Quasisymmetric functions and Γ-partitions

Darij Grinberg

version 0.3, October 18, 2015

The goal of this note is to give a new proof (and restatement) of Malvenuto's and Reutenauer's antipode formula for quasisymmetric functions [MalReu98, Theorem 3.1], and to generalize it in a way which also encompasses Jochemko's enumerative results in [Joch13, Theorem 2.8]. Along the way, we will give a self-contained introduction into quasisymmetric functions and reprove some of their basic properties.

[... add some material about the chain lemma as used in my original proof of Zabrocki's conjecture.]

This note is work in progress and the proofs are mostly just sketched. Nevertheless please let me know of any errors you find! (Just don't act surprised.)

0.1. Acknowledgments

Katharina Jochemko's work on order polynomials [Joch13] and my attempts at generalizing that work gave the present paper its first impetus.

1. Quasisymmetric functions

The ring QSym of quasisymmetric functions (along with its further structure, such as it being a Hopf algebra, having an internal coproduct, dendriform operations etc.) has been the object of decades of research, probably beginning with Gessel [Gessel84] [... check what he actually does in that paper] and continued, among other places, in the 7-part NCSF study [... add references]. Introductions into the theory of this ring can be found in [Malve93], [GriRei15, Chapter 5], [Hazewi08, §11], [Stan99, §7.19], [Hivert99], [Gessel84] and [MalReu98, §1] [... update references to ever-expanding Reiner notes]; applications appear in [ABS03]. We recall the definitions and the facts that we need. [... check self-containedness.] First, we make some conventions which we will use throughout the text:

Definition 1.1. In the following, we use the notations $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{N}_+ = \{1, 2, 3, \ldots\}$.

Definition 1.2. If a and b are two integers such that $a \geq b + 1$, then $\{a, a+1, \ldots, b\}$ means the empty set. (In particular, $\{1, 2, \ldots, -1\}$ means $\varnothing$.)

Definition 1.3. If $\mathcal{A}$ is any statement, then $[\mathcal{A}]$ will mean the integer $\begin{cases} 1, & \text{if } \mathcal{A} \text{ is true;} \\ 0, & \text{if } \mathcal{A} \text{ is false} \end{cases}$.

Definition 1.4. Let $\mathbf{k}$ be any commutative ring. We consider $\mathbf{k}$ to be fixed for the following (unless otherwise stated). (The reader can assume that $\mathbf{k} = \mathbb{Z}$ without losing anything; we will briefly use $\mathbf{k} = \mathbb{Q}$ in a proof.)

Let $x_1, x_2, x_3, \ldots$ be a countable family of pairwise distinct symbols; we will use these symbols as (commuting) indeterminates.

In the following, a *monomial* means a monomial in these commuting indeterminates $x_1, x_2, x_3, \ldots$ (without coefficient). (We view monomials as formal expressions[1]; thus, e.g., we distinguish between the monomials 1 and x_1 even if $\mathbf{k} = 0$.) For instance, $x_1^3 x_5 x_9$, $x_2 x_4^2$, x_7 and 1 are monomials. We regard, e.g., the monomials $x_1^0 x_2$ and x_2 as identical. The *total degree* of a monomial is defined as the sum of the exponents of all indeterminates in the monomial. For example, the total degree of the monomial $x_2 x_3^4 x_{15}^2$ is $1 + 4 + 2 = 7$, while the total degree of the monomial 1 is (empty sum) $= 0$.

Two monomials $\mathfrak{m}$ and $\mathfrak{n}$ are said to be *pack-equivalent* if there exists an $\ell \in \mathbb{N}$, a list $(a_1, a_2, \ldots, a_\ell) \in \mathbb{N}_+^\ell$ of positive integers, and two strictly increasing sequences $(i_1 < i_2 < \cdots < i_\ell)$ and $(j_1 < j_2 < \cdots < j_\ell)$ of positive integers such that $\mathfrak{m} = x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_\ell}^{a_\ell}$ and $\mathfrak{n} = x_{j_1}^{a_1} x_{j_2}^{a_2} \cdots x_{j_\ell}^{a_\ell}$. For instance, the monomials $x_1 x_2^2$, $x_1 x_3^2$, $x_4 x_{19}^2$ are pairwise pack-equivalent (all having $\ell = 2$ and $(a_1, a_2) = (1, 2)$), while the monomials $x_1 x_3^2$ and $x_3 x_1^2$ are not pack-equivalent. We will collect some (mostly trivial) properties of pack-equivalent monomials in Proposition 1.6; as for now, let us remark that pack-equivalent monomials can be intuitively regarded as monomials which can be obtained from each other by "pulling apart" or "pushing together" the variables (without moving them past each other or merging them).

A power series $f \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ is said to be *bounded-degree* if there exists an $n \in \mathbb{N}$ such that every monomial having total degree $\geq n$ has coefficient 0 in f. For instance, the power series $\sum_i x_i = x_1 + x_2 + x_3 + \cdots$ is bounded-degree. (Here and in the following, summation indices are supposed to be positive integers if not otherwise specified.) Also, the power series $x_1 + x_2 + x_3 + \cdots + x_1 x_2 + x_2 x_3 + x_3 x_4 + \cdots$ is bounded-degree, while the power series $\dfrac{1}{1 - x_2} = 1 + x_2 + x_2^2 + x_2^3 + \cdots$ is not.

Let $\mathbf{k}[[x_1, x_2, x_3, \ldots]]_{\text{bdd}}$ denote the set of bounded-degree power series in $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$. This set is a $\mathbf{k}$-subalgebra of $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ (by Proposition 1.7 **(a)** below).

A power series $f \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ is said to be *quasisymmetric* if for any two pack-equivalent monomials $\mathfrak{m}$ and $\mathfrak{n}$, the coefficient of $\mathfrak{m}$ in f equals the coefficient of $\mathfrak{n}$ in f. For instance, the power series

$$\sum_{i<j} x_i x_j^2 = x_1 x_2^2 + x_1 x_3^2 + x_1 x_4^2 + \cdots + x_2 x_3^2 + x_2 x_4^2 + \cdots + x_3 x_4^2 + \cdots$$

is quasisymmetric, while the power series

$$\sum_{i<j} x_i x_{j+1}^2 = x_1 x_3^2 + x_1 x_4^2 + x_1 x_5^2 + \cdots + x_2 x_4^2 + x_2 x_5^2 + \cdots + x_3 x_5^2 + \cdots$$

is not (its coefficients before $x_1 x_2^2$ and before $x_1 x_3^2$ are not equal, despite $x_1 x_2^2$ and $x_1 x_3^2$ being pack-equivalent). The set of all quasisymmetric power series in $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ is a $\mathbf{k}$-subalgebra of $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ (by Proposition 1.7 **(b)** further below).

We denote by $\operatorname{QSym}_{\mathbf{k}}$ the set of all quasisymmetric power series in $\mathbf{k}[[x_1, x_2, x_3, \ldots]]_{\text{bdd}}$. This is a $\mathbf{k}$-subalgebra of $\mathbf{k}[[x_1, x_2, x_3, \ldots]]_{\text{bdd}}$ (by Proposition 1.7 **(c)** further below). This $\mathbf{k}$-subalgebra $\operatorname{QSym}_{\mathbf{k}}$ is called the $\mathbf{k}$*-algebra of quasisymmetric functions over* $\mathbf{k}$ (even though its elements are not "functions" in any literal sense).

If $\mathbf{k} = \mathbb{Z}$, then the ring $\operatorname{QSym}_{\mathbf{k}} = \operatorname{QSym}_{\mathbb{Z}}$ is simply denoted by QSym. The ring QSym is called the *ring of quasisymmetric functions.*

We endow the ring $\mathbf{k}$ with the discrete topology. To define a topology on the $\mathbf{k}$-algebra $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$, we (temporarily) regard every power series in $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ as the family of its coefficients. Thus, $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ becomes a product of infinitely many copies of $\mathbf{k}$ (one for each monomial). This allows us to define a product topology on $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$. This product topology is the topology that we will be using whenever we make statements about convergence in $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ or write down infinite sums of power series. A sequence $(a_n)_{n\in\mathbb{N}}$ of power series converges to a power series a with respect to this topology if and only if for every monomial $\mathfrak{m}$, all sufficiently high $n \in \mathbb{N}$ satisfy

$$(\text{the coefficient of } \mathfrak{m} \text{ in } a_n) = (\text{the coefficient of } \mathfrak{m} \text{ in } a).$$

Note that this is **not** the topology obtained by taking the completion of $\mathbf{k}[x_1, x_2, x_3, \ldots]$ with respect to the standard grading (in which all x_i have degree 1). In fact, the latter completion is not even $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ as a set.

Many authors work only in the situation when $\mathbf{k} = \mathbb{Z}$; this usually does not render their results any less general, because $\operatorname{QSym}_{\mathbf{k}} \cong \operatorname{QSym}_{\mathbb{Z}} \otimes \mathbf{k}$ for every commutative ring $\mathbf{k}$ (by Proposition 1.11 further below).

Before we proceed any further, let us state some basic facts. The first is a collection of properties of pack-equivalent monomials.

Definition 1.5. **(a)** A monomial is said to be *packed* if it has the form $x_1^{a_1} x_2^{a_2} \cdots x_\ell^{a_\ell}$ for some $\ell \in \mathbb{N}$ and some $(a_1, a_2, \ldots, a_\ell) \in \mathbb{N}_+^\ell$.

(b) If $\mathfrak{m}$ is a monomial, then $\operatorname{Supp} \mathfrak{m}$ will be the finite subset

$$\{i \in \mathbb{N}_+ \mid \text{the variable } x_i \text{ appears with a positive exponent in } \mathfrak{m}\}$$

[1]More precisely, a monomial is an element of the free abelian monoid on the set $\{x_1, x_2, x_3, \ldots\}$ (where $x_1, x_2, x_3, \ldots$ are countably many distinct indeterminates).

of $\mathbb{N}_+$.

Proposition 1.6. **(a)** Let $\mathfrak{m}$ be a monomial. Then, the monomial $\mathfrak{m}$ is packed if and only if there exists an $\ell \in \mathbb{N}$ such that $\operatorname{Supp} \mathfrak{m} = \{1, 2, \ldots, \ell\}$.

(b) Let $\mathfrak{m}$ be a monomial. Then, there exists a unique packed monomial $\mathfrak{n}$ which is pack-equivalent to $\mathfrak{m}$. This monomial $\mathfrak{n}$ is called the *packing* of $\mathfrak{m}$ and denoted by $\operatorname{pack} \mathfrak{m}$.

(c) For any $\ell \in \mathbb{N}$, any list $(a_1, a_2, \ldots, a_\ell) \in \mathbb{N}_+^\ell$ of positive integers, and any strictly increasing sequence $(i_1 < i_2 < \cdots < i_\ell)$ of positive integers, we have $\operatorname{pack} \left(x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_\ell}^{a_\ell} \right) = x_1^{a_1} x_2^{a_2} \cdots x_\ell^{a_\ell}$.

(d) Let $\mathfrak{m}$ and $\mathfrak{n}$ be two monomials. Then, $\mathfrak{m}$ and $\mathfrak{n}$ are pack-equivalent if and only if $\operatorname{pack} \mathfrak{m} = \operatorname{pack} \mathfrak{n}$.

(e) Being pack-equivalent is an equivalence relation.

Properties like Proposition 1.6 are normally used without explicit mention in literature; the idea of formalizing them using the notion of pack-equivalence originates in work of Hivert, Novelli and Thibon [NovThi05].

Proof of Proposition 1.6 (sketched). **(a)** Let us first show the following logical implication:

$$\begin{aligned} &(\text{the monomial } \mathfrak{m} \text{ is packed}) \\ &\Longrightarrow \ (\text{there exists an } \ell \in \mathbb{N} \text{ such that } \operatorname{Supp} \mathfrak{m} = \{1, 2, \ldots, \ell\}) . \end{aligned} \tag{1}$$

Proof of (1): Assume that the monomial $\mathfrak{m}$ is packed. Recall that the monomial $\mathfrak{m}$ is packed if and only if $\mathfrak{m}$ has the form $x_1^{a_1} x_2^{a_2} \cdots x_\ell^{a_\ell}$ for some $\ell \in \mathbb{N}$ and some $(a_1, a_2, \ldots, a_\ell) \in \mathbb{N}_+^\ell$ (due to the definition of "packed"). Thus, $\mathfrak{m}$ has the form $x_1^{a_1} x_2^{a_2} \cdots x_\ell^{a_\ell}$ for some $\ell \in \mathbb{N}$ and some $(a_1, a_2, \ldots, a_\ell) \in \mathbb{N}_+^\ell$ (since the monomial $\mathfrak{m}$ is packed). Denote this ℓ and this $(a_1, a_2, \ldots, a_\ell)$ by m and $(b_1, b_2, \ldots, b_m)$. Thus, m is an element of $\mathbb{N}$, and $(b_1, b_2, \ldots, b_m)$ is an element of $\mathbb{N}_+^m$ such that $\mathfrak{m} = x_1^{b_1} x_2^{b_2} \cdots x_m^{b_m}$.

For every $i \in \{1, 2, \ldots, m\}$, the variable x_i appears in the monomial $\mathfrak{m}$ with exponent b_i (because $\mathfrak{m} = x_1^{b_1} x_2^{b_2} \cdots x_m^{b_m}$). This exponent b_i is positive (since $b_i \in \mathbb{N}_+$ (since $(b_1, b_2, \ldots, b_m) \in \mathbb{N}_+^m$)). Therefore XXX

Now, the definition of $\operatorname{Supp} \mathfrak{m}$ yields

$$\operatorname{Supp} \mathfrak{m}$$
$$=$$

XXX

[...] □

We owe the reader a proof of the following fact:

Proposition 1.7. **(a)** The set $\mathbf{k}[[x_1, x_2, x_3, \ldots]]_{\text{bdd}}$ is a $\mathbf{k}$-subalgebra of $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$.

(b) The set of all quasisymmetric power series in $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ is a $\mathbf{k}$-subalgebra of $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$.

(c) The subset $\operatorname{QSym}_{\mathbf{k}}$ of $\mathbf{k}[[x_1, x_2, x_3, \ldots]]_{\text{bdd}}$ is a $\mathbf{k}$-subalgebra of $\mathbf{k}[[x_1, x_2, x_3, \ldots]]_{\text{bdd}}$.

Proof of Proposition 1.7 (sketched). **(a)** [...]

(b) Let $\operatorname{QSYM}_{\mathbf{k}}$ denote the set of all quasisymmetric power series in $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$. In order to prove Proposition 1.7 **(b)**, we need to show that $\operatorname{QSYM}_{\mathbf{k}}$ is a $\mathbf{k}$-subalgebra of $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$. In order to do that, it is enough to check that $fg \in \operatorname{QSYM}_{\mathbf{k}}$ for every $f \in \operatorname{QSYM}_{\mathbf{k}}$ and $g \in \operatorname{QSYM}_{\mathbf{k}}$ (because it is clear that $\operatorname{QSYM}_{\mathbf{k}}$ is a $\mathbf{k}$-submodule of $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$, and contains the power series 1).

So let $f \in \operatorname{QSYM}_{\mathbf{k}}$ and $g \in \operatorname{QSYM}_{\mathbf{k}}$ be arbitrary. Then, f and g are quasisymmetric power series.

Now, let $\mathfrak{m}$ and $\mathfrak{n}$ be two pack-equivalent monomials. We are going to show that the coefficient of $\mathfrak{m}$ in fg equals the coefficient of $\mathfrak{n}$ in fg.

For every power series $h \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]_{\text{bdd}}$ and every monomial $\mathfrak{p}$, let $[\mathfrak{p}]\, h$ denote the coefficient of $\mathfrak{p}$ in h. By the definition of the product of two power series, we have

$$[\mathfrak{m}]\,(fg) = \sum_{\substack{\mathfrak{u} \text{ and } \mathfrak{v} \text{ are monomials;} \\ \mathfrak{u}\mathfrak{v}=\mathfrak{m}}} ([\mathfrak{u}]\, f)\,([\mathfrak{v}]\, g)$$

[...]

(c) It is clear that $\operatorname{QSym}_{\mathbf{k}}$ is a $\mathbf{k}$-submodule of $\mathbf{k}[[x_1, x_2, x_3, \ldots]]_{\text{bdd}}$, and contains the power series 1. Hence, in order to prove Proposition 1.7 **(c)**, we only need to check that $fg \in \operatorname{QSym}_{\mathbf{k}}$ for every $f \in \operatorname{QSym}_{\mathbf{k}}$ and $g \in \operatorname{QSym}_{\mathbf{k}}$.

So let $f \in \operatorname{QSym}_{\mathbf{k}}$ and $g \in \operatorname{QSym}_{\mathbf{k}}$ be arbitrary. Then, f and g belong to $\mathbf{k}[[x_1, x_2, x_3, \ldots]]_{\text{bdd}}$, [...] □

We will later give an alternative proof of this theorem, as a consequence of our results on Γ-partitions later on. [... when?] [... also part **(b)**]

A natural question is how $\operatorname{QSym}_{\mathbf{k}}$ looks like, e.g., whether it has a basis as a $\mathbf{k}$-module. It is clear that every symmetric power series in $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ is quasisymmetric, so the $\mathbf{k}$-algebra of symmetric functions is a $\mathbf{k}$-subalgebra of $\operatorname{QSym}_{\mathbf{k}}$. However, $\operatorname{QSym}_{\mathbf{k}}$ is bigger (when $\mathbf{k} \neq 0$), containing non-symmetric power series such as $\sum_{i<j} x_i x_j^2$.

A $\mathbf{k}$-basis of $\operatorname{QSym}_{\mathbf{k}}$ can be easily found:

Definition 1.8. In the following, a *composition* will mean a tuple of positive integers. For example, $(3,1)$ and $(1,2,1)$ are compositions, and so is the empty tuple $()$.

Let Comp denote the set of all compositions.

If α is a composition, then the sum of the elements of α is called the *size* of α. We write $\alpha \models n$ to say that α is a composition of size n. Also, if α is any tuple of integers (e.g., a composition), then the length of α (that is, the number of entries of α) is called $\ell(\alpha)$.

Let α be a composition. Write α in the form $(\alpha_1, \alpha_2, \ldots, \alpha_\ell) \in \mathbb{N}_+^\ell$. Then, we denote by M_α the power series

$$\sum_{i_1 < i_2 < \cdots < i_\ell} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell} \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$$

This power series M_α is called the *α-th monomial quasisymmetric function*.

For instance, $M_{()} = 1$ (here, the sum is over all choices of $i_1 < i_2 < \cdots < i_\ell$ with $\ell = 0$; but there is only one such choice, namely the "empty choice"), while

$$M_{(2)} = \sum_i x_i^2 = x_1^2 + x_2^2 + x_3^2 + \cdots;$$
$$M_{(1,3)} = \sum_{i<j} x_i x_j^3 = x_1 x_2^3 + x_1 x_3^3 + x_1 x_4^3 + \cdots + x_2 x_3^3 + x_2 x_4^3 + \cdots + x_3 x_4^3 + \cdots;$$
$$M_{(2,2)} = \sum_{i<j} x_i^2 x_j^2 = x_1^2 x_2^2 + x_1^2 x_3^2 + x_1^2 x_4^2 + \cdots + x_2^2 x_3^2 + x_2^2 x_4^2 + \cdots + x_3^2 x_4^2 + \cdots.$$

Proposition 1.9. Let α be a composition. Then, $M_\alpha \in \operatorname{QSym}_{\mathbf{k}}$.

Proof of Proposition 1.9 (sketched). [...] □

Proposition 1.10. The family $(M_\alpha)_{\alpha \in \operatorname{Comp}}$ is a basis of the $\mathbf{k}$-module $\operatorname{QSym}_{\mathbf{k}}$.

Proof of Proposition 1.10 (sketched). The family $(M_\alpha)_{\alpha \in \operatorname{Comp}}$ is linearly independent, since its elements are nonzero and have no monomials in common (pairwise). Thus it remains to prove that this family spans the $\mathbf{k}$-module $\operatorname{QSym}_{\mathbf{k}}$.

Let $f \in \operatorname{QSym}_{\mathbf{k}}$. For every $(\beta_1, \beta_2, \beta_3, \ldots) \in \mathbb{N}^{\mathbb{N}_+}$, let $\operatorname{coeff}_{\beta_1, \beta_2, \beta_3, \ldots} f$ denote the coefficient of the monomial $x_1^{\beta_1} x_2^{\beta_2} x_3^{\beta_3} \cdots$ in f. Then,

$$f - \sum_{(\alpha_1, \alpha_2, \ldots, \alpha_\ell) \in \operatorname{Comp}} \left(\operatorname{coeff}_{\alpha_1, \alpha_2, \ldots, \alpha_\ell, 0, 0, 0, \ldots} f\right) \cdot M_{(\alpha_1, \alpha_2, \ldots, \alpha_\ell)}$$

is quasisymmetric, but its coefficient before each packed monomial is 0. But a quasisymmetric power series whose coefficient before each packed monomial is 0 must be identically 0 (since the definition of "quasisymmetric" shows that every of its coefficients equals its coefficient before some packed monomial). Hence, $f - \sum_{(\alpha_1, \alpha_2, \ldots, \alpha_\ell) \in \operatorname{Comp}} \left(\operatorname{coeff}_{\alpha_1, \alpha_2, \ldots, \alpha_\ell, 0, 0, 0, \ldots} f\right) \cdot M_{(\alpha_1, \alpha_2, \ldots, \alpha_\ell)}$ is identically 0, so that

$$f = \sum_{(\alpha_1, \alpha_2, \ldots, \alpha_\ell) \in \operatorname{Comp}} \left(\operatorname{coeff}_{\alpha_1, \alpha_2, \ldots, \alpha_\ell, 0, 0, 0, \ldots} f\right) \cdot M_{(\alpha_1, \alpha_2, \ldots, \alpha_\ell)}$$

lies in the $\mathbf{k}$-span of $(M_\alpha)_{\alpha \in \text{Comp}}$. This proves Proposition 1.10. □

As a consequence of the above proposition, we can see that the dependence of $\text{QSym}_{\mathbf{k}}$ on the ground ring $\mathbf{k}$ is fairly simple:

Proposition 1.11. We have $\text{QSym}_{\mathbf{k}} \cong \mathbf{k} \otimes \text{QSym}_{\mathbb{Z}}$ as $\mathbf{k}$-algebras.

Proof of Proposition 1.11 (sketched). [...] □

A result similar to Proposition 1.10 also holds for power series without the bounded-degree requirement:

Proposition 1.12. **(a)** For any family $(\lambda_\alpha)_{\alpha \in \text{Comp}}$, the (infinite) sum $\sum_{\alpha \in \text{Comp}} \lambda_\alpha M_\alpha$ converges coefficientwise (i.e., for each monomial $\mathfrak{m}$, the sum of the coefficients of $\mathfrak{m}$ in $\lambda_\alpha M_\alpha$ over all $\alpha \in \text{Comp}$ converges with respect to the discrete topology). It is a well-defined quasisymmetric power series in $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$.

(b) If f is a quasisymmetric power series in $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$, then there exists a unique family $(\lambda_\alpha)_{\alpha \in \text{Comp}}$ such that $f = \sum_{\alpha \in \text{Comp}} \lambda_\alpha M_\alpha$.

Proof of Proposition 1.12 (sketched). Let $\text{QSYM}_{\mathbf{k}}$ denote the set of all quasisymmetric power series in $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$. [...] □

To find a more interesting basis of $\text{QSym}_{\mathbf{k}}$ than $(M_\alpha)_{\alpha \in \text{Comp}}$, we need to introduce the notion of descents and of descent sets. First, let us relate compositions to subsets of $\{1, 2, \ldots, n-1\}$:

Definition 1.13. Let $n \in \mathbb{N}$. In the following, a *composition of n* means a composition of size n.

(a) If α is a composition of n, then we define a subset $D(\alpha)$ of $\{1, 2, \ldots, n-1\}$ by

$$\begin{aligned} D(\alpha) &= \{\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \ldots, \alpha_1 + \alpha_2 + \cdots + \alpha_{\ell-1}\} \\ &= \{\alpha_1 + \alpha_2 + \cdots + \alpha_i \mid i \in \{1, 2, \ldots, \ell - 1\}\}, \end{aligned}$$

where α is written in the form $(\alpha_1, \alpha_2, \ldots, \alpha_\ell) \in \mathbb{N}_+^\ell$. (Keep in mind that we are using Definition 1.2 if $n = 0$.) This subset $D(\alpha)$ is well-defined (by Proposition 1.14 **(a)** below). For instance,

$$\begin{aligned} D(()) &= \varnothing \subseteq \{1, 2, \ldots, 0 - 1\} = \varnothing; \\ D((4)) &= \varnothing \subseteq \{1, 2, \ldots, 4 - 1\}; \\ D((2, 3)) &= \{2\} \subseteq \{1, 2, \ldots, 5 - 1\}; \\ D((1, 4, 2, 3, 1)) &= \{1, 5, 7, 10\} \subseteq \{1, 2, \ldots, 11 - 1\}; \\ D((1, 4, 2, 3, 2)) &= \{1, 5, 7, 10\} \subseteq \{1, 2, \ldots, 12 - 1\}. \end{aligned}$$

Thus, we have defined a subset $D(\alpha)$ of $\{1, 2, \ldots, n-1\}$ for every composition α of n. In other words, we have defined a map D from the set of compositions of n to the set of subsets of $\{1, 2, \ldots, n-1\}$.

(b) Conversely, given a subset S of $\{1, 2, \ldots, n-1\}$, define a $\operatorname{comp}_n S$ of n as follows:

- If $n = 0$, then set $\operatorname{comp}_n S = ()$.
- Otherwise, write the set S in the form $\{s_1, s_2, \ldots, s_k\}$ with $s_1 < s_2 < \cdots < s_k$. Furthermore, set $s_0 = 0$ and $s_{k+1} = n$. Thus, a $(k+2)$-tuple $(s_0, s_1, s_2, \ldots, s_{k+1})$ of nonnegative integers is defined. Then, define $\operatorname{comp}_n S$ as the composition

  $$(s_1 - s_0, s_2 - s_1, s_3 - s_2, \ldots, s_k - s_{k-1}, s_{k+1} - s_k)$$

  of n.

This composition $\operatorname{comp}_n S$ of n is well-defined (by Proposition 1.14 **(b)** below). (Note that this composition $\operatorname{comp}_n S$ depends on both S and n.) For instance,

$$\begin{aligned} \operatorname{comp}_3 \{1,2\} &= (1,1,1); \\ \operatorname{comp}_4 \{1,2\} &= (1,1,2); \\ \operatorname{comp}_9 \{3,5,6\} &= (3,2,1,3); \\ \operatorname{comp}_9 \varnothing &= (9). \end{aligned}$$

Thus, we have defined a composition $\operatorname{comp}_n S$ of n for every subset S of $\{1, 2, \ldots, n-1\}$. In other words, we have defined a map comp_n from the set of subsets of $\{1, 2, \ldots, n-1\}$ to the subset of compositions of n.

(c) We now have defined a map D from the set of compositions of n to the set of subsets of $\{1, 2, \ldots, n-1\}$, and a map comp_n in the opposite direction. These two maps are mutually inverse (by Proposition 1.14 **(c)** below), and thus the compositions of n are in 1-to-1 correspondence with the subsets of $\{1, 2, \ldots, n-1\}$. Hence, the number of compositions of n equals 2^{n-1} if $n \geq 1$ (and 1 if $n = 0$).

Proposition 1.14. Let $n \in \mathbb{N}$.

(a) If α is a composition of n, then the subset of $D(\alpha)$ of $\{1, 2, \ldots, n-1\}$ defined in Definition 1.13 **(a)** is well-defined.

(b) If S is a subset of $\{1, 2, \ldots, n-1\}$, then the composition $\operatorname{comp}_n S$ of n defined in Definition 1.13 **(b)** is well-defined.

(c) The maps D and comp_n defined in Definition 1.13 are mutually inverse.

Proof of Proposition 1.14 (sketched). [...] □

Definition 1.15. Let α be a composition. Let n be the size of α (so that α is a composition of n). The *α-th fundamental quasisymmetric function* is defined as the power series

$$\sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_n; \\ i_j < i_{j+1} \text{ for every } j \in D(\alpha)}} x_{i_1} x_{i_2} \cdots x_{i_n} \in \mathbf{k}\left[\left[x_1, x_2, x_3, \ldots\right]\right].$$

It is denoted by L_α.

For instance, $D\left(\left(1,3,2\right)\right) = \left\{1,4\right\}$, so that

$$\begin{aligned} L_{(1,3,2)} &= \sum_{\substack{i_1 \leq i_2 \leq i_3 \leq i_4 \leq i_5 \leq i_6; \\ i_j < i_{j+1} \text{ for every } j \in D((1,3,2))}} x_{i_1} x_{i_2} x_{i_3} x_{i_4} x_{i_5} x_{i_6} \\ &= \sum_{\substack{i_1 \leq i_2 \leq i_3 \leq i_4 \leq i_5 \leq i_6; \\ i_1 < i_2 \text{ and } i_4 < i_5}} x_{i_1} x_{i_2} x_{i_3} x_{i_4} x_{i_5} x_{i_6} = \sum_{i_1 < i_2 \leq i_3 \leq i_4 < i_5 \leq i_6} x_{i_1} x_{i_2} x_{i_3} x_{i_4} x_{i_5} x_{i_6}. \end{aligned}$$

Proposition 1.16. Let α be a composition. We have $L_\alpha \in \operatorname{QSym}_{\mathbf{k}}$.

Proof of Proposition 1.16 (sketched). [...] □

To understand the relation between the family $\left(L_\alpha\right)_{\alpha \in \operatorname{Comp}}$ and the basis $\left(M_\alpha\right)_{\alpha \in \operatorname{Comp}}$ (and, in particular, to see that the former family also is a basis), we need to introduce a partial order on the set of compositions of a given integer:

Definition 1.17. Let $n \in \mathbb{N}$. Let α and β be two compositions of n. We say that α *refines* β if $D\left(\beta\right) \subseteq D\left(\alpha\right)$. For example, the composition $\left(2,1,3,4,2\right)$ of 12 refines $\left(3,3,6\right)$, since $D\left(\left(3,3,6\right)\right) = \left\{3,6\right\} \subseteq \left\{2,3,6,10\right\} = D\left(\left(2,1,3,4,2\right)\right)$.

The relation "refines" is a partial order on the set of all compositions of n (since D is a bijection, and since inclusion is a partial order on the set of subsets of $\left\{1,2,\ldots,n-1\right\}$). It is called the *refinement order*. It turns the set of all compositions of n into a poset, which is order-isomorphic to the lattice of subsets of $\left\{1,2,\ldots,n-1\right\}$ under reverse inclusion.

The idea behind the refinement order is that a composition α refines another composition β of the same size if and only one can obtain α from β by breaking apart entries (e.g., breaking apart the first 3 in the composition $\left(1,3,2,1,3\right)$ into 1 and 2 results in $\left(1,1,2,2,1,3\right)$). More formally, one can easily show that a composition α of size n refines a composition $\beta = \left(\beta_1, \beta_2, \ldots, \beta_k\right)$ of size n if there exists a composition $\left(\beta_{i,1}, \beta_{i,2}, \ldots, \beta_{i,m_i}\right)$ of β_i for every $i \in \left\{1,2,\ldots,k\right\}$ such that

$$\alpha = \left(\beta_{1,1}, \beta_{1,2}, \ldots, \beta_{1,m_1}, \ \beta_{2,1}, \beta_{2,2}, \ldots, \beta_{2,m_2}, \ \ldots, \ \beta_{k,1}, \beta_{k,2}, \ldots, \beta_{k,m_k}\right).$$

Proposition 1.18. Let $n \in \mathbb{N}$. Let α be a composition of n. Then,

$$L_\alpha = \sum_{\substack{\beta \models n; \\ \beta \text{ refines } \alpha}} M_\beta .$$

Proposition 1.18 appears in [GriRei15, Proposition 5.17].[2]

Before we prove Proposition 1.18, we rewrite the definition of M_α in terms similar to how we defined L_α:

Proposition 1.19. Let $n \in \mathbb{N}$. Let α be a composition of n. Then,

$$M_\alpha = \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_n; \\ i_j < i_{j+1} \text{ for every } j \in D(\alpha); \\ i_j = i_{j+1} \text{ for every } j \in \{1,2,\ldots,n-1\} \setminus D(\alpha)}} x_{i_1} x_{i_2} \cdots x_{i_n} .$$

Proof of Proposition 1.19 (sketched). [...] □

Proof of Proposition 1.18 (sketched). [... add this proof?] □

Corollary 1.20. The family $(L_\alpha)_{\alpha \in \mathrm{Comp}}$ is a basis of the $\mathbf{k}$-module $\mathrm{QSym}_{\mathbf{k}}$.

Proof of Corollary 1.20 (sketched). [... add this proof?] □

2. Γ-partitions

Much of the use of the ring $\mathrm{QSym}_{\mathbf{k}}$ stems from the fact that to every "P-partition" (a notion which we will define in Section 5) one can assign a certain generating function in $\mathrm{QSym}_{\mathbf{k}}$. This was discovered by Stanley and Gessel [... refs], and generalizes both the definition of L_α for α a composition and the definition of a Schur function. See [GriRei15, proof of Corollary 5.29] and [Stan99, mainly §7.19] for applications of this notion.

In [MalReu98], Malvenuto and Reutenauer have generalized the notion of a P-partition to the notion of a Γ-partition, for Γ a finite directed graph whose arcs are partitioned into "strict" and "weak" arcs. (Malvenuto and Reutenauer denote the graph by G instead of Γ; we will not follow their notation for reasons that should become clear in Section [... add ref to section with group action].

[2]To be more precise, L_α is defined as $\sum_{\substack{\beta \models n; \\ \beta \text{ refines } \alpha}} M_\beta$ in [GriRei15, Definition 5.15], and then [GriRei15, Proposition 5.17] states that $L_\alpha = \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_n; \\ i_j < i_{j+1} \text{ for every } j \in D(\alpha)}} x_{i_1} x_{i_2} \cdots x_{i_n}$ (thus showing that the L_α defined in [GriRei15, Definition 5.15] is identical to the L_α that we defined above).

We will also allow the set of weak arcs and the set of strict arcs to be non-disjoint, and make a few other non-substantial generalizations just to simplify the statements of our results.)

We begin by introducing the notion of graphs we will be using:

Definition 2.1. In the following, a *directed graph* means a pair (V, A), where V is a set and A is a subset of $V \times V$. If $\mathbf{D} = (V, A)$ is a directed graph, then the elements of V are called the *vertices* of $\mathbf{D}$, and the elements of A are called the *arcs* of $\mathbf{D}$. A *cycle* of a directed graph $\mathbf{D} = (V, A)$ is defined to mean a sequence $(v_1, v_2, \ldots, v_k)$ of elements of V such that $k > 1$ and $v_k = v_1$ and such that, for every $i \in \{1, 2, \ldots, k-1\}$, the pair (v_i, v_{i+1}) is an arc of $\mathbf{D}$. A directed graph $\mathbf{D}$ is said to be *acyclic* if there are no cycles of $\mathbf{D}$. A directed graph $\mathbf{D} = (V, A)$ is said to be *finite* if the set V is finite.

Note that directed graphs (in this sense) can have no double arcs, and their arcs carry no other information than what vertices they connect.

Next, we introduce the concept of a weak-strict digraph, mostly following [MalReu98] (except that we allow the set of weak arcs and the set of strict arcs to have common elements):

Definition 2.2. A *weak-strict digraph* means a triple $\Gamma = (V, A_w, A_s)$, where V, A_w and A_s are three sets such that $(V, A_w \cup A_s)$ is a finite directed graph. (We don't require A_w and A_s to be disjoint.) Given a weak-strict diagraph $\Gamma = (V, A_w, A_s)$, the elements of A_w are called the *weak arcs* of our weak-strict digraph Γ, and the elements of A_s are called the *strict arcs* of Γ. The elements of V are called the *vertices* of Γ, and the set V is called the *vertex set* of Γ.

To any weak-strict digraph $\Gamma = (V, A_w, A_s)$, we can assign the finite directed graph $(V, A_w \cup A_s)$. This finite directed graph will be simply referred to as the "directed graph Γ" (although, of course, it does not carry all the information that the weak-strict digraph Γ carries). Every notion defined for a finite directed graph can thus be applied to a weak-strict digraph $\Gamma = (V, A_w, A_s)$ simply by applying it to this directed graph $(V, A_w \cup A_s)$. This way, for example, the notion of an acyclic weak-strict digraph is well-defined. So a weak-strict digraph (V, A_w, A_s) is acyclic if and only if the directed graph $(V, A_w \cup A_s)$ is acyclic.

The reason why we have required finiteness in this definition is simply to avoid assuming it in all our statements below.

We next define some transformations on weak-strict digraphs:

Definition 2.3. **(a)** If p is a pair of two objects, then we define a new pair $\overline{p}$ by $\overline{p} = (b, a)$, where p is written in the form (a, b). This pair $\overline{p}$ will be called the *reverse* of the pair p. Thus, in particular, $\overline{p}$ is defined whenever p is an arc of a directed graph.

(b) If B is a set of arcs of a directed graph D, then $\overline{B}$ will denote the set of the reverses of all arcs from B. In other words,

$$\overline{B} = \{(v, u) \mid (u, v) \in B\}.$$

Next, we define the notion of "Γ-partitions" for a weak-strict digraph Γ:

Definition 2.4. Let $\Gamma = (V, A_w, A_s)$ be a weak-strict digraph. Let X be a poset. We use the notations $\leq$ and $<$ for the smaller relation and the smaller-or-equal relation, respectively, of the poset X. A Γ-*partition to* X will mean a function $f : V \to X$ such that:

(a) every weak arc (u, v) of Γ satisfies $f(u) \leq f(v)$;

(b) every strict arc (u, v) of Γ satisfies $f(u) < f(v)$.

In the following, we regard $\mathbb{N}_+$ as a poset using the standard smaller relation on $\mathbb{N}_+$ (so this poset is actually a totally ordered set). When we don't explicitly mention X and just speak of "Γ-partitions", we always mean Γ-partitions to this poset $\mathbb{N}_+$.

The notion of a Γ-partition generalizes Stanley's notions of a P-partition and of a (P, ω)-partition (we will see how in Section 5). While the added generality it provides is not of much use in applications (this paper could have been written about (P, ω)-partitions just as well!), it is helpful in simplifying the proofs (the main advantage being that a weak-strict digraph is easier to "tweak", or modify, than a poset) and a little tad easier to apply in many cases (e.g., to writing the fundamental quasisymmetric functions – see Section 5 below). In [MalReu98], Malvenuto and Reutenauer put this notion to use in studying plethysms of quasisymmetric functions.

Let us connect Γ-partitions to $\mathrm{QSym}_{\mathbf{k}}$ by defining a certain generating function, an idea which goes back to Malvenuto and Reutenauer[3]:

Definition 2.5. Let $\Gamma = (V, A_w, A_s)$ be a weak-strict digraph.

Define a formal power series $F_\Gamma \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ by

$$F_\Gamma = \sum_{f \text{ is a } \Gamma\text{-partition}} \prod_{p \in V} x_{f(p)}.$$

This is well-defined [... explain].

Before Malvenuto and Reutenauer, various special cases of F_Γ have been studied, some of which are almost as general as F_Γ. The most important one is the

[3] A remark about our notation: What we call F_Γ in the following definition has been denoted by $\Gamma(G)$ in the paper [MalReu98] by Malvenuto and Reutenauer, where their G stands for the weak-strict digraph that we call Γ, and their Γ is a symbol signifying this generating function. We found it difficult to adher to standard notation in this subject, since we will later want to use the letter G for a group.

P-partition enumerator $F_{P,\omega}$ which we will introduce in Section 5. It first appeared in Gessel's [Gessel84], and in some special cases has been anticipated in the work of Thomas [Thomas77].

A warning is in order here. My notation imitates [GriRei15]; it (sadly) conflicts with other notation in literature. In particular, what we denote by F_Γ is not a generalization of what Stanley calls $F_{P,\omega}$ in [Stan11, §3.15] (but rather is a generalization of what he calls $K_{P,\omega}$ in [Stan99, §7.19]).

Example 2.6. Before we study F_Γ in general, let us illustrate its definition on a particular case.

For this example, let Γ be the weak-strict digraph with vertex set $\{1,2,3,4,5\}$, weak arcs $(1,3)$ and $(2,5)$, and strict arcs $(1,2)$, $(3,4)$ and $(4,5)$. Then, the Γ-partitions are the maps $f : \{1,2,3,4,5\} \to \mathbb{N}_+$ satisfying $f(1) \leq f(3)$, $(2) \leq f(5)$, $f(1) < f(2)$, $f(3) < f(4)$ and $f(4) < f(5)$. Hence,

$$F_\Gamma = \sum_{f \text{ is a } \Gamma\text{-partition}} \prod_{p \in \{1,2,3,4,5\}} x_{f(p)} = \sum_{\substack{f:\{1,2,3,4,5\}\to\mathbb{N}_+; \\ f(1)\leq f(3);\ f(2)\leq f(5); \\ f(1)<f(2);\ f(3)<f(4); \\ f(4)<f(5)}} x_{f(1)}x_{f(2)}x_{f(3)}x_{f(4)}x_{f(5)}$$

$$= \sum_{\substack{(i_1,i_2,i_3,i_4,i_5)\in\mathbb{N}_+^5; \\ i_1\leq i_3;\ i_2\leq i_5;\ i_1<i_2;\ i_3<i_4;\ i_4<i_5}} x_{i_1}x_{i_2}x_{i_3}x_{i_4}x_{i_5}.$$

Our first observation is:

Proposition 2.7. Let $\Gamma = (V, A_w, A_s)$ be a weak-strict digraph. Then, $F_\Gamma \in \operatorname{QSym}_{\mathbf{k}}$.

Proposition 2.7 can be proven by a standard argument, similar to [LMvW13, proof of Proposition 3.3.19]. It can actually generalized to so-called scheduling problems and extended to quasisymmetric functions in non-commuting variables – see [BreKli14, §4] for the definitions of these notions –, and still be proven by the same argument. We are going to give a slight variation of this argument, which has the advantage of proving the following explicit formula:

Proposition 2.8. Let $\Gamma = (V, A_w, A_s)$ be a weak-strict digraph.

Let us say that a Γ-partition f is *packed* if there exists some $\ell \in \mathbb{N}$ such that $f(V) = \{1,2,\ldots,\ell\}$.

(a) If f is any Γ-partition, then f is packed if and only if $f(V) = \{1,2,\ldots,|V|\}$.

(b) If f is any packed Γ-partition, and if we write ℓ for $|f(V)|$, then

$$\left(\left|f^{-1}(\{1\})\right|, \left|f^{-1}(\{2\})\right|, \ldots, \left|f^{-1}(\{\ell\})\right|\right)$$

is a composition of $|V|$. This composition will be called $\operatorname{ev} f$.

(c) We have

$$F_\Gamma = \sum_{f \text{ is a packed } \Gamma\text{-partition}} M_{\operatorname{ev} f}.$$

Proof of Proposition 2.8 (sketched). [...] □

Proof of Proposition 2.7 (sketched). [...] □

The quasisymmetric function F_Γ already generalizes a number of quasisymmetric functions appearing in literature; but it is so far not more general than what has been classically constructed for P-partitions by Gessel (see Section 5 below). But Malvenuto and Reutenauer have extended it further:

Definition 2.9. For any set Y, we denote by $\mathbf{1}$ the map $Y \to \mathbb{N}_+$ that sends everything to $1 \in \mathbb{N}_+$. (The domain Y of this map is going to be clear from the context.)

Definition 2.10. Let $\Gamma = (V, A_w, A_s)$ be a weak-strict digraph. Let $w : V \to \mathbb{N}_+$ be any map.

Define a formal power series $F_{(\Gamma,w)} \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ by

$$F_{(\Gamma,w)} = \sum_{f \text{ is a } \Gamma\text{-partition}} \prod_{p \in V} x_{f(p)}^{w(p)}.$$

This is well-defined [... explain].

Note that the definition of $F_{(\Gamma,w)}$ would no longer make sense if we would allow w to be a map $V \to \mathbb{N}$ rather than a map $V \to \mathbb{N}_+$ (not every sum converges!).

Example 2.11. The weak-strict digraph Γ considered in Example 2.6 satisfies $F_{(\Gamma,w)} = \sum_{\substack{(i_1,i_2,i_3,i_4,i_5) \in \mathbb{N}_+^5; \\ i_1 \leq i_3;\ i_2 \leq i_5;\ i_1 < i_2;\ i_3 < i_4;\ i_4 < i_5}} x_{i_1}^{w(1)} x_{i_2}^{w(2)} x_{i_3}^{w(3)} x_{i_4}^{w(4)} x_{i_5}^{w(5)}$ for any map $w : \{1, 2, 3, 4, 5\} \to \mathbb{N}_+$.

In analogy to Proposition 2.7, we have:

Proposition 2.12. Let $\Gamma = (V, A_w, A_s)$ be a weak-strict digraph. Let $w : V \to \mathbb{N}_+$ be any map. Then, $F_{(\Gamma,w)} \in \operatorname{QSym}_{\mathbf{k}}$.

We have an analogue of Proposition 2.8 **(b)** and **(c)**:

Proposition 2.13. Let $\Gamma = (V, A_w, A_s)$ be a weak-strict digraph. Let $w : V \to \mathbb{N}_+$ be any map.

(b) If f is any packed Γ-partition, and if we write ℓ for $|f(V)|$, then

$$\left(\sum_{p \in f^{-1}(\{1\})} w(p), \sum_{p \in f^{-1}(\{2\})} w(p), \ldots, \sum_{p \in f^{-1}(\{\ell\})} w(p) \right)$$

is a composition of $\sum_{p \in V} w(p)$. This composition will be called $\operatorname{ev}_w f$.

(c) We have

$$F_{(\Gamma,w)} = \sum_{f \text{ is a packed } \Gamma\text{-partition}} M_{\operatorname{ev}_w f}.$$

(This proposition has no part **(a)** since there is nothing to generalize about Proposition 2.8 **(a)**.)

Proof of Proposition 2.13 (sketched). **(b)** [...]

(c) For any finite subset S of $\mathbb{Z}$, there exists a unique strictly increasing bijection $r : \{1, 2, \ldots, |S|\} \to S$. This strictly increasing bijection will be denoted by r_S. Clearly, its inverse r_S^{-1} is also strictly increasing (in the sense that whenever s and t are elements of S satisfying $s < t$, then $r_S^{-1}(s) < r_S^{-1}(t)$).

Let us define the notion of a *packing* of a Γ-partition.

In fact, let f be a Γ-partition. Then, $f(V)$ is a finite subset of $\mathbb{Z}$, and so the map $r_{f(V)}$ is a strictly increasing bijection $\{1, 2, \ldots, |f(V)|\} \to f(V)$. As we know, $r_{f(V)}^{-1}$ also is a strictly increasing bijection. Now, $r_{f(V)}^{-1} \circ f$ is well-defined (since the range of f is $f(V)$, which coincides with the domain of $r_{f(V)}^{-1}$) and is also a Γ-partition (since f is a Γ-partition and $r_{f(V)}^{-1}$ is strictly increasing). Moreover, $\left(r_{f(V)}^{-1} \circ f\right)(V) = r_{f(V)}^{-1}(f(V)) = \{1, 2, \ldots, |f(V)|\}$ (since $r_{f(V)}$ is a bijection $\{1, 2, \ldots, |f(V)|\} \to f(V)$). Hence, $r_{f(V)}^{-1} \circ f$ is a packed Γ-partition (due to Proposition 2.8 **(a)**). We call $r_{f(V)}^{-1} \circ f$ the *packing* of the Γ-partition f.

Now, forget that we fixed f. Thus, for every Γ-partition f, we have defined the packing of the Γ-partition f. This packing is a packed Γ-partition.

Now,

$$\begin{aligned} F_{(\Gamma,w)} &= \sum_{f \text{ is a } \Gamma\text{-partition}} \prod_{p \in V} x_{f(p)}^{w(p)} = \sum_{\substack{g \text{ is a packed} \\ \Gamma\text{-partition}}} \sum_{\substack{f \text{ is a } \Gamma\text{-partition;} \\ \text{the packing of } f \text{ is } g}} \prod_{p \in V} x_{f(p)}^{w(p)} \\ &= \sum_{\substack{f \text{ is a packed} \\ \Gamma\text{-partition}}} \sum_{\substack{g \text{ is a } \Gamma\text{-partition;} \\ \text{the packing of } g \text{ is } f}} \prod_{p \in V} x_{g(p)}^{w(p)} \end{aligned} \qquad (2)$$

(here, we renamed the indices g and f in the two sums as f and g).

Now, let us fix a packed Γ-partition f. We are going to show that

$$\sum_{\substack{g \text{ is a } \Gamma\text{-partition;} \\ \text{the packing of } g \text{ is } f}} \prod_{p \in V} x_{g(p)}^{w(p)} = M_{\operatorname{ev}_w f}. \tag{3}$$

First of all, let $\ell = |f(V)|$. Thus,

$$\operatorname{ev}_w f = \left(\sum_{p \in f^{-1}(\{1\})} w(p), \sum_{p \in f^{-1}(\{2\})} w(p), \ldots, \sum_{p \in f^{-1}(\{\ell\})} w(p) \right). \tag{4}$$

Let us write α_k for $\sum_{p \in f^{-1}(\{k\})} w(p)$ whenever $k \in \{1, 2, \ldots, \ell\}$. Then, (4) rewrites as

$$\operatorname{ev}_w f = (\alpha_1, \alpha_2, \ldots, \alpha_\ell).$$

Hence,

$$\begin{aligned} M_{\operatorname{ev}_w f} = M_{(\alpha_1, \alpha_2, \ldots, \alpha_\ell)} &= \sum_{i_1 < i_2 < \cdots < i_\ell} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell} \qquad \left(\text{by the definition of } M_{(\alpha_1, \alpha_2, \ldots, \alpha_\ell)}\right) \\ &= \sum_{i_1 < i_2 < \cdots < i_\ell} \prod_{k \in \{1, 2, \ldots, \ell\}} x_{i_k}^{\alpha_k}. \end{aligned} \tag{5}$$

But if i_1, i_2, $\ldots$, i_ℓ are some positive integers satisfying $i_1 < i_2 < \cdots < i_\ell$, then $i_k = r_{\{i_1, i_2, \ldots, i_\ell\}}(k)$ for every $k \in \{1, 2, \ldots, \ell\}$ (by the definition of $r_{\{i_1, i_2, \ldots, i_\ell\}}$). Thus, (5) becomes

$$\begin{aligned} M_{\operatorname{ev}_w f} &= \sum_{i_1 < i_2 < \cdots < i_\ell} \prod_{k \in \{1, 2, \ldots, \ell\}} \underbrace{x_{i_k}^{\alpha_k}}_{\substack{= x_{r_{\{i_1, i_2, \ldots, i_\ell\}}(k)}^{\alpha_k} \\ (\text{since } i_k = r_{\{i_1, i_2, \ldots, i_\ell\}}(k))}} = \sum_{i_1 < i_2 < \cdots < i_\ell} \prod_{k \in \{1, 2, \ldots, \ell\}} x_{r_{\{i_1, i_2, \ldots, i_\ell\}}(k)}^{\alpha_k} \\ &= \sum_{\substack{I \subseteq \mathbb{N}_+; \\ |I| = \ell}} \prod_{k \in \{1, 2, \ldots, \ell\}} x_{r_I(k)}^{\alpha_k} \end{aligned} \tag{6}$$

(here, we substituted I for $\{i_1, i_2, \ldots, i_\ell\}$, because the map

$$\left\{ (i_1, i_2, \ldots, i_\ell) \in \mathbb{N}_+^\ell \mid i_1 < i_2 < \cdots < i_\ell \right\} \to \{ I \subseteq \mathbb{N}_+ \mid |I| = \ell \},$$
$$(i_1, i_2, \ldots, i_\ell) \mapsto \{i_1, i_2, \ldots, i_\ell\}$$

is a bijection).

On the other hand, recall that f is a packed Γ-partition, so that Proposition 2.8 **(a)** yields $f(V) = \{1, 2, \ldots, |f(V)|\} = \{1, 2, \ldots, \ell\}$ (since $|f(V)| = \ell$). If g is a

Γ-partition such that the packing of g is f, then $|g(V)| = \ell$ [4]. Hence,

$$\sum_{\substack{g \text{ is a } \Gamma\text{-partition;} \\ \text{the packing of } g \text{ is } f}} \prod_{p \in V} x_{g(p)}^{w(p)} = \sum_{\substack{I \subseteq \mathbb{N}_+; \\ |I| = \ell}} \sum_{\substack{g \text{ is a } \Gamma\text{-partition;} \\ \text{the packing of } g \text{ is } f; \\ g(V) = I}} \prod_{p \in V} x_{g(p)}^{w(p)}. \qquad (7)$$

Now, fix a subset I of $\mathbb{N}_+$ satisfying $|I| = \ell$. We are going to show that there exists exactly one Γ-partition g such that the packing of g is f and such that $g(V) = I$; this will be the Γ-partition $r_I \circ f$.

Indeed, it is first clear that $r_I \circ f$ is a Γ-partition (since f is a Γ-partition, and since r_I is strictly increasing). Moreover,

$$\begin{aligned} (r_I \circ f)(V) &= r_I \left(\underbrace{f(V)}_{=\{1,2,\ldots,\ell\}} \right) = r_I(\{1,2,\ldots,\ell\}) \\ &= r_I(\{1,2,\ldots,|I|\}) \qquad (\text{since } \ell = |I|) \\ &= I \end{aligned}$$

(by the definition of r_I). Now, the packing of $r_I \circ f$ is (by its definition) equal to

$$\begin{aligned} r_{(r_I \circ f)(V)}^{-1} \circ (r_I \circ f) &= r_I^{-1} \circ (r_I \circ f) \qquad (\text{since } (r_I \circ f)(V) = I) \\ &= f. \end{aligned}$$

Altogether, we thus have shown that $r_I \circ f$ is a Γ-partition g such that the packing of g is f and such that $g(V) = I$. We now need to prove that it is the only such Γ-partition.

So fix any Γ-partition g such that the packing of g is f and such that $g(V) = I$. The packing of g is $r_{g(V)}^{-1} \circ g$ (by the definition of the packing of g). Since the packing of g is the map f, this rewrites as follows: The map f is $r_{g(V)}^{-1} \circ g$. Thus, $f = r_{g(V)}^{-1} \circ g = r_I^{-1} \circ g$ (since $g(V) = I$). In other words, $r_I \circ f = g$, so that $g = r_I \circ f$.

Let us now forget that we fixed g. We thus have shown that if g is any Γ-partition such that the packing of g is f and such that $g(V) = I$, then $g = r_I \circ f$. Hence, there exists at most one Γ-partition g such that the packing of g is f and such that $g(V) = I$. Combining this with the fact that $r_I \circ f$ is a Γ-partition g such that the packing of g is f and such that $g(V) = I$, we obtain the following:

[4] *Proof.* Let g be a Γ-partition such that the packing of g is f. Then, $f = (\text{the packing of } g) = r_{g(V)}^{-1} \circ g$, so that $f(V) = \left(r_{g(V)}^{-1} \circ g\right)(V) = r_{g(V)}^{-1}(g(V))$ and thus

$$|f(V)| = \left| r_{g(V)}^{-1}(g(V)) \right| = |g(V)| \qquad \left(\text{since } r_{g(V)}^{-1} \text{ is a bijection}\right)$$

and thus $|g(V)| = |f(V)| = \ell$, qed.

There exists exactly one Γ-partition g such that the packing of g is f and such that $g(V) = I$; this Γ-partition is $r_I \circ f$. Hence, the sum $\sum_{\substack{g \text{ is a } \Gamma\text{-partition;} \\ \text{the packing of } g \text{ is } f; \\ g(V)=I}} \prod_{p\in V} x_{g(p)}^{w(p)}$ has precisely one term, namely the term for $g = r_I \circ f$. Thus, this sum becomes

$$\sum_{\substack{g \text{ is a } \Gamma\text{-partition;} \\ \text{the packing of } g \text{ is } f; \\ g(V)=I}} \prod_{p\in V} x_{g(p)}^{w(p)} = \prod_{p\in V} x_{(r_I\circ f)(p)}^{w(p)} = \prod_{p\in V} x_{r_I(f(p))}^{w(p)} = \prod_{k\in\{1,2,\ldots,\ell\}} \underbrace{\prod_{\substack{p\in V; \\ f(p)=k}}}_{=\prod_{p\in f^{-1}(\{k\})}} \underbrace{x_{r_I(f(p))}^{w(p)}}_{=x_{r_I(k)}^{w(p)} \text{ (since } f(p)=k)}$$

$$\left(\text{since } f(p) \in f(V) = \{1,2,\ldots,\ell\} \text{ for every } p \in V\right)$$

$$= \prod_{k\in\{1,2,\ldots,\ell\}} \underbrace{\prod_{p\in f^{-1}(\{k\})} x_{r_I(k)}^{w(p)}}_{\substack{=x_{r_I(k)}^{\sum_{p\in f^{-1}(\{k\})} w(p)} = x_{r_I(k)}^{\alpha_k} \\ \text{(since } \sum_{p\in f^{-1}(\{k\})} w(p)=\alpha_k \\ \text{(by the definition of } \alpha_k))}} = \prod_{k\in\{1,2,\ldots,\ell\}} x_{r_I(k)}^{\alpha_k}. \tag{8}$$

Now, forget that we fixed I. We thus have shown that every subset I of $\mathbb{N}_+$ satisfying $|I| = \ell$, the equality (8) holds. Now, (7) becomes

$$\sum_{\substack{g \text{ is a } \Gamma\text{-partition;} \\ \text{the packing of } g \text{ is } f}} \prod_{p\in V} x_{g(p)}^{w(p)} = \sum_{\substack{I\subseteq\mathbb{N}_+; \\ |I|=\ell}} \underbrace{\sum_{\substack{g \text{ is a } \Gamma\text{-partition;} \\ \text{the packing of } g \text{ is } f; \\ g(V)=I}} \prod_{p\in V} x_{g(p)}^{w(p)}}_{\substack{=\prod_{k\in\{1,2,\ldots,\ell\}} x_{r_I(k)}^{\alpha_k} \\ \text{(by (8))}}}$$

$$= \sum_{\substack{I\subseteq\mathbb{N}_+; \\ |I|=\ell}} \prod_{k\in\{1,2,\ldots,\ell\}} x_{r_I(k)}^{\alpha_k} = M_{\operatorname{ev}_w f} \qquad \text{(by (5))}.$$

This proves (3).

Now, forget that we fixed f. We thus have shown that every packed Γ-partition f satisfies (3). The equality (2) becomes

$$F_{(\Gamma,w)} = \sum_{\substack{f \text{ is a packed} \\ \Gamma\text{-partition}}} \underbrace{\sum_{\substack{g \text{ is a } \Gamma\text{-partition;} \\ \text{the packing of } g \text{ is } f}} \prod_{p\in V} x_{g(p)}^{w(p)}}_{=M_{\operatorname{ev}_w f}} = \sum_{\substack{f \text{ is a packed} \\ \Gamma\text{-partition}}} M_{\operatorname{ev}_w f}.$$

This proves Proposition 2.13 **(c)**. □

Proof of Proposition 2.12 (sketched). [...] □

We notice that Proposition 2.7 and Proposition 2.8 are particular cases of Proposition 2.12 and Proposition 2.13, respectively, obtained by setting w equal to the map $\mathbf{1} : V \to \mathbb{N}_+$. In fact, we have the following:

Proposition 2.14. Let Γ be a weak-strict digraph.
(a) We have $F_\Gamma = F_{(\Gamma,\mathbf{1})}$.
(b) For any packed Γ-partition f, we have $\operatorname{ev} f = \operatorname{ev}_{\mathbf{1}} f$.

Proof of Proposition 2.14 (sketched). [...] □

Let us give two families of examples of $F_{(\Gamma,w)}$, exhibiting the M_α and the L_α as particular cases of this construction.

Proposition 2.15. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ be a composition. Let $\operatorname{Path}_s \alpha$ denote the weak-strict digraph

$$(\{1,2,\ldots,\ell\}, \varnothing, \{(i,i+1) \mid i \in \{1,2,\ldots,\ell-1\}\}).$$

(Colloquially speaking, this is a directed graph which consists of a single path on ℓ vertices, with all its arcs being strict.) Let $w(\alpha)$ be the map $\{1,2,\ldots,\ell\} \to \mathbb{N}_+$ which sends every $i \in \{1,2,\ldots,\ell\}$ to α_i. Then, $F_{(\operatorname{Path}_s \alpha, w(\alpha))} = M_\alpha$.

Proposition 2.16. Let α be a composition. Let n be the size of α. Let $\operatorname{DPath} \alpha$ denote the weak-strict digraph

$$(\{1,2,\ldots,n\}, \{(i,i+1) \mid i \in \{1,2,\ldots,n-1\} \setminus D(\alpha)\}, \{(i,i+1) \mid i \in D(\alpha)\}).$$

(Colloquially speaking, this is a directed graph which consists of a single path on n vertices, with the i-th arc being strict if $i \in D(\alpha)$ and weak otherwise.) Then, $F_{\operatorname{DPath} \alpha} = L_\alpha$.

Proof of Proposition 2.16 (sketched). [...] □

Proof of Proposition 2.14 (sketched). [...] □

The following proposition demonstrates that the skew Schur functions of classical symmetric function theory are particular cases of the F_Γ construction as well:

Proposition 2.17. Let λ/μ be a skew partition. (For this and all other terminology of symmetric function theory, see, e.g., [Stan99, Chapter 7], [GriRei15, §2] and other sources.) Let V denote the Young diagram of λ/μ (regarded as a set). We draw Young diagrams in English notation, so that cell (i,j) is one unit lower than cell $(i-1,j)$ but one unit to the right of cell $(i,j-1)$. Let A_w denote the set of all $(a,b) \in V^2$ such that the cell b is the right neighbor of a. Let A_s denote the set of all $(a,b) \in V^2$ such that the cell b is the lower neighbor of a. Then, $F_{(V,A_w,A_s)}$ is the Schur function $s_{\lambda/\mu}$.

Proof of Proposition 2.17 (sketched). [...] □

The dual immaculate quasisymmetric functions introduced in [BBSSZ13, §3.7] are also particular case of the F_Γ construction:

Proposition 2.18. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ be a composition. Define the *diagram* of α to be the set of all $(i,j) \in \mathbb{N}_+^2$ such that $j \leq \alpha_i$. The elements of this set are called the *cells* of this diagram. More generally, the same notions that are defined for Young diagrams (such as rows, columns, descents, English notation, etc.) can be defined for diagrams of compositions analogously. We draw diagrams of compositions in English notation (but of course, this no longer means that the rows weakly decrease in length from top to bottom).

Let V denote the diagram of α. Let A_w denote the set of all $(a,b) \in V^2$ such that the cell b is the right neighbor of a. Let A_s denote the set of all $(a,b) \in V^2$ such that the cell b is the lower neighbor of a and both cells lie **in the first column** (i.e., they have their second coordinates equal to 1). Then, $F_{(V,A_w,A_s)}$ is the dual immaculate function $\mathfrak{S}_\alpha$ defined in [BBSSZ13, §3.7].

Proof of Proposition 2.18 (sketched). [... Use [BBSSZ13, Proposition 3.36] and compare the notion of an immaculate tableau with that of a Γ-partition; then recall Proposition 2.8.] □

Let us reap the first harvest from the notion of Γ-partitions. We will introduce the disjoint union of two weak-strict digraphs and show that the F_Γ corresponding to that disjoint union is the product of the F_Γ's corresponding to the two weak-strict digraphs:

Definition 2.19. Let $\Gamma_0 = (V_0, A_{w0}, A_{s0})$ and $\Gamma_1 = (V_1, A_{w1}, A_{s1})$ be two weak-strict digraphs. The *disjoint union* of Γ_0 and Γ_1 is defined as the weak-strict digraph (V, A_w, A_s), where

$$V = \{(v,0) \mid v \in V_0\} \cup \{(v,1) \mid v \in V_1\};$$
$$A_w = \{((a,0),(b,0)) \mid (a,b) \in A_{w0}\} \cup \{((a,1),(b,1)) \mid (a,b) \in A_{w1}\};$$
$$A_s = \{((a,0),(b,0)) \mid (a,b) \in A_{s0}\} \cup \{((a,1),(b,1)) \mid (a,b) \in A_{s1}\}.$$

In more friendly terms, the disjoint union of Γ_0 and Γ_1 is defined as the weak-strict digraph whose vertex set is the disjoint union of the vertex sets of Γ_0 and Γ_1, and whose set of weak (resp. strict) arcs is the union of the sets of weak (resp. strict) arcs of Γ_0 and Γ_1 (with their endpoints relabelled accordingly). We denote the disjoint union of Γ_0 and Γ_1 by $\Gamma_0\Gamma_1$. When V_0 and V_1 are disjoint, we identify V with the set-theoretic union $V_0 \cup V_1$ (by equating each $v \in V_0$ with $(v, 0) \in V$, each $v \in V_1$ with $(v, 1) \in V$) and similarly identifying A_w with the union $A_{w0} \cup A_{w1}$ (by equation $(a, b) \in A_{w0}$ with $((a, 0), (b, 0)) \in A_w$, etc.) and A_s with the union $A_{s0} \cup A_{s1}$; in this case, we thus simply have $\Gamma_0\Gamma_1 = (V_0 \cup V_1, A_{w0} \cup A_{w1}, A_{s0} \cup A_{s1})$.

Proposition 2.20. Let Γ_0 and Γ_1 be two weak-strict digraphs. Then, $F_{\Gamma_0\Gamma_1} = F_{\Gamma_0} \cdot F_{\Gamma_1}$.

Proposition 2.21. Let $\Gamma_0 = (V_0, A_{w0}, A_{s0})$ and $\Gamma_1 = (V_1, A_{w1}, A_{s1})$ be two weak-strict digraphs such that V_0 and V_1 are disjoint. Identify the disjoint union $\Gamma_0\Gamma_1$ with $(V_0 \cup V_1, A_{w0} \cup A_{w1}, A_{s0} \cup A_{s1})$. Let $w : V_0 \cup V_1 \to \mathbb{N}_+$ be a map. Then,

$$F_{\Gamma_0\Gamma_1, w} = F_{\Gamma_0, w|_{V_0}} \cdot F_{\Gamma_1, w|_{V_1}}.$$

Proof of Proposition 2.20 (sketched). [...] □

Proof of Proposition 2.21 (sketched). [...] □

[...]
[... Second proof of $\operatorname{QSym}_{\mathbf{k}}$ being closed under mult!]

3. The Hopf algebra structure on $\operatorname{QSym}_{\mathbf{k}}$

We have so far been regarding $\operatorname{QSym}_{\mathbf{k}}$ as an algebra – with algebra structure inherited from $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$. In this section, we are going to endow $\operatorname{QSym}_{\mathbf{k}}$ with a further structure – that of a Hopf algebra.

We will not give an introduction into Hopf algebras here, as this has already been done by various authors in various references – for example, [Montg93], [GriRei15, §1], [Manchon04, §1-§2], [Abe77], [Sweed69], [DNR01] and [Fresse14, Chapter 7] each include the definitions and the basic properties of bialgebras and Hopf algebras, and we will not need anything beyond that. [... check that the latter claim is true!] (While most authors discussing Hopf algebras limit themselves to Hopf algebras over fields, the notion can be defined over any commutative ring in the same way, and all properties of Hopf algebras which are sufficiently elementary to be used in the present papers continue to hold in this generality.)

[... introduce coalgebras and/or give references]

Let us define the coalgebra structure on $\operatorname{QSym}_{\mathbf{k}}$. This is commonly done (e.g., in [GriRei15, §5]) using what is usually called "alphabet doubling trick"; we are going to give a more down-to-earth definition instead:

Definition 3.1. **(a)** In the following, the tensor sign $\otimes$ without subscript always means $\otimes$.

(b) We define a $\mathbf{k}$-linear map $\Delta : \operatorname{QSym}_{\mathbf{k}} \to \operatorname{QSym}_{\mathbf{k}} \otimes \operatorname{QSym}_{\mathbf{k}}$ by requiring that every composition $(\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ satisfy

$$\Delta\left(M_{(\alpha_1,\alpha_2,\ldots,\alpha_\ell)}\right) = \sum_{k=0}^{\ell} M_{(\alpha_1,\alpha_2,\ldots,\alpha_k)} \otimes M_{(\alpha_{k+1},\alpha_{k+2},\ldots,\alpha_\ell)}.$$

This is clearly well-defined, because the $M_{(\alpha_1,\alpha_2,\ldots,\alpha_\ell)}$ with $(\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ ranging over all compositions form a $\mathbf{k}$-basis of $\operatorname{QSym}_{\mathbf{k}}$ (by Proposition 1.10). The map Δ is called the *coproduct* of $\operatorname{QSym}_{\mathbf{k}}$.

(c) We further define a $\mathbf{k}$-linear map $\varepsilon : \operatorname{QSym}_{\mathbf{k}} \to \mathbf{k}$ by requiring that every composition $(\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ satisfy

$$\varepsilon\left(M_{(\alpha_1,\alpha_2,\ldots,\alpha_\ell)}\right) = [\ell = 0]$$

(where we are using the notation of Definition 1.3). This is clearly well-defined, because the $M_{(\alpha_1,\alpha_2,\ldots,\alpha_\ell)}$ with $(\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ ranging over all compositions form a $\mathbf{k}$-basis of $\operatorname{QSym}_{\mathbf{k}}$ (by Proposition 1.10). The map ε is called the *counity* of $\operatorname{QSym}_{\mathbf{k}}$.

We state some basic properties of these maps Δ and ε:

Proposition 3.2. **(a)** The triple $(\operatorname{QSym}_{\mathbf{k}}, \Delta, \varepsilon)$ is a $\mathbf{k}$-coalgebra. That is, the diagrams

$$\begin{array}{ccc} \operatorname{QSym}_{\mathbf{k}} & \xrightarrow{\Delta} & \operatorname{QSym}_{\mathbf{k}} \otimes \operatorname{QSym}_{\mathbf{k}} \\ {\scriptstyle \Delta}\downarrow & & \downarrow{\scriptstyle \Delta \otimes \mathrm{id}} \\ \operatorname{QSym}_{\mathbf{k}} \otimes \operatorname{QSym}_{\mathbf{k}} & \xrightarrow[\mathrm{id} \otimes \Delta]{} & \operatorname{QSym}_{\mathbf{k}} \otimes \operatorname{QSym}_{\mathbf{k}} \otimes \operatorname{QSym}_{\mathbf{k}} \end{array} , \qquad (9)$$

$$\begin{array}{ccc} \operatorname{QSym}_{\mathbf{k}} & \xrightarrow{\Delta} & \operatorname{QSym}_{\mathbf{k}} \otimes \operatorname{QSym}_{\mathbf{k}} \\ & {\scriptstyle i_1}\searrow & \downarrow{\scriptstyle \varepsilon \otimes \mathrm{id}} \\ & & \mathbf{k} \otimes \operatorname{QSym}_{\mathbf{k}} \end{array} \qquad \text{and} \qquad (10)$$

$$\begin{array}{ccc} \operatorname{QSym}_{\mathbf{k}} & \xrightarrow{\Delta} & \operatorname{QSym}_{\mathbf{k}} \otimes \operatorname{QSym}_{\mathbf{k}} \\ & {\scriptstyle i_2}\searrow & \downarrow{\scriptstyle \mathrm{id} \otimes \varepsilon} \\ & & \operatorname{QSym}_{\mathbf{k}} \otimes \mathbf{k} \end{array} \qquad (11)$$

are commutative, where $i_1 : \operatorname{QSym}_{\mathbf{k}} \to \mathbf{k} \otimes \operatorname{QSym}_{\mathbf{k}}$ and $i_2 : \operatorname{QSym}_{\mathbf{k}} \to \operatorname{QSym}_{\mathbf{k}} \otimes \mathbf{k}$ are the canonical $\mathbf{k}$-module isomorphisms.

(b) The $\mathbf{k}$-coalgebra $(\operatorname{QSym}_{\mathbf{k}}, \Delta, \varepsilon)$ combined with the $\mathbf{k}$-algebra structure on $\operatorname{QSym}_{\mathbf{k}}$ forms a $\mathbf{k}$-bialgebra. In other words, the maps $\Delta : \operatorname{QSym}_{\mathbf{k}} \to \operatorname{QSym}_{\mathbf{k}} \otimes \operatorname{QSym}_{\mathbf{k}}$ and $\varepsilon : \operatorname{QSym}_{\mathbf{k}} \to \mathbf{k}$ are $\mathbf{k}$-algebra homomorphisms.

This proposition would have more or less fallen into our hands had we used the alphabet doubling trick; with our definition, it is less obvious. However, we can obtain it easily once we understand how Δ and ε act on the $F_{(\Gamma,w)}$. To do so, first we need a definition:

Definition 3.3. Let $\mathbf{D} = (V, A)$ be a directed graph. An *admissible cut* of $\mathbf{D}$ will denote a pair (P, Q), where P and Q are subsets of V satisfying $P \cup Q = V$ and $P \cap Q = \varnothing$ such that there exists no $(q, p) \in A$ satisfying $p \in P$ and $q \in Q$. We denote by $\operatorname{Adm} \mathbf{D}$ the set of all admissible cuts of $\mathbf{D}$.

Definition 3.4. Let $\Gamma = (V, A_w, A_s)$ be a weak-strict digraph. An *admissible cut* of Γ will mean an admissible cut of the directed graph Γ (that is, of the directed graph $(V, A_w \cup A_s)$). We denote by $\operatorname{Adm} \Gamma$ the set of all admissible cuts of Γ.

Let P be a subset of V. Then, $\Gamma \mid_P$ will denote the weak-strict digraph $(P,\ A_w \cap (P \times P),\ A_s \cap (P \times P))$. (This is simply the weak-strict digraph obtained by keeping only the vertices from P and the weak and strict arcs connecting these vertices to each other.)

The notion of admissible cuts is classical, although more usually applied to trees rather than to directed graphs (see, e.g., [Manchon04, §II.9.3], albeit the definition there is in our opinion not the best). We will use it here to compute the coproduct and the counity on $F_{(\Gamma,w)}$:

Proposition 3.5. Let $\Gamma = (V, A_w, A_s)$ be a weak-strict digraph. Let $w : V \to \mathbb{N}_+$ be any map.

(a) We have

$$\Delta\left(F_{(\Gamma,w)}\right) = \sum_{(P,Q)\in \operatorname{Adm}\Gamma} F_{(\Gamma|_P, w|_P)} \otimes F_{\left(\Gamma|_Q, w|_Q\right)}.$$

(b) We have

$$\varepsilon\left(F_{(\Gamma,w)}\right) = [|V| = 0].$$

Proof of Proposition 3.5 (sketched). [... use packed stuff here] □

Now,

Proof of Proposition 3.2 (sketched). [... do this using Γ-partitions!] □

We have so far shown that $\operatorname{QSym}_{\mathbf{k}}$ is a $\mathbf{k}$-bialgebra. Is it a Hopf algebra? There is a sense in which the answer is "obviously yes", because $\operatorname{QSym}_{\mathbf{k}}$ is a case of what is called a connected graded $\mathbf{k}$-bialgebra, and such $\mathbf{k}$-bialgebras are automatically Hopf algebras (see [GriRei15, Proposition 1.36] for a proof, and [Manchon04, Corollary II.3.2] for a more general result). We will, however, reprove this result in a different way, and derive an explicit formula for the antipode on $F_{(\Gamma,w)}$ for a wide class of weak-strict digraphs Γ.

4. The antipode of F_Γ

Definition 4.1. Let α be a composition. Let $n = |\alpha|$. Then, we define a composition $\omega(\alpha)$ of n by

$$\omega(\alpha) = \operatorname{comp}_n(\{1, 2, \ldots, n-1\} \setminus \{n - i \mid i \in D(\alpha)\}).$$

Proposition 4.2. Let α be a composition. Then, $\omega(\omega(\alpha)) = \alpha$.

Proof of Proposition 4.2 (sketched). [...] □

Let us now state the existence of an antipode on $\operatorname{QSym}_{\mathbf{k}}$ and an explicit formula for it:

Theorem 4.3. Define a $\mathbf{k}$-linear map $S : \operatorname{QSym}_{\mathbf{k}} \to \operatorname{QSym}_{\mathbf{k}}$ by requiring that every composition α satisfy

$$S(L_\alpha) = (-1)^{|\alpha|} L_{\omega(\alpha)}.$$

This is clearly well-defined, because the L_α with α ranging over the compositions form a basis of $\operatorname{QSym}_{\mathbf{k}}$ (by Corollary 1.20).

Then, S is an antipode of the $\mathbf{k}$-bialgebra $\operatorname{QSym}_{\mathbf{k}}$.

We will actually prove a much more general statement about how S acts. First, we define two ways to construct new weak-strict digraphs from a given one:

Definition 4.4. Let $\Gamma = (V, A_w, A_s)$ be a weak-strict digraph. We define a new weak-strict digraph Γ' by

$$\Gamma' = \left(V, A_w, \overline{A_s}\right)$$

(that is, it is the result of reversing all strict arcs in Γ). We define a further weak-strict digraph $\omega(\Gamma)$ by

$$\omega(\Gamma) = \left(V, \overline{A_s}, \overline{A_w}\right)$$

(that is, the weak-strict digraph obtained from Γ by reversing all arcs and exchanging strict and weak arcs). Notice that $\Gamma'' = \Gamma$ and $\omega(\omega(\Gamma)) = \Gamma$.

Theorem 4.5. Let $\Gamma = (V, A_w, A_s)$ be a weak-strict digraph. Let $w : V \to \mathbb{N}_+$ be any map. Assume that Γ' is acyclic. Then,

$$S\left(F_{(\Gamma,w)}\right) = (-1)^{|V|} F_{(\omega(\Gamma),w)}.$$

Corollary 4.6. Let $\Gamma = (V, A_w, A_s)$ be a weak-strict digraph. Assume that Γ' is acyclic. Then,

$$S(F_\Gamma) = (-1)^{|V|} F_{\omega(\Gamma)}.$$

Of course, Corollary 4.6 is obtained from Theorem 4.5 by setting $w = \mathbf{1}$.

Theorem 4.5 is more or less equivalent to [MalReu98, Theorem 3.1], while Corollary 4.6 is more or less equivalent to [MalReu98, Lemma 3.2]. We are saying "more or less" since the statements in [MalReu98] are formulated in a slightly different way (mainly, instead of the weight map w in Theorem 4.5, the authors of [MalReu98] are considering vertices connected by "undirected edges" – in our opinion, a slightly awkward technique) and burdened by some unnecessary conditions which are easy to lift (the statements in [MalReu98] require A_w to be distinct from A_s, and Γ to be acyclic as well as Γ'). We will see in somewhat more detail how to derive Theorem 4.5 from [MalReu98, Theorem 3.1] in the Second proof of Theorem 4.5 below.

Also, Corollary 4.6 is easily seen to be equivalent to the rather well-known [GriRei15, Corollary 5.27], a cornerstone of P-partition theory (we will say more about this in Section 5). The main difference between these two facts is that we are using Γ-partitions instead of P-partitions; the condition that Γ' be acyclic is precisely what is needed to reduce the former to the latter. However, it seems to us that the setting of Γ-partitions is more natural.

We are going to give three proofs of Theorem 4.5. The reader is advised to only read the first one (which, to our knowledge, is new and probably the most enlightening) unless interested in the history of the subject. The other two proofs are merely there to show that Theorem 4.5 is not very novel: the second proof derives it from [MalReu98, Theorem 3.1], whereas the third derives it from [GriRei15, Corollary 5.27].

The vehicle that carries us to the proof of Theorems 4.3 and to our first proof of Theorem 4.5 is the following fact (which, per se, involves neither Hopf algebras nor quasisymmetric functions):

Proposition 4.7. Let $\Gamma = (V, A_w, A_s)$ be a weak-strict digraph. Assume that Γ' is acyclic. Let X be a totally ordered set (regarded as a poset). Let A be a topological commutative ring. For every $v \in V$ and $u \in X$, let $z_{v,u}$ be an element of A. Assume that, for every subset S of V, the sum $\sum\limits_{f \in X^S} \prod\limits_{v \in S} z_{v,f(v)}$ converges with respect to the topology on A. (Notice that this happens, for example, when $X = \mathbb{N}_+$, $A = \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ and $z_{v,u} = x_u$.)

For every subset S of V, define an element $Z_{S,X} \in A$ by

$$Z_{S,X} = \sum_{\substack{f \text{ is a } \Gamma|_S\text{-partition} \\ \text{to } X}} \prod_{v \in S} z_{v,f(v)}.$$

(This sum converges, since it is a subsum of the convergent sum $\sum_{f \in X^S} \prod_{v \in S} z_{v,f(v)}$.) Also, for every subset S of V, define an element $Z'_{S,X} \in A$ by

$$Z^{\omega}_{S,X} = \sum_{\substack{f \text{ is a } \omega(\Gamma)|_S\text{-partition} \\ \text{to } X}} \prod_{v \in S} z_{v,f(v)}.$$

(This sum converges, since it is a subsum of the convergent sum $\sum_{f \in X^S} \prod_{v \in S} z_{v,f(v)}$.)

(a) We have

$$\sum_{(P,Q) \in \operatorname{Adm} \Gamma} (-1)^{|Q|} Z_{P,X} Z^{\omega}_{Q,X} = [V = \varnothing].$$

(b) We have

$$\sum_{(P,Q) \in \operatorname{Adm} \Gamma} (-1)^{|P|} Z^{\omega}_{P,X} Z_{Q,X} = [V = \varnothing].$$

We shall prove Proposition 4.7 with the help of a sign-reversed involution. The idea of using sign-reversing involutions to identify antipodes of Hopf algebras is natural and not new (see, e.g., [GriRei15, proof of Theorem 5.11], and, more recently, [BenSag14]), but the crux of such involutive proofs is the precise construction of the involution. The one that we will use will proceed by "exiling" a certain element of V from P into Q or vice versa; this is again far from being a novel motive, but the choice of the element to be exiled is (in our opinion) interesting.

Proof of Proposition 4.7 (sketched). **(a)** Proposition 4.7 **(a)** can be checked immediately in the case when $V = \varnothing$. Hence, for the rest of this proof, we can WLOG assume that $V \neq \varnothing$. Let us assume this.

Let us fix a $(P, Q) \in \operatorname{Adm} \Gamma$ first. We have

$$Z_{P,X} = \sum_{\substack{f \text{ is a } \Gamma|_P\text{-partition} \\ \text{to } X}} \prod_{v \in P} z_{v,f(v)} = \sum_{\substack{f: P \to X; \\ f \text{ is a } \Gamma|_P\text{-partition to } X}} \prod_{v \in P} z_{v,f(v)}$$

and

$$Z^{\omega}_{Q,X} = \sum_{\substack{f \text{ is a } \omega(\Gamma)|_Q\text{-partition} \\ \text{to } X}} \prod_{v\in Q} z_{v,f(v)} = \sum_{\substack{f:Q\to X; \\ f \text{ is a } \omega(\Gamma)|_Q\text{-partition to } X}} \prod_{v\in Q} z_{v,f(v)}$$
$$= \sum_{\substack{g:Q\to X; \\ g \text{ is a } \omega(\Gamma)|_Q\text{-partition to } X}} \prod_{v\in Q} z_{v,g(v)}.$$

Multiplying these two equalities, we obtain

$$Z_{P,X} Z^{\omega}_{Q,X} = \left(\sum_{\substack{f:P\to X; \\ f \text{ is a } \Gamma|_P\text{-partition to } X}} \prod_{v\in P} z_{v,f(v)} \right) \left(\sum_{\substack{g:Q\to X; \\ g \text{ is a } \omega(\Gamma)|_Q\text{-partition to } X}} \prod_{v\in Q} z_{v,g(v)} \right)$$
$$= \sum_{\substack{f:P\to X; \\ f \text{ is a } \Gamma|_P\text{-partition to } X}} \sum_{\substack{g:Q\to X; \\ g \text{ is a } \omega(\Gamma)|_Q\text{-partition to } X}} \left(\prod_{v\in P} z_{v,f(v)} \right) \left(\prod_{v\in Q} z_{v,g(v)} \right)$$
$$= \sum_{\substack{(f,g); \\ f:P\to X;\ g:Q\to X; \\ f \text{ is a } \Gamma|_P\text{-partition to } X; \\ g \text{ is a } \omega(\Gamma)|_Q\text{-partition to } X}} \left(\prod_{v\in P} z_{v,f(v)} \right) \left(\prod_{v\in Q} z_{v,g(v)} \right)$$
$$= \sum_{\substack{h:V\to X; \\ h|_P \text{ is a } \Gamma|_P\text{-partition to } X; \\ h|_Q \text{ is a } \omega(\Gamma)|_Q\text{-partition to } X}} \left(\prod_{v\in P} \underbrace{z_{v,(h|_P)(v)}}_{=z_{v,h(v)}} \right) \left(\prod_{v\in Q} \underbrace{z_{v,(h|_Q)(v)}}_{=z_{v,h(v)}} \right)$$
$$\left(\begin{array}{c} \text{here, we substituted } (h\mid_P, h\mid_Q) \text{ for } (f,g)\text{,} \\ \text{since the pairs } (f,g) \text{ of maps } f:P\to X \text{ and } g:Q\to X \text{ are in} \\ \text{1-to-1 correspondence with the maps } h:V\to X\text{, and the} \\ \text{former pair } (f,g) \text{ can be reconstructed from its respective} \\ h \text{ by } (f,g) = (h\mid_P, h\mid_Q) \end{array} \right)$$
$$= \sum_{\substack{h:V\to X; \\ h|_P \text{ is a } \Gamma|_P\text{-partition to } X; \\ h|_Q \text{ is a } \omega(\Gamma)|_Q\text{-partition to } X}} \underbrace{\left(\prod_{v\in P} z_{v,h(v)} \right) \left(\prod_{v\in Q} z_{v,h(v)} \right)}_{=\prod_{v\in V} z_{v,h(v)}}$$
$$= \sum_{\substack{h:V\to X; \\ h|_P \text{ is a } \Gamma|_P\text{-partition to } X; \\ h|_Q \text{ is a } \omega(\Gamma)|_Q\text{-partition to } X}} \prod_{v\in V} z_{v,h(v)}. \qquad (12)$$

Now, forget that we fixed (P,Q). We thus have proven (12) for every $(P,Q) \in$

Adm Γ. Now,

$$\sum_{(P,Q)\in\operatorname{Adm}\Gamma}(-1)^{|Q|}\underbrace{Z_{P,X}Z^{\omega}_{Q,X}}_{\substack{=\sum_{\substack{h:V\to X;\\ h|_P \text{ is a } \Gamma|_P\text{-partition to } X;\\ h|_Q \text{ is a } \omega(\Gamma)|_Q\text{-partition to } X}}\prod_{v\in V} z_{v,h(v)}\\ \text{(by (12))}}}$$

$$=\sum_{(P,Q)\in\operatorname{Adm}\Gamma}(-1)^{|Q|}\sum_{\substack{h:V\to X;\\ h|_P \text{ is a } \Gamma|_P\text{-partition to } X;\\ h|_Q \text{ is a } \omega(\Gamma)|_Q\text{-partition to } X}}\prod_{v\in V} z_{v,h(v)}$$

$$=\sum_{h:V\to X}\prod_{v\in V} z_{v,h(v)}\sum_{\substack{(P,Q)\in\operatorname{Adm}\Gamma;\\ h|_P \text{ is a } \Gamma|_P\text{-partition to } X;\\ h|_Q \text{ is a } \omega(\Gamma)|_Q\text{-partition to } X}}(-1)^{|Q|}.$$

Hence, in order to prove Proposition 4.7 **(a)**, it is enough to show that every map $h : V \to X$ satisfies

$$\sum_{\substack{(P,Q)\in\operatorname{Adm}\Gamma;\\ h|_P \text{ is a } \Gamma|_P\text{-partition to } X;\\ h|_Q \text{ is a } \omega(\Gamma)|_Q\text{-partition to } X}}(-1)^{|Q|}=0 \tag{13}$$

(because once (13) will be proven, we will be able to conclude that

$$\sum_{(P,Q)\in\operatorname{Adm}\Gamma}(-1)^{|Q|}Z_{P,X}Z^{\omega}_{Q,X}$$
$$=\sum_{h:V\to X}\prod_{v\in V} z_{v,h(v)}\underbrace{\sum_{\substack{(P,Q)\in\operatorname{Adm}\Gamma;\\ h|_P \text{ is a } \Gamma|_P\text{-partition to } X;\\ h|_Q \text{ is a } \omega(\Gamma)|_Q\text{-partition to } X}}(-1)^{|Q|}}_{\substack{=0\\ \text{(by (13))}}}=\sum_{h:V\to X}\prod_{v\in V} z_{v,h(v)}0$$
$$=0=[V=\varnothing] \qquad (\text{since } V\neq\varnothing)$$

). It is (13) that we will now be proving.

So let $h : V \to X$ be any map. We need to show that (13) holds.

We will use the notations $<$, $\leq$, $>$ and $\geq$ for the smaller relation, the smaller-or-equal relation, the greater relation and the greater-or-equal relation, respectively, of the poset X.

Let us first define a certain set $\mathfrak{C}$. Namely, we let $\mathfrak{C}$ be the set of all pairs (P, Q) of subsets of V which satisfy the following five properties:

Property C1: For any $a \in V$ and any $b \in P$ satisfying $(a, b) \in A_w$, we have $h(a) \leq h(b)$.

Property C2: For any $a \in V$ and any $b \in P$ satisfying $(a, b) \in A_s$, we have $h(a) < h(b)$.

Property C3: For any $a \in Q$ and any $b \in V$ satisfying $(a,b) \in A_w$, we have $h(a) > h(b)$.

Property C4: For any $a \in Q$ and any $b \in V$ satisfying $(a,b) \in A_s$, we have $h(a) \geq h(b)$.

Property C5: We have $P \cup Q = V$ and $P \cap Q = \varnothing$.

Now, our first claim is the following:

Claim 1: Let (P,Q) be a pair of subsets of V. Then, (P,Q) belongs to $\mathfrak{C}$ if and only if the following three properties hold:

Property S1: We have $(P,Q) \in \operatorname{Adm}\Gamma$.

Property S2: The map $h\mid_P$ is a Γ-partition to X.

Property S3: The map $h\mid_Q$ is a $\omega(\Gamma)\mid_Q$-partition to X.

Proof of Claim 1: Let us first assume that the Properties S1, S2 and S3 hold.

We have $(P,Q) \in \operatorname{Adm}\Gamma$ (since Property S1 holds). In other words, (P,Q) is an admissible cut of the directed graph $\Gamma = (V, A_w \cup A_s)$. In other words, P and Q are subsets of V satisfying $P \cup Q = V$ and $P \cap Q = \varnothing$ such that

$$\text{there exists no } (q,p) \in A_w \cup A_s \text{ satisfying } p \in P \text{ and } q \in Q. \tag{14}$$

As a consequence, Property C5 holds, so that $Q = V \setminus P$ and $P = V \setminus Q$.

We know that $h\mid_P$ is a $\Gamma\mid_P$-partition to X (since Property S2 holds). In other words,

$$\text{every } (u,v) \in P \times P \text{ satisfying } (u,v) \in A_w \text{ satisfies } h(u) \leq h(v), \tag{15}$$

and

$$\text{every } (u,v) \in P \times P \text{ satisfying } (u,v) \in A_s \text{ satisfies } h(u) < h(v). \tag{16}$$

We know that $h\mid_Q$ is a $\omega(\Gamma)\mid_Q$-partition to X (since Property S3 holds). In other words,

$$\text{every } (u,v) \in Q \times Q \text{ satisfying } (u,v) \in \overline{A_s} \text{ satisfies } h(u) \leq h(v), \tag{17}$$

and

$$\text{every } (u,v) \in Q \times Q \text{ satisfying } (u,v) \in \overline{A_w} \text{ satisfies } h(u) < h(v). \tag{18}$$

Now, let us prove that Property C1 holds. In fact, let $a \in V$ and $b \in P$ be such that $(a,b) \in A_w$. If we had $a \in Q$, then $(a,b) \in A_w \subseteq A_w \cup A_s$ would contradict (14) (applied to $q = a$ and $p = b$). Hence, we cannot have $a \in Q$. We thus have $a \in V \setminus Q = P$, and now $h(a) \leq h(b)$ follows from (15) (applied to $u = a$ and $v = b$). This proves Property C1.

Similarly we can prove Property C2 (using (16) instead of (15)).

Let us now show that Property C3 holds. In fact, let $a \in Q$ and $b \in V$ be such that $(a,b) \in A_w$. If we had $b \in P$, then $(a,b) \in A_w \subseteq A_w \cup A_s$ would contradict (14) (applied to $q = a$ and $p = b$). Hence, we cannot have $b \in P$. We thus have

$b \in V \setminus P = Q$. But $(a,b) \in A_w$ shows that $(b,a) = \overline{(a,b)} \in \overline{A_w}$, so that (18) (applied to $u = b$ and $v = a$) yields $h(b) < h(a)$. In other words, $h(a) > h(b)$. This proves Property C3.

Analogously we can prove Property C4 (but this time we would have to derive $h(b) \leq h(a)$ from (17) instead of deriving $h(b) < h(a)$ from (18)).

We thus know that the pair (P, Q) satisfies all five Properties C1, C2, C3, C4 and C5. By the definition of $\mathfrak{C}$, this means precisely that the pair (P, Q) belongs to $\mathfrak{C}$.

Now, forget that we assumed that the Properties S1, S2 and S3 hold. We thus have shown that

$$\left(\text{if the Properties S1, S2 and S3 hold, then } (P,Q) \text{ belongs to } \mathfrak{C}\right). \tag{19}$$

Now, let us assume that (P, Q) belongs to $\mathfrak{C}$. By the definition of $\mathfrak{C}$, this means that the pair (P, Q) satisfies all five Properties C1, C2, C3, C4 and C5.

From Property C5, we conclude that $P \cup Q = V$ and $P \cap Q = \varnothing$.

Let us now see that there exists no $(q, p) \in A_w \cup A_s$ satisfying $p \in P$ and $q \in Q$. In fact, assume the contrary. Then, there exists a $(q, p) \in A_w \cup A_s$ satisfying $p \in P$ and $q \in Q$. Consider such a (q, p). We have $(q, p) \in A_w \cup A_s$. Let us assume (for the sake of contradiction) that $(q, p) \in A_w$. Then, Property C3 (applied to $(a,b) = (q,p)$) yields $h(q) > h(p)$. But Property C1 (applied to $(a,b) = (q,p)$) yields $h(q) \leq h(p)$, which contradicts $h(q) > h(p)$. This contradiction shows that our assumption (that $(q,p) \in A_w$) was wrong. Hence, we cannot have $(q, p) \in A_w$. So we have $(q,p) \notin A_w$. Since $(q,p) \in A_w \cup A_s$, this yields $(q,p) \in (A_w \cup A_s) \setminus A_w \subseteq A_s$. Thus, Property C2 (applied to $(a,b) = (q,p)$) yields $h(q) < h(p)$. But Property C4 (applied to $(a,b) = (q,p)$) yields $h(q) \geq h(p)$, which contradicts $h(q) < h(p)$. This contradiction shows that our assumption was wrong. We thus have proven that there exists no $(q, p) \in A_w \cup A_s$ satisfying $p \in P$ and $q \in Q$. This (combined with the fact that $P \cup Q = V$ and $P \cap Q = \varnothing$) shows that (P, Q) is an admissible cut of the directed graph $(V, A_w \cup A_s)$. In other words, (P, Q) is an admissible cut of the directed graph Γ (since the directed graph Γ is defined as $(V, A_w \cup A_s)$). In other words, $(P,Q) \in \operatorname{Adm}\Gamma$. Property S1 thus holds.

Next, let us prove Property S2. Let (u, v) be a weak arc of $\Gamma\mid_P$. Then, $(u,v) \in A_w \cap (P \times P)$ (because of how $\Gamma\mid_P$ is defined). In other words, $(u,v) \in A_w$, $u \in P$ and $v \in P$. Property C1 (applied to $a = u$ and $b = v$) yields $h(u) \leq h(v)$. Since $u \in P$ and $v \in P$, we can rewrite this as $(h\mid_P)(u) \leq (h\mid_P)(v)$. Now forget that we fixed (u, v). We have thus proven that every weak arc (u, v) of $\Gamma\mid_P$ satisfies $(h\mid_P)(u) \leq (h\mid_P)(v)$. Combining this with the fact that every strict arc (u, v) of $\Gamma\mid_P$ satisfies $(h\mid_P)(u) < (h\mid_P)(v)$ (the proof of this is analogous, but we now have to use Property C2 instead of Property C1), we conclude that $h\mid_P$ is a $\Gamma\mid_P$-partition to X. In other words, Property S2 holds.

Let us prove Property S3 now. Let (u, v) be a weak arc of $\omega(\Gamma)\mid_Q$. Then, $(u,v) \in \overline{A_s} \cap (Q \times Q)$ (because $\omega(\Gamma) = \left(V, \overline{A_s}, \overline{A_w}\right)$ and thus $\omega(\Gamma)\mid_Q = \left(Q, \overline{A_s} \cap (Q \times Q), \overline{A_w} \cap (Q \times Q)\right)$). In other words, $(u,v) \in \overline{A_s}$, $u \in Q$

and $v \in Q$. Since $(u, v) \in \overline{A_s}$, we have $(v, u) \in A_s$. Property C4 (applied to $a = v$ and $b = u$) thus yields $h(v) \geq h(u)$. In other words, $h(u) \leq h(v)$. Since $u \in Q$ and $v \in Q$, we can rewrite this as $\left(h\mid_Q\right)(u) \leq \left(h\mid_Q\right)(v)$. Now forget that we fixed (u, v). We have thus proven that every weak arc (u, v) of $\omega(\Gamma)\mid_Q$ satisfies $\left(h\mid_Q\right)(u) \leq \left(h\mid_Q\right)(v)$. Combining this with the fact that every strict arc (u, v) of $\omega(\Gamma)\mid_Q$ satisfies $(h\mid_P)(u) < (h\mid_P)(v)$ (the proof of this is analogous, but we now have to use Property C3 instead of Property C4), we conclude that $h\mid_Q$ is a $\omega(\Gamma)\mid_Q$-partition to X. In other words, Property S3 holds.

We have thus proven that all three Properties S1, S2 and S3 hold. Now, forget that we assumed that (P, Q) belongs to $\mathfrak{C}$. We thus have shown that

$$(\text{if } (P, Q) \text{ belongs to } \mathfrak{C}\text{, then the Properties S1, S2 and S3 hold}).$$

Combined with (19), this proves that (P, Q) belongs to $\mathfrak{C}$ if and only if the Properties S1, S2 and S3 hold. Claim 1 is thus proven.

Clearly, the summation sign $\sum\limits_{\substack{(P,Q)\in \operatorname{Adm}\Gamma;\\ h|_P \text{ is a } \Gamma|_P\text{-partition to } X;\\ h|_Q \text{ is a } \omega(\Gamma)|_Q\text{-partition to } X}}$ in (13) can be rewritten as $\sum\limits_{\substack{(P,Q) \text{ is a pair of subsets of } V;\\ \text{the Properties S1, S2 and S3 hold}}}$. Due to Claim 1, the latter summation sign can be replaced by $\sum\limits_{\substack{(P,Q) \text{ is a pair of subsets of } V;\\ (P,Q) \text{ belongs to } \mathfrak{C}}} = \sum\limits_{(P,Q)\in\mathfrak{C}}$. Hence, (13) is equivalent to

$$\sum_{(P,Q)\in\mathfrak{C}} (-1)^{|Q|} = 0. \tag{20}$$

Thus, instead of proving (13), it is enough to prove (20). Let us introduce the notation $\operatorname{snd} U$ for the second component of an ordered pair U (so that $\operatorname{snd}(P, Q) = Q$ for any ordered pair (P, Q)). Then, (20) further rewrites as

$$\sum_{U\in\mathfrak{C}} (-1)^{|\operatorname{snd} U|} = 0. \tag{21}$$

Hence, our goal is now to prove (21).

In order to prove this, it clearly is enough to construct a map $\Theta : \mathfrak{C} \to \mathfrak{C}$ satisfying the following three properties:

Property Th1: We have $\Theta \circ \Theta = \operatorname{id}_{\mathfrak{C}}$.

Property Th2: Every $U \in \mathfrak{C}$ satisfies $(-1)^{|\operatorname{snd}(\Phi(U))|} = -(-1)^{|\operatorname{snd} U|}$.

Property Th3: Every $U \in \mathfrak{C}$ satisfies $\Theta(U) \neq U$.

In fact, once a map Θ satisfying the Properties Th1, Th2 and Th3 is constructed, we can quickly conclude that (21) holds by fixing any total order on the set $\mathfrak{C}$ (which will be referred to using the $>$ and $<$ symbols) and arguing that

$$\begin{aligned}
\sum_{U\in\mathfrak{C}} (-1)^{|\operatorname{snd} U|} &= \sum_{\substack{U\in\mathfrak{C};\\ \Theta(U)>U}} (-1)^{|\operatorname{snd} U|} + \sum_{\substack{U\in\mathfrak{C};\\ \Theta(U)<U}} (-1)^{|\operatorname{snd} U|} \\
&\qquad \left(\text{because every } U\in\mathfrak{C} \text{ satisfies } \Theta(U)\neq U \text{ (by Property Th3)}\right) \\
&= \sum_{\substack{U\in\mathfrak{C};\\ \Theta(U)>U}} (-1)^{|\operatorname{snd} U|} + \underbrace{\sum_{\substack{U\in\mathfrak{C};\\ \Theta(\Theta(U))<\Theta(U)}}}_{\substack{=\sum_{\substack{U\in\mathfrak{C};\\ U<\Theta(U)}} \\ \text{(since Property 1} \\ \text{yields } \Theta(\Theta(U))=U)}} \underbrace{(-1)^{|\operatorname{snd}(\Theta(U))|}}_{\substack{=-(-1)^{|\operatorname{snd} U|} \\ \text{(by Property Th2)}}} \\
&\qquad \left(\begin{array}{c}\text{here, we have substituted } \Theta(U) \text{ for } U \text{ in the second sum} \\ \text{(since the map } \Theta:\mathfrak{C}\to\mathfrak{C} \text{ is a bijection (by Property Th1))}\end{array}\right) \\
&= \sum_{\substack{U\in\mathfrak{C};\\ \Theta(U)>U}} (-1)^{|\operatorname{snd} U|} - \underbrace{\sum_{\substack{U\in\mathfrak{C};\\ U<\Theta(U)}}}_{=\sum_{\substack{U\in\mathfrak{C};\\ \Theta(U)>U}}} (-1)^{|\operatorname{snd} U|} \\
&= \sum_{\substack{U\in\mathfrak{C};\\ \Theta(U)>U}} (-1)^{|\operatorname{snd} U|} - \sum_{\substack{U\in\mathfrak{C};\\ \Theta(U)>U}} (-1)^{|\operatorname{snd} U|} = 0.
\end{aligned}$$

Hence, it remains to construct a map Θ satisfying the Properties Th1, Th2 and Th3.

Before we can define our map Θ, we shall introduce some more notations.

We say that an element $v \in V$ is a *swing voter* if it satisfies the following four properties:

Property SW1: For any $u \in V$ satisfying $(u, v) \in A_w$, we have $h(v) \geq h(u)$.

Property SW2: For any $u \in V$ satisfying $(u, v) \in A_s$, we have $h(v) > h(u)$.

Property SW3: For any $u \in V$ satisfying $(v, u) \in A_w$, we have $h(v) > h(u)$.

Property SW4: For any $u \in V$ satisfying $(v, u) \in A_s$, we have $h(v) \geq h(u)$.

Our next goal is to show that there exists at least one swing voter.

In fact, recall that the set V is nonempty, so that $h(V)$ is nonempty as well. Thus, $h(V)$ is a nonempty finite subset of the totally ordered set X. Hence, the maximum element $\max(h(V))$ of $h(V)$ is well-defined. Let M be the set of all $w \in V$ such that $h(w) = \max(h(V))$. Then, M is nonempty.

Consider the weak-strict digraph $\Gamma' \mid_M$ as a directed graph. This directed graph $\Gamma' \mid_M$ is acyclic (since Γ' is acyclic), and its vertex set M is nonempty and finite. Hence, there exists a vertex $v \in M$ such that no arc of $\Gamma' \mid_M$ starts[5] at v (because it is known that whenever $\mathbf{D}$ is an acyclic finite directed graph with

[5] We say that an arc *starts* at v if it has the form (v, u) for some $u \in V$.

nonempty vertex set, there exists at least one vertex v of $\mathbf{D}$ such that no arc of $\mathbf{D}$ starts at v). Consider this v. We are going to show that v is a swing voter.

First of all, v belongs to the set M of all $w \in V$ such that $h(w) = \max(h(V))$. As a consequence,

$$\text{every } u \in V \text{ satisfies } h(u) \leq h(v), \tag{22}$$

and

$$\text{every } u \in V \text{ such that } h(u) \geq h(v) \text{ must belong to } M. \tag{23}$$

From (22), we conclude immediately that v satisfies Property SW1 and Property SW4.

Let us next prove that Property SW2 holds. Let $u \in V$ be such that $(u, v) \in A_s$. Assume (for the sake of contradiction) that $h(u) \geq h(v)$. Then, (23) shows that u belongs to M. But $(u, v) \in A_s$, so that $(v, u) = \overline{(u, v)} \in \overline{A_s} \subseteq A_w \cup \overline{A_s}$. Thus, (v, u) is an arc of Γ', and therefore also an arc of $\Gamma' \mid_M$ (since u and v belong to M). This contradicts the fact that no arc of $\Gamma' \mid_M$ starts at v. This contradiction shows that we were wrong to assume that $h(u) \geq h(v)$. Hence, we must have $h(v) > h(u)$. This proves Property SW2.

The proof of Property SW3 is exactly analogous, with the only difference that the relation $(v, u) \in A_w \cup \overline{A_s}$ is now proven using $(v, u) \in A_w \subseteq A_w \cup \overline{A_s}$.

We now have shown that our v satisfies all four Properties SW1, SW2, SW3, and SW4. In other words, v is a swing voter. Hence, there exists a swing voter.

Let us now fix a swing voter v. Here comes a final piece of notation before define the map Θ: Namely, we denote the symmetric difference of two sets A and B by $A \triangle B$ (this is the set $(A \cup B) \setminus (A \cap B)$).

Now, let $(P, Q) \in \mathfrak{C}$ be an arbitrary. We then define $\Theta(P, Q)$ as $(P \triangle \{v\}, Q \triangle \{v\})$. We must prove that this map Θ is well-defined; in other words, we must prove that $(P \triangle \{v\}, Q \triangle \{v\})$ belongs to $\mathfrak{C}$ for every $(P, Q) \in \mathfrak{C}$. In order to do so, we fix some $(P, Q) \in \mathfrak{C}$. By the definition of $\mathfrak{C}$, this means that (P, Q) is a pair of subsets of V satisfying the Properties C1, C2, C3, C4 and C5.

We will now show that the Properties C1, C2, C3, C4 and C5 with P and Q replaced by $P \triangle \{v\}$ and $Q \triangle \{v\}$ are also satisfied.

Let us start with Property C5. Since Property C5 itself is satisfied, we have $P \cup Q = V$ and $P \cap Q = \varnothing$. Therefore, v belongs to exactly one of the two sets P and Q. From this, it is easy to conclude (by checking both possible cases) that $(P \triangle \{v\}) \cup (Q \triangle \{v\}) = V$ and $(P \triangle \{v\}) \cap (Q \triangle \{v\}) = \varnothing$. In other words, Property C5 with P and Q replaced by $P \triangle \{v\}$ and $Q \triangle \{v\}$ is satisfied.

Let now $a \in V$ and $b \in P \triangle \{v\}$ be such that $(a, b) \in A_w$. We are going to prove that $h(a) \leq h(b)$. If $b \in P$, then this follows from Property C1. Hence, for the rest of the proof of $h(a) \leq h(b)$, we can WLOG assume that we don't have $b \in P$. Assume this. So we have $b \in P \triangle \{v\} \subseteq P \cup \{v\}$ but we don't have $b \in P$. Thus, we must have $b \in (P \cup \{v\}) \setminus P = \{v\}$, so that $b = v$. Thus, $\left(a, \underbrace{v}_{=b}\right) = (a, b) \in A_w$. Now, Property SW1 (applied to $u = a$) yields

$h(v) \geq h(a)$. Hence, $h(a) \leq h\left(\underbrace{v}_{=b}\right) = h(b)$. This completes the proof of $h(a) \leq h(b)$. Now, forget that we fixed a and b. We thus have proven that for any $a \in V$ and any $b \in P \triangle \{v\}$ satisfying $(a,b) \in A_w$, we have $h(a) \leq h(b)$. In other words, Property C1 with P and Q replaced by $P \triangle \{v\}$ and $Q \triangle \{v\}$ is satisfied.

Similarly we can prove that Property C2 with P and Q replaced by $P \triangle \{v\}$ and $Q \triangle \{v\}$ is satisfied (but now we have to apply Property SW2 instead of Property SW1). Similarly we can prove that Property C3 with P and Q replaced by $P \triangle \{v\}$ and $Q \triangle \{v\}$ is satisfied (but now we have to rule out the case $a \in Q$ instead of the case $b \in P$, and to apply Property SW3 instead of Property SW1). Similarly we can prove that Property C4 with P and Q replaced by $P \triangle \{v\}$ and $Q \triangle \{v\}$ is satisfied (but now we have to rule out the case $a \in Q$ instead of the case $b \in P$, and to apply Property SW4 instead of Property SW1).

Altogether, we now know that the Properties C1, C2, C3, C4 and C5 with P and Q replaced by $P \triangle \{v\}$ and $Q \triangle \{v\}$ are also satisfied. Since $(P \triangle \{v\}, Q \triangle \{v\})$ is a pair of subsets of V, this yields that $(P \triangle \{v\}, Q \triangle \{v\})$ belongs to $\mathfrak{C}$ (because of the definition of $\mathfrak{C}$). We thus have proven that the map Θ is well-defined.

It remains to prove that this map Θ satisfies the Properties Th1, Th2 and Th3. It is clear that Θ satisfies Property Th2 (in fact, if $U \in \mathfrak{C}$ is arbitrary, then we can write U in the form $U = (P,Q)$ for some P and Q, and then we have $\Theta(U) = \Theta(P,Q) = (P \triangle \{v\}, Q \triangle \{v\})$ (by the definition of Θ), so that

$$\left|\operatorname{snd}\left(\underbrace{\Theta(U)}_{=(P\triangle\{v\},Q\triangle\{v\})}\right)\right| = \left|\underbrace{\operatorname{snd}(P\triangle\{v\},Q\triangle\{v\})}_{=Q\triangle\{v\}}\right| = |Q\triangle\{v\}|$$
$$\equiv |Q| + \underbrace{|\{v\}|}_{=1} = \left|\underbrace{Q}_{=\operatorname{snd} U}\right| + 1 = |\operatorname{snd} U| + 1 \bmod 2,$$

so that $(-1)^{|\operatorname{snd}(\Phi(U))|} = (-1)^{|\operatorname{snd} U|+1} = -(-1)^{|\operatorname{snd} U|}$), and therefore also satisfies Property Th3 (since Property Th2 yields $(-1)^{|\operatorname{snd}(\Phi(U))|} = -(-1)^{|\operatorname{snd} U|} \neq (-1)^{|\operatorname{snd} U|}$ for every $U \in \mathfrak{C}$). It therefore remains to prove Property Th1. In order to do so, we fix $U \in \mathfrak{C}$. Write U in the form (P,Q). Then,

$$\Theta(U) = \Theta(P,Q) = (P\triangle\{v\}, Q\triangle\{v\}) \qquad \text{(by the definition of } \Theta\text{)},$$

so that

$$\begin{aligned}\Theta(\Theta(U)) &= \Theta(P\triangle\{v\}, Q\triangle\{v\}) \\ &= \left(\underbrace{(P\triangle\{v\})\triangle\{v\}}_{=P}, \underbrace{(Q\triangle\{v\})\triangle\{v\}}_{=Q}\right) \qquad \text{(by the definition of } \Theta\text{)} \\ &= (P,Q) = U.\end{aligned}$$

Since this holds for every $U \in \mathfrak{C}$, we thus have $\Theta \circ \Theta = \operatorname{id}_{\mathfrak{C}}$. Hence, Property Th1 is proven.

We now know that the map Θ satisfies all three Properties Th1, Th2 and Th3. Thus, as we have seen, (21) holds, and this completes the proof of Proposition 4.7 **(a)** (as we have seen).

(b) Let X^{op} denote the opposite poset of X; this is the poset with the same ground set as X, but whose smaller relation is the greater relation of X. Of course, X^{op} is a totally ordered set (since X is a totally ordered set).

We denote by $\widetilde{\Gamma}$ the weak-strict digraph (V, A_s, A_w). (This is obtained from Γ by interchanging the set of weak arcs with the set of strict arcs.) It is easy to see that the $\omega(\Gamma)\mid_P$-partitions to X are the same thing as the $\widetilde{\Gamma}$-partitions to X^{op}. Moreover, the $\Gamma\mid_Q$-partitions to X are the same as the $\omega\left(\widetilde{\Gamma}\right)\mid_Q$-partitions to X^{op}. Finally, $\operatorname{Adm}\widetilde{\Gamma} = \operatorname{Adm}\Gamma$. Hence, Proposition 4.7 **(a)** (applied to A_s, A_w, $\widetilde{\Gamma}$ and X^{op} instead of A_w, A_s, Γ and X) yields

$$\sum_{(P,Q)\in\operatorname{Adm}\Gamma} (-1)^{|Q|} Z^{\omega}_{P,X} Z_{Q,X} = [V = \varnothing].$$

But since every $(P,Q) \in \operatorname{Adm}\Gamma$ satisfies $|Q| = |V| - |P|$ (because $P \cup Q = V$ and $P \cap Q = \varnothing$), the left hand side of this rewrites as

$$\begin{aligned} &\sum_{(P,Q)\in\operatorname{Adm}\Gamma} \underbrace{(-1)^{|Q|}}_{=(-1)^{|V|-|P|}=(-1)^{|V|}(-1)^{|P|}} Z^{\omega}_{P,X} Z_{Q,X} \\ &= (-1)^{|V|} \sum_{(P,Q)\in\operatorname{Adm}\Gamma} (-1)^{|P|} Z^{\omega}_{P,X} Z_{Q,X}. \end{aligned}$$

The rest is clear.

[... detail proof of (b)] □

Proof of Theorem 4.3. Let $m : \operatorname{QSym}_{\mathbf{k}} \otimes \operatorname{QSym}_{\mathbf{k}} \to \operatorname{QSym}_{\mathbf{k}}$ be the multiplication map of the $\mathbf{k}$-algebra $\operatorname{QSym}_{\mathbf{k}}$ (that is, the $\mathbf{k}$-linear map sending every $a \otimes b$ to ab). Let $u : \mathbf{k} \to \operatorname{QSym}_{\mathbf{k}}$ be the unity map of the $\mathbf{k}$-algebra $\operatorname{QSym}_{\mathbf{k}}$ (that is, the $\mathbf{k}$-linear map sending 1 to $1_{\operatorname{QSym}_{\mathbf{k}}}$).

We need to prove that S is an antipode of the $\mathbf{k}$-bialgebra $\operatorname{QSym}_{\mathbf{k}}$. For this, it is clearly enough to show that

$$m \circ \left(S \otimes \operatorname{id}_{\operatorname{QSym}_{\mathbf{k}}}\right) \circ \Delta = u \circ \varepsilon \tag{24}$$

and

$$m \circ \left(\operatorname{id}_{\operatorname{QSym}_{\mathbf{k}}} \otimes S\right) \circ \Delta = u \circ \varepsilon. \tag{25}$$

We first introduce a certain class of weak-strict digraphs. Namely, a *dipath* will mean a weak-strict digraph of the form $\operatorname{DPath}\alpha$ for a composition α. For any dipath Γ, the weak-strict digraph $\omega(\Gamma)$ also is a dipath, and in fact we have

$\omega(\Gamma) = \operatorname{DPath}(\omega(\alpha))$ for the composition α satisfying $\Gamma = \operatorname{DPath}\alpha$. Hence, if $\Gamma = (V, A_w, A_s)$ is a dipath, then

$$S(F_\Gamma) = (-1)^{|V|} F_{\omega(\Gamma)} \tag{26}$$

(because the formula $S(L_\alpha) = (-1)^{|\alpha|} L_{\omega(\alpha)}$, applied to the composition α satisfying $\Gamma = \operatorname{DPath}\alpha$, rewrites as $S(F_\Gamma) = (-1)^{|V|} F_{\omega(\Gamma)}$ once we recall Proposition 2.16).

Let us now prove (24). Indeed, let α be any composition. Let Γ be the weak-strict digraph $\operatorname{DPath}\alpha$. Proposition 2.16 yields $F_{\operatorname{DPath}\alpha} = L_\alpha$, so that $L_\alpha = F_{\operatorname{DPath}\alpha} = F_\Gamma = F_{(\Gamma,\mathbf{1})}$ and thus

$$\Delta(L_\alpha) = \Delta\left(F_{(\Gamma,\mathbf{1})}\right) = \sum_{(P,Q)\in\operatorname{Adm}\Gamma} \underbrace{F_{(\Gamma|_P,\mathbf{1}|_P)}}_{=F_{\Gamma|_P}} \otimes \underbrace{F_{(\Gamma|_Q,\mathbf{1}|_Q)}}_{=F_{\Gamma|_Q}} \qquad \text{(by Proposition 3.5 \textbf{(a)})}$$
$$= \sum_{(P,Q)\in\operatorname{Adm}\Gamma} F_{\Gamma|_P} \otimes F_{\Gamma|_Q}.$$

Hence,

$$\begin{aligned}
&\left(m \circ \left(S \otimes \operatorname{id}_{\operatorname{QSym}_{\mathbf{k}}}\right) \circ \Delta\right)(L_\alpha) \\
&= \left(m \circ \left(S \otimes \operatorname{id}_{\operatorname{QSym}_{\mathbf{k}}}\right)\right)\left(\underbrace{\Delta(L_\alpha)}_{=\sum_{(P,Q)\in\operatorname{Adm}\Gamma} F_{\Gamma|_P}\otimes F_{\Gamma|_Q}}\right) \\
&= \left(m \circ \left(S \otimes \operatorname{id}_{\operatorname{QSym}_{\mathbf{k}}}\right)\right)\left(\sum_{(P,Q)\in\operatorname{Adm}\Gamma} F_{\Gamma|_P} \otimes F_{\Gamma|_Q}\right) \\
&= \sum_{(P,Q)\in\operatorname{Adm}\Gamma} \underbrace{S\left(F_{\Gamma|_P}\right)}_{\substack{=(-1)^{|P|}F_{\omega(\Gamma|_P)} \\ \text{(by (26) (applied to} \\ \Gamma|_P \text{ instead of } \Gamma\text{) (since } \Gamma|_P \text{ is a dipath} \\ \text{(since } \Gamma \text{ is a dipath and } (P,Q)\in\operatorname{Adm}\Gamma)))}} \cdot F_{\Gamma|_Q} \\
&= \sum_{(P,Q)\in\operatorname{Adm}\Gamma} (-1)^{|P|} F_{\omega(\Gamma|_P)} F_{\Gamma|_Q}.
\end{aligned}$$

But Γ' is acyclic (since Γ is a dipath). Let X be the totally ordered set $\mathbb{N}_+$ (with the usual ordering). For every $v \in V$ and

[...]

XXX

[... detail this pf] □

[...]

First proof of Theorem 4.5 (sketched). [...]

[...] □

Second proof of Theorem 4.5 (sketched). Let us define one more notion: Let a *weak-strict-undirected digraph* be defined as a quadruple (V, A_s, A_w, E) such that (V, A_s, A_w) is a weak-strict digraph and (V, E) is a simple undirected graph. If Γ is a weak-strict-undirected digraph, define two new weak-strict-undirected digraphs $\Gamma' = \left(V, A_w, \overline{A_s}, E\right)$ and $\omega(\Gamma) = \left(V, \overline{A_s}, \overline{A_w}, E\right)$ (these definitions imitate the analogous definitions for weak-strict digraphs, and leave E unchanged).

We assume that $\mathbf{k} = \mathbb{Z}$ to match the setting of [MalReu98] (we don't lose anything from this assumption, because of Proposition 1.11). Thus, $\operatorname{QSym}_{\mathbf{k}} = \operatorname{QSym}$.

We define the conjugation map $\omega : \operatorname{QSym} \to \operatorname{QSym}$ as in [MalReu98, §3]. It is well-known that

$$S(f) = (-1)^d \omega(f) \tag{27}$$

for every homogeneous $f \in \operatorname{QSym}$ having degree d.

We can WLOG assume that the directed graph Γ is acyclic[6]. Assume this. Notice also that $A_w \cap A_s = \varnothing$ (since Γ' is acyclic).

Define the set $\widetilde{V}$ as $\{(p, i) \mid p \in V,\ 1 \leq i \leq w(p)\}$. Roughly speaking, this set $\widetilde{V}$ is obtained from V by breaking up each element $p \in V$ into $w(p)$ distinct elements. Embed V into V by sending every $p \in V$ to $(p, 1) \in \widetilde{V}$. Then, A_w and A_s are subsets of $\widetilde{V} \times \widetilde{V}$. Therefore, $\left(\widetilde{V}, A_w, A_s\right)$ becomes a weak-strict digraph. Denote this weak-strict digraph by $\widetilde{\Gamma}$. Now, define a set $E \subseteq \binom{\widetilde{V}}{2}$ by

$$E = \{\{(p, i), (p, i+1)\} \mid p \in V,\ 1 \leq i \leq w(p) - 1\}.$$

Then, $\left(\widetilde{V}, A_w, A_s, E\right)$ is a weak-strict-undirected digraph. It is easily seen that both $\left(\widetilde{V}, A_w, A_s, E\right)$ and $\left(\widetilde{V}, A_w, A_s, E\right)'$ are acyclic. We can therefore apply [MalReu98, Theorem 3.1] to the set of equalities and inequalities induced by $\left(\widetilde{V}, A_w, A_s, E\right)$ (that is, $x_u \leq x_v$ for each $(u, v) \in A_w$, as well as $x_u < x_v$ for each $(u, v) \in A_s$, as well as $x_u = x_v$ for each $\{u, v\} \in E$). The result is basically

[6] In fact, assume that it is not. Thus, there exists a cycle consisting of weak arcs and strict arcs in Γ. This cycle cannot consist of weak arcs alone (because Γ' is acyclic), so its existence forces an unsatisfiable chain of inequalities upon any Γ-partition. Hence, there exist no Γ-partitions; thus, $F_{(\Gamma,w)} = 0$. Also, the cycle that we have just found cannot consist of strict arcs alone (because Γ' is acyclic again), so its reversal is a cycle consisting of weak arcs and strict arcs in $\omega(\Gamma)$ which cannot consist of weak arcs alone. This latter cycle forces an unsatisfiable chain of inequalities upon any $\omega(\Gamma)$-partition; thus, $F_{(\omega(\Gamma),w)} = 0$. Hence, Theorem 4.5 boils down to $0 = (-1)^{|V|} 0$ in this case, which is obvious.

$\omega\left(F_{(\Gamma,w)}\right) = (-1)^{|E|} F_{(\omega(\Gamma),w)}$. Now, $S\left(F_{(\Gamma,w)}\right) = (-1)^{|V|} F_{(\omega(\Gamma),w)}$ follows from this by applying (27) and by noticing that $|E| = \sum_{p\in V} w(p) - |V|$. Theorem 4.5 is proven. □

Third proof of Theorem 4.5 (sketched). We will prove Theorem 4.5 by a series of reductions to simpler cases.

First, we notice that $A_w \cap A_s = \varnothing$ (since Γ' is acyclic).

For any arc a, let $\overline{a}$ denote the reverse of the arc a.

Second, if $a \in A_w$ is any arc, then Theorem 4.5 for the weak-strict digraph $\Gamma = (V, A_w, A_s)$ follows if we can prove Theorem 4.5 for the weak-strict digraph $(V, A_w \setminus \{a\}, A_s)$ and Theorem 4.5 for the weak-strict digraph $(V, A_w \setminus \{a\}, A_s \cup \{\overline{a}\})$. In fact, the Γ-partitions are precisely the $(V, A_w \setminus \{a\}, A_s)$-partitions which are not $(V, A_w \setminus \{a\}, A_s \cup \{\overline{a}\})$-partitions (just look at the definitions); hence,

$$F_{(\Gamma,w)} = F_{((V,A_w\setminus\{a\},A_s),w)} - F_{((V,A_w\setminus\{a\},A_s\cup\{\overline{a}\}),w)},$$

and similarly

$$F_{(\omega(\Gamma),w)} = F_{(\omega(V,A_w\setminus\{a\},A_s),w)} - F_{(\omega(V,A_w\setminus\{a\},A_s\cup\{\overline{a}\}),w)}.$$

Moreover, both $(V, A_w \setminus \{a\}, A_s)'$ and $(V, A_w \setminus \{a\}, A_s \cup \{\overline{a}\})'$ are acyclic (since Γ' is acyclic), so that Theorem 4.5 applies to them. Thus, Theorem 4.5 for the weak-strict digraph $\Gamma = (V, A_w, A_s)$ follows if we can prove Theorem 4.5 for the weak-strict digraph $(V, A_w \setminus \{a\}, A_s)$ and Theorem 4.5 for the weak-strict digraph $(V, A_w \setminus \{a\}, A_s \cup \{\overline{a}\})$. Hence, we have reduced Theorem 4.5 for the weak-strict digraph $\Gamma = (V, A_w, A_s)$ to two cases of Theorem 4.5 for weak-strict digraphs with a smaller number of weak arcs than Γ (but with the same number of vertices). Hence, by induction over $|A_w|$, we can reduce the general case to the case when $A_w = \varnothing$.

We thus WLOG assume that $A_w = \varnothing$. Hence, the weak-strict digraph Γ is acyclic (since Γ' is acyclic).

We now want to get rid of w (or, more precisely, assume WLOG that $w = \mathbf{1}$). In fact, assume that $w \neq \mathbf{1}$. Then, there exists some $p \in V$ such that $w(p) \neq 1$. Consider this p. Clearly, $w(p) > 1$. Now, let p_1 be a new vertex not in V, and set V_1 be the set $V \cup \{p_1\}$. Let $w_1 : V_1 \to \mathbb{N}_+$ be the map which sends every $q \in V \setminus \{p\}$ to $w(q)$, sends p to $w(p) - 1$ and sends p_1 to 1. Let Γ_1 be the weak-strict digraph $(V_1, \varnothing, A_s)$; let $\Gamma_{\to 1}$ be the weak-strict digraph $(V_1, \varnothing, A_s \cup \{(p, p_1)\})$; let $\Gamma_{\leftarrow 1}$ be the weak-strict digraph $(V_1, \varnothing, A_s \cup \{(p_1, p)\})$. It is easy to see that all three weak-strict digraphs Γ_1', $\Gamma_{\to 1}'$ and $\Gamma_{\leftarrow 1}'$ are acyclic, but we have

$$F_{(\Gamma,w)} = F_{(\Gamma_1,w_1)} - F_{(\Gamma_{\to 1},w_1)} - F_{(\Gamma_{\leftarrow 1},w_1)}$$

and similarly

$$F_{(\omega(\Gamma),w)} = F_{(\omega(\Gamma_{\to 1}),w_1)} - F_{(\omega(\Gamma_{\leftarrow 1}),w_1)} - F_{(\omega(\Gamma_1),w_1)}$$

(because the map ω turns strict arcs into weak arcs!). Thus, Theorem 4.5 for the weak-strict digraph Γ follows from Theorem 4.5 for the weak-strict digraphs Γ_1, $\Gamma_{\to 1}$ and $\Gamma_{\leftarrow 1}$ (keep in mind that $|V_1| = |V| + 1$, whence the signs switch). While these weak-strict digraphs Γ_1, $\Gamma_{\to 1}$ and $\Gamma_{\leftarrow 1}$ have more vertices than Γ, they have the same number $\sum_{p \in V} w(p)$, and thus we can use this as a recursive proof to transform our problem until w becomes $\mathbf{1}$ (and of course the fact that $A_w = \varnothing$ does not break in the process). So we can WLOG assume that $w = \mathbf{1}$. Assume this.

The directed graph Γ is acyclic, and hence there exists a poset structure on V such that $a > b$ for every $(a, b) \in A_s$. Consider this poset structure. Give this poset a natural labelling, i.e., a labelling of the vertices of Γ by the integers 1, 2, ..., $|V|$ such that $a > b$ as integers whenever $a > b$ in V. Hence, V is a labelled poset. It is easy to see that $F_{(\Gamma,w)} = F_V(\mathbf{x})$ and $F_{(\omega(\Gamma),w)} = F_{V^{\text{opp}}}(\mathbf{x})$ using the notations of [GriRei15, Section 5]. Thus, the equality that needs to be proven, $S\left(F_{(\Gamma,w)}\right) = (-1)^{|V|} F_{(\omega(\Gamma),w)}$, rewrites as $S(F_V(\mathbf{x})) = (-1)^{|V|} F_{V^{\text{opp}}}(\mathbf{x})$. This follows from [GriRei15, Corollary 5.27]. Theorem 4.5 is thus proven. □

[...]

5. P-partitions and special cases

Let us now relate the notion of Γ-partitions, which we so far have been studying, with the far more classical notion of P-partitions. This allows applying our results to reprove existing results in literature, but also conversely deriving our results from known results.

The theory of P-partitions (or, to be more precise, (P, ω)-partitions, of which P-partitions are but a particular case) goes back to Richard Stanley's thesis [Stan71], where it was introduced as a common generalization for the enumerations of multiple combinatorial objects (such as usual partitions, plane partitions and various others); this allowed Stanley to derive several results of enumerative combinatorics from a rather manageable collection of (fairly uncomplicated) theorems. We will skip the (too restrictive) notion of P-partitions for a poset P, and introduce (P, ω)-partitions right away:

Definition 5.1. Let P be a finite poset. Let X be a poset. Let $\omega : P \to \{1, 2, \ldots, |P|\}$ be a bijection. We will use the symbols $\leq$ and $<$ for the smaller relation and the smaller-or-equal relation, respectively, of a poset. (Which poset we mean will be clear from the types of the objects on both sides of these symbols.) A *(P, ω)-partition to X* will mean a function $f : P \to X$ such that:

(a) every two elements u and v of P satisfying $u < v$ and $\omega(u) < \omega(v)$ satisfy $f(u) \leq f(v)$;

(b) every two elements u and v of P satisfying $u < v$ and $\omega(u) > \omega(v)$ satisfy $f(u) < f(v)$.

As before, we regard $\mathbb{N}_+$ as a poset using the standard smaller relation on $\mathbb{N}_+$ (so this poset is actually a totally ordered set). When we don't explicitly mention X and just speak of "(P, ω)-partitions", we always mean (P, ω)-partitions to this poset $\mathbb{N}_+$.

Remark 5.2. Conflicting definitions of the notion of a "(P, ω)-partition" exist in literature. We follow the notation of Reiner [GriRei15, §5.2] (except that Reiner subsumes both the poset P and the bijection ω into one object P, which he calls a "labelled poset"), while Stanley (in [Stan11, §3.15] and in [Stan71, Definition 2.2]) has a definition which differs from ours in requiring $f(u) \geq f(v)$ (instead of $f(u) \geq f(v)$) in **(a)**, requiring $f(u) > f(v)$ (instead of $f(u) < f(v)$) in **(b)**, and assuming that $X = \mathbb{N}$ (not $\mathbb{N}_+$). Our notion of "(P, ω)-partition" is what Stanley calls a "reverse (P, ω)-partition" in [Stan99, §7.19]. Our definition of a (P, ω)-partition is essentially equivalent to that given by Gessel in [Gessel84, §2] and that given by Malvenuto in [Malve93, §4.2] (except that both Gessel and Malvenuto require X to be infinite, and equate u with $\omega(u)$). The definition given in [LMvW13, Definition 3.3.13] differs from ours only in having ω map P injectively into a totally ordered set rather than bijectively into $\{1, 2, \ldots, |P|\}$.

It should be fairly obvious that (P, ω)-partitions are a particular case of Γ-partitions.

[... proposition, proof]

Conversely, however, Γ-partitions are a particular case of (P, ω)-partitions for those weak-strict digraphs Γ for which Γ' is acyclic.

[... proposition, proof]

The connection between (P, ω)-partitions and quasisymmetric functions uses a generating function $F_{P,\omega}$ similar to our generating function F_Γ defined in Definition 2.5 (and, in fact, being a particular case thereof):

[... Definition]

[... Why particular case]

Remark 5.3. The quasisymmetric function $F_{P,\omega}$ for a finite poset P and a bijection $\omega : P \to \{1, 2, \ldots, |P|\}$ has been first introduced by Stanley in [Stan71, §21], however under a different name (he called it $\{P, \omega\}$, or more precisely, he called $\{P, \omega\}$ the quasisymmetric function that would be $F_{P,\omega}$ if we had defined the notion of (P, ω)-partition as he defined it); it has later been denoted by $K_{P,\omega}$ in [Stan99, §7.19]. The reader should be warned that it is **not** related to the function called $F_{P,\omega}$ in Stanley's [Stan11, §3.15] (or the function called $F(P, \omega)$ in [Stan71]). The quasisymmetric function $F_{P,\omega}$ is denoted F_P in [GriRei15, §5.2] (where the poset P and the bijection ω are subsumed into one object P), denoted by $F(P, \omega)$ in [LMvW13, Definition 3.3.18], and denoted by

$\Gamma(P)$ in [Gessel84, §2] and in [Malve93, §4.2] (the map ω is assumed to be id in both of these references).

Historically, the concept of $F_{P,\omega}$ has been anticipated in the work of Glânffrwd Thomas ([Thomas77], [Thomas76]), who defined a particular case (in which P is a subset of the lattice $\mathbb{Z} \times \mathbb{Z}$, and ω is a "column-strict labelling", so that $F_{P,\omega}$ becomes a generalization of Schur functions) using the formalism of Baxter operators.

In view of Proposition [... ref], we might wonder what we have gain from generalizing (P, ω)-partitions to Γ-partitions, given that the main results (Theorem 4.5 and Corollary 4.6) include the very condition that forces Γ-partitions to be just (P, ω)-partitions (the acyclicity of Γ'), and therefore follow from known theorems about the latter [... formulate!]. It is our opinion, however, that the notion of a Γ-partition is more natural and more malleable than that of a (P, ω)-partition. In particular, in weak-strict digraphs, the symmetry between the notion of weak arcs and that of strict arcs is more obvious, and it is easier to add or remove an arc from a weak-strict digraph than to add or remove relations to a poset. Also, it is somewhat simpler to present L_α as a particular case of F_Γ (see Proposition 2.16) than to present it as a particular case of $F_{(P,\omega)}$; here is how the latter works:

[...]

Definition 5.4. Let $n \in \mathbb{N}$. We denote by S_n the symmetric group on the set $\{1, 2, \ldots, n\}$. Let $\sigma \in S_n$. Then, an element $i \in \{1, 2, \ldots, n-1\}$ is said to be a *descent* of σ if $\sigma(i) > \sigma(i+1)$. The set of all descents of σ is called the *descent set* of σ, and denoted by $\operatorname{Des}\sigma$. The composition $\operatorname{comp}_n(\operatorname{Des}\sigma)$ is called the *descent composition* of σ. For instance, if σ is the permutation in S_6 written as $(3, 1, 4, 2, 6, 5)$ in one-line notation, then the descent set of σ is $\{1, 3, 5\}$, and the descent composition of σ is $(1, 2, 2, 1)$.

[...]

6. Γ-partitions up to group action

We are now ready to introduce a group action into the picture:

Corollary 6.1. Let $\Gamma = (V, A_w, A_s)$ be a weak-strict digraph. Let $w : V \to \mathbb{N}_+$ be any map. Assume that Γ' is acyclic.

An *automorphism* of a weak-strict digraph means an automorphism of its vertex set which preserves both the set of weak arcs and the set of strict arcs.

Given a group G, we say that the group G *acts on* Γ *by automorphisms* if we are given an action of G on the vertex set V of Γ such that the action of each $g \in G$ is an automorphism of Γ.

Let G be a finite group which acts on Γ by automorphisms and which preserves w (that is, $w(gp) = w(p)$ for every $g \in G$ and $p \in V$).

Let $\operatorname{Par}\Gamma$ denote the set of all Γ-partitions. Clearly, G acts on this set $\operatorname{Par}\Gamma$ too.

(a) We have $\prod_{p\in V} x_{f(p)}^{w(p)} = \prod_{p\in V} x_{(gf)(p)}^{w(p)}$ for every $f \in \operatorname{Par}\Gamma$ and every $g \in G$. Hence, we can define a formal power series $F_{(\Gamma,w),G} \in \mathbb{Z}[[x_1, x_2, x_3, \ldots]]$ by

$$F_{(\Gamma,w),G} = \sum_{f\in S} \prod_{p\in V} x_{f(p)}^{w(p)}$$

by picking a system of representatives S of the orbits $(\operatorname{Par}\Gamma)\diagup G$, and this series will not depend on the choice of S. This series satisfies $F_{(\Gamma,w),G} \in \operatorname{QSym}$.

(b) Let $\rho : G \to S_V$ denote the action of G on the vertex-set V of Γ, written as a group homomorphism from G into the symmetric group S_V. A Γ-partition $f \in \operatorname{Par}\Gamma$ is said to be *even* if every $g \in G$ satisfying $gf = f$ must satisfy $\operatorname{sgn}(\rho(g)) = 1$. Let $(\operatorname{Par}\Gamma)^+$ denote the set of all even Γ-partitions. Then, the action of G on $\operatorname{Par}\Gamma$ restricts to an action on $(\operatorname{Par}\Gamma)^+$. Hence, we can define a formal power series $F^+_{(\Gamma,w),G} \in \mathbb{Z}[[x_1, x_2, x_3, \ldots]]$ by

$$F^+_{(\Gamma,w),G} = \sum_{f\in T} \prod_{p\in V} x_{f(p)}^{w(p)}$$

by picking a system of representatives T of the orbits $(\operatorname{Par}\Gamma)^+\diagup G$, and this series will not depend on the choice of T. This series satisfies $F^+_{(\Gamma,w),G} \in \operatorname{QSym}$.

(c) We have

$$S\left(F_{(\Gamma,w),G}\right) = (-1)^{|V|} F^+_{(\omega(\Gamma),w),G}.$$

Proof sketch. For every $g \in G$, define a weak-strict digraph Γ^g as follows: Let V^g be the set V modulo the equivalence relation ($p \sim gp$ for every $g \in G$ and $p \in V$). This set V^g will be the vertex set of Γ^g. The set of all weak arcs of Γ^g will be $\left\{\left([u]_g, [v]_g\right) \mid (u,v) \in A_w\right\}$ (where $[p]_g$ denotes the projection of $p \in V$ onto V^g). The set of all strict arcs of Γ^g will be $\left\{\left([u]_g, [v]_g\right) \mid (u,v) \in A_s\right\}$. The resulting weak-strict digraph Γ^G is easily seen to satisfy the property that $(\Gamma^g)'$ is acyclic. (This is mostly due to the fact that G acts on Γ by automorphisms, thus G acts on Γ' by automorphisms, and consequently each orbit of g on V' is an independent set of Γ' (since Γ' is acyclic).)

Furthermore, for every $g \in G$, define a map $w^g : V^g \to \mathbb{N}_+$ by $w^g\left([p]_g\right) = \left|[p]_g\right| \cdot w(g)$.

By the usual trick that goes into the proof of Burnside's lemma,

$$\sum_{f\in S}\prod_{p\in V} x_{f(p)}^{w(p)} = \sum_{f \text{ is a } \Gamma\text{-partition}} \frac{1}{|Gf|}\prod_{p\in V} x_{f(p)}^{w(p)} = \sum_{f \text{ is a } \Gamma\text{-partition}} \frac{|\operatorname{Stab}_G f|}{|G|}\prod_{p\in V} x_{f(p)}^{w(p)}$$
$$= \frac{1}{|G|}\sum_{g\in G} \underbrace{\sum_{\substack{f \text{ is a } \Gamma\text{-partition;}\\ gf=f}}\prod_{p\in V} x_{f(p)}^{w(p)}}_{=\sum_{f \text{ is a } \Gamma^g\text{-partition}}\prod_{p\in V^g} x_{f(p)}^{w^g(p)}=F_{(\Gamma^g,w^g)}} = \frac{1}{|G|}\sum_{g\in G} F_{(\Gamma^g,w^g)}.$$

This proves **(a)**. Similarly, one can show **(b)**, and finally these formulas reduce **(c)** to Theorem 4.5 **(b)** (because $\omega(\Gamma^g) = (\omega(\Gamma))^g$ for every $g \in G$).

7. An enumerative corollary

Let us now reap some enumerative harvest:

Corollary 7.1. Let $\Gamma = (V, A_w, A_s)$ be a weak-strict digraph. Assume that Γ' is acyclic.

An *automorphism* of a weak-strict digraph means an automorphism of its vertex set which preserves both the set of weak arcs and the set of strict arcs.

Let G be a finite group which acts on Γ by automorphisms.

For every $n \in \mathbb{N}$, let $\operatorname{Par}_n \Gamma$ denote the set of all Γ-partitions to $\{1,2,\ldots,n\}$. Clearly, G acts on this set $\operatorname{Par}_n \Gamma$ too.

(a) Let $W_{\Gamma,G}(n) = |(\operatorname{Par}_n \Gamma) \diagup G|$. Then, $W_{\Gamma,G}(n)$ agrees with a polynomial of degree $|V|$ in n for $n \geq 0$.

(b) Let $\rho : G \to S_V$ denote the action of G on the vertex-set V of Γ, written as a group homomorphism from G into the symmetric group S_V. A Γ-partition $f \in \operatorname{Par}_n \Gamma$ is said to be *even* if every $g \in G$ satisfying $gf = f$ must satisfy $\operatorname{sgn}(\rho(g)) = 1$. Let $(\operatorname{Par}_n \Gamma)^+$ denote the set of all even Γ-partitions to $\{1,2,\ldots,n\}$. Let $W^+_{\Gamma,G}(n) = \left|(\operatorname{Par}_n \Gamma)^+ \diagup G\right|$. Then, $W^+_{\Gamma,G}(n)$ agrees with a polynomial of degree $|V|$ in n for $n \geq 0$.

(c) We have

$$W_{\Gamma,G}(-n) = (-1)^{|V|} W^+_{\omega(\Gamma),G}(n).$$

Notice that this yields [Joch13, Theorem 2.8] when Γ is set to $(P, \operatorname{cov} P, \varnothing)$ with $\operatorname{cov} P$ being the set of all covering relations of P. This also yields the twin of [Joch13, Theorem 2.8] when Γ is set to $(P, \varnothing, \operatorname{cov} P)$.

Definition 7.2. We use standard Hopf-algebraic notations for the Hopf algebra $\operatorname{QSym}_{\mathbf{k}}$. In particular, S denotes the antipode of $\operatorname{QSym}_{\mathbf{k}}$, and $*$ denotes the convolution product on linear maps from $\operatorname{QSym}_{\mathbf{k}}$ to any algebra.

Definition 7.3. If $f \in \mathrm{QSym}$, and if a_1, a_2, ..., a_k are some elements of a commutative ring, then the evaluation $f\left(a_1, a_2, \ldots, a_k, \underbrace{0, 0, 0, \ldots}_{\text{just zeroes}}\right)$ will be denoted by $f(a_1, a_2, \ldots, a_k)$ (as is usual for symmetric functions).

Let $\varepsilon^\times$ denote the "second counit" of QSym; this is the ring homomorphism $\mathrm{QSym} \to \mathbb{Z}$ which sends every $f \in \mathrm{QSym}$ to $f(1)$.

Proof sketch for Corollary 7.1. We use the standard(?) tactic to derive polynomiality and combinatorial reciprocity phenomena from Hopf algebras. (For example, this is how [Joch13, Theorem 2.3] follows from [GriRei15, Proposition 7.7, Corollary 5.27 and Theorem 5.19].)

Comparing the definitions of $W_{\Gamma,G}(n)$ and $F_{(\Gamma,\mathbf{1}),G}$ (recall that $\mathbf{1}$ is the map $V \to \mathbb{N}_+$ sending everything to 1), we easily observe that $W_{\Gamma,G}(n) = F_{(\Gamma,\mathbf{1}),G}\left(\underbrace{1, 1, \ldots, 1}_{n \text{ ones}}\right)$. But every $f \in \mathrm{QSym}$ satisfies

$$f\left(\underbrace{1, 1, \ldots, 1}_{n \text{ ones}}\right) = \left(\Delta^{(n)} f\right)\left(\underbrace{(1), (1), \ldots, (1)}_{n \text{ times the alphabet } (1)}\right),$$

where $\Delta^{(n)}$ denotes the n-fold iterated coproduct $\mathrm{QSym} \to \mathrm{QSym}^{\otimes n}$ (depending on your tastes you might be calling this $\Delta^{(n-1)}$) and where elements of $\mathrm{QSym}^{\otimes n}$ are supposed to be evaluated on n-tuples of alphabets by evaluating each tensorand on the corresponding alphabet and then multiplying. In other words, every $f \in \mathrm{QSym}$ satisfies

$$f\left(\underbrace{1, 1, \ldots, 1}_{n \text{ ones}}\right) = \left(\varepsilon^\times\right)^{\otimes n}\left(\Delta^{(n)} f\right) = \left(\varepsilon^\times\right)^{*n} f,$$

where $\left(\varepsilon^\times\right)^{*n}$ denotes the n-th power of $\varepsilon^\times$ in the convolution algebra $\mathrm{Hom}(\mathrm{QSym}, \mathbb{Z})$ (the algebra of all $\mathbb{Z}$-linear maps $\mathrm{QSym} \to \mathbb{Z}$). Thus,

$$W_{\Gamma,G}(n) = F_{(\Gamma,\mathbf{1}),G}\left(\underbrace{1, 1, \ldots, 1}_{n \text{ ones}}\right) = \left(\varepsilon^\times\right)^{*n} F_{(\Gamma,\mathbf{1}),G}.$$

Now, why is this polynomial in n ? Well, the counit ε of the coalgebra QSym is the unity of the convolution algebra $\mathrm{Hom}(\mathrm{QSym}, \mathbb{Z})$, and we can write $\varepsilon^\times = \varepsilon + (\varepsilon^\times - \varepsilon)$. The element $\varepsilon^\times - \varepsilon \in \mathrm{Hom}(\mathrm{QSym}, \mathbb{Z})$ is locally nilpotent: specifically, for every homogeneous $f \in \mathrm{QSym}$ of degree d, we have $(\varepsilon^\times - \varepsilon)^{*e} f = 0$ for every integer $e > d$. Since $F_{(\Gamma,\mathbf{1}),G}$ is homogeneous of degree $|V|$, this yields that $(\varepsilon^\times - \varepsilon)^{*e} F_{(\Gamma,\mathbf{1}),G} = 0$ for every integer $e > |V|$. Now,

$$W_{\Gamma,G}(n) = \Big(\underbrace{\varepsilon^\times}_{=\varepsilon+(\varepsilon^\times-\varepsilon)}\Big)^{*n} F_{(\Gamma,\mathbf{1}),G} = \underbrace{\left(\varepsilon + \left(\varepsilon^\times - \varepsilon\right)\right)^{*n}}_{\substack{=\sum_{k=0}^{n}\binom{n}{k}(\varepsilon^\times-\varepsilon)^{*k} \\ \text{(by the binomial formula, since } \varepsilon \text{ is the unity} \\ \text{of } \operatorname{Hom}(\operatorname{QSym},\mathbb{Z}))}} F_{(\Gamma,\mathbf{1}),G}$$

$$= \sum_{k=0}^{n}\binom{n}{k}\left(\varepsilon^\times - \varepsilon\right)^{*k} F_{(\Gamma,\mathbf{1}),G} = \sum_{k=0}^{|V|}\binom{n}{k}\left(\varepsilon^\times - \varepsilon\right)^{*k} F_{(\Gamma,\mathbf{1}),G} \qquad (28)$$

$$\left(\begin{array}{c}\text{we have cut the sum at } k = |V| \text{ since} \\ (\varepsilon^\times - \varepsilon)^{*e} F_{(\Gamma,\mathbf{1}),G} = 0 \text{ for every integer } e > |V|\end{array}\right).$$

This clearly is a polynomial in n of degree at most $|V|$. That the degree is not less than $|V|$ follows from asymptotics (we have $W_{\Gamma,G}(n) \geq \binom{n}{|V|}$). This proves Corollary 7.1 **(a)**.

The proof of **(b)** is not much different. Let us now come to **(c)**. Recall that $W_{\Gamma,G}(n) = (\varepsilon^\times)^{*n} F_{(\Gamma,\mathbf{1}),G}$. But it is a general fact that if H is a Hopf algebra and A is an algebra, and $f : H \to A$ is an algebra homomorphism, then $f^{*(-1)} = f \circ S$ (where $f^{*(-1)}$ is the inverse of f with respect to convolution). Since $(\varepsilon^\times)^{*n} : \operatorname{QSym} \to \mathbb{Z}$ is an algebra homomorphism (this follows, e. g., from $(\varepsilon^\times)^{*n} f = f\Big(\underbrace{1, 1, \ldots, 1}_{n \text{ ones}}\Big)$, or from general Hopf algebra lore), this yields that $\left((\varepsilon^\times)^{*n}\right)^{*(-1)} = (\varepsilon^\times)^{*n} \circ S$. In other words, $(\varepsilon^\times)^{*(-n)} = (\varepsilon^\times)^{*n} \circ S$. Now, since $W_{\Gamma,G}(n) = (\varepsilon^\times)^{*n} F_{(\Gamma,\mathbf{1}),G}$, we have

$$W_{\Gamma,G}(-n) = \left(\varepsilon^\times\right)^{*(-n)} F_{(\Gamma,\mathbf{1}),G}$$

(we used the polynomiality of $(\varepsilon^\times)^{*n} F_{(\Gamma,\mathbf{1}),G}$ here), so that

$$W_{\Gamma,G}(-n) = \underbrace{\left(\varepsilon^\times\right)^{*(-n)}}_{=(\varepsilon^\times)^{*n}\circ S} F_{(\Gamma,\mathbf{1}),G} = \left(\varepsilon^\times\right)^{*n}\left(\underbrace{S\left(F_{(\Gamma,\mathbf{1}),G}\right)}_{\substack{=(-1)^{|V|}F^+_{(\omega(\Gamma),\mathbf{1}),G} \\ \text{(by Corollary 6.1 (c))}}}\right)$$

$$= (-1)^{|V|}\underbrace{\left(\varepsilon^\times\right)^{*n}\left(F^+_{(\omega(\Gamma),\mathbf{1}),G}\right)}_{\substack{=W^+_{(\omega(\Gamma),\mathbf{1}),G}(n) \\ \text{(by an analogue of (28))}}} = (-1)^{|V|} W^+_{(\omega(\Gamma),\mathbf{1}),G}(n).$$

This proves part **(c)**.

References

[Abe77] Eiichi Abe, *Hopf Algebras*, CUP 1977.

[ABS03] Marcelo Aguiar, Nantel Bergeron, Frank Sottile, *Combinatorial Hopf algebras and generalized Dehn-Sommerville relations*, Compositio Mathematica, vol. 142, Issue 01, January 2006, pp. 1–30.
Also available as arXiv:math/0310016.
Newer version at `http://www.math.tamu.edu/~maguiar/CHalgebra.pdf`

[BBSSZ13] Chris Berg, Nantel Bergeron, Franco Saliola, Luis Serrano, Mike Zabrocki, *A lift of the Schur and Hall-Littlewood bases to non-commutative symmetric functions*, Canadian Journal of Mathematics, `http://dx.doi.org/10.4153/CJM-2013-013-0`
Also available as arXiv:1208.5191v3.
`http://arxiv.org/abs/1208.5191v3`

[BenSag14] Carolina Benedetti, Bruce Sagan, *Antipodes and involutions*, arXiv:1410.5023v1.
`http://arxiv.org/abs/1410.5023v1`

[BreKli14] Felix Breuer, Caroline J. Klivans, *Scheduling Problems*, arXiv:1401.2978v1.
`http://arxiv.org/abs/1401.2978v1`

[DNR01] Sorin Dăscălescu, Constantin Năstăsescu, Şerban Raianu, *Hopf Algebras*, Marcel Dekker 2001.

[EbrFar08] Kurusch Ebrahimi-Fard, Dominique Manchon, *Dendriform Equations*, Journal of Algebra 322 (2009), pp. 4053–4079.
`http://www.sciencedirect.com/science/article/pii/S0021869309003524`
An older version is also available as arXiv:0805.0762v2:
`http://arxiv.org/abs/0805.0762v2`

[Fresse14] Benoit Fresse, *Homotopy of operads and Grothendieck-Teichmüller groups, Part I: From operads to Grothendieck-Teichmüller groups*, preprint, March 2, 2014.
`http://math.univ-lille1.fr/~fresse/OperadGT-December2012Preprint.pdf`

[Gessel84] Ira M. Gessel, *Multipartite P-partitions and Inner Products of Skew Schur Functions*, Contemporary Mathematics, vol. 34, 1984, pp.

289–301.
http://people.brandeis.edu/~gessel/homepage/papers/multipartite.pdf

[GriRei15] Darij Grinberg, Victor Reiner, *Hopf algebras in Combinatorics*, June 1, 2015, arXiv:1409.8356v2.
http://www.math.umn.edu/~reiner/Classes/HopfComb.pdf

[Hazewi08] Michiel Hazewinkel, *Witt vectors. Part 1*, arXiv:0804.3888v1.
http://arxiv.org/abs/0804.3888v1
Also appears as a chapter in: Michiel Hazewinkel, *Handbook of Algebra: Volume 6*, Elsevier 2009.

[Hivert99] Florent Hivert, *Combinatoire des fonctions quasi-symétriques*, PhD thesis, defended 1999, January the 15.
https://www.lri.fr/~hivert/PAPER/these.ps

[Joch13] Katharina Jochemko, *Order polynomials and Pólya's enumeration theorem*, arXiv:1310.0838v2.
http://arxiv.org/abs/1310.0838v2

[LMvW13] Kurt Luoto, Stefan Mykytiuk and Stephanie van Willigenburg, *An introduction to quasisymmetric Schur functions – Hopf algebras, quasisymmetric functions, and Young composition tableaux*, May 23, 2013.
http://www.math.ubc.ca/~steph/papers/QuasiSchurBook.pdf

[Malve93] Claudia Malvenuto, *Produits et coproduits des fonctions quasi-symétriques et de l'algèbre des descentes*, thesis, defended November 1993.
http://www1.mat.uniroma1.it/people/malvenuto/Thesis.pdf

[MalReu98] Claudia Malvenuto, Christophe Reutenauer, *Plethysm and conjugation of quasi-symmetric functions*, Discrete Mathematics, Volume 193, Issues 1–3, 28 November 1998, Pages 225–233.
http://www.sciencedirect.com/science/article/pii/S0012365X98001423

[Manchon04] Dominique Manchon, *Hopf algebras, from basics to applications to renormalization*, Comptes Rendus des Rencontres Mathematiques de Glanon 2001 (published in 2003).
http://arxiv.org/abs/math/0408405v2

[Montg93] Susan Montgomery, *Hopf Algebras and their Actions on Rings*, Regional Conference Series in Mathematics Nr. 82, AMS 1993.

[NovThi05] Jean-Christophe Novelli, Jean-Yves Thibon, *Hopf algebras and dendriform structures arising from parking functions*, Fundamenta Mathematicae 193 (2007), 189–241. A preprint also appears on arXiv as arXiv:math/0511200v1.

[Stan11] Richard Stanley, *Enumerative Combinatorics, volume 1*, Cambridge University Press, 2011.
`http://math.mit.edu/~rstan/ec/ec1/`

[Stan99] Richard Stanley, *Enumerative Combinatorics, volume 2*, Cambridge University Press, 1999.

[Stan71] Richard Stanley, *Ordered Structures and Partitions*, Memoirs of the American Mathematical Society, No. 119, American Mathematical Society, Providence, R.I., 1972.
`http://www-math.mit.edu/~rstan/pubs/pubfiles/9.pdf`

[Sweed69] Moss E. Sweedler, *Hopf Algebras*, W. A. Benjamin 1969.

[Thomas77] Glânffrwd P. Thomas, *Frames, Young Tableaux, and Baxter Sequences*, Advances in Mathematics, Volume 26, Issue 3, December 1977, pp. 275–289.
`http://www.sciencedirect.com/science/article/pii/0001870877900421`

[Thomas76] Glânffrwd P. Thomas, *Further Results on Baxter Sequences and Generalized Schur Functions*, Actes de la Table Ronde du C.N.R.S. tenue à l'Université Louis-Pasteur de Strasbourg, 26 au 30 avril 1976, Lecture Notes in Mathematics Volume 579 (1977), pp. 155-167.