} \) of permutations) to a basis of \( A \) (namely, to the basis \( (u_{\alpha}) \) bijectively; thus, \( \gamma \) is an isomorphism of \( \mathbb{Q} \)-vector spaces.

The diagram

$$\begin{array}{ccc}
\text{QSym} & \xrightarrow{p_{st}} & A_{st} \\
\phi_{st} \downarrow & & \downarrow \gamma \\
& & A
\end{array}$$

is commutative (as one can easily check by tracing an arbitrary basis element \( F_L \) of \( \text{QSym} \) through the diagram). Since the maps \( p_{st} \) and \( \phi_{st} \) in this diagram are \( \mathbb{Q} \)-algebra homomorphisms, and since \( p_{st} \) is surjective, we thus conclude that \( \gamma \) is also a \( \mathbb{Q} \)-algebra homomorphism . Since \( \gamma \) is an isomorphism of \( \mathbb{Q} \)-vector spaces, we thus conclude that \( \gamma \) is a \( \mathbb{Q} \)-algebra isomorphism \( A_{st} \to A \). This proves Theorem 3.10 (b).

---

\( ^{29} \) Proof. Let \( a, b \in A_{st} \). We shall show that \( \gamma (ab) = \gamma (a) \gamma (b) \).

There exist \( a', b' \in \text{QSym} \) such that \( a = p_{st} (a') \) and \( b = p_{st} (b') \) (since \( p_{st} \) is surjective).

Consider these \( a', b' \). Then, \( \gamma \left( \frac{a}{p_{st} (a')} \right) = \gamma (p_{st} (a')) = \phi_{st} (a') \) (since the diagram is commutative) and \( \gamma (b) = \phi_{st} (b') \) (similarly). But from \( a = p_{st} (a') \) and \( b = p_{st} (b') \), we obtain
5. Dendriform structures

Next, we shall study how the ideal $\mathcal{K}_{\text{Epk}}$ interacts with some additional structure on $\text{QSym}$, viz. the dendriform operations $\prec$ and $\succeq$ and the “runic” operations $\Phi$ and $\Psi$. These operations were introduced in [Grinbe16]. Our study shall lead us to the notions of “left-shuffle-compatibility” and “right-shuffle-compatibility”, which are two properties similar to shuffle-compatibility (which, when combined, are stronger than shuffle-compatibility). We shall prove that Epk is left-shuffle-compatible and right-shuffle-compatible; similar studies can probably be made for other descent statistics.

5.1. Four operations on $\text{QSym}$

We begin with some definitions. We will use some notations from [Grinbe16], but we set $k = \mathbb{Q}$ because we are working over the ring $\mathbb{Q}$ in this paper. Monomials always mean formal expressions of the form $x_1^{a_1}x_2^{a_2}x_3^{a_3} \cdots$ with $a_1 + a_2 + a_3 + \cdots < \infty$ (see [Grinbe16, Section 2] for details). If $m$ is a monomial, then $\text{Supp } m$ will denote the finite subset

$$\{i \in \{1, 2, 3, \ldots\} \mid \text{ the exponent with which } x_i \text{ occurs in } m \text{ is } > 0\}$$

of $\{1, 2, 3, \ldots\}$. Next, we define four binary operations

$$\prec$$ (called “dendriform less-than”; but it’s an operation, not a relation),

$$\succeq$$ (called “dendriform greater-or-equal”; but it’s an operation, not a relation),

$$\Phi$$ (called “belghor”),

$$\Psi$$ (called “tvimadur”)

We now prove some properties of these operations.

\[ ab = p_{\text{st}} (a') p_{\text{st}} (b') = p_{\text{st}} (a'b') \quad (\text{since } p_{\text{st}} \text{ is a } \mathbb{Q}\text{-algebra homomorphism}), \quad \text{so that} \]
\[ \gamma (ab) = \gamma (p_{\text{st}} (a'b')) = \phi_{\text{st}} (a'b') \quad (\text{since the diagram is commutative}) \]
\[ = \phi_{\text{st}} (a') \phi_{\text{st}} (b') \quad (\text{since } \phi_{\text{st}} \text{ is a } \mathbb{Q}\text{-algebra homomorphism}) \]
\[ = \gamma (a) \gamma (b) \cdot \]

Now, forget that we fixed $a, b$. We thus have proven that $\gamma (ab) = \gamma (a) \gamma (b)$ for all $a, b \in A_{\text{st}}$. Similarly, $\gamma (1) = 1$. Hence, $\gamma$ is a $\mathbb{Q}$-algebra homomorphism (since $\gamma$ is $\mathbb{Q}$-linear).
on the ring $\mathbb{k}[[x_1, x_2, x_3, \ldots]]$ of power series by first defining how they act on monomials:

\[
\begin{align*}
m &\prec n = \begin{cases} m \cdot n, & \text{if } \min(Supp \, m) < \min(Supp \, n); \\ 0, & \text{if } \min(Supp \, m) \geq \min(Supp \, n) \end{cases}; \\
m &\succeq n = \begin{cases} m \cdot n, & \text{if } \min(Supp \, m) \geq \min(Supp \, n); \\ 0, & \text{if } \min(Supp \, m) < \min(Supp \, n) \end{cases}; \\
m &\Phi n = \begin{cases} m \cdot n, & \text{if } \max(Supp \, m) \leq \min(Supp \, n); \\ 0, & \text{if } \max(Supp \, m) > \min(Supp \, n) \end{cases}; \\
m &\% n = \begin{cases} m \cdot n, & \text{if } \max(Supp \, m) < \min(Supp \, n); \\ 0, & \text{if } \max(Supp \, m) \geq \min(Supp \, n) \end{cases};
\end{align*}
\]

and then requiring that they all be $\mathbb{k}$-bilinear and continuous (so their action on pairs of arbitrary power series can be computed by “opening the parentheses”). These operations $\prec$, $\succeq$, $\Phi$ and $\%$ all restrict to the subset $\text{QSym}$ of $\mathbb{k}[[x_1, x_2, x_3, \ldots]]$ (this is proven in [Grinbe16, detailed version, Section 3]). They furthermore satisfy numerous relations:

- The dendriform operations satisfy the four rules

\[
a \prec b + a \succeq b = ab; \\
(a \prec b) \prec c = a \prec (bc); \\
(a \succeq b) \prec c = a \succeq (b \prec c); \\
a \succeq (b \succeq c) = (ab) \succeq c
\]

for all $a, b, c \in \mathbb{k}[[x_1, x_2, x_3, \ldots]]$. (In other words, they turn $\mathbb{k}[[x_1, x_2, x_3, \ldots]]$ into what is called a dendriform algebra.)

- For any $a \in \mathbb{k}[[x_1, x_2, x_3, \ldots]]$, we have

\[
\begin{align*}
1 \prec a &= 0; \\
a \prec 1 &= a - \varepsilon(a); \\
1 \succeq a &= a; \\
a \succeq 1 &= \varepsilon(a),
\end{align*}
\]

where $\varepsilon(a)$ denotes the constant term of the power series $a$.

- The binary operation $\Phi$ is associative and unital (with 1 serving as the unity).

- The binary operation $\%$ is associative and unital (with 1 serving as the unity).

These relations are all easy to prove (by linearity, it suffices to verify them on monomials only, and this verification is straightforward). A proof of the associativity of $\Phi$ was given in [Grinbe16, detailed version, Proposition 3.4].
Recall that we are using the notations $M_\alpha$ for the monomial quasisymmetric functions and $F_\alpha$ for the fundamental quasisymmetric functions.

- For any two nonempty compositions $\alpha$ and $\beta$, we have $M_\alpha \ast M_\beta = M_{[\alpha, \beta]} + M_{\alpha \circ \beta}$, where $[\alpha, \beta]$ and $\alpha \circ \beta$ are two compositions defined by
  
  $$(a_1, a_2, \ldots, a_\ell, \beta_1, \beta_2, \ldots, \beta_m) = (a_1, a_2, \ldots, a_\ell, \beta_1, \beta_2, \ldots, \beta_m);$$
  $$(a_1, a_2, \ldots, a_\ell) \circ (\beta_1, \beta_2, \ldots, \beta_m) = (a_1, a_2, \ldots, a_{\ell-1}, a_\ell + \beta_1, \beta_2, \beta_3, \ldots, \beta_m).$$

- For any two compositions $\alpha$ and $\beta$, we have $M_\alpha \triangleright M_\beta = M_{[\alpha, \beta]}$.

- For any two compositions $\alpha$ and $\beta$, we have $F_\alpha \ast F_\beta = F_{\alpha \circ \beta}$. (Here, $\alpha \circ \beta$ is defined to be $\alpha$ if $\beta$ is the empty composition, and is defined to be $\beta$ if $\alpha$ is the empty composition.)

- For any two compositions $\alpha$ and $\beta$, we have $F_\alpha \triangleright F_\beta = F_{[\alpha, \beta]}$.

Furthermore, we shall use two theorems from [Grinbe16, detailed version, Section 3]:

**Theorem 5.1.** Let $S$ denote the antipode of the Hopf algebra $\text{QSym}$. Let us use Sweedler’s notation $\sum_{(b)} b_{(1)} \otimes b_{(2)}$ for $\Delta(b)$, where $b$ is any element of $\text{QSym}$. Then,

$$\sum_{(b)} \left( S\left( b_{(1)} \right) \ast a \right) b_{(2)} = a \prec b$$

for any $a \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ and $b \in \text{QSym}$.

**Theorem 5.2.** Let $S$ denote the antipode of the Hopf algebra $\text{QSym}$. Let us use Sweedler’s notation $\sum_{(b)} b_{(1)} \otimes b_{(2)}$ for $\Delta(b)$, where $b$ is any element of $\text{QSym}$. Then,

$$\sum_{(b)} \left( S\left( b_{(1)} \right) \triangleright a \right) b_{(2)} = b \succeq a$$

for any $a \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ and $b \in \text{QSym}$.

(Notice that Theorem 5.2 differs from [Grinbe16, detailed version, Theorem 3.15] in that we are writing $b \succeq a$ instead of $a \preceq b$. But this is the same thing, since $a \preceq b = b \succeq a$ for all $a, b \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$.)

### 5.2. Ideals
**Definition 5.3.** Let $A$ be a $k$-module equipped with some binary operation $*$ (written infix).

(a) If $B$ and $C$ are two $k$-submodules of $A$, then $B * C$ shall mean the $k$-submodule of $A$ spanned by all elements of the form $b * c$ with $b \in B$ and $c \in C$.

(b) A $k$-submodule $M$ of $A$ is said to be a *left $*$-ideal* if and only if it satisfies $A * M \subseteq M$.

(c) A $k$-submodule $M$ of $A$ is said to be a *right $*$-ideal* if and only if it satisfies $M * A \subseteq M$.

(d) A $k$-submodule $M$ of $A$ is said to be a *$*$-ideal* if and only if it is both a left $*$-ideal and a right $*$-ideal.

**Theorem 5.4.** Let $M$ be an ideal of $\text{QSym}$. Let $A = \text{QSym}$.

(a) If $A \Diamond M \subseteq M$, then $M \prec A \subseteq M$.

(b) If $A \nabla M \subseteq M$, then $A \succ M \subseteq M$.

(c) If $A \nabla M \subseteq M$ and $A \Diamond M \subseteq M$, then $M$ is a $\prec$-ideal and a $\succ$-ideal of $\text{QSym}$.

**Proof of Theorem 5.4**

(a) Assume that $A \Diamond M \subseteq M$. If $a \in M$ and $b \in A$, then

$$a \prec b = \sum_{(b)} \left( \sum_{a \in A} \left( S \left( b_{(1)} \right) \Phi \left( a \in A \right) b_{(2)} \right) \right)$$

(by Theorem 5.1)

$$\in (A \Diamond M) A \subseteq M A A \subseteq M$$

(since $M$ is an ideal of $A$).

Thus, $M \prec A \subseteq M$. This proves Theorem 5.4 (a).

(b) Assume that $A \nabla M \subseteq M$. If $a \in M$ and $b \in A$, then

$$b \succ a = \sum_{(b)} \left( \sum_{a \in A} \left( S \left( b_{(1)} \right) \nabla \left( a \in A \right) b_{(2)} \right) \right)$$

(by Theorem 5.2)

$$\in (A \nabla M) A \subseteq MA \subseteq M$$

(since $M$ is an ideal of $A$).

Thus, $A \succ M \subseteq M$. This proves Theorem 5.4 (b).

(c) Assume that $A \nabla M \subseteq M$ and $A \Diamond M \subseteq M$. Then, Theorem 5.4 (b) yields $A \succ M \subseteq M$. Thus, $M$ is a left $\succ$-ideal.

Now, any $b \in M$ and $a \in A$ satisfy

$$a \prec b = \underbrace{a}_{\in A} \underbrace{b}_{\in M} - \underbrace{a}_{\in A} \underbrace{b}_{\in M}$$

(by (25))

$$\in \underbrace{AM}_{\subseteq M} - \underbrace{A}_{\subseteq M} \underbrace{M}_{\subseteq M} - \underbrace{M}_{\subseteq M}$$

(since $M$ is an ideal of $A$)
In other words, $A \prec M \subseteq M$. In other words, $M$ is a left $\prec$-ideal.

But Theorem 5.4 (a) yields $M \prec A \subseteq M$. In other words, $M$ is a right $\prec$-ideal. Any $a \in M$ and $b \in A$ satisfy

$$a \succeq b = \underbrace{a \prec \underbrace{b \in QSym}}_{\in M} - \underbrace{a \prec \underbrace{b \in QSym}}_{\in A} \quad (\text{by (25)})$$

$$\quad \in \underbrace{MA \subseteq M \prec A \subseteq M - M \subseteq M.}$$

(since $M$ is an ideal of $A$)

In other words, $M \succeq A \subseteq M$. In other words, $M$ is a right $\succeq$-ideal.

Hence, $M$ is a $\prec$-ideal (since $M$ is a left $\prec$-ideal and a right $\prec$-ideal) and a $\succeq$-ideal (since $M$ is a left $\succeq$-ideal and a right $\succeq$-ideal). This proves Theorem 5.4 (c).

Another simple fact is the following:

**Proposition 5.5.** Let $M$ be simultaneously a $\prec$-ideal and a $\succeq$-ideal of $QSym$. Then, $M$ is an ideal of $QSym$.

**Proof of Proposition 5.5.** Any $a \in M$ and $b \in QSym$ satisfy

$$ab = \underbrace{a \prec b \in QSym}_{\in M} + \underbrace{a \succeq b \in QSym}_{\in QSym} \quad (\text{by (25)})$$

$$\quad \in \underbrace{M \prec QSym \subseteq M + M \subseteq M}_{\subseteq M} \quad (\text{since } M \text{ is a } \prec\text{-ideal of } QSym)$$

$$\quad + \underbrace{M \succeq QSym \subseteq M + M \subseteq M}_{\subseteq M} \quad (\text{since } M \text{ is a } \succeq\text{-ideal of } QSym).$$

In other words, $M$ is an ideal of $QSym$. This proves Proposition 5.5.

---

**Question 5.6.** Proposition 5.5 says that if a $Q$-vector subspace $M$ of $QSym$ is simultaneously a $\prec$-ideal and an $\succeq$-ideal, then it is also an ideal. Similarly, if $M$ is an ideal and a $\prec$-ideal, then it is a $\succeq$-ideal. Can we state any other such criteria?

---

**5.3. Application to $\mathcal{K}_{Epk}$**

We now claim the following:

**Theorem 5.7.** The ideal $\mathcal{K}_{Epk}$ of $QSym$ is a $\prec$-ideal, a $\succeq$-ideal, a $\prec$-ideal and a $\succeq$-ideal of $QSym$.

**Proof of Theorem 5.7.** Let $A = QSym$. Corollary 4.4 shows that $\mathcal{K}_{Epk}$ is an ideal of $QSym$.

Let us recall the binary relation $\rightarrow$ on the set of compositions defined in Proposition 4.5.

[Proof of Claim 1: Write the composition $J$ in the form $J = (j_1, j_2, \ldots, j_m)$. Write the composition $G$ in the form $G = (g_1, g_2, \ldots, g_p)$.

We have $J \rightarrow K$. In other words, there exists an $\ell \in \{2, 3, \ldots, m\}$ such that $j_\ell > 2$ and $K = (j_1, j_2, \ldots, j_{\ell-1}, 1, j_\ell - 1, j_{\ell+1}, j_{\ell+2}, \ldots, j_m)$ (by the definition of the relation $\rightarrow$). Consider this $\ell$. Clearly, $\ell > 1$ (since $\ell \in \{2, 3, \ldots, m\}$), so that $p + \ell > p + 1 \geq 1$.

From $G = (g_1, g_2, \ldots, g_p)$ and $J = (j_1, j_2, \ldots, j_m)$, we obtain

$$[G, J] = (g_1, g_2, \ldots, g_p, j_1, j_2, \ldots, j_m).$$

(30)

From $G = (g_1, g_2, \ldots, g_p)$ and $K = (j_1, j_2, \ldots, j_{\ell-1}, 1, j_\ell - 1, j_{\ell+1}, j_{\ell+2}, \ldots, j_m)$, we obtain

$$[G, K] = (g_1, g_2, \ldots, g_p, j_1, j_2, \ldots, j_{\ell-1}, 1, j_\ell - 1, j_{\ell+1}, j_{\ell+2}, \ldots, j_m).$$

(31)

From looking at (30) and (31), we conclude immediately that the composition $[G, K]$ is obtained from $[G, J]$ by “splitting” the entry $j_\ell > 2$ into two consecutive entries $1$ and $j_\ell - 1$, and that this entry $j_\ell$ was not the first entry (indeed, this entry is the $(p + \ell)$-th entry, but $p + \ell > 1$). Hence, $[G, J] \rightarrow [G, K]$ (by the definition of the relation $\rightarrow$). This proves Claim 1.]

Claim 2: We have $A \not\leq K_{Epk} \subseteq K_{Epk}$.

[Proof of Claim 2: We must show that $a \not\leq m \in K_{Epk}$ for every $a \in A$ and $m \in K_{Epk}$. So let us fix $a \in A$ and $m \in K_{Epk}$.

Proposition 4.5 shows that the Q-vector space $K_{Epk}$ is spanned by all differences of the form $F_j - F_K$, where $J$ and $K$ are two compositions satisfying $J \rightarrow K$. Hence, we can WLOG assume that $m$ is such a difference (because the relation $a \not\leq m \in K_{Epk}$, which we must prove, is $Q$-linear in $m$). Assume this. Thus, $m = F_j - F_K$ for some two compositions $J$ and $K$ satisfying $J \rightarrow K$. Consider these $J$ and $K$.

From $J \rightarrow K$, we easily conclude that the composition $J$ is nonempty. Thus, $|J| \neq 0$. But from $J \rightarrow K$, we also obtain $|J| = |K|$. Hence, $|K| = |J| \neq 0$. Thus, the composition $K$ is nonempty.

Recall that the family $(F_L)_{L \in A}$ is a composition is a basis of the Q-vector space $QSym = A$. Hence, we can WLOG assume that $a$ belongs to this family (since the relation $a \not\leq m \in K_{Epk}$, which we must prove, is $Q$-linear in $a$). Assume this. Thus, $a = F_G$ for some composition $G$. Consider this $G$.

If $G$ is the empty composition, then $a = F_G = 1$, and therefore $a \not\leq m = 1 \not\leq m \in K_{Epk}$ holds. Thus, for the rest of this proof, we WLOG assume that $G$ is not the empty composition. Thus, $G$ is nonempty.
Recall that for any two compositions \( \alpha \) and \( \beta \), we have \( F_\alpha \not\asymp F_\beta = F_{[\alpha, \beta]} \). Applying this to \( \alpha = G \) and \( \beta = J \), we obtain \( F_G \not\asymp F_J = F_{[G, J]} \). Similarly, \( F_G \not\asymp F_K = F_{[G, K]} \).

But Claim 1 yields \( [G, J] \to [G, K] \). Hence, the difference \( F_{[G, J]} - F_{[G, K]} \) is one of the differences which span the ideal \( \mathcal{K}_{Epk} \) according to Proposition 4.3. Thus, in particular, this difference lies in \( \mathcal{K}_{Epk} \). In other words, \( F_{[G, J]} - F_{[G, K]} \in \mathcal{K}_{Epk} \).

Now,

\[
\begin{align*}
\alpha \not\asymp m & = F_G \not\asymp (F_J - F_K) = F_G \not\asymp F_J - F_G \not\asymp F_K \\
& = F_{[G, J]} - F_{[G, K]} \in \mathcal{K}_{Epk}.
\end{align*}
\]

This proves Claim 2.]

Claim 3: Let \( J \) and \( K \) be two compositions satisfying \( J \to K \). Let \( G \) be a further composition. Then, \( [J, G] \to [K, G] \).

[Proof of Claim 3: This is proven in the same way as we proved Claim 1, with the only difference that \( j_\ell \) is now the \( \ell \)-th entry of \( [J, G] \) and not the \( (p + \ell) \)-th entry (but this is still sufficient, since \( \ell > 1 \)).]

Claim 4: We have \( \mathcal{K}_{Epk} \not\asymp A \subseteq \mathcal{K}_{Epk} \).

[Proof of Claim 4: This is proven in the same way as we proved Claim 2, with the only difference that now we need to use Claim 3 instead of Claim 1.]

Combining Claim 2 and Claim 4, we conclude that \( \mathcal{K}_{Epk} \) is a \( \not\asymp \)-ideal of \( A = \text{QSym} \).

Claim 5: Let \( J \) and \( K \) be two nonempty compositions satisfying \( J \to K \).

Let \( G \) be a further nonempty composition. Then, \( G \circ J \to G \circ K \).

[Proof of Claim 5: Write the composition \( J \) in the form \( J = (j_1, j_2, \ldots, j_m) \). Write the composition \( G \) in the form \( G = (g_1, g_2, \ldots, g_p) \). Thus, \( p > 0 \) (since the composition \( G \) is nonempty).

We have \( J \to K \). In other words, there exists an \( \ell \in \{2, 3, \ldots, m\} \) such that

\[
\begin{align*}
& j_\ell > 2 \quad \text{and} \quad K = (j_1, j_2, \ldots, j_{\ell-1}, 1, j_\ell - 1, j_{\ell+1}, j_{\ell+2}, \ldots, j_m) \quad \text{(by the definition of the relation \( \to \))} \\
& \text{Consider this } \ell. \quad \text{Clearly, } \ell \geq 2 \quad (\text{since } \ell \in \{2, 3, \ldots, m\}), \quad \text{so that} \\
& p + \ell - 1 > 0 + 2 - 1 = 1.
\end{align*}
\]

From \( G = (g_1, g_2, \ldots, g_p) \) and \( J = (j_1, j_2, \ldots, j_m) \), we obtain

\[
G \circ J = (g_1, g_2, \ldots, g_{p-1}, g_p + j_1, j_2, j_3, \ldots, j_m).
\]

(32)

From \( G = (g_1, g_2, \ldots, g_p) \) and \( K = (j_1, j_2, \ldots, j_{\ell-1}, 1, j_\ell - 1, j_{\ell+1}, j_{\ell+2}, \ldots, j_m) \), we obtain

\[
G \circ K = (g_1, g_2, \ldots, g_{p-1}, g_p + j_1, j_2, j_3, \ldots, j_{\ell-1}, 1, j_\ell - 1, j_{\ell+1}, j_{\ell+2}, \ldots, j_m).
\]

(33)
(notice that the \(g_p + j_1\) term is not a \(g_p + 1\) term, because \(\ell \geq 2\)).

From looking at (32) and (33), we conclude immediately that the composition \(G \circ K\) is obtained from \(G \circ J\) by “splitting” the entry \(j_\ell > 2\) into two consecutive entries 1 and \(j_\ell - 1\), and that this entry \(j_\ell\) was not the first entry (indeed, this entry is the \((p + \ell - 1)\)-th entry, but \(p + \ell - 1 > 1\)). Hence, \(G \circ J \to G \circ K\) (by the definition of the relation \(\to\)). This proves Claim 5.]

Claim 6: We have \(A \vartriangleleft K_{\mathbb{E}p_k} \subseteq K_{\mathbb{E}p_k}\).

[Proof of Claim 6: This is proven in the same way as we proved Claim 2, with the only difference that now we need to use Claim 5 instead of Claim 1 and that we need to use the formula \(F_\alpha \vartriangleleft F_\beta = F_{\alpha \circ \beta}\) instead of \(F_\alpha \not\triangleright F_\beta = F_{[\alpha, \beta]}\).]

Claim 7: Let \(J\) and \(K\) be two nonempty compositions satisfying \(J \to K\).

Let \(G\) be a further nonempty composition. Then, \(J \circ G \to K \circ G\).

[Proof of Claim 7: Write the composition \(J\) in the form \(J = (j_1, j_2, \ldots, j_m)\). Write the composition \(G\) in the form \(G = (g_1, g_2, \ldots, g_p)\). Thus, \(p > 0\) (since the composition \(G\) is nonempty).

We have \(J \to K\). In other words, there exists an \(\ell \in \{2, 3, \ldots, m\}\) such that \(j_\ell > 2\) and \(K = (j_1, j_2, \ldots, j_{\ell - 1}, 1, j_\ell - 1, j_{\ell + 1}, j_{\ell + 2}, \ldots, j_m)\) (by the definition of the relation \(\to\)). Consider this \(\ell\). Clearly, \(\ell \geq 2\) (since \(\ell \in \{2, 3, \ldots, m\}\)), so that \(\ell > 1\).

From \(G = (g_1, g_2, \ldots, g_p)\) and \(J = (j_1, j_2, \ldots, j_m)\), we obtain

\[
J \circ G = (j_1, j_2, \ldots, j_{m - 1}, j_\ell + 1, g_1, g_2, g_3, \ldots, g_p).
\]

(34)

Now, we distinguish between the following two cases:

Case 1: We have \(\ell = m\).

Case 2: We have \(\ell \neq m\).

Let us first consider Case 1. In this case, we have \(\ell = m\). Thus, \(m = \ell > 2 > 1\) and \(j_m + g_1 = \underbrace{j_\ell}_{\ell > 1} + \underbrace{g_1}_{\geq 0} > 2\).

From \(G = (g_1, g_2, \ldots, g_p)\) and

\[
K = (j_1, j_2, \ldots, j_{\ell - 1}, 1, j_\ell - 1, j_{\ell + 1}, j_{\ell + 2}, \ldots, j_m)
= (j_1, j_2, \ldots, j_{m - 1}, 1, j_m - 1) \quad \text{(since \(\ell = m\)),}
\]

we obtain

\[
K \circ G = (j_1, j_2, \ldots, j_{m - 1}, 1, (j_m - 1) + g_1, g_2, g_3, \ldots, g_p)
= (j_1, j_2, \ldots, j_{m - 1}, 1, j_m + g_1 - 1, g_2, g_3, \ldots, g_p).
\]

(35)

From looking at (34) and (35), we conclude immediately that the composition \(K \circ G\) is obtained from \(J \circ G\) by “splitting” the entry \(j_m + g_1 > 2\) into two
consecutive entries 1 and \( j_m + g_1 - 1 \), and that this entry \( j_m + g_1 \) was not the first entry (indeed, this entry is the \( m \)-th entry, but \( m > 1 \)). Hence, \( J \odot G \rightarrow K \odot G \) (by the definition of the relation \( \rightarrow \)). This proves Claim 7 in Case 1.

Let us next consider Case 2. In this case, we have \( \ell \neq m \). Hence, \( \ell \in \{2, 3, \ldots, m-1\} \) (since \( \ell \in \{2, 3, \ldots, m\} \)).

From \( G = (g_1, g_2, \ldots, g_p) \) and \( K = (j_1, j_2, \ldots, j_{\ell-1}, 1, j_{\ell}-1, j_{\ell+1}, j_{\ell+2}, \ldots, j_m) \), we obtain

\[
K \odot G = (j_1, j_2, \ldots, j_{\ell-1}, 1, j_{\ell}-1, j_{\ell+1}, j_{\ell+2}, \ldots, j_m + g_1, g_2, g_3, \ldots, g_p)
\]  

(36)

(notice that the \( j_m + g_1 \) term is not a \((j_\ell - 1) + g_1\) term, because \( \ell \neq m \)).

From looking at (34) and (36), we conclude immediately that the composition \( K \odot G \) is obtained from \( J \odot G \) by “splitting” the entry \( j_\ell > 2 \) into two consecutive entries 1 and \( j_{\ell-1} \), and that this entry \( j_\ell \) was not the first entry (indeed, this entry is the \( \ell \)-th entry, but \( \ell > 1 \)). Hence, \( J \odot G \rightarrow K \odot G \) (by the definition of the relation \( \rightarrow \)). This proves Claim 7 in Case 2.

We have now proven Claim 7 in both Cases 1 and 2. Thus, Claim 7 always holds.

**Claim 8:** We have \( \mathcal{K}_{EPK} \triangleleft A \subseteq \mathcal{K}_{EPK} \).

[Proof of Claim 8: This is proven in the same way as we proved Claim 6, with the only difference that now we need to use Claim 7 instead of Claim 5.]

Combining Claim 6 and Claim 8, we conclude that \( \mathcal{K}_{EPK} \) is a \( \triangleleft \)-ideal of \( A = QSym \).

Finally, Theorem 5.4 (c) (applied to \( M = \mathcal{K}_{EPK} \)) shows that \( \mathcal{K}_{EPK} \) is a \( \prec \)-ideal and a \( \succeq \)-ideal of \( QSym \).

Thus, altogether, we have proven that \( \mathcal{K}_{EPK} \) is a \( \prec \)-ideal, a \( \triangleright \)-ideal, a \( \prec \)-ideal and a \( \succeq \)-ideal of \( QSym \). This proves Theorem 5.7.

**Question 5.8.** What other descent statistics \( st \) have the property that \( \mathcal{K}_{st} \) is an \( \prec \)-ideal, \( \triangleright \)-ideal, \( \prec \)-ideal and/or \( \succeq \)-ideal? We will see some answers in Subsection 5.6, but a more systematic study would be interesting.

### 5.4. Dendriform shuffle-compatibility

We have seen (in Proposition 4.3) that the kernel \( \mathcal{K}_{st} \) of a descent statistic \( st \) is an ideal of \( QSym \) if and only if \( st \) is shuffle-compatible. It is natural to ask whether similar combinatorial interpretations exist for when the kernel \( \mathcal{K}_{st} \) of a descent statistic \( st \) is a \( \triangleleft \)-ideal, a \( \triangleright \)-ideal, a \( \prec \)-ideals or a \( \succeq \)-ideal. In this section, we shall prove such interpretations.

We begin by introducing dendriform shuffles (i.e., left shuffles and right shuffles). There is a well-known notion of dendriform shuffles of words. Specialized to permutations, it can be defined in a simple way as follows:
Definition 5.9. If $\pi$ and $\sigma$ are two disjoint nonempty permutations, then:

- A left shuffle of $\pi$ and $\sigma$ means a shuffle $\tau$ of $\pi$ and $\sigma$ such that the first letter of $\tau$ is the first letter of $\pi$.
- A right shuffle of $\pi$ and $\sigma$ means a shuffle $\tau$ of $\pi$ and $\sigma$ such that the first letter of $\tau$ is the first letter of $\sigma$.
- We let $S_\prec (\pi, \sigma)$ denote the set of all left shuffles of $\pi$ and $\sigma$.
- We let $S_\succ (\pi, \sigma)$ denote the set of all right shuffles of $\pi$ and $\sigma$.

We have restricted ourselves to nonempty permutations in this definition because it is not clear what the “first letter” of an empty permutation would be (and either way, allowing empty permutations do not change much in our analysis).

The following theorem is analogous to Theorem 3.6:

Theorem 5.10. Let $\pi$ be a nonempty permutation with descent composition $J$. Let $\sigma$ be a nonempty permutation with descent composition $K$. Assume that the permutations $\pi$ and $\sigma$ are disjoint, and that

\[ \text{the first entry of } \pi \text{ is greater than the first entry of } \sigma. \]  

(37)

For any composition $L$, let $c^{L_\prec}_{J,K}$ be the number of permutations with descent composition $L$ among the left shuffles of $\pi$ and $\sigma$, and let $c^{L_\succ}_{J,K}$ be the number of permutations with descent composition $L$ among the right shuffles of $\pi$ and $\sigma$. Then,

\[ F_J \prec F_K = \sum_L c^{L_\prec}_{J,K} F_L \]

and

\[ F_J \succ F_K = \sum_L c^{L_\succ}_{J,K} F_L. \]

Note the condition (37), which is not present in Theorem 3.6 and which makes Theorem 5.10 somewhat harder to apply.

Theorem 5.10 can be proven similarly to [GriRei17, (5.2.6)], but in lieu of the application of [GriRei17, Lemma 5.2.17], it requires the following fact:

Proposition 5.11. We shall use the notations of [GriRei17, Section 5.2]. Let $P$ and $Q$ be two disjoint labelled posets, each of which has a minimum element. Assume that $\min P > \min Q$. Consider the disjoint union $P \sqcup Q$ of $P$ and $Q$.

(a) Add a further relation $\min P < \min Q$ to $P \sqcup Q$; denote the resulting labelled poset by $P \prec Q$. Then, $F_P (x) \prec F_Q (x) = F_{P \prec Q} (x)$. 

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(b) Add a further relation $\min P > \min Q$ to $P \sqcup Q$; denote the resulting labelled poset by $P \succeq Q$. Then, $F_p(x) \succeq F_Q(x) = F_{P \succeq Q}(x)$.

Also, the following simple fact is used:

**Lemma 5.12.** We shall use the notations of [GriRei17, Section 5.2]. Let $P$ and $Q$ be two disjoint posets, each of which has a minimum element. Consider the disjoint union $P \sqcup Q$ of $P$ and $Q$ as the set-theoretic union $P \cup Q$. Assume that $P$ and $Q$ are subsets of $\mathbb{P}$ (the set of positive integers); thus, any linear extension of $P$ or of $Q$ or of $P \sqcup Q$ is a permutation (a word with letters in $\mathbb{P}$).

(a) Add a further relation $\min P < \min Q$ to $P \sqcup Q$; denote the resulting poset by $P \prec Q$. Then,

$$\mathcal{L}(P \prec Q) = \bigsqcup_{\pi \in \mathcal{L}(P); \sigma \in \mathcal{L}(Q)} S_{\prec}(\pi, \sigma).$$

(b) Add a further relation $\min P > \min Q$ to $P \sqcup Q$; denote the resulting poset by $P \succ Q$. Then,

$$\mathcal{L}(P \preceq Q) = \bigsqcup_{\pi \in \mathcal{L}(P); \sigma \in \mathcal{L}(Q)} S_{\preceq}(\pi, \sigma).$$

We leave the details to the reader.

We can rewrite Theorem 5.10 as follows:\[31\]

**Corollary 5.13.** Let $\pi$ and $\sigma$ be two disjoint nonempty permutations. Assume that the first entry of $\pi$ is greater than the first entry of $\sigma$.

Then,

$$F_{\text{Comp } \pi} \prec F_{\text{Comp } \sigma} = \sum_{\chi \in S_{\prec}(\pi, \sigma)} F_{\text{Comp } \chi}$$

and

$$F_{\text{Comp } \pi} \preceq F_{\text{Comp } \sigma} = \sum_{\chi \in S_{\preceq}(\pi, \sigma)} F_{\text{Comp } \chi}.$$ 

Now, let us define dendriform analogues to the concept of shuffle-compatibility:

**Definition 5.14.** Let $st$ be a permutation statistic.

(a) We say that $st$ is left-shuffle-compatible if for any two disjoint nonempty permutations $\pi$ and $\sigma$ having the property that

the first entry of $\pi$ is greater than the first entry of $\sigma$,

(38)

Recall that for any permutation $\varphi$, we have let $\text{Comp } \varphi$ denote the descent composition of $\varphi$.\[31\]
the multiset \[ \{ st(\tau) \mid \tau \in S_{\prec}(\pi, \sigma) \} \]
depends only on \( st(\pi), st(\sigma), |\pi| \) and \( |\sigma| \).

(b) We say that \( st \) is \textit{weakly left-shuffle-compatible} if for any two disjoint nonempty permutations \( \pi \) and \( \sigma \) having the property that

\[
each entry of \pi is greater than each entry of \sigma,
\]
the multiset \( \{ st(\tau) \mid \tau \in S_{\prec}(\pi, \sigma) \} \)
depends only on \( st(\pi), st(\sigma), |\pi| \) and \( |\sigma| \).

(c) We say that \( st \) is \textit{right-shuffle-compatible} if for any two disjoint nonempty permutations \( \pi \) and \( \sigma \) having the property that

\[
the first entry of \pi is greater than the first entry of \sigma,
\]
the multiset \( \{ st(\tau) \mid \tau \in S_{\prec}(\pi, \sigma) \} \)
depends only on \( st(\pi), st(\sigma), |\pi| \) and \( |\sigma| \).

(d) We say that \( st \) is \textit{weakly right-shuffle-compatible} if for any two disjoint nonempty permutations \( \pi \) and \( \sigma \) having the property that

\[
each entry of \pi is greater than each entry of \sigma,
\]
the multiset \( \{ st(\tau) \mid \tau \in S_{\prec}(\pi, \sigma) \} \)
depends only on \( st(\pi), st(\sigma), |\pi| \) and \( |\sigma| \).

Then, the following analogues to the first part of Proposition 4.3 holds:

**Theorem 5.15.** Let \( st \) be a descent statistic. Then, the following three statements are equivalent:

- **Statement A:** The statistic \( st \) is left-shuffle-compatible.
- **Statement B:** The statistic \( st \) is weakly left-shuffle-compatible.
- **Statement C:** The set \( K_{st} \) is an \( \prec \)-ideal of \( \text{QSym} \).

**Theorem 5.16.** Let \( st \) be a descent statistic. Then, the following three statements are equivalent:

- **Statement A:** The statistic \( st \) is right-shuffle-compatible.
- **Statement B:** The statistic \( st \) is weakly right-shuffle-compatible.
- **Statement C:** The set \( K_{st} \) is an \( \succeq \)-ideal of \( \text{QSym} \).

Let us prove Theorem 5.15 directly, without using shuffle algebras:
Proof of Theorem 5.15 (sketched). The implication A—>B is obvious.

Proof of the implication B—>C: Assume that Statement B holds. Thus, the statistic st is weakly left-shuffle-compatible.

Let us show that the set $\mathcal{K}_{st}$ is a $\prec$-ideal of $QSym$. Indeed, it suffices to show that every two st-equivalent compositions $J$ and $K$ and every further composition $L$ satisfy

$$(F_J - F_K) \prec F_L \in \mathcal{K}_{st} \quad \text{and} \quad F_L \prec (F_J - F_K) \in \mathcal{K}_{st} \quad (40)$$

(because of the definition of $\mathcal{K}_{st}$). So let $J$ and $K$ be two st-equivalent compositions, and let $L$ be a further composition. If $J = K$, then (40) follows immediately from realizing that $F_J - F_K = 0$; thus, we WLOG assume that $J \neq K$. But $|J| = |K|$ (since $J$ and $K$ are st-equivalent). Hence, $|J| = |K| > 0$ (since otherwise, we would have $|J| = |K| = 0$, which would imply that both $J$ and $K$ would be the empty composition, contradicting $J \neq K$). Thus, the power series $F_J$ and $F_K$ are homogeneous of degree $|J| = |K| > 0$; consequently, $\epsilon(F_J) = 0$ and $\epsilon(F_K) = 0$. Hence, $\epsilon(J) = \epsilon(K) = 0$.

The compositions $J$ and $K$ are nonempty (since $|J| = |K| > 0$). If $L$ is empty, then (40) holds for easy reasons (indeed, we have $F_L = 1$ in this case, and therefore (27) yields

$$(F_J - F_K) \prec F_L = (F_J - F_K) - \epsilon(F_J - F_K) = F_J - F_K \in \mathcal{K}_{st},$$

and similarly (26) leads to $F_L \prec (F_J - F_K) \in \mathcal{K}_{st})$. Hence, we WLOG assume that $L$ is nonempty.

Pick three disjoint permutations $\varphi$, $\psi$ and $\sigma$ having descent compositions $J$, $K$ and $L$, respectively, and having the property that

each entry of $\varphi$ is greater than each entry of $\sigma$

and

each entry of $\psi$ is greater than each entry of $\sigma$.

(Such permutations $\varphi$, $\psi$ and $\sigma$ exist, since the set $P$ is infinite.)

The permutations $\varphi$ and $\psi$ are st-equivalent (since their descent compositions $J$ and $K$ are st-equivalent). In other words, $|\varphi| = |\psi|$ and $st(\varphi) = st(\psi)$.

The statistic $st$ is weakly left-shuffle-compatible. Thus, the multiset $\{st(\tau) \mid \tau \in S_{\prec}(\pi, \sigma)\}$ (where $\pi$ is a nonempty permutation disjoint from $\sigma$ and having the property that each entry of $\pi$ is greater than each entry of $\sigma$) depends only on $st(\pi)$ and $|\pi|$ (by the definition of “weakly left-shuffle-compatible”). Therefore, the multisets $\{st(\tau) \mid \tau \in S_{\prec}(\varphi, \sigma)\}$ and $\{st(\tau) \mid \tau \in S_{\prec}(\psi, \sigma)\}$ are equal (since

$\text{Recall that } \sigma \text{ is fixed here, which is why we don't have to say that it depends on } st(\sigma) \text{ and } |\sigma| \text{ as well.}
Consider this \( \alpha \). Clearly, each \( \chi \in S_{\prec}(\phi, \sigma) \) satisfies

\[
(\chi \text{ and } \alpha(\chi) \text{ are st-equivalent})
\]

(because of (41) and since \( |\chi| = |\phi| + |\sigma| = |\psi| + |\sigma| = |\alpha(\chi)| \) and therefore

\[
(\text{Comp } \chi \text{ and } \text{Comp } (\alpha(\chi)) \text{ are st-equivalent})
\]

(since st is a descent statistic) and thus \( F_{\text{Comp } \chi} - F_{\text{Comp } (\alpha(\chi))} \in K_{st} \) (by the definition of \( K_{st} \)) and therefore

\[
F_{\text{Comp } \chi} \equiv F_{\text{Comp } (\alpha(\chi))} \mod K_{st}.
\]

The first claim of Corollary 5.13 yields

\[
F_{\text{Comp } \phi} \prec F_{\text{Comp } \sigma} = \sum_{\chi \in S_{\prec}(\phi, \sigma)} F_{\text{Comp } \chi} \quad \text{ and }
\]

\[
F_{\text{Comp } \psi} \prec F_{\text{Comp } \sigma} = \sum_{\chi \in S_{\prec}(\psi, \sigma)} F_{\text{Comp } \chi}.
\]

Hence,

\[
F_{\text{Comp } \phi} \prec F_{\text{Comp } \sigma} = \sum_{\chi \in S_{\prec}(\phi, \sigma)} \left( F_{\text{Comp } \chi} \equiv F_{\text{Comp } (\alpha(\chi))} \mod K_{st} \right)
= \sum_{\chi \in S_{\prec}(\psi, \sigma)} F_{\text{Comp } \chi}
\]

(here, we have substituted \( \chi \) for \( \alpha(\chi) \) in the sum, since the map \( \alpha : S_{\prec}(\phi, \sigma) \to S_{\prec}(\psi, \sigma) \) is a bijection)

\[
= F_{\text{Comp } \psi} \prec F_{\text{Comp } \sigma} \mod K_{st}.
\]

Since \( \text{Comp } \phi = J \), \( \text{Comp } \psi = K \) and \( \text{Comp } \sigma = L \) (by the definition of \( \phi, \psi \) and \( \sigma \), this rewrites as \( F_J \prec F_L \equiv F_K \prec F_L \mod K_{st} \). In other words, \( F_J \prec F_L - F_K \prec F_L \in K_{st} \). In other words, \( (F_J - F_K) \prec F_L \in K_{st} \). This proves the first claim of (40). The second is proven similarly. Altogether, we thus conclude that \( K_{st} \) is a \( \prec \)-ideal of QSym. In other words, Statement C holds. This proves the implication \( B \Rightarrow C \).

**Proof of the implication C \Rightarrow A**: Assume that Statement C holds. Thus, the set \( K_{st} \) is an \( \prec \)-ideal of QSym.
Let X be the codomain of the map st. Let \( Q[X] \) be the free \( Q \)-vector space with basis \( ([x])_{x \in X} \). Then, we can define a \( Q \)-linear map \( st : \text{QSym} \to Q[X], F_j \mapsto [st f] \). This map \( st \) sends each of the generators of \( K_{st} \) to 0 (by the definition of \( K_{st} \)), and therefore sends the whole \( K_{st} \) to 0. In other words, \( st (K_{st}) = 0 \).

Now, consider any two disjoint nonempty permutations \( \pi \) and \( \sigma \) having the property that

the first entry of \( \pi \) is greater than the first entry of \( \sigma \).

Also, consider two further disjoint nonempty permutations \( \pi' \) and \( \sigma' \) having the property that

the first entry of \( \pi' \) is greater than the first entry of \( \sigma' \)

and satisfying \( st(\pi) = st(\pi'), st(\sigma) = st(\sigma'), |\pi| = |\pi'| \) and \( |\sigma| = |\sigma'| \). We shall show that \( \{ st(\tau) \mid \tau \in S_s(\pi, \sigma) \} = \{ st(\tau) \mid \tau \in S_s(\pi', \sigma') \} \) as multisets. This will show that the multiset \( \{ st(\tau) \mid \tau \in S_s(\pi, \sigma) \} \) depends only on \( st(\pi), st(\sigma), |\pi| \) and \( |\sigma| \).

From \( st(\pi) = st(\pi') \) and \( |\pi| = |\pi'| \), we conclude that \( \pi \) and \( \pi' \) are st-equivalent. In other words, Comp \( \pi \) and Comp \( \pi' \) are st-equivalent. Hence, \( F_{\text{Comp}} \pi - F_{\text{Comp}} \pi' \in K_{st} \) (by the definition of \( K_{st} \)), so that \( F_{\text{Comp}} \pi \equiv F_{\text{Comp}} \pi' \mod K_{st} \).

Similarly, \( F_{\text{Comp}} \sigma \equiv F_{\text{Comp}} \sigma' \mod K_{st} \). These two congruences, combined, yield \( F_{\text{Comp}} \pi \lesssim F_{\text{Comp}} \sigma \equiv F_{\text{Comp}} \pi' \lesssim F_{\text{Comp}} \sigma' \mod K_{st} \). (Indeed, we can conclude \( a \lesssim c \equiv b \lesssim d \mod K_{st} \) whenever we have \( a \equiv b \mod K_{st} \) and \( c \equiv d \mod K_{st} \); this is because we know that \( K_{st} \) is a \( \lesssim \)-ideal of \( \text{QSym} \).)

From \( F_{\text{Comp}} \pi \lesssim F_{\text{Comp}} \sigma \equiv F_{\text{Comp}} \pi' \lesssim F_{\text{Comp}} \sigma' \mod K_{st} \), we obtain

\[
\text{st} \left( F_{\text{Comp}} \pi \lesssim F_{\text{Comp}} \sigma \right) = \text{st} \left( F_{\text{Comp}} \pi' \lesssim F_{\text{Comp}} \sigma' \right) \quad (43)
\]

(since \( st(K_{st}) = 0 \)).

The first claim of Corollary 5.13 yields

\[
F_{\text{Comp}} \pi \lesssim F_{\text{Comp}} \sigma \equiv \sum_{\chi \in S_s(\pi, \sigma)} F_{\text{Comp}} \chi.
\]

Applying the map \( st \) to both sides of this equality, we find

\[
\text{st} \left( F_{\text{Comp}} \pi \lesssim F_{\text{Comp}} \sigma \right) = \text{st} \left( \sum_{\chi \in S_s(\pi, \sigma)} F_{\text{Comp}} \chi \right) = \sum_{\chi \in S_s(\pi, \sigma)} \text{st} \left( F_{\text{Comp}} \chi \right) = \sum_{\chi \in S_s(\pi, \sigma)} \left[ st(\chi) \right].
\]

Similarly,

\[
\text{st} \left( F_{\text{Comp}} \pi' \lesssim F_{\text{Comp}} \sigma' \right) = \sum_{\chi \in S_s(\pi', \sigma')} \left[ st(\chi) \right].
\]
But the left-hand sides of the last two equalities are equal (because of (43)); therefore, the right-hand sides must be equal as well. In other words,

$$\sum_{\chi \in S_{\prec}((\pi,\sigma)} [st \chi] = \sum_{\chi \in S_{\prec}((\pi',\sigma')} [st \chi].$$

This shows exactly that $$\{st(\chi) \mid \chi \in S_{\prec}(\pi,\sigma)\} = \{st(\chi) \mid \chi \in S_{\prec}(\pi',\sigma')\}$$ as multisets. In other words, $$\{st(\tau) \mid \tau \in S_{\prec}(\pi,\sigma)\} = \{st(\tau) \mid \tau \in S_{\prec}(\pi',\sigma')\}$$ as multisets. Thus, we have proven that the multiset $$\{st(\tau) \mid \tau \in S_{\prec}(\pi,\sigma)\}$$ depends only on $$st(\pi), st(\sigma), |\pi|$$ and $$|\sigma|$$. Hence, the statistic $$st$$ is left-shuffle-compatible. In other words, Statement A holds. This proves the implication $$C \Rightarrow A$$. Now that we have proven all three implications $$A \Rightarrow B, B \Rightarrow C$$ and $$C \Rightarrow A,$$ the proof of Theorem 5.15 is complete.

**Proof of Theorem 5.16.** The proof of Theorem 5.16 is analogous to the above proof of Theorem 5.15.

As a consequence of the above two theorems, we can see that any descent statistic that is weakly left-shuffle-compatible and weakly right-shuffle-compatible must automatically be shuffle-compatible. Note that this is only true for descent statistics! As far as arbitrary permutation statistics are concerned, this is false; for example, the number of inversions is weakly left-shuffle-compatible and weakly right-shuffle-compatible but not shuffle-compatible.

On the other hand, every permutation statistic that is left-shuffle-compatible and right-shuffle-compatible must automatically be shuffle-compatible (whether or not it is a descent statistic). This follows from a careful look at the definitions of “left-shuffle-compatible” and “right-shuffle-compatible” and the observation that $$S_{\prec}(\pi,\sigma) = S_{\succ}(\sigma,\pi)$$ and $$S_{\prec}(\pi,\sigma) = S_{\prec}(\sigma,\pi)$$ for any two disjoint permutations $$\pi$$ and $$\sigma$$.

**Corollary 5.17.** The descent statistic $$Ep_k$$ is left-shuffle-compatible and right-shuffle-compatible.

**Proof of Corollary 5.17.** To prove that $$Ep_k$$ is left-shuffle-compatible, combine Theorem 5.15 with Theorem 5.7. Similarly for right-shuffle-compatibility.

Using Theorem 5.10, we can try to state an analogue of Theorem 3.10 (a). Let us first define the notion of dendriform algebras:

$$\text{Proof.}$$ Let $$st$$ be a descent statistic that is weakly left-shuffle-compatible and weakly right-shuffle-compatible. We must prove that $$st$$ is shuffle-compatible.

The implication $$B \Rightarrow C$$ in Theorem 5.15 shows that the set $$K_{st}$$ is an $$\prec$$-ideal of QSym. Similarly, the set $$K_{st}$$ is an $$\succ$$-ideal of QSym. Hence, Proposition 5.5 (applied to $$M = K_{st}$$) yields that $$K_{st}$$ is an ideal of QSym. By Proposition 4.3, this shows that $$st$$ is shuffle-compatible.
Definition 5.18. (a) A dendriform algebra over a field $k$ means a $k$-algebra $A$ equipped with two further $k$-bilinear binary operations $\prec$ and $\succeq$ (these are operations, not relations, despite the symbols) from $A \times A$ to $A$ that satisfy the four rules

\[
\begin{align*}
  a \prec b + a \succeq b &= ab; \\
  (a \prec b) \prec c &= a \prec (bc); \\
  (a \succeq b) \prec c &= a \succeq (b \prec c); \\
  a \succeq (b \succeq c) &= (ab) \succeq c
\end{align*}
\]

for all $a, b, c \in A$. (Depending on the situation, it is useful to also impose a few axioms that relate the unity $1$ of the $k$-algebra $A$ with the operations $\prec$ and $\succeq$. For example, we could require $1 \prec a = a$ for each $a \in A$. For what we are going to do in the following, it does not matter whether we make this requirement.)

(b) If $A$ and $B$ are two dendriform algebras over $k$, then a dendriform algebra homomorphism from $A$ to $B$ means a $k$-algebra homomorphism $\phi : A \to B$ preserving the operations $\prec$ and $\succeq$ (that is, satisfying $\phi (a \prec b) = \phi (a) \prec \phi (b)$ and $\phi (a \succeq b) = \phi (a) \succeq \phi (b)$ for all $a, b \in A$). (Some authors only require it to be a $k$-linear map instead of being a $k$-algebra homomorphism; this boils down to the question whether $\phi (1)$ must be $1$ or not. This does not make a difference for us here.)

Of course, $\text{QSym}$ (with its two operations $\prec$ and $\succeq$) thus becomes a dendriform algebra over $Q$.

Notice that if $A$ and $B$ are two dendriform algebras over $k$, then the kernel of any dendriform algebra homomorphism $A \to B$ is an $\prec$-ideal and a $\succeq$-ideal of $A$. Conversely, if $A$ is a dendriform algebra over $k$, and $I$ is simultaneously an $\prec$-ideal and a $\succeq$-ideal of $A$, then $A/I$ canonically becomes a dendriform algebra, and the canonical projection $A \to A/I$ becomes a dendriform algebra homomorphism.

Therefore, Theorem 5.15 and Theorem 5.16 (and the $A_{\text{st}} \cong \text{QSym} / K_{\text{st}}$ isomorphism from Proposition 4.3) yield the following:

Corollary 5.19. If a descent statistic $\text{st}$ is left-shuffle-compatible and right-shuffle-compatible, then its shuffle algebra $A_{\text{st}}$ canonically becomes a dendriform algebra.

We furthermore have the following analogue of Theorem 3.10, which easily follows from Theorem 5.15 and Theorem 5.16:

Theorem 5.20. Let $\text{st}$ be a descent statistic.

(a) The descent statistic $\text{st}$ is left-shuffle-compatible and right-shuffle-compatible if and only if there exists a homomorphism $\phi_{\text{st}} : \text{QSym} \to A$
of dendriform algebras, where $A$ is a dendriform algebra with basis $(u_\alpha)$ indexed by st-equivalence classes $\alpha$ of compositions, such that $\phi_{st}(F_L) = u_\alpha$ whenever $L \in \alpha$.

(b) In this case, the $Q$-linear map $A_{st} \to A$ given by

$$\left[\pi\right]_{st} \mapsto u_\alpha = \phi_{st}(F_{\text{Comp}(\pi)})$$

where $\text{Comp}(\pi) = \alpha$ is an isomorphism of dendriform algebras.

Question 5.21. Can the $Q$-algebra $\text{Pow} N$ from Definition 2.10 be endowed with two binary operations $\prec$ and $\succeq$ that make it into a dendriform algebra? Can we then find an analogue of Proposition 2.13 along the following lines?

Let $(P, \gamma)$, $(Q, \delta)$ and $(P \sqcup Q, \epsilon)$ be as in Proposition 2.13. Assume that each of the posets $P$ and $Q$ has a minimum element; denote these elements by $\text{min} P$ and $\text{min} Q$, respectively. Define two posets $P \prec Q$ and $P \succeq Q$ as in Proposition 5.11. Then, we hope to have

$$\Gamma_Z (P, \gamma) \prec \Gamma_Z (Q, \delta) = \Gamma_Z (P \prec Q, \epsilon)$$

and

$$\Gamma_Z (P, \gamma) \succeq \Gamma_Z (Q, \delta) = \Gamma_Z (P \succeq Q, \epsilon).$$

Ideally, this would be a generalization of Proposition 5.11.

5.5. Criteria for $K_{st}$ to be a stack ideal

We have so far studied the combinatorial significance of when the kernel $K_{st}$ of a statistic $st$ is a $\prec$-ideal or a $\succeq$-ideal of $QSym$. What about $\Phi$-ideals and $\chi$-ideals? It turns out that the answer to this question is given (on the level of compositions) by the following (easily verified) proposition:

Proposition 5.22. Let $st$ be a descent statistic.

(a) The set $K_{st}$ is a left $\Phi$-ideal of $QSym$ if and only if $st$ has the following property: If $J$ and $K$ are two $st$-equivalent nonempty compositions, and if $G$ is any nonempty composition, then $G \odot J$ and $G \odot K$ are $st$-equivalent.

(b) The set $K_{st}$ is a right $\Phi$-ideal of $QSym$ if and only if $st$ has the following property: If $J$ and $K$ are two $st$-equivalent nonempty compositions, and if $G$ is any nonempty composition, then $J \odot G$ and $K \odot G$ are $st$-equivalent.

(c) The set $K_{st}$ is a left $\chi$-ideal of $QSym$ if and only if $st$ has the following property: If $J$ and $K$ are two $st$-equivalent nonempty compositions, and if $G$ is any nonempty composition, then $[G, J]$ and $[G, K]$ are $st$-equivalent.

(d) The set $K_{st}$ is a right $\chi$-ideal of $QSym$ if and only if $st$ has the following property: If $J$ and $K$ are two $st$-equivalent nonempty compositions, and if $G$ is any nonempty composition, then $[J, G]$ and $[K, G]$ are $st$-equivalent.

Proposition 5.22 allows us to give a new proof of Theorem 5.7, which makes
no use of Proposition 4.5. Instead, it will rely on analyzing $\text{Epk}([A, B])$ and $\text{Epk}(A \odot B)$ when $A$ and $B$ are two nonempty compositions.

First, we introduce a notation: If $S$ is a set of integers, and $p$ is an integer, then $S + p$ shall denote the set $\{s + p \mid s \in S\}$.

We shall use the following simple lemma:

**Lemma 5.23.** Let $A$ and $B$ be two nonempty compositions. Let $n = |A|$.

(a) We have $\text{Epk}([A, B]) = (\text{Epk} A) \cup ((\text{Epk} B + n) \setminus \{n + 1\})$.

(b) We have $\text{Epk}(A \odot B) = ((\text{Epk} A) \setminus \{n\}) \cup (\text{Epk} B + n)$.

**Proof of Lemma 5.23.** Let $m = |B|$. Consider any $n$-permutation $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ satisfying $\text{Comp} \alpha = A$. (Such $\alpha$ exists, since $n = |A|$.) Consider any $m$-permutation $\beta = (\beta_1, \beta_2, \ldots, \beta_m)$ satisfying $\text{Comp} \beta = B$. (Such $\beta$ exists, since $m = |B|$.) From $\text{Comp} \alpha = A$, we obtain $\text{Epk} \alpha = \text{Epk} A$. Similarly, $\text{Epk} \beta = \text{Epk} B$.

(a) WLOG assume that $\alpha_i > \beta_j$ for all $i \in [n]$ and $j \in [m]$. (Indeed, we can achieve this by choosing a positive integer $g$ that is larger than each entry of $\beta$, and adding $g$ to each entry of $\alpha$.) Thus, in particular, the entries of $\alpha$ are distinct from the entries of $\beta$. Also, $\alpha_n > \beta_1$ (since $\alpha_i > \beta_j$ for all $i \in [n]$ and $j \in [m]$).

Let $\gamma$ be the $(n + m)$-permutation $(\alpha_1, \alpha_2, \ldots, \alpha_n, \beta_1, \beta_2, \ldots, \beta_m)$. Then, the descents of $\gamma$ are obtained as follows:

- Each descent of $\alpha$ is a descent of $\gamma$.
- The number $n$ is a descent of $\gamma$ (since $\alpha_n > \beta_1$).
- Adding $n$ to each descent of $\beta$ yields a descent of $\gamma$ (that is, if $i$ is a descent of $\beta$, then $i + n$ is a descent of $\gamma$).

These are all the descents of $\gamma$. Thus,

$$\text{Des} \gamma = \text{Des} \alpha \cup \{n\} \cup (\text{Des} \beta + n).$$

Hence,

$$\text{Comp} (\text{Des} \gamma) = \text{Comp} (\text{Des} \alpha \cup \{n\} \cup (\text{Des} \beta + n))$$

$$= [\text{Comp} (\text{Des} \alpha), \text{Comp} (\text{Des} \beta)]$$

(because of how $\text{Comp} S$ is defined for a set $S$). Since $\text{Comp} (\text{Des} \pi) = \text{Comp} \pi$ for any permutation $\pi$, this rewrites as

$$\text{Comp} \gamma = [\text{Comp} \alpha, \text{Comp} \beta].$$

In view of $\text{Comp} \alpha = A$ and $\text{Comp} \beta = B$, this rewrites as $\text{Comp} \gamma = [A, B]$. Thus, $\text{Epk}([A, B]) = \text{Epk} \gamma$.

On the other hand, recall again that $\gamma = (\alpha_1, \alpha_2, \ldots, \alpha_n, \beta_1, \beta_2, \ldots, \beta_m)$ and $\alpha_n > \beta_1$. Thus, the exterior peaks of $\gamma$ are obtained as follows:
• Each exterior peak of \( \alpha \) is an exterior peak of \( \gamma \). (This includes \( n \), if \( n \) is an exterior peak of \( \alpha \), because \( \alpha_n > \beta_1 \).)

• Adding \( n \) to each exterior peak of \( \beta \) yields an exterior peak of \( \gamma \), except for the number \( n + 1 \), which is not an exterior peak of \( \gamma \) (since \( \alpha_n > \beta_1 \)).

These are all the exterior peaks of \( \gamma \). Thus,

\[
\text{Epk} \gamma = (\text{Epk} \alpha) \cup ((\text{Epk} \beta + n) \setminus \{n + 1\}).
\]

In view of \( \text{Epk} \alpha = \text{Epk} A \), \( \text{Epk} \beta = \text{Epk} B \) and \( \text{Epk} \gamma = \text{Epk} ([A, B]) \), this rewrites as

\[
\text{Epk} ([A, B]) = (\text{Epk} A) \cup ((\text{Epk} B + n) \setminus \{n + 1\}).
\]

This proves Lemma 5.23 (a).

(b) WLOG assume that \( \alpha_i < \beta_j \) for all \( i \in [n] \) and \( j \in [m] \). (Indeed, we can achieve this by choosing a positive integer \( g \) that is larger than each entry of \( \alpha \), and adding \( g \) to each entry of \( \beta \).) Thus, in particular, the entries of \( \alpha \) are distinct from the entries of \( \beta \). Also, \( \alpha_n < \beta_1 \) (since \( \alpha_i < \beta_j \) for all \( i \in [n] \) and \( j \in [m] \)).

Let \( \gamma \) be the \((n + m)\)-permutation \((\alpha_1, \alpha_2, \ldots, \alpha_n, \beta_1, \beta_2, \ldots, \beta_m)\). Then, the descents of \( \gamma \) are obtained as follows:

• Each descent of \( \alpha \) is a descent of \( \gamma \).

• Adding \( n \) to each descent of \( \beta \) yields a descent of \( \gamma \) (that is, if \( i \) is a descent of \( \beta \), then \( i + n \) is a descent of \( \gamma \)).

These are all the descents of \( \gamma \) (in particular, \( n \) is not a descent of \( \gamma \), since \( \alpha_n < \beta_1 \)). Thus,

\[
\text{Des} \gamma = \text{Des} \alpha \cup (\text{Des} \beta + n).
\]

Hence,

\[
\text{Comp} (\text{Des} \gamma) = \text{Comp} (\text{Des} \alpha \cup (\text{Des} \beta + n))
\]

\[
= \text{Comp} (\text{Des} \alpha) \odot \text{Comp} (\text{Des} \beta)
\]

(because of how \( \text{Comp} S \) is defined for a set \( S \)). Since \( \text{Comp} (\text{Des} \pi) = \text{Comp} \pi \) for any permutation \( \pi \), this rewrites as

\[
\text{Comp} \gamma = \text{Comp} \alpha \odot \text{Comp} \beta.
\]

In view of \( \text{Comp} \alpha = A \) and \( \text{Comp} \beta = B \), this rewrites as \( \text{Comp} \gamma = A \odot B \). Thus, \( \text{Epk} (A \odot B) = \text{Epk} \gamma \).

On the other hand, recall again that \( \gamma = (\alpha_1, \alpha_2, \ldots, \alpha_n, \beta_1, \beta_2, \ldots, \beta_m) \) and \( \alpha_n < \beta_1 \). Thus, the exterior peaks of \( \gamma \) are obtained as follows:

• Each exterior peak of \( \alpha \) is an exterior peak of \( \gamma \), except for the number \( n \), which is not an exterior peak of \( \gamma \) (since \( \alpha_n < \beta_1 \)).
• Adding \( n \) to each exterior peak of \( \beta \) yields an exterior peak of \( \gamma \). (This includes \( n + 1 \), if 1 is an exterior peak of \( \beta \), because \( a_n < \beta_1 \).)

These are all the exterior peaks of \( \gamma \). Thus,

\[
\text{Epk } \gamma = ((\text{Epk } \alpha) \setminus \{n\}) \cup (\text{Epk } \beta + n).
\]

In view of \( \text{Epk } \alpha = \text{Epk } A \), \( \text{Epk } \beta = \text{Epk } B \) and \( \text{Epk } \gamma = \text{Epk } (A \odot B) \), this rewrites as

\[
\text{Epk } (A \odot B) = ((\text{Epk } A) \setminus \{n\}) \cup (\text{Epk } B + n).
\]

This proves Lemma 5.23(b). \( \square \)

We can now easily prove Theorem 5.7 again:

**Second proof of Theorem 5.7 (sketched).** Let \( A = \text{QSym} \). Corollary 4.4 shows that \( \mathcal{K}_{\text{Epk}} \) is an ideal of \( \text{QSym} \).

We now argue the following claim:

**Claim 2:** We have \( A \not\asymp \mathcal{K}_{\text{Epk}} \subseteq \mathcal{K}_{\text{Epk}} \).

**[Proof of Claim 2]:** We must show that \( \mathcal{K}_{\text{Epk}} \) is a left \( \not\asymp \)-ideal of \( \text{QSym} \). According to Proposition 5.22(c), this boils down to proving that if \( J \) and \( K \) are two \( \text{Epk} \)-equivalent nonempty compositions, and if \( G \) is any nonempty composition, then \([G, J]\) and \([G, K]\) are \( \text{Epk} \)-equivalent.

So let \( J \) and \( K \) be two \( \text{Epk} \)-equivalent nonempty compositions. Thus, \(|J| = |K| > 0\) and \( \text{Epk } J = \text{Epk } K \).

Define a positive integer \( n \) by \( n = |G| \). Lemma 5.23(a) (applied to \( A = G \) and \( B = J \)) yields

\[
\text{Epk } ([G, J]) = (\text{Epk } G) \cup ((\text{Epk } J + n) \setminus \{n + 1\}). \tag{44}
\]

Similarly,

\[
\text{Epk } ([G, K]) = (\text{Epk } G) \cup ((\text{Epk } K + n) \setminus \{n + 1\}). \tag{45}
\]

The right hand sides of (44) and (45) are equal (since \( \text{Epk } J = \text{Epk } K \)). Hence, the left hand sides are equal as well. In other words, \( \text{Epk } ([G, J]) = \text{Epk } ([G, K]) \).

Combining this with

\[
\]

we conclude that \([G, J]\) and \([G, K]\) are \( \text{Epk} \)-equivalent. As we have said, this concludes the proof of Claim 2.

Similarly to Claim 2, we can show the following three claims:

---

34These claims are numbered Claim 2, Claim 4, Claim 6 and Claim 8, in order to match the numbering of the corresponding claims in the first proof of Theorem 5.7 above.
Claim 4: We have $\mathcal{K}_{Epk} \not\subseteq A \subseteq \mathcal{K}_{Epk}$.

Claim 6: We have $A \not\subseteq \mathcal{K}_{Epk} \subseteq \mathcal{K}_{Epk}$.

Claim 8: We have $\mathcal{K}_{Epk} \not\subseteq A \subseteq \mathcal{K}_{Epk}$.

(Of course, in proving Claims 4, 6 and 8, we need to use the other three parts of Proposition 5.22 instead of Proposition 5.22 (c), and we occasionally need to use Lemma 5.23 (b) instead of Lemma 5.23 (a).)

Combining Claim 2 and Claim 4, we conclude that $\mathcal{K}_{Epk}$ is a $\prec$-ideal of $A$.

Combining Claim 6 and Claim 8, we conclude that $\mathcal{K}_{Epk}$ is a $\succ$-ideal of $A$.

Thus, Theorem 5.4 (c) (applied to $M = \mathcal{K}_{Epk}$) shows that $\mathcal{K}_{Epk}$ is a $\prec$-ideal and a $\succ$-ideal of $\text{QSym}$. This proves Theorem 5.7 again.

5.6. Left/right-shuffle-compatibility of other statistics

Let us now briefly analyze the kernels $\mathcal{K}_{st}$ of some other descent statistics, following the same approach that we took in our above second proof of Theorem 5.7 again.

5.6.1. The descent set $\text{Des}$

First of all, the following is obvious:

**Proposition 5.24.** The ideal $\mathcal{K}_{\text{Des}}$ of $\text{QSym}$ is the trivial ideal 0, and is a $\not\subseteq$-ideal, a $\not\supset$-ideal, a $\prec$-ideal and a $\succ$-ideal of $\text{QSym}$.

**Corollary 5.25.** The descent statistic $\text{Des}$ is left-shuffle-compatible and right-shuffle-compatible.

**Proof of Corollary 5.25.** Corollary 5.25 can be derived from Proposition 5.24 in the same way as Corollary 5.17 was derived from Theorem 5.7.

5.6.2. The descent number $\text{des}$

The permutation statistic $\text{des}$ (called the *descent number*) is defined as follows: For each permutation $\pi$, we set $\text{des} \; \pi = |\text{Des} \; \pi|$ (that is, $\text{des} \; \pi$ is the number of all descents of $\pi$). It was proven in [GesZhu17, Theorem 4.6 (a)] that this statistic $\text{des}$ is shuffle-compatible. Furthermore, $\text{des}$ is clearly a descent statistic. Hence, Proposition 4.3 (applied to $st = \text{des}$) shows that $\mathcal{K}_{\text{des}}$ is an ideal of $\text{QSym}$. We now claim the following:

**Proposition 5.26.** The ideal $\mathcal{K}_{\text{des}}$ of $\text{QSym}$ is a $\not\subseteq$-ideal, a $\not\supset$-ideal, a $\prec$-ideal and a $\succ$-ideal of $\text{QSym}$. 

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Corollary 5.27. The descent statistic des is left-shuffle-compatible and right-shuffle-compatible.

The proofs rely on the following fact (similar to Lemma 5.23):

Lemma 5.28. Let $A$ and $B$ be two nonempty compositions. Let $n = |A|$.

(a) We have $\text{des}([A, B]) = \text{des} A + \text{des} B + 1$.

(b) We have $\text{des}(A \odot B) = \text{des} A + \text{des} B$.

Proof of Lemma 5.28. If $I$ is a nonempty composition, then $\text{des} I$ equals the length of $I$ minus 1. Lemma 5.28 follows easily from this. □

Proof of Proposition 5.26. Analogous to the above second proof of Theorem 5.7, but using Lemma 5.28 instead of Lemma 5.23. □

Proof of Corollary 5.27. Corollary 5.27 can be derived from Proposition 5.26 in the same way as Corollary 5.17 was derived from Theorem 5.7. □

5.6.3. The major index maj

The permutation statistic $\text{maj}$ (called the major index) is defined as follows: For each permutation $\pi$, we set $\text{maj} \pi = \sum_{i \in \text{Des} \pi} i$ (that is, $\text{maj} \pi$ is the sum of all descents of $\pi$). It was proven in [GesZhu17, Theorem 3.1 (a)] that this statistic $\text{maj}$ is shuffle-compatible. Furthermore, $\text{maj}$ is clearly a descent statistic. Hence, Proposition 4.3 (applied to $\text{st} = \text{maj}$) shows that $\mathcal{K}_{\text{maj}}$ is an ideal of $\text{QSym}$. We now claim the following:

Proposition 5.29. The ideal $\mathcal{K}_{\text{maj}}$ of $\text{QSym}$ is a right $\prec$-ideal and a right $\preceq$-ideal, but neither a $\prec$-ideal nor a $\preceq$-ideal of $\text{QSym}$.

Corollary 5.30. The descent statistic $\text{maj}$ is neither left-shuffle-compatible nor right-shuffle-compatible.

The proofs rely on the following fact (similar to Lemma 5.23):

Lemma 5.31. Let $A$ and $B$ be two nonempty compositions. Let $n = |A|$.

(a) We have $\text{maj}([A, B]) = \text{maj} A + \text{maj} B + n \cdot (\text{des} B + 1)$.

(b) We have $\text{maj}(A \odot B) = \text{maj} A + \text{maj} B + n \cdot \text{des} B$.

Proof of Lemma 5.31. If $I = (i_1, i_2, \ldots, i_k)$ is a nonempty composition, then

$$\text{maj} I = i_1 + (i_1 + i_2) + (i_1 + i_2 + i_3) + \cdots + (i_1 + i_2 + \cdots + i_{k-1})$$

$$= (k-1)i_1 + (k-2)i_2 + \cdots + (k-k)i_k.$$ 

Lemma 5.31 follows easily from this. □
Proof of Proposition 5.29. To prove that $K_{maj}$ is a right $\prec$-ideal of QSym, we proceed as in the proof of Claim 2 in the second proof of Theorem 5.7 but using Lemma 5.31 instead of Lemma 5.23. Similarly, we can show that $K_{maj}$ is a right $\Phi$-ideal of QSym.

To prove that $K_{maj}$ is not a $\preceq$-ideal of QSym (and not even a left $\prec$-ideal of QSym), it suffices to find some $m \in K_{maj}$ and some $a \in QSym$ such that $a \prec m \notin K_{maj}$. For example, we can take $m = F_{(1,1,2)} - F_{(3,1)}$ and $a = F_{(1)}$; then, $a \prec m = F_{(1,1,1,2)} - F_{(1,3,1)} \notin K_{maj}$. The same values of $m$ and $a$ also satisfy $a \succeq m \notin K_{maj}$, $m \prec a \notin K_{maj}$ and $m \succeq a \notin K_{maj}$; thus, $K_{maj}$ is not a $\succeq$-ideal of QSym either. Proposition 5.29 is now proven.

Proof of Corollary 5.30. Again, this follows from Proposition 5.29.

5.6.4. The joint statistic $(\text{des, maj})$

The next permutation statistic we shall study is the so-called joint statistic $(\text{des, maj})$. This statistic is defined as the permutation statistic that sends each permutation $\pi$ to the ordered pair $(\text{des } \pi, \text{maj } \pi)$. (Calling it $(\text{des, maj})$ is thus a slight abuse of notation.) It was proven in [GesZhu17, Theorem 4.5 (a)] that this statistic $(\text{des, maj})$ is shuffle-compatible. Furthermore, $(\text{des, maj})$ is clearly a descent statistic. Hence, Proposition 4.3 (applied to $st = (\text{des, maj})$) shows that $K_{(\text{des, maj})}$ is an ideal of QSym. We now claim the following:

Proposition 5.32. The ideal $K_{(\text{des, maj})}$ of QSym is a $\prec$-ideal, a $\Phi$-ideal, a $\prec$-ideal and a $\succeq$-ideal of QSym.

Corollary 5.33. The descent statistic $(\text{des, maj})$ is left-shuffle-compatible and right-shuffle-compatible.

Proof of Proposition 5.32. Analogous to the above second proof of Theorem 5.7, but using Lemma 5.28 together with Lemma 5.31 instead of Lemma 5.23.

Proof of Corollary 5.33. Corollary 5.33 can be derived from Proposition 5.32 in the same way as Corollary 5.17 was derived from Theorem 5.7.

5.6.5. The left peak set $Lpk$

Recall the permutation statistic $Lpk$ (the left peak set) defined in Definition 1.10. It was proven in [GesZhu17, Theorem 4.9 (a)] that this statistic $Lpk$ is shuffle-compatible. Furthermore, $Lpk$ is clearly a descent statistic. Hence, Proposition 4.3 (applied to $st = Lpk$) shows that $K_{Lpk}$ is an ideal of QSym. We now claim the following:

Proposition 5.34. The ideal $K_{Lpk}$ of QSym is a $\prec$-ideal, a $\Phi$-ideal, a $\prec$-ideal and a $\succeq$-ideal of QSym.
Corollary 5.35. The descent statistic Lpk is left-shuffle-compatible and right-shuffle-compatible.

The proofs rely on the following fact (similar to Lemma 5.23):

Lemma 5.36. Let A and B be two nonempty compositions. Let n = |A|.
(a) We have Lpk ([A, B]) = (Lpk A) \cup ((Lpk B + n) \setminus \{n + 1\}) \cup \{n \mid n - 1 \notin \text{Des } A\}.
(b) We have Lpk (A \odot B) = (Lpk A) \cup (Lpk B + n).

Proof of Lemma 5.36. Not unlike the proof of Lemma 5.23 (but left to the reader).

Proof of Proposition 5.34. Analogous to the above second proof of Theorem 5.7, but using Lemma 5.36 instead of Lemma 5.23. This time, however, the analogue of Claim 4 will be false (i.e., we don’t have \( K_{Lpk} \not\subseteq \mathcal{K}_{Lpk} \)), because the formula for Lpk ([A, B]) in Lemma 5.36 (a) depends on Des A. Thus, \( K_{Lpk} \) is merely a left \( \prec \)-ideal, not a \( \preceq \)-ideal. (But this does not prevent us from applying Theorem 5.4 (c), because that theorem does not require \( M \preceq A \subseteq M \).)

Proof of Corollary 5.35. Corollary 5.35 can be derived from Proposition 5.34 in the same way as Corollary 5.17 was derived from Theorem 5.7.

5.6.6. The right peak set Rpk

Recall the permutation statistic Rpk (the right peak set) defined in Definition 1.10. It was proven in [GesZhu17, §4.4] that this statistic Rpk is shuffle-compatible. Furthermore, Rpk is clearly a descent statistic. Hence, Proposition 4.3 (applied to \( st = Rpk \)) shows that \( K_{Rpk} \) is an ideal of QSym. We now claim the following:

Proposition 5.37. The ideal \( K_{Rpk} \) of QSym is a \( \prec \)-ideal, a right \( \Phi \)-ideal, a left \( \prec \)-ideal and a left \( \succeq \)-ideal, but neither a \( \prec \)-ideal nor a \( \succeq \)-ideal of QSym.

Corollary 5.38. The descent statistic Rpk is neither left-shuffle-compatible nor right-shuffle-compatible.

The proofs rely on the following fact (similar to Lemma 5.23):

Lemma 5.39. Let A and B be two nonempty compositions. Let n = |A| and m = |B|.
(a) We have Rpk ([A, B]) = (Rpk A) \cup (Rpk B + n).
(b) We have Rpk (A \odot B) = ((Rpk A) \setminus \{n\}) \cup (Rpk B + n) \cup \{n + 1 \mid 1 \in \text{Des } B \text{ or } m = 1\}.

Proof of Lemma 5.39. Not unlike the proof of Lemma 5.23 (but left to the reader).
Proof of Proposition 5.37. To prove that $\mathcal{K}_{Rpk}$ is a left and a right $\Phi$-ideal, we proceed as in the above second proof of Theorem 5.7 but using Lemma 5.39 instead of Lemma 5.23. This time, however, the analogue of Claim 6 will be false (i.e., we don’t have $A \Phi \mathcal{K}_{Rpk} \subseteq \mathcal{K}_{Rpk}$), because the formula for $Rpk(A \odot B)$ in Lemma 5.39 (b) depends on Des $B$. Thus, $\mathcal{K}_{Rpk}$ is merely a right $\Phi$-ideal, not a $\Phi$-ideal. This prevents us from applying Theorem 5.4 (c). However, we can apply Theorem 5.4 (b) instead, and obtain $QSym \geq \mathcal{K}_{Rpk} \subseteq \mathcal{K}_{Rpk}$. In other words, $\mathcal{K}_{Rpk}$ is a left $\geq$-ideal of $QSym$. Using (25), we thus easily see that $\mathcal{K}_{Rpk}$ is a left $\leq$-ideal of $QSym$ as well.

To prove that $\mathcal{K}_{Rpk}$ is not a $\prec$-ideal of $QSym$ (and not even a right $\prec$-ideal of $QSym$), it suffices to find some $m \in \mathcal{K}_{Rpk}$ and some $a \in QSym$ such that $m \prec a \notin \mathcal{K}_{Rpk}$. For example, we can take $m = F_{(1,2)} - F_{(3)}$ and $a = F_{(1)}$; then,

$$m \prec a = F_{(3,2)} + F_{(2,3)} + F_{(2,2,1)} - F_{(1,2,2)} - F_{(1,1,3)} - F_{(1,1,2,1)} \notin \mathcal{K}_{Rpk}.$$

The same values of $m$ and $a$ also satisfy $m \succcurlyeq a \notin \mathcal{K}_{Rpk}$; thus, $\mathcal{K}_{Rpk}$ is not a $\succcurlyeq$-ideal of $QSym$ either. Proposition 5.37 is now proven.

Proof of Corollary 5.38. Follows from Proposition 5.37.

5.6.7. The peak set $P_k$

Recall the permutation statistic $P_k$ (the peak set) defined in Definition 1.10. It was proven in [GesZhu17, Theorem 4.7 (a)] that this statistic $P_k$ is shuffle-compatible. Furthermore, $P_k$ is clearly a descent statistic. Hence, Proposition 4.3 (applied to $st = P_k$) shows that $\mathcal{K}_{P_k}$ is an ideal of $QSym$. We now claim the following:

**Proposition 5.40.** The ideal $\mathcal{K}_{P_k}$ of $QSym$ is a left $\neq$-ideal, a right $\Phi$-ideal, a left $\prec$-ideal and a left $\succcurlyeq$-ideal, but neither a $\prec$-ideal nor a $\succcurlyeq$-ideal of $QSym$.

**Corollary 5.41.** The descent statistic $P_k$ is neither left-shuffle-compatible nor right-shuffle-compatible.

The proofs rely on the following fact (similar to Lemma 5.23):

**Lemma 5.42.** Let $A$ and $B$ be two nonempty compositions. Let $n = |A|$ and $m = |B|$.

(a) We have $P_k([A, B]) = (P_k A) \cup (P_k B + n) \cup \{n \mid n - 1 \notin \text{Des } A \text{ and } n > 1\}$.

(b) We have $P_k(A \odot B) = (P_k A) \cup (P_k B + n) \cup \{n + 1 \mid 1 \in \text{Des } B\}$.

Proof of Lemma 5.42. Not unlike the proof of Lemma 5.23 (but left to the reader).
Proof of Proposition 5.40. To prove that $\mathcal{K}_{P_k}$ is a left $\mathcal{X}$-ideal and a right $\mathcal{Y}$-ideal, we proceed as in the above second proof of Theorem 5.7, but using Lemma 5.39 instead of Lemma 5.23. This time, however, the analogues of Claim 4 and Claim 6 will be false (i.e., neither $\mathcal{K}_{P_k} \not\subseteq \mathcal{K}_{P_k}$ nor $A \not\subseteq \mathcal{K}_{P_k}$ will hold), because the formula for $P_k([A, B])$ in Lemma 5.42 (a) depends on Des $A$ whereas the formula for $P_k(A \odot B)$ in Lemma 5.42 (b) depends on Des $B$. Again, this prevents us from applying Theorem 5.4 (c). However, we can apply Theorem 5.4 (b) instead, and obtain $\text{QSym} \geq \mathcal{K}_{P_k} \subseteq \mathcal{K}_{P_k}$. In other words, $\mathcal{K}_{P_k}$ is a left $\geq$-ideal of QSym. Using (25), we thus easily see that $\mathcal{K}_{P_k}$ is a left $\prec$-ideal of QSym as well.

To prove that $\mathcal{K}_{P_k}$ is not a $\prec$-ideal of QSym (and not even a right $\prec$-ideal of QSym), it suffices to find some $m \in \mathcal{K}_{P_k}$ and some $a \in \text{QSym}$ such that $m \prec a \notin \mathcal{K}_{P_k}$. For example, we can take $m = F(1,2) - F(3)$ and $a = F(1)$; then,

$$m \prec a = F(3,2) + F(2,3) + F(2,2,1) - F(1,2,2) - F(1,1,3) - F(1,1,2,1) \notin \mathcal{K}_{P_k}.$$ 

The same values of $m$ and $a$ also satisfy $m \succeq a \notin \mathcal{K}_{P_k}$; thus, $\mathcal{K}_{P_k}$ is not a $\succeq$-ideal of QSym either. Proposition 5.40 is now proven. □

Proof of Corollary 5.41. Follows from Proposition 5.40. □

References


