The filtered Tutte polynomial

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June 26, 2016

Contents

In this note, we shall prove a generalization of a conjecture made in [LiPost15, Question 21.7].

1. Introduction

TODO 1.1. Write an introduction.

2. Matroids

Definition 2.1. In the following, $\mathbb{N}$ shall always denote the set $\{0, 1, 2, \ldots\}$. For any set E, we let $\mathcal{P}(E)$ denote the powerset of E.

We recall the basic properties of matroids. We will actually use rather little from the theory of matroids; [Schrij13, §10.1–§10.2] is a perfectly sufficient reference for our purposes.

Let us first give a definition of a matroid.

Definition 2.2. A *matroid* means a pair $(E, \mathcal{I})$ of a finite set E and a set $\mathcal{I} \subseteq \mathcal{P}(E)$ satisfying the following axioms:

- We have $\varnothing \in \mathcal{I}$.

- If $Y \in \mathcal{I}$ and $Z \in \mathcal{P}(E)$ are such that $Z \subseteq Y$, then $Z \in \mathcal{I}$.
- If $Y \in \mathcal{I}$ and $Z \in \mathcal{I}$ are such that $|Y| < |Z|$, then there exists some $x \in Z \setminus Y$ such that $Y \cup \{x\} \in \mathcal{I}$.

When $(E, \mathcal{I})$ is a matroid, a subset S of E is said to be *independent* (for this matroid) if and only if $S \in \mathcal{I}$.

When $(E, \mathcal{I})$ is a matroid, the set E is called the *ground set* of the matroid $(E, \mathcal{I})$.

Definition 2.2 is how a matroid is defined in [Schrij13, §10.1] and in [Martin15, Definition 3.15] (where it is called a "(matroid) independence system"). There exist other definitions of a matroid, which turn out to be equivalent.

Definition 2.3. Let $M = (E, \mathcal{I})$ be a matroid. Let $S \in \mathcal{P}(E)$. A *basis* of S (for the matroid M) means a maximum-size independent (for M) subset of S.

Definition 2.4. Let $M = (E, \mathcal{I})$ be a matroid. Then, a function $r_M : \mathcal{P}(E) \to \mathbb{N}$ is defined by

$$r_M(S) = \max\{|Z| \mid Z \in \mathcal{I} \text{ and } Z \subseteq S\} \qquad \text{for every } S \subseteq E. \tag{1}$$

The axioms of a matroid show that if $S \subseteq E$, and if Z is a basis of S (for M), then $r_M(S) = |Z|$.

The function r_M is called the *rank function* of M. For every $S \in \mathcal{P}(E)$, the number $r_M(S)$ is called the *rank* of S (for M). Clearly,

$$r_M(S) \leq |S| \qquad \text{for every } S \subseteq E. \tag{2}$$

Moreover, $r_M(\varnothing) = 0$ and

$$r_M(S) \leq r_M(T) \qquad \text{for every } S \subseteq T \subseteq E. \tag{3}$$

The following facts about bases for matroids are fundamental (and easy to prove), and will be used without explicit mention:

Proposition 2.5. Let $M = (E, \mathcal{I})$ be a matroid. Let $S \in \mathcal{P}(E)$.

(a) There exists at least one basis of S (for M).

(b) If U is any independent (for M) subset of S, then there exists a basis T of S such that $U \subseteq T$. (In other words, every independent subset of S can be extended to a basis of S.)

(c) The bases of S are precisely the inclusion-maximal independent subsets of S.

(d) Every basis of S has the same size.

(e) If U is any independent subset of S, then $|U| \leq r_M(S)$.

Corollary 2.6. Let $M = (E, \mathcal{I})$ be a matroid. Then,

$$\mathcal{I} = \{S \in \mathcal{P}(E) \mid r_M(S) = |S|\}.$$

Proof of Corollary 2.6. Let $U \in \mathcal{I}$. Thus, U is an independent set for M (since the independent sets for M are the elements of $\mathcal{I}$). Consequently, U is an independent subset of U. Thus, U is a maximum-size independent subset of U (since U is a maximum-size subset of U). In other words, U is a basis of U (by the definition of a "basis"). Hence, $r_M(U) = |U|$. Thus, we have $U \in I \subseteq \mathcal{P}(E)$ and $r_M(U) = |U|$. In other words, $U \in \{S \in \mathcal{P}(E) \mid r_M(S) = |S|\}$.

Let us now forget that we fixed U. We thus have shown that $U \in \{S \in \mathcal{P}(E) \mid r_M(S) = |S|\}$ for every $U \in \mathcal{I}$. In other words,

$$\mathcal{I} \subseteq \{S \in \mathcal{P}(E) \mid r_M(S) = |S|\}. \tag{4}$$

On the other hand, let $V \in \{S \in \mathcal{P}(E) \mid r_M(S) = |S|\}$. Thus, $V \in \mathcal{P}(E)$ and $r_M(V) = |V|$.

We know that there exists at least one basis of V (by Proposition 2.5 **(b)**, applied to $S = V$). Fix such a basis, and denote it by Q. Then, $r_M(V) = |Q|$ (since Q is a basis of V). Hence, $|Q| = r_M(V) = |V|$. Combined with $Q \subseteq V$, this shows that $Q = V$. But Q is a basis of V, and thus an independent set. In other words, $Q \in \mathcal{I}$. Hence, $V = Q \in \mathcal{I}$.

Let us now forget that we fixed V. We thus have shown that $V \in \mathcal{I}$ for every $V \in \{S \in \mathcal{P}(E) \mid r_M(S) = |S|\}$. In other words,

$$\{S \in \mathcal{P}(E) \mid r_M(S) = |S|\} \subseteq \mathcal{I}.$$

Combining this with (4), we obtain $\mathcal{I} = \{S \in \mathcal{P}(E) \mid r_M(S) = |S|\}$. This proves Corollary 2.6. $\square$

Corollary 2.7. Let M and M' be two matroids with one and the same ground set E. Assume that $r_M = r_{M'}$. Then, $M = M'$.

Proof of Corollary 2.7. Write the matroids M and M' as $(E, \mathcal{I})$ and $(E, \mathcal{I}')$, respectively. Then, Corollary 2.6 shows that

$$\mathcal{I} = \left\{ S \in \mathcal{P}(E) \mid \underbrace{r_M}_{= r_{M'}}(S) = |S| \right\} = \{S \in \mathcal{P}(E) \mid r_{M'}(S) = |S|\}. \tag{5}$$

But Corollary 2.6 (applied to M' and $\mathcal{I}'$ instead of M and $\mathcal{I}$) shows that

$$\mathcal{I}' = \{S \in \mathcal{P}(E) \mid r_{M'}(S) = |S|\}.$$

Comparing this with (5), we obtain $\mathcal{I} = \mathcal{I}'$. Hence, $M = \left(E, \underbrace{\mathcal{I}}_{= \mathcal{I}'} \right) = (E, \mathcal{I}') = M'$. This proves Corollary 2.7. $\square$

Definition 2.8. Let $M = (E, \mathcal{I})$ be a matroid. Then, a function $n_M : \mathcal{P}(E) \to \mathbb{N}$ is defined by

$$n_M(S) = |S| - r_M(S) \qquad \text{for every } S \subseteq E. \tag{6}$$

(This is well-defined due to (2).)

The function n_M is called the *nullity function* of M. For every $S \in \mathcal{P}(E)$, the number $n_M(S)$ is called the *nullity* of S (for M).

Proposition 2.9. Let $M = (E, \mathcal{I})$ be a matroid.

(a) We have $n_M(\varnothing) = 0$.

(b) We have

$$n_M(S) \leq n_M(T) \qquad \text{for every } S \subseteq T \subseteq E. \tag{7}$$

Proof of Proposition 2.9. **(a)** This follows from the definition of n_M and from $r_M(\varnothing) = 0$.

(b) Let $S \subseteq T \subseteq E$. Pick a basis Z of S (this is possible, since a basis of S exists). Then, $r_M(S) = |Z|$. But Z is a basis of S, and thus an independent set. Hence, we can extend Z to a basis of T (according to Proposition 2.5 **(b)**). In other words, there exists a basis Y of T such that $Z \subseteq Y$. Pick such a Y. Since Y is a basis of T, we have $r_M(T) = |Y|$.

The set Y is a basis of T, thus a maximum-size independent subset of T.

But $Y \setminus Z \subseteq T \setminus S$ [1]. Hence, $|Y \setminus Z| \leq |T \setminus S| = |T| - |S|$ (since $S \subseteq T$). Now,

$$\begin{aligned} \underbrace{r_M(T)}_{=|Y|} - \underbrace{r_M(S)}_{=|Z|} &= |Y| - |Z| = |Y \setminus Z| \qquad (\text{since } Z \subseteq Y) \\ &\leq |T| - |S|. \end{aligned}$$

In other words, $|S| - r_M(S) \leq |T| - r_M(T)$. Since $n_M(S) = |S| - r_M(S)$ and $n_M(T) = |T| - r_M(T)$ (by the definition of n_M), this rewrites as $n_M(S) \leq n_M(T)$. This proves Proposition 2.9 **(b)**. ☐

Next, we shall define loops and coloops in a matroid (following, for example, [Martin15, Definition 3.35]):

[1] *Proof.* Let $y \in Y \setminus Z$. Hence, $y \notin Z$.

Assume (for the sake of contradiction) that $y \in S$. The set $Z \cup y$ is a subset of Y (since $Z \subseteq Y$ and $y \in Y \setminus Z \subseteq Y$) and thus is independent (since Y is independent). Moreover, $|Z \cup y| = |Z| + 1$ (since $y \notin Z$). Furthermore, $Z \cup y \subseteq S$ (since $Z \subseteq S$ and $y \in S$). Hence, $Z \cup y$ is an independent subset of S. Therefore, $|Z \cup y| \leq |Z|$ (since Z is a maximum-size independent subset of S). This contradicts $|Z \cup y| = |Z| + 1$. This contradiction shows that our assumption (that $y \in S$) was wrong. Hence, we have $y \notin S$. Combined with $y \in Y \setminus Z \subseteq Y \subseteq T$, this yields $y \in T \setminus S$.

Now, we have proven this for every $y \in Y \setminus Z$. In other words, $Y \setminus Z \subseteq T \setminus S$, qed.

Definition 2.10. Let $M = (E, \mathcal{I})$ be a matroid. Let $e \in E$. We write $r_M(e)$ for $r_M(\{e\})$. Also, if S is any subset of E, then we abbreviate the sets $S \setminus \{e\}$ and $S \cup \{e\}$ by $S \setminus e$ and $S \cup e$, respectively.

(a) The element e is said to be a *loop* (of M) if and only if no basis of E (for M) contains e.

(b) The element e is said to be a *coloop* (of M) if and only if every basis of E (for M) contains e.

Proposition 2.11. Let $M = (E, \mathcal{I})$ be a matroid. Let $e \in E$.

(a) The element e is a loop (of M) if and only if $r_M(e) = 0$.

(b) The element e is a coloop (of M) if and only if $r_M(E) = r_M(E \setminus e) + 1$.

(c) If $r_M(E \setminus e) \neq r_M(E)$, then the element e is a coloop.

Proof of Proposition 2.11. **(a)** $\Longrightarrow$: Assume that the element e is a loop. We need to show that $r_M(e) = 0$.

Let Z be a basis of $\{e\}$ (for M). Thus, $r_M(\{e\}) = |Z|$.

The set Z is a basis of $\{e\}$, thus an independent set. Clearly, $Z \subseteq E$. Thus, the independent set Z can be extended to a basis of E. In other words, there exists a basis W of E such that $Z \subseteq W$. Consider this W.

The element e is a loop. Thus, no basis of E contains e (by the definition of a "loop"). In particular, W does not contain e (since W is a basis of E). Hence, Z does not contain e (since $Z \subseteq W$). Combining this with $Z \subseteq \{e\}$, we obtain $Z \subseteq \{e\} \setminus e = \varnothing$. Hence, $Z = \varnothing$ and thus $|Z| = 0$. Now, $r_M(e) = r_M(\{e\}) = |Z| = 0$. This finishes the proof of the $\Longrightarrow$ direction of Proposition 2.11 **(a)**.

$\Longleftarrow$: Assume that $r_M(e) = 0$. We need to show that the element e is a loop.

Let B be a basis of E (for M) such that $e \in B$. From $e \in B$, we obtain $\{e\} \subseteq B$. The set B is independent (since it is a basis of E). Hence, the set $\{e\}$ is independent (since $\{e\} \subseteq B$). Since $\{e\} \subseteq \{e\}$, this entails that $|\{e\}| \leq r_M(\{e\})$ (by Proposition 2.5 **(e)**, applied to $U = \{e\}$ and $S = \{e\}$). Hence, $r_M(\{e\}) \geq |\{e\}| = 1$. This contradicts $r_M(\{e\}) = r_M(e) = 0$.

Now, let us forget that we fixed B. We thus have found a contradiction for every basis B of E satisfying $e \in B$. Hence, there exists no basis B of E satisfying $e \in B$. In other words, no basis of E contains e. In other words, e is a loop (by the definition of a "loop"). This proves the $\Longleftarrow$ direction of Proposition 2.11 **(a)**.

(c) Assume that $r_M(E \setminus e) \neq r_M(E)$. We need to show that the element e is a coloop.

From $E \setminus e \subseteq E$, we obtain $r_M(E \setminus e) \leq r_M(E)$. Combined with $r_M(E \setminus e) \neq r_M(E)$, this yields $r_M(E \setminus e) < r_M(E)$.

Let B be a basis of E. Thus, $r_M(E) = |B|$.

Assume (for the sake of contradiction) that $e \notin B$. Thus, $B \subseteq E \setminus e$. But B is a basis of E and thus an independent subset. Hence, from $B \subseteq E \setminus e$, we obtain $|B| \leq r_M(E \setminus e)$ (by Proposition 2.5 **(e)**, applied to $U = B$ and $S = E \setminus e$). Thus,

$$|B| \leq r_M(E \setminus e) < r_M(E) = |B|,$$

which is absurd. This contradiction shows that our assumption (that $e \notin B$) was wrong. Hence, we must have $e \in B$.

Now, let us forget that we fixed B. We thus have shown that $e \in B$ for every basis B of E. In other words, every basis of E (for M) contains e. In other words, e is a coloop (by the definition of a "coloop"). This proves Proposition 2.11 **(c)**.

(b) $\Longrightarrow$: Assume that the element e is a coloop. We need to show that $r_M(E) = r_M(E \setminus e) + 1$.

Pick a basis B of E (for M). Thus, $r_M(E) = |B|$.

Recall that e is a coloop. In other words, every basis of E (for M) contains e (by the definition of a "coloop").

But B is a basis of E, and thus an independent set. Hence, $B \setminus e$ is also an independent set (since $B \setminus e \subseteq B$). Since $\underbrace{B}_{\subseteq E} \setminus e \subseteq E \setminus e$, we thus have $|B \setminus e| \leq r_M(E \setminus e)$ (by Proposition 2.5 **(e)**, applied to $U = B \setminus e$ and $S = E \setminus e$). Thus, $r_M(E \setminus e) \geq |B \setminus e| \geq \underbrace{|B|}_{= r_M(E)} - 1 = r_M(E) - 1$.

Now, pick any basis C of $E \setminus e$ (for M). Thus, $r_M(E \setminus e) = |C|$.

The set C is a basis of $E \setminus e$, thus an independent set. Hence, we can extend C to a basis of E (according to Proposition 2.5 **(b)**). In other words, there exists a basis D of E such that $C \subseteq D$. Pick such a D. Since D is a basis of E, we have $r_M(E) = |D|$. Since D is a basis of E, we have $e \in D$ (since every basis of E contains e). But $e \notin C$ (since $C \subseteq E \setminus e$). Hence, $C \neq D$ (since $e \notin C$ but $e \in D$). Combined with $C \subseteq D$, this entails $|C| < |D|$. Hence, $r_M(E \setminus e) = |C| < |D| = r_M(E)$. In other words, $r_M(E \setminus e) \leq r_M(E) - 1$. Combined with $r_M(E \setminus e) \geq r_M(E) - 1$, this yields $r_M(E \setminus e) = r_M(E) - 1$. In other words, $r_M(E) = r_M(E \setminus e) + 1$. This proves the $\Longrightarrow$ direction of Proposition 2.11 **(b)**.

$\Longleftarrow$: Assume that $r_M(E) = r_M(E \setminus e) + 1$. We must show that the element e is a coloop.

From $r_M(E) = r_M(E \setminus e) + 1 > r_M(E \setminus e)$, we have $r_M(E \setminus e) \neq r_M(E)$. Hence, e is a coloop (by 2.11 **(c)**). The $\Longleftarrow$ direction of Proposition 2.11 **(b)** is thus proven. ☐

Proposition 2.12. Let $M = (E, \mathcal{I})$ be a matroid. Let $e \in E$.

(a) If e is a loop, then $r_M(S \cup e) = r_M(S)$ for every $S \in \mathcal{P}(E)$.

(b) If e is a coloop, then $r_M(S \cup e) = r_M(S) + 1$ for every $S \in \mathcal{P}(E)$ satisfying $e \notin S$.

(c) If e is a loop, then $n_M(S \cup e) = n_M(S) + 1$ for every $S \in \mathcal{P}(E)$ satisfying $e \notin S$.

(d) If e is a coloop, then $n_M(S \cup e) = n_M(S)$ for every $S \in \mathcal{P}(E)$.

Proof of Proposition 2.12. **(a)** Assume that e is a loop. Let $S \in \mathcal{P}(E)$.

Pick a basis X of $S \cup e$ (for M). Then, $r_M(S \cup e) = |X|$.

The set X is a basis of $S \cup e$, and thus is independent.

The element e is a loop. Thus, $r_M(e) = 0$ (by Proposition 2.11 **(a)**).

Assume (for the sake of contradiction) that $e \in X$. Then, $\{e\} \subseteq X$, so that $\{e\}$ is an independent set (since X is an independent set). Hence, Proposition 2.5 **(e)** (applied to $\{e\}$ and $\{e\}$ instead of S and U) shows that $|\{e\}| \leq r_M(\{e\})$. Thus, $r_M(\{e\}) \geq |\{e\}| = 1$. This contradicts $r_M(\{e\}) = r_M(e) = 0$.

This contradiction shows that our assumption (that $e \in X$) was wrong. In other words, we have $e \notin X$. Hence, $X \subseteq (S \cup e) \setminus e \subseteq S$. Thus, Proposition 2.5 **(e)** (applied to $U = X$) shows that $|X| \leq r_M(S)$. But $S \subseteq S \cup e$ shows that $r_M(S) \leq r_M(S \cup e) = |X|$. Combining this with $|X| \leq r_M(S)$, we obtain $|X| = r_M(S)$. Hence, $r_M(S \cup e) = |X| = r_M(S)$. This proves Proposition 2.12 **(a)**.

(b) Assume that e is a coloop. Let $S \in \mathcal{P}(E)$ be such that $e \notin S$.

The element e is a coloop. Thus, $r_M(E) = r_M(E \setminus e) + 1$ (according to Proposition 2.11 **(b)**).

Applying (7) to $T = S \cup e$, we obtain $n_M(S) \leq n_M(S \cup e)$. Since $n_M(S) = |S| - r_M(S)$ and $n_M(S \cup e) = |S \cup e| - r_M(S \cup e)$ (by the definition of $n_M(S \cup e)$), this rewrites as $|S| - r_M(S) \leq |S \cup e| - r_M(S \cup e)$. Hence,

$$\begin{aligned} r_M(S) &\leq |S| - \left(\underbrace{|S \cup e|}_{\substack{=|S|+1 \\ (\text{since } e \notin S)}} - r_M(S \cup e) \right) = |S| - (|S| + 1 - r_M(S \cup e)) \\ &= r_M(S \cup e) - 1. \end{aligned} \tag{8}$$

On the other hand, let us pick a basis X of $S \cup e$ (for M). Then, $r_M(S \cup e) = |X|$.

The set X is a basis of $S \cup e$, and thus is independent. Also, $X \subseteq E$. Hence, the independent set X can be extended to a basis of E. In other words, there exists a basis Z of E such that $X \subseteq Z$. Consider this Z.

Since Z is a basis of E, we have $r_M(E) = |Z|$.

The set Z is independent (since it is a basis of E). If we had $Z \subseteq E \setminus e$, then we would have $|Z| \leq r_M(E \setminus e)$ (by Proposition 2.5 **(e)**, applied to Z and $E \setminus e$ instead of U and S), which would contradict $|Z| = r_M(E) = r_M(E \setminus e) + 1 > r_M(E \setminus e)$. Hence, we cannot have $Z \subseteq E \setminus e$. In other words, we must have $e \in Z$.

From $X \subseteq Z$ and $e \in Z$, we obtain $X \cup e \subseteq Z$. Thus, the set $X \cup e$ is independent (since Z is independent). From $\underbrace{X}_{\subseteq S \cup e} \cup e \subseteq S \cup e \cup e = S \cup e$, we therefore obtain $|X \cup e| \leq r_M(S \cup e)$ (by Proposition 2.5 **(e)**, applied to $X \cup e$ and $S \cup e$ instead of U and S).

Now, $|X \cup e| \leq r_M(S \cup e) = |X|$. Combined with $|X| \leq |X \cup e|$ (since $X \subseteq X \cup e$), this shows that $|X \cup e| = |X|$. Hence, $e \in X$.

From $X \subseteq S \cup e$, we obtain $X \setminus e \subseteq S$. Also, $X \setminus e$ is independent (since X is independent, and since $X \setminus e \subseteq X$). Hence, $X \setminus e$ is an independent subset of S. Therefore, $|X \setminus e| \leq r_M(S)$ (by Proposition 2.5 **(e)**, applied to Z and $E \setminus e$ instead of U and S). Thus, $r_M(S) \geq |X \setminus e| = |X| - 1$ (since $e \in X$). Since

$r_M(S \cup e) = |X|$, this rewrites as $r_M(S) \geq r_M(S \cup e) - 1$. Combining this with (8), we obtain $r_M(S) = r_M(S \cup e) - 1$. In other words, $r_M(S \cup e) = r_M(S) + 1$. This proves Proposition 2.12 **(b)**.

(c) Assume that e is a loop. Let $S \in \mathcal{P}(E)$ be such that $e \notin S$.

The definition of $n_M(S)$ shows that $n_M(S) = |S| - r_M(S)$.

The definition of $n_M(S \cup e)$ shows that

$$n_M(S \cup e) = \underbrace{|S \cup e|}_{\substack{=|S|+1 \\ (\text{since } e \notin S)}} - \underbrace{r_M(S \cup e)}_{\substack{=r_M(S) \\ (\text{by Proposition 2.12 } \textbf{(a)})}} = |S| + 1 - r_M(S)$$
$$= \underbrace{|S| - r_M(S)}_{=n_M(S)} + 1 = n_M(S) + 1.$$

This proves Proposition 2.12 **(c)**.

(d) Assume that e is a coloop. Let $S \in \mathcal{P}(E)$. We need to show that $n_M(S \cup e) = n_M(S)$. If $S \cup e = S$, then this is obvious. Hence, for the rest of this proof, we WLOG assume that $S \cup e \neq S$. Thus, $e \notin S$.

The definition of $n_M(S)$ shows that $n_M(S) = |S| - r_M(S)$.

The definition of $n_M(S \cup e)$ shows that

$$n_M(S \cup e) = \underbrace{|S \cup e|}_{\substack{=|S|+1 \\ (\text{since } e \notin S)}} - \underbrace{r_M(S \cup e)}_{\substack{=r_M(S)+1 \\ (\text{by Proposition 2.12 } \textbf{(b)})}} = |S| + 1 - (r_M(S) + 1)$$
$$= |S| - r_M(S) = n_M(S).$$

This proves Proposition 2.12 **(d)**. □

Definition 2.13. Let $M = (E, \mathcal{I})$ be a matroid. A *basis* of M will mean a basis of E (for M). We let $\mathcal{B}(M)$ denote the set of all bases of M. It is well-known that M can be uniquely reconstructed from $\mathcal{B}(M)$.

Proposition 2.14. Let $M = (E, \mathcal{I})$ be a matroid. We have

$$\mathcal{B}(M) = \{B \in \mathcal{I} \mid |B| = r_M(E)\}.$$

Proof of Proposition 2.14. Let $Z \in \mathcal{B}(M)$. Thus, Z is a basis of M (since $\mathcal{B}(M)$ is the set of all bases of M). In other words, Z is a basis of E (for M) (by the definition of a "basis of M"). In other words, Z is a maximum-size independent (for M) subset of E. Thus, Z is an independent subset of E. In other words, $Z \in \mathcal{I}$. Also, $r_M(E) = |Z|$ (since Z is a basis of E). Combined with $Z \in \mathcal{I}$, this yields $Z \in \{B \in \mathcal{I} \mid |B| = r_M(E)\}$.

Let us now forget that we fixed Z. We thus have shown that $Z \in \{B \in \mathcal{I} \mid |B| = r_M(E)\}$ for every $Z \in \mathcal{B}(M)$. In other words,

$$\mathcal{B}(M) \subseteq \{B \in \mathcal{I} \mid |B| = r_M(E)\}. \tag{9}$$

On the other hand, let $Y \in \{B \in \mathcal{I} \mid |B| = r_M(E)\}$. Thus, $Y \in \mathcal{I}$ is such that $|Y| = r_M(E)$. The set Y is independent (since $Y \in \mathcal{I}$). Hence, then there exists a basis T of E such that $Y \subseteq T$ (by Proposition 2.5 **(b)**, applied to $S = E$ and $U = Y$). Consider this T. We have $|T| = r_M(E)$ (since T is a basis of E) and thus $|T| = r_M(E) = |Y|$. Combined with $Y \subseteq T$, this yields $Y = T$. Thus, Y is a basis of E (since T is a basis of E). In other words, Y is a basis of M (by the definition of a "basis of M"). In other words, $Y \in \mathcal{B}(M)$ (since $\mathcal{B}(M)$ is the set of all bases of M).

Let us now forget that we fixed Y. We thus have shown that $Y \in \mathcal{B}(M)$ for every $Y \in \{B \in \mathcal{I} \mid |B| = r_M(E)\}$. In other words,

$$\{B \in \mathcal{I} \mid |B| = r_M(E)\} \subseteq \mathcal{B}(M).$$

Combined with (9), this yields $\mathcal{B}(M) = \{B \in \mathcal{I} \mid |B| = r_M(E)\}$. Proposition 2.14 is thus proven. □

Next, let us define the dual matroid of a matroid (following [Martin15, Definition 3.32]):

Definition 2.15. Let $M = (E, \mathcal{I})$ be a matroid. There exists a unique matroid $N = (E, \mathcal{J})$ with ground set E such that $\mathcal{B}(N) = \{E \setminus B \mid B \in \mathcal{B}(M)\}$. This matroid N is called the *dual matroid* of M, and is denoted by M^*. For every $S \in \mathcal{P}(E)$, we have

$$r_{M^*}(S) = |S| + r_M(E \setminus S) - r_M(E) \tag{10}$$

(by [Schrij13, Theorem 10.3]). It is furthermore easy to see that $(M^*)^* = M$.

Proposition 2.16. Let $M = (E, \mathcal{I})$ be a matroid. Let $e \in E$. Then, e is a loop of M if and only if e is a coloop of M^*.

Proof of Proposition 2.16. The equality (10) (applied to $S = E$) shows that

$$\begin{aligned} r_{M^*}(E) &= |E| + r_M\left(\underbrace{E \setminus E}_{=\varnothing}\right) - r_M(E) = |E| + \underbrace{r_M(\varnothing)}_{=0} - r_M(E) \\ &= |E| - r_M(E). \end{aligned}$$

Also, (10) (applied to $S = E \setminus e$) shows that

$$\begin{aligned} r_{M^*}(E \setminus e) &= \underbrace{|E \setminus e|}_{\substack{=|E|-1 \\ (\text{since } e \in E)}} + r_M\left(\underbrace{E \setminus (E \setminus e)}_{=\{e\}}\right) - r_M(E) = |E| - 1 + \underbrace{r_M(\{e\})}_{=r_M(e)} - r_M(E) \\ &= |E| - 1 + r_M(e) - r_M(E). \end{aligned}$$

Write the matroid M^* in the form $M^* = (E, \mathcal{J})$. Proposition 2.11 **(b)** (applied to M^* and $\mathcal{J}$ instead of M and $\mathcal{I}$) shows that e is a coloop of M^* if and only if $r_{M^*}(E) = r_{M^*}(E \setminus e) + 1$. Hence, we have the following chain of logical equivalences:

$$\begin{aligned}
&(e \text{ is a coloop of } M^*) \\
\iff &\left(\underbrace{r_{M^*}(E)}_{=|E|-r_M(E)} = \underbrace{r_{M^*}(E \setminus e)}_{=|E|-1+r_M(e)-r_M(E)} + 1 \right) \\
\iff &(|E| - r_M(E) = |E| - 1 + r_M(e) - r_M(E) + 1) \\
\iff &(r_M(e) = 0) \iff (e \text{ is a loop of } M)
\end{aligned}$$

(by Proposition 2.11 **(a)**). This proves Proposition 2.16. □

Definition 2.17. Let $M = (E, \mathcal{I})$ be a matroid. Let Y be a subset of E. Then, the pair $(Y, \mathcal{I} \cap \mathcal{P}(Y))$ is a matroid with ground set Y. This matroid is called the *restriction* of M to Y, and is denoted by $M\mid_Y$.

Definition 2.18. Let $M = (E, \mathcal{I})$ be a matroid. Let Z be a subset of E.

(a) The *deletion* of Z from M is defined to be the matroid $M\mid_{E\setminus Z}$ (that is, the restriction of M to $E \setminus Z$). This is a matroid with ground set Z, and is denoted by $M \setminus Z$.

(b) The *contraction* of Z in M is defined to be the matroid $(M^* \setminus Z)^*$. This is a matroid with ground set Z, and is denoted by M/Z.

We shall first show formulas for ranks in $M \setminus Z$ and M/Z:

Proposition 2.19. Let $M = (E, \mathcal{I})$ be a matroid. Let Z be a subset of E. Let $S \in \mathcal{P}(E \setminus Z)$.

(a) We have $r_{M\setminus Z}(S) = r_M(S)$.

(b) We have $r_{M/Z}(S) = r_M(S \cup Z) - r_M(Z)$.

Proof of Proposition 2.19. **(a)** We have $S \in \mathcal{P}(E \setminus Z)$, thus $S \subseteq E \setminus Z \subseteq E$. Thus,

$$r_M(S) = \max\{|Y| \mid Y \in \mathcal{I} \text{ and } Y \subseteq S\}. \tag{11}$$

(This is merely the equality (1), with the index Z renamed as Y.)

But the definition of $M \setminus Z$ yields $M \setminus Z = M\mid_{E\setminus Z} = (E \setminus Z, \mathcal{I} \cap \mathcal{P}(E \setminus Z))$ (by the definition of $M\mid_{E\setminus Z}$). Hence, we can apply (11) to $M \setminus Z$, $E \setminus Z$ and $\mathcal{I} \cap \mathcal{P}(E \setminus Z)$ instead of M, E and $\mathcal{I}$ (since $S \subseteq E \setminus Z$). We thus obtain

$$r_{M\setminus Z}(S) = \max\{|Y| \mid Y \in \mathcal{I} \cap \mathcal{P}(E \setminus Z) \text{ and } Y \subseteq S\}. \tag{12}$$

For any subset Y of S, the statements $Y \in \mathcal{I}$ and $Y \in \mathcal{I} \cap \mathcal{P}(E \setminus Z)$ are equivalent[2]. Thus, (11) becomes

$$r_M(S) = \max \left\{ |Y| \mid \underbrace{Y \in \mathcal{I}}_{\substack{\text{this is equivalent to} \\ Y \in \mathcal{I} \cap \mathcal{P}(E \setminus Z)}} \text{ and } Y \subseteq S \right\}$$
$$= \max \{ |Y| \mid Y \in \mathcal{I} \cap \mathcal{P}(E \setminus Z) \text{ and } Y \subseteq S \} = r_{M \setminus Z}(S)$$

(by (12)). This proves Proposition 2.19 **(a)**.

(b) We have $S \in \mathcal{P}(E \setminus Z)$, thus $S \subseteq E \setminus Z \subseteq E$. Combined with $Z \subseteq E$, this yields $Z \cup S \subseteq E$. From $S \subseteq E \setminus Z$, we obtain $|(E \setminus Z) \setminus S| = |E \setminus Z| - |S|$. Hence, $\left| \underbrace{E \setminus (Z \cup S)}_{=(E \setminus Z) \setminus S} \right| = |(E \setminus Z) \setminus S| = |E \setminus Z| - |S|$.

Now, (10) (applied to $E \setminus (Z \cup S)$ instead of S) shows that

$$r_{M^*}(E \setminus (Z \cup S)) = \underbrace{|E \setminus (Z \cup S)|}_{=|E \setminus Z| - |S|} + r_M \left(\underbrace{E \setminus (E \setminus (Z \cup S))}_{\substack{= Z \cup S \\ (\text{since } Z \cup S \subseteq E)}} \right) - r_M(E)$$
$$= |E \setminus Z| - |S| + r_M(Z \cup S) - r_M(E). \qquad (13)$$

The definition of M/Z yields $M/Z = (M^* \setminus Z)^*$. The ground set of $(M^* \setminus Z)^*$ is the ground set of $M^* \setminus Z$, which is $E \setminus Z$ (since the ground set of M^* is E). Now, from $M/Z = (M^* \setminus Z)^*$, we obtain $r_{M/Z}(S) = r_{(M^* \setminus Z)^*}(S)$.

Let us write the matroid M^* as $(E, \mathcal{K})$.

But let us write the matroid $M^* \setminus Z$ as $(E \setminus Z, \mathcal{J})$. Then, (10) (applied to $M^* \setminus Z$, $E \setminus Z$ and $\mathcal{J}$ instead of M, E and $\mathcal{I}$) shows that

$$r_{(M^* \setminus Z)^*}(S) = |S| + r_{M^* \setminus Z}((E \setminus Z) \setminus S) - r_{M^* \setminus Z}(E \setminus Z)$$

[2] *Proof.* Let Y be a subset of S. We need to show that the statements $Y \in \mathcal{I}$ and $Y \in \mathcal{I} \cap \mathcal{P}(E \setminus Z)$ are equivalent. Since the statement $Y \in \mathcal{I} \cap \mathcal{P}(E \setminus Z)$ clearly implies the statement $Y \in \mathcal{I}$, we only need to show that the statement $Y \in \mathcal{I}$ implies the statement $Y \in \mathcal{I} \cap \mathcal{P}(E \setminus Z)$. Let us do this now.

Let us assume that $Y \in \mathcal{I}$. But $Y \subseteq S \subseteq E \setminus Z$, so that $Y \in \mathcal{P}(E \setminus Z)$. Combined with $Y \in \mathcal{I}$, this entails $Y \in \mathcal{I} \cap \mathcal{P}(E \setminus Z)$. Now, let us forget that we assumed that $Y \in \mathcal{I}$. We thus have shown that the statement $Y \in \mathcal{I}$ implies the statement $Y \in \mathcal{I} \cap \mathcal{P}(E \setminus Z)$. This completes our proof.

(since the ground set of $M^* \setminus Z$ is $E \setminus Z$). Hence,

$$\begin{aligned}
r_{M/Z}(S) &= r_{(M^*\setminus Z)^*}(S) \\
&= |S| + \underbrace{r_{M^*\setminus Z}((E\setminus Z)\setminus S)}_{\substack{=r_{M^*}((E\setminus Z)\setminus S) \\ \text{(by Proposition 2.19 (a)} \\ \text{(applied to } M^*, \mathcal{K} \text{ and } (E\setminus Z)\setminus S \\ \text{instead of } M, \mathcal{I} \text{ and } S))}} - \underbrace{r_{M^*\setminus Z}(E\setminus Z)}_{\substack{=r_{M^*}(E\setminus Z) \\ \text{(by Proposition 2.19 (a)} \\ \text{(applied to } M^*, \mathcal{K} \text{ and } (E\setminus Z)\setminus S \\ \text{instead of } M, \mathcal{I} \text{ and } S))}} \\
&= |S| + r_{M^*}\left(\underbrace{(E\setminus Z)\setminus S}_{=E\setminus(Z\cup S)}\right) - r_{M^*}(E\setminus Z) \\
&= |S| + \underbrace{r_{M^*}(E\setminus(Z\cup S))}_{\substack{=|E\setminus Z|-|S|+r_M(Z\cup S)-r_M(E) \\ \text{(by (13))}}} - \underbrace{r_{M^*}(E\setminus Z)}_{\substack{=|E\setminus Z|+r_M(E\setminus(E\setminus Z))-r_M(E) \\ \text{(by (10)} \\ \text{(applied to } E\setminus Z \text{ instead of } S))}} \\
&= |S| + (|E\setminus Z| - |S| + r_M(Z\cup S) - r_M(E)) \\
&\qquad - (|E\setminus Z| + r_M(E\setminus(E\setminus Z)) - r_M(E)) \\
&= r_M\left(\underbrace{Z\cup S}_{=S\cup Z}\right) - r_M\left(\underbrace{E\setminus(E\setminus Z)}_{\substack{=Z \\ \text{(since } Z\subseteq E)}}\right) = r_M(S\cup Z) - r_M(Z).
\end{aligned}$$

This proves Proposition 2.19 **(b)**. □

Definition 2.20. Let $M = (E, \mathcal{I})$ be a matroid. Let $e \in E$.

(a) We denote the matroid $M \setminus \{e\}$ by $M \setminus e$. This is a matroid with ground set $E \setminus \{e\} = E \setminus e$.

(b) We denote the matroid $M / \{e\}$ by M/e. This is a matroid with ground set $E \setminus \{e\} = E \setminus e$.

Proposition 2.21. Let $M = (E, \mathcal{I})$ be a matroid. Let $e \in E$. Assume that e is not a coloop of M. Then,

$$\mathcal{B}(M\setminus e) = \{B \in \mathcal{B}(M) \mid e \notin B\}.$$

Proposition 2.21 shows that our definition of $M \setminus e$ (in the case when e is not a coloop of M) is equivalent to the definition in [Martin15, Definition 3.36].

Proof of Proposition 2.21. The definition of $M \setminus e$ shows that

$$M\setminus e = M\setminus\{e\} = M\mid_{E\setminus\{e\}} = (E\setminus\{e\}, \mathcal{I}\cap\mathcal{P}(E\setminus\{e\})) = (E\setminus e, \mathcal{I}\cap\mathcal{P}(E\setminus e)).$$

Hence, Proposition 2.14 (applied to $M \setminus e$, $E \setminus e$ and $\mathcal{I} \cap \mathcal{P}(E \setminus e)$ instead of M, E and $\mathcal{I}$) shows that

$$\mathcal{B}(M \setminus e) = \left\{ B \in \mathcal{I} \cap \mathcal{P}(E \setminus e) \ \mid \ |B| = r_{M \setminus e}(E \setminus e) \right\}.$$

But $E \setminus e = E \setminus \{e\} \in \mathcal{P}(E \setminus \{e\})$. Hence, Proposition 2.19 **(a)** (applied to $Z = \{e\}$ and $S = E \setminus e$) shows that $r_{M \setminus \{e\}}(E \setminus e) = r_M(E \setminus e)$.

But if we had $r_M(E \setminus e) \neq r_M(E)$, then the element e would be a coloop of M (by Proposition 2.11 **(c)**), which would contradict the fact that e is not a coloop of M. Hence, we cannot have $r_M(E \setminus e) \neq r_M(E)$. In other words, we have $r_M(E \setminus e) = r_M(E)$. Hence, $r_{M \setminus \{e\}}(E \setminus e) = r_M(E \setminus e) = r_M(E)$-

Now,

$$\begin{aligned}
\mathcal{B}(M \setminus e) &= \left\{ B \in \mathcal{I} \cap \mathcal{P}(E \setminus e) \ \mid \ |B| = \underbrace{r_{M \setminus e}(E \setminus e)}_{= r_{M \setminus \{e\}}(E \setminus e) = r_M(E)} \right\} \\
&= \{ B \in \mathcal{I} \cap \mathcal{P}(E \setminus e) \ \mid \ |B| = r_M(E) \} \\
&= \underbrace{\{ B \in \mathcal{I} \ \mid \ |B| = r_M(E) \}}_{\substack{= \mathcal{B}(M) \\ \text{(by Proposition 2.14)}}} \cap \mathcal{P}(E \setminus e) \\
&= \mathcal{B}(M) \cap \mathcal{P}(E \setminus e) = \left\{ B \in \mathcal{B}(M) \ \mid \ \underbrace{B \in \mathcal{P}(E \setminus e)}_{\text{this is equivalent to } B \subseteq E \setminus e} \right\} \\
&= \left\{ B \in \mathcal{B}(M) \ \mid \ \underbrace{B \subseteq E \setminus e}_{\text{this is equivalent to } e \notin B} \right\} = \{ B \in \mathcal{B}(M) \ \mid \ e \notin B \}.
\end{aligned}$$

This proves Proposition 2.21. □

Proposition 2.22. Let $M = (E, \mathcal{I})$ be a matroid. Let $e \in E$. Assume that e is not a loop of M. Then,

$$\mathcal{B}(M / e) = \{ B \setminus e \ \mid \ B \in \mathcal{B}(M) \text{ and } e \in B \}.$$

Proposition 2.22 shows that our definition of M/e (in the case when e is not a loop of M) is equivalent to the definition in [Martin15, Definition 3.36].

Proof of Proposition 2.22. Proposition 2.16 shows that e is a loop of M if and only if e is a coloop of M^*. Thus, e is not a coloop of M^* (since e is not a loop of M).

But the definition of M^* shows that

$$\mathcal{B}(M^*) = \{ E \setminus B \ \mid \ B \in \mathcal{B}(M) \} = \{ E \setminus C \ \mid \ C \in \mathcal{B}(M) \}$$

(here, we renamed the index B as C).

Write the matroid M^* as $(E, \mathcal{K})$. Proposition 2.21 (applied to M^* and $\mathcal{K}$ instead of M and $\mathcal{I}$) shows that

$$\begin{aligned}
\mathcal{B}(M^* \setminus e) &= \left\{ B \in \underbrace{\mathcal{B}(M^*)}_{=\{E \setminus C \ \mid \ C \in \mathcal{B}(M)\}} \ \mid \ e \notin B \right\} \\
&= \{B \in \{E \setminus C \ \mid \ C \in \mathcal{B}(M)\} \ \mid \ e \notin B\} \\
&= \left\{ E \setminus C \ \mid \ C \in \mathcal{B}(M) \text{ and } \underbrace{e \notin E \setminus C}_{\substack{\text{this is equivalent to } e \in C \\ \text{(since } e \in E\text{)}}} \right\} \\
&= \{E \setminus C \ \mid \ C \in \mathcal{B}(M) \text{ and } e \in C\}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
M/e = M/\{e\} &= \Bigg(\underbrace{M^* \setminus \{e\}}_{=M^* \setminus e} \Bigg)^* \qquad \text{(by the definition of } M/\{e\}\text{)} \\
&= (M^* \setminus e)^*,
\end{aligned}$$

so that

$$\begin{aligned}
\mathcal{B}(M/e) = \mathcal{B}\left((M^* \setminus e)^*\right) &= \left\{ (E \setminus e) \setminus B \ \mid \ B \in \underbrace{\mathcal{B}(M^* \setminus e)}_{=\{E \setminus C \ \mid \ C \in \mathcal{B}(M) \text{ and } e \in C\}} \right\} \\
&= \{(E \setminus e) \setminus B \ \mid \ B \in \{E \setminus C \ \mid \ C \in \mathcal{B}(M) \text{ and } e \in C\}\} \\
&= \left\{ \underbrace{(E \setminus e) \setminus (E \setminus C)}_{\substack{=C \setminus e \\ \text{(since } e \in C\text{)}}} \ \mid \ C \in \mathcal{B}(M) \text{ and } e \in C \right\} \\
&= \{C \setminus e \ \mid \ C \in \mathcal{B}(M) \text{ and } e \in C\} \\
&= \{B \setminus e \ \mid \ B \in \mathcal{B}(M) \text{ and } e \in B\}
\end{aligned}$$

(here, we renamed the index C as B). This proves Proposition 2.22. □

Proposition 2.23. Let $M = (E, \mathcal{I})$ be a matroid. Let $e \in E$.
(a) If e is a loop of M, then $M/e = M \setminus e$.
(b) If e is a coloop of M, then $M/e = M \setminus e$.

Proof of Proposition 2.23. Both matroids $M/\{e\}$ and $M\setminus\{e\}$ have the ground set $E\setminus\{e\}$.

(a) Assume that e is a loop of M. Proposition 2.11 **(a)** therefore shows that $r_M(e)=0$.

Let $S\in\mathcal{P}(E\setminus\{e\})$. Proposition 2.19 **(b)** (applied to $Z=\{e\}$) yields

$$\begin{aligned} r_{M/\{e\}}(S) &= r_M\left(\underbrace{S\cup\{e\}}_{=S\cup e}\right) - \underbrace{r_M(\{e\})}_{=r_M(e)=0} \\ &= r_M(S\cup e) = r_M(S) \qquad \text{(by Proposition 2.12 **(a)**)}. \end{aligned}$$

But Proposition 2.19 **(a)** (applied to $Z=\{e\}$) shows that $r_{M\setminus\{e\}}(S)=r_M(S)$. Comparing this with $r_{M/\{e\}}(S)=r_M(S)$, we obtain $r_{M/\{e\}}(S)=r_{M\setminus\{e\}}(S)$.

Now, let us forget that we fixed S. We thus have shown that $r_{M/\{e\}}(S)=r_{M\setminus\{e\}}(S)$ for every $S\in\mathcal{P}(E\setminus\{e\})$. In other words, $r_{M/\{e\}}=r_{M\setminus\{e\}}$. Hence, Corollary 2.7 (applied to $M/\{e\}$, $M\setminus\{e\}$ and $E\setminus\{e\}$ instead of M, M' and E) shows that $M/\{e\}=M\setminus\{e\}$. Thus, $M/e=M/\{e\}=M\setminus\{e\}=M\setminus e$. This proves Proposition 2.23 **(a)**.

(b) Assume that e is a coloop of M. In other words, e is a coloop of $(M^*)^*$ (since $(M^*)^*=M$).

Write the matroid M^* in the form $(E,\mathcal{K})$. Proposition 2.16 (applied to M^* and $\mathcal{K}$ instead of M and $\mathcal{I}$) shows that e is a loop of M^* if and only if e is a coloop of $(M^*)^*$. Hence, e is a loop of M^* (since e is a coloop of $(M^*)^*$). Hence, Proposition 2.23 **(a)** (applied to M^* and $\mathcal{K}$ instead of M and $\mathcal{I}$) shows that $M^*/e=M^*\setminus e$.

On the other hand,

$$\begin{aligned} M^*/e = M^*/\{e\} &= \left((M^*)^*\setminus\{e\}\right)^* \qquad \text{(by the definition of } M^*/\{e\}\text{)} \\ &= \left(\underbrace{(M^*)^*}_{=M}\setminus e\right)^* = (M\setminus e)^*. \end{aligned}$$

Now,

$$\begin{aligned} M/e = M/\{e\} &= \left(\underbrace{M^*\setminus\{e\}}_{=M^*\setminus e=M^*/e}\right)^* \qquad \text{(by the definition of } M/\{e\}\text{)} \\ &= \left(\underbrace{M^*/e}_{=(M\setminus e)^*}\right)^* = \left((M\setminus e)^*\right)^* = M\setminus e. \end{aligned}$$

This proves Proposition 2.23 **(b)**. □

3. The Tutte polynomial

Let us recall the definition of the Tutte polynomial of a matroid (according to [Martin15, Definition 4.1]):

Definition 3.1. Let $M = (E, \mathcal{I})$ be a matroid. The *Tutte polynomial* of this matroid M is defined to be the polynomial

$$\sum_{A \subseteq E} (x-1)^{r_M(E) - r_M(A)} (y-1)^{|A| - r_M(A)} \in \mathbb{Z}[x, y].$$

This polynomial is denoted by $T_M = T_M(x, y)$.

This polynomial has many properties; in particular, it encodes a lot of information about M. The most relevant property is the following proposition, which gives a recursive way to compute it (starting with $T_{(\varnothing,\varnothing)} = 1$) without ever having to subtract 1:

Proposition 3.2. Let $M = (E, \mathcal{I})$ be a matroid. Let $e \in E$.

(a) If e is a loop, then $T_M = yT_{M \setminus e}$.

(b) If e is a coloop, then $T_M = xT_{M/e}$.

(c) If e is neither a loop nor a coloop, then $T_M = T_{M \setminus e} + T_{M/e}$.

As a corollary, we obtain the following:

Corollary 3.3. Let $M = (E, \mathcal{I})$ be a matroid. Then, $T_M \in \mathbb{N}[x, y]$.

Here, $\mathbb{N}[x, y]$ denotes the set of all polynomials $P \in \mathbb{Z}[x, y]$ whose all coefficients are nonnegative integers.

Proposition 3.2 is well-known (e.g., it is [Martin15, Theorem 4.5]); we shall not prove it right now, but it will be a consequence of some results further below.

4. The filtered Tutte polynomial

Our goal in this note is to define a generalization of the Tutte polynomial T_M for matroids M equipped with a *filtration*, i.e., a weakly increasing chain $\varnothing = E_0 \subseteq E_1 \subseteq \cdots \subseteq E_m = E$ of subsets of E. This generalization will be called the *filtered Tutte polynomial*. Let us first define filtered matroids:

Definition 4.1. A *filtered matroid* means a pair $(M, \mathbf{E})$, where $M = (E, \mathcal{I})$ is a matroid, and where $\mathbf{E}$ is a finite list $(E_0, E_1, \ldots, E_m)$ of subsets of E such that

$$\varnothing = E_0 \subseteq E_1 \subseteq \cdots \subseteq E_m = E.$$

Definition 4.2. Let $\mathbf{M} = (M, \mathbf{E})$ be a filtered matroid, with $M = (E, \mathcal{I})$. Let $e \in E$. Then, $\mathbf{E} \setminus e$ shall denote the list $(E_0 \setminus e, E_1 \setminus e, \ldots, E_m \setminus e)$, where the list $\mathbf{E}$ is written as $(E_0, E_1, \ldots, E_m)$. We let $\mathbf{M} \setminus e$ and $\mathbf{M}/e$ denote the filtered matroids $(M \setminus e, \mathbf{E} \setminus e)$ and $(M/e, \mathbf{E} \setminus e)$.

Definition 4.3. Let $\mathbf{M} = (M, \mathbf{E})$ be a filtered matroid, with $M = (E, \mathcal{I})$. Write the list $\mathbf{E}$ as $(E_0, E_1, \ldots, E_m)$. The *filtered Tutte polynomial* of this filtered matroid $\mathbf{M}$ is defined to be the polynomial

$$\sum_{A \subseteq E} \left(\prod_{i=1}^{m} (x_i - 1)^{r_M(A \cup E_i) - r_M(A \cup E_{i-1})} \right) \left(\prod_{i=1}^{m} (y_i - 1)^{n_M(A \setminus E_{i-1}) - n_M(A \setminus E_i)} \right)$$
$$\in \mathbb{Z}\left[x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_m\right].$$

This polynomial is well-defined[3], and is denoted by $T_{\mathbf{M}}$.

Filtered Tutte polynomials have the following properties:

Proposition 4.4. Let $\mathbf{M} = (M, \mathbf{E})$ be a filtered matroid, with $M = (E, \mathcal{I})$. Write the list $\mathbf{E}$ as $(E_0, E_1, \ldots, E_m)$. Then, in the polynomial ring $\mathbb{Z}[x, y]$, we have

$$T_{\mathbf{M}} \left(\underbrace{x, x, \ldots, x}_{m \text{ times}}, \underbrace{y, y, \ldots, y}_{m \text{ times}} \right) = T_M(x, y).$$

Proposition 4.5. Let $M = (E, \mathcal{I})$ be a matroid, and let $\mathbf{E}$ be the list $(\varnothing, E)$. Then, $(M, \mathbf{E})$ is a filtered matroid, and satisfies

$$T_{(M,\mathbf{E})} = T_M(x_1, y_1).$$

[3] *Proof.* We only need to show the following two statements:

Statement 1: We have $r_M(A \cup E_i) - r_M(A \cup E_{i-1}) \in \mathbb{N}$ for every $i \in \{1, 2, \ldots, m\}$.

Statement 2: We have $n_M(A \setminus E_{i-1}) - n_M(A \setminus E_i) \in \mathbb{N}$ for every $i \in \{1, 2, \ldots, m\}$.

Proof of Statement 1: Let $i \in \{1, 2, \ldots, m\}$. We have $\varnothing = E_0 \subseteq E_1 \subseteq \cdots \subseteq E_m = E$ (since $(M, \mathbf{E})$ is a filtered matroid) and thus $E_{i-1} \subseteq E_i$. Hence, $A \cup \underbrace{E_{i-1}}_{\subseteq E_i} \subseteq A \cup E_i$. Therefore, (3) (applied to $S = A \cup E_{i-1}$ and $T = A \cup E_i$) shows that $r_M(A \cup E_{i-1}) \leq r_M(A \cup E_i)$. In other words, $r_M(A \cup E_i) - r_M(A \cup E_{i-1}) \in \mathbb{N}$. This proves Statement 1.

Proof of Statement 2: Let $i \in \{1, 2, \ldots, m\}$. We have $\varnothing = E_0 \subseteq E_1 \subseteq \cdots \subseteq E_m = E$ (since $(M, \mathbf{E})$ is a filtered matroid) and thus $E_{i-1} \subseteq E_i$. In other words, $E_i \supseteq E_{i-1}$. Hence, $A \setminus \underbrace{E_i}_{\supseteq E_{i-1}} \subseteq A \setminus E_{i-1}$. Therefore, (7) (applied to $S = A \setminus E_i$ and $T = A \setminus E_{i-1}$) shows that $n_M(A \setminus E_i) \leq n_M(A \setminus E_{i-1})$. In other words, $n_M(A \setminus E_{i-1}) - n_M(A \setminus E_i) \in \mathbb{N}$. This proves Statement 2.

Proposition 4.6. We have $T_{((\varnothing,\varnothing),(\varnothing))} = 1$.

Proposition 4.7. Let $\mathbf{M} = (M, \mathbf{E})$ be a filtered matroid, with $M = (E, \mathcal{I})$. Write the list $\mathbf{E}$ as $(E_0, E_1, \ldots, E_m)$. Assume that $E_{m-1} = E_m$. Let $\mathbf{E}'$ be the list $(E_0, E_1, \ldots, E_{m-1})$. Then, $(M, \mathbf{E}')$ is a filtered matroid, and satisfies

$$T_{(M,\mathbf{E})} = T_{(M,\mathbf{E}')}.$$

(Here, we regard $\mathbb{Z}[x_1, x_2, \ldots, x_{m-1}, y_1, y_2, \ldots, y_{m-1}]$ as a subring of $\mathbb{Z}[x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_m]$ in the obvious way.)

Proposition 4.8. Let $\mathbf{M} = (M, \mathbf{E})$ be a filtered matroid, with $M = (E, \mathcal{I})$. Write the list $\mathbf{E}$ as $(E_0, E_1, \ldots, E_m)$. Let $e \in E$.

(a) If e is a loop, then $T_{\mathbf{M}} = y_k T_{\mathbf{M} \setminus e}$, where $k \in \{1, 2, \ldots, m\}$ is such that $e \in E_k \setminus E_{k-1}$.

(b) If e is a coloop, then $T_{\mathbf{M}} = x_k T_{\mathbf{M}/e}$, where $k \in \{1, 2, \ldots, m\}$ is such that $e \in E_k \setminus E_{k-1}$.

(c) If e belongs to $E_m \setminus E_{m-1}$ and is neither a loop nor a coloop, then $T_{\mathbf{M}} = T_{\mathbf{M} \setminus e} + T_{\mathbf{M}/e}$.

We do not know whether Proposition 4.8 can be extended to the case when e does not belong to $E_m \setminus E_{m-1}$.

Proposition 4.9. Let $\mathbf{M} = (M, \mathbf{E})$ be a filtered matroid. Write the list $\mathbf{E}$ as $(E_0, E_1, \ldots, E_m)$. Then, $T_{\mathbf{M}} \in \mathbb{N}[x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_m]$.

Here, $\mathbb{N}[x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_m]$ denotes the set of all polynomials $P \in \mathbb{Z}[x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_m]$ whose all coefficients are nonnegative integers.

Proposition 4.10. Let $\mathbf{M} = (M, \mathbf{E})$ be a filtered matroid, with $M = (E, \mathcal{I})$. Write the list $\mathbf{E}$ as $(E_0, E_1, \ldots, E_m)$.

Then, $(M^*, \mathbf{E})$ is a filtered matroid as well, and satisfies

$$T_{(M^*,\mathbf{E})} = T_{\mathbf{M}}(y_1, y_2, \ldots, y_m, x_1, x_2, \ldots, x_m).$$

We shall now prove the properties listed above.

Proof of Proposition 4.4. By the telescope principle, we have

$$\begin{aligned} &\sum_{i=1}^{m} \left(r_M(A \cup E_i) - r_M(A \cup E_{i-1})\right) \\ &= r_M\Big(A \cup \underbrace{E_m}_{=E}\Big) - r_M\Big(A \cup \underbrace{E_0}_{=\varnothing}\Big) = r_M\Big(\underbrace{A \cup E}_{\substack{=E \\ (\text{since } A \subseteq E)}}\Big) - r_M\Big(\underbrace{A \cup \varnothing}_{=A}\Big) \\ &= r_M(E) - r_M(A). \end{aligned} \tag{14}$$

By the telescope principle, we have

$$\begin{aligned}
&\sum_{i=1}^{m}\left(n_M\left(A\setminus E_{i-1}\right)-n_M\left(A\setminus E_i\right)\right)\\
&=n_M\left(A\setminus \underbrace{E_0}_{=\varnothing}\right)-n_M\left(A\setminus \underbrace{E_m}_{=E}\right)=n_M\left(\underbrace{A\setminus\varnothing}_{=A}\right)-n_M\left(\underbrace{A\setminus E}_{\substack{=\varnothing\\ (\text{since } A\subseteq E)}}\right)\\
&=n_M\left(A\right)-\underbrace{n_M\left(\varnothing\right)}_{=0}=n_M\left(A\right)=\left|A\right|-r_M\left(A\right)
\end{aligned} \tag{15}$$

(by the definition of $n_M\left(A\right)$).

The definition of $T_{\mathbf{M}}$ yields

$$\begin{aligned}
&T_{\mathbf{M}}\left(\underbrace{x,x,\ldots,x}_{m\text{ times}},\underbrace{y,y,\ldots,y}_{m\text{ times}}\right)\\
&=\sum_{A\subseteq E}\underbrace{\left(\prod_{i=1}^{m}\left(x-1\right)^{r_M(A\cup E_i)-r_M(A\cup E_{i-1})}\right)}_{=(x-1)^{\sum_{i=1}^{m}\left(r_M(A\cup E_i)-r_M(A\cup E_{i-1})\right)}}\underbrace{\left(\prod_{i=1}^{m}\left(y-1\right)^{n_M(A\setminus E_{i-1})-n_M(A\setminus E_i)}\right)}_{=(y-1)^{\sum_{i=1}^{m}\left(n_M(A\setminus E_{i-1})-n_M(A\setminus E_i)\right)}}\\
&=\sum_{A\subseteq E}\left(x-1\right)^{\sum_{i=1}^{m}\left(r_M(A\cup E_i)-r_M(A\cup E_{i-1})\right)}\left(y-1\right)^{\sum_{i=1}^{m}\left(n_M(A\setminus E_{i-1})-n_M(A\setminus E_i)\right)}\\
&=\sum_{A\subseteq E}\left(x-1\right)^{r_M(E)-r_M(A)}\left(y-1\right)^{|A|-r_M(A)}\qquad\text{(by (14) and (15))}\\
&=T_M\qquad\text{(by the definition of } T_M\text{)}\\
&=T_M\left(x,y\right).
\end{aligned}$$

This proves Proposition 4.4. □

Proof of Proposition 4.5. We have $\varnothing=\varnothing\subseteq E=E$. Thus, $\left(M,\mathbf{E}\right)$ is a filtered matroid. It remains to show that $T_{(M,\mathbf{E})}=T_M\left(x_1,y_1\right)$.

Proposition 4.4 (applied to $m=1$, $\left(E_0,E_1,\ldots,E_m\right)=\left(\varnothing,E\right)$ and $\mathbf{M}=\left(M,\mathbf{E}\right)$) yields $T_{(M,\mathbf{E})}\left(\underbrace{x,x,\ldots,x}_{1\text{ times}},\underbrace{y,y,\ldots,y}_{1\text{ times}}\right)=T_M\left(x,y\right)=T_M$. Thus,

$$T_M=T_{(M,\mathbf{E})}\left(\underbrace{x,x,\ldots,x}_{1\text{ times}},\underbrace{y,y,\ldots,y}_{1\text{ times}}\right)=T_{(M,\mathbf{E})}\left(x,y\right).$$

Substituting x_1 and y_1 for x and y on both sides of this equality, we obtain $T_M(x_1, y_1) = \left(T_{(M,\mathbf{E})}(x, y)\right)(x_1, y_1) = T_{(M,\mathbf{E})}$. In other words, $T_{(M,\mathbf{E})} = T_M(x_1, y_1)$. This completes the proof of Proposition 4.5. □

Proof of Proposition 4.6. Proposition 4.6 follows straightforwardly from the definition. □

Proof of Proposition 4.7. Since $(M, \mathbf{E})$ is a filtered matroid, we have $\varnothing = E_0 \subseteq E_1 \subseteq \cdots \subseteq E_m = E$. Combined with $E_{m-1} = E_m$, this yields $\varnothing = E_0 \subseteq E_1 \subseteq \cdots \subseteq E_{m-1} = E$. Thus, $(M, \mathbf{E}')$ is a filtered matroid. It remains to prove that $T_{(M,\mathbf{E})} = T_{(M,\mathbf{E}')}$.

The definition of $T_{(M,\mathbf{E}')}$ yields

$$T_{(M,\mathbf{E}')} = \sum_{A \subseteq E} \left(\prod_{i=1}^{m-1} (x_i - 1)^{r_M(A \cup E_i) - r_M(A \cup E_{i-1})} \right) \left(\prod_{i=1}^{m-1} (y_i - 1)^{n_M(A \setminus E_{i-1}) - n_M(A \setminus E_i)} \right).$$

But

$$\begin{aligned}
&\prod_{i=1}^{m} (x_i - 1)^{r_M(A \cup E_i) - r_M(A \cup E_{i-1})} \\
&= \left(\prod_{i=1}^{m-1} (x_i - 1)^{r_M(A \cup E_i) - r_M(A \cup E_{i-1})} \right) \underbrace{(x_i - 1)^{r_M(A \cup E_m) - r_M(A \cup E_{m-1})}}_{\substack{=1 \\ \text{(since } E_{m-1} = E_m \text{, so that} \\ r_M(A \cup E_{m-1}) = r_M(A \cup E_m) \text{, so that} \\ r_M(A \cup E_m) - r_M(A \cup E_{m-1}) = 0)}} \\
&= \prod_{i=1}^{m-1} (x_i - 1)^{r_M(A \cup E_i) - r_M(A \cup E_{i-1})}
\end{aligned}$$

and

$$\begin{aligned}
&\prod_{i=1}^{m} (y_i - 1)^{n_M(A \setminus E_{i-1}) - n_M(A \setminus E_i)} \\
&= \left(\prod_{i=1}^{m-1} (y_i - 1)^{n_M(A \setminus E_{i-1}) - n_M(A \setminus E_i)} \right) \underbrace{(y_i - 1)^{n_M(A \setminus E_{m-1}) - n_M(A \setminus E_m)}}_{\substack{=1 \\ \text{(since } E_{m-1} = E_m \text{, so that} \\ n_M(A \setminus E_{m-1}) = n_M(A \setminus E_m) \text{, so that} \\ n_M(A \setminus E_{m-1}) - n_M(A \setminus E_m) = 0)}} \\
&= \prod_{i=1}^{m-1} (y_i - 1)^{n_M(A \setminus E_{i-1}) - n_M(A \setminus E_i)}.
\end{aligned}$$

Now, the definition of $T_{(M,\mathbf{E})}$ yields

$$\begin{aligned} T_{(M,\mathbf{E})} &= \sum_{A\subseteq E} \underbrace{\left(\prod_{i=1}^{m} (x_i-1)^{r_M(A\cup E_i)-r_M(A\cup E_{i-1})}\right)}_{=\prod_{i=1}^{m-1}(x_i-1)^{r_M(A\cup E_i)-r_M(A\cup E_{i-1})}} \underbrace{\left(\prod_{i=1}^{m} (y_i-1)^{n_M(A\setminus E_{i-1})-n_M(A\setminus E_i)}\right)}_{=\prod_{i=1}^{m-1}(y_i-1)^{n_M(A\setminus E_{i-1})-n_M(A\setminus E_i)}} \\ &= \sum_{A\subseteq E} \left(\prod_{i=1}^{m-1} (x_i-1)^{r_M(A\cup E_i)-r_M(A\cup E_{i-1})}\right) \left(\prod_{i=1}^{m-1} (y_i-1)^{n_M(A\setminus E_{i-1})-n_M(A\setminus E_i)}\right) \\ &= T_{(M,\mathbf{E}')}. \end{aligned}$$

This completes the proof of Proposition 4.7. $\square$

We shall now prepare for the proof of Proposition 4.8.

First, we introduce the *Iverson bracket notation*: If $\mathcal{S}$ is any logical statement, then $[\mathcal{S}]$ shall mean the integer $\begin{cases} 1, & \text{if } \mathcal{S} \text{ is true;} \\ 0, & \text{if } \mathcal{S} \text{ is false} \end{cases}$.

Next, we shall show some lemmas:

Lemma 4.11. Let $M = (E, \mathcal{I})$ be a matroid. Let $e \in E$. Let $A \in \mathcal{P}(E)$ be such that $e \notin A$. Let $K \in \mathcal{P}(E)$. Then,

$$r_M((A\cup e)\cup K) = r_{M/e}(A\cup(K\setminus e)) + r_M(e).$$

Proof of Lemma 4.11. We have $e \notin A$, thus $A \subseteq E \setminus \{e\}$.

Let $S = A \cup (K \setminus e)$. Then, $S = \underbrace{A}_{\subseteq E\setminus\{e\} = E\setminus e} \cup \left(\underbrace{K}_{\subseteq E} \setminus e\right) \subseteq (E\setminus e)\cup(E\setminus e) = E\setminus e$. In other words, $S \in \mathcal{P}(E\setminus\{e\})$. Thus, Proposition 2.19 **(b)** (applied to $Z = \{e\}$) shows that $r_{M/\{e\}}(S) = r_M(S\cup\{e\}) - r_M(\{e\})$. Since $M/\{e\} = M/e$, $S\cup\{e\} = S\cup e$ and $r_M(\{e\}) = r_M(e)$, this rewrites as $r_{M/e}(S) = r_M(S\cup e) - r_M(e)$.

But

$$\underbrace{S}_{=A\cup(K\setminus e)} \cup e = A \cup \underbrace{(K\setminus e)\cup e}_{=K\cup e} = A\cup K\cup e.$$

Now,

$$\begin{aligned} r_{M/e}\left(\underbrace{A\cup(K\setminus e)}_{=S}\right) &= r_{M/e}(S) = r_M\left(\underbrace{S\cup e}_{=A\cup K\cup e = (A\cup e)\cup K}\right) - r_M(e) \\ &= r_M((A\cup e)\cup K) - r_M(e). \end{aligned}$$

In other words, $r_M((A\cup e)\setminus K) = r_{M/e}(A\cup(K\setminus e)) + r_M(e)$. This proves Lemma 4.11. $\square$

Lemma 4.12. Let $M = (E, \mathcal{I})$ be a matroid. Let $e \in E$. Let $A \in \mathcal{P}(E)$ be such that $e \notin A$. Let $K \in \mathcal{P}(E)$.

(a) If $e \in K$, then

$$\begin{aligned} & n_M((A \cup e) \setminus K) \\ & = n_{M/e}(A \setminus (K \setminus e)) + r_M((A \setminus K) \cup e) - r_M(A \setminus K) - r_M(e). \end{aligned}$$

(b) If $e \notin K$, then

$$n_M((A \cup e) \setminus K) = n_{M/e}(A \setminus (K \setminus e)) + 1 - r_M(e).$$

(c) If e is a loop, then

$$n_M((A \cup e) \setminus K) = n_{M/e}(A \setminus (K \setminus e)) + [e \notin K].$$

(d) If e is a coloop, then

$$n_M((A \cup e) \setminus K) = n_{M/e}(A \setminus (K \setminus e)).$$

(e) If $K = E$, then

$$n_M((A \cup e) \setminus K) = n_{M/e}(A \setminus (K \setminus e)).$$

Proof of Lemma 4.12. Let $S = A \setminus (K \setminus e)$. Then, $S \subseteq A \subseteq E \setminus \{e\}$. In other words, $S \in \mathcal{P}(E \setminus \{e\})$. Thus, Proposition 2.19 **(b)** (applied to $Z = \{e\}$) shows that $r_{M/\{e\}}(S) = r_M(S \cup \{e\}) - r_M(\{e\})$. Since $M/\{e\} = M/e$, $S \cup \{e\} = S \cup e$ and $r_M(\{e\}) = r_M(e)$, this rewrites as $r_{M/e}(S) = r_M(S \cup e) - r_M(e)$.

But $S = A \setminus (K \setminus e) \subseteq (A \setminus K) \cup e$. Hence,

$$\underbrace{S}_{\subseteq (A \setminus K) \cup e} \cup e \subseteq (A \setminus K) \cup e \cup e = (A \setminus K) \cup e.$$

Combining this with $\underbrace{(A \setminus K)}_{\subseteq A \setminus (K \setminus e) = S} \cup e \subseteq S \cup e$, this yields $S \cup e = (A \setminus K) \cup e$. Now,

$$r_{M/e}\left(\underbrace{A \setminus (K \setminus e)}_{=S}\right) = r_M(S) = r_M\left(\underbrace{S \cup e}_{=(A \setminus K) \cup e}\right) - r_M(e) = r_M((A \setminus K) \cup e) - r_M(e).$$

On the other hand,

$$\underbrace{n_M((A\cup e)\setminus K)}_{\substack{=|(A\cup e)\setminus K|-r_M((A\cup e)\setminus K)\\ \text{(by the definition of } n_M)}} - \underbrace{n_{M/e}(A\setminus(K\setminus e))}_{\substack{=|A\setminus(K\setminus e)|-r_{M/e}(A\setminus(K\setminus e))\\ \text{(by the definition of } n_{M/e})}}$$

$$= (|(A\cup e)\setminus K| - r_M((A\cup e)\setminus K)) - \left(|A\setminus(K\setminus e)| - \underbrace{r_{M/e}(A\setminus(K\setminus e))}_{=r_M((A\setminus K)\cup e)-r_M(e)}\right)$$
$$= (|(A\cup e)\setminus K| - r_M((A\cup e)\setminus K)) - (|A\setminus(K\setminus e)| - (r_M((A\setminus K)\cup e) - r_M(e)))$$
$$= |(A\cup e)\setminus K| - r_M((A\cup e)\setminus K) - |A\setminus(K\setminus e)| + r_M((A\setminus K)\cup e) - r_M(e). \tag{16}$$

(a) Assume that $e\in K$. Combined with $e\notin A$, this readily yields $(A\cup e)\setminus K = A\setminus(K\setminus e)$. Also, $(A\cup e)\setminus K = A\setminus K$ (since $e\in K$). Hence, (16) becomes

$$n_M((A\cup e)\setminus K) - n_{M/e}(A\setminus(K\setminus e))$$
$$= \left|\underbrace{(A\cup e)\setminus K}_{=A\setminus(K\setminus e)}\right| - r_M\left(\underbrace{(A\cup e)\setminus K}_{=A\setminus K}\right) - |A\setminus(K\setminus e)| + r_M((A\setminus K)\cup e) - r_M(e)$$
$$= |A\setminus(K\setminus e)| - r_M(A\setminus K) - |A\setminus(K\setminus e)| + r_M((A\setminus K)\cup e) - r_M(e)$$
$$= r_M((A\setminus K)\cup e) - r_M(A\setminus K) - r_M(e).$$

In other words,

$$n_M((A\cup e)\setminus K) = n_{M/e}(A\setminus(K\setminus e)) + r_M((A\setminus K)\cup e) - r_M(A\setminus K) - r_M(e).$$

This proves Lemma 4.12 **(a)**.

(b) Assume that $e\notin K$. Combined with $e\notin A$, this readily yields $(A\cup e)\setminus K = (A\setminus(K\setminus e))\cup e$. Thus, $|(A\cup e)\setminus K| = |(A\setminus(K\setminus e))\cup e| = |A\setminus(K\setminus e)|+1$ (since $e\notin A\setminus(K\setminus e)$ (since $e\notin A$)). Also, $(A\cup e)\setminus K = (A\setminus K)\cup e$ (since $e\notin K$). Hence, (16) becomes

$$n_M((A\cup e)\setminus K) - n_{M/e}(A\setminus(K\setminus e))$$
$$= \underbrace{|(A\cup e)\setminus K|}_{=|A\setminus(K\setminus e)|+1} - r_M\left(\underbrace{(A\cup e)\setminus K}_{=(A\setminus K)\cup e}\right) - |A\setminus(K\setminus e)| + r_M((A\setminus K)\cup e) - r_M(e)$$
$$= |A\setminus(K\setminus e)| + 1 - r_M((A\setminus K)\cup e) - |A\setminus(K\setminus e)| + r_M((A\setminus K)\cup e) - r_M(e)$$
$$= 1 - r_M(e).$$

In other words,

$$n_M((A\cup e)\setminus K) = n_{M/e}(A\setminus(K\setminus e)) + 1 - r_M(e).$$

This proves Lemma 4.12 **(b)**.

(c) Assume that e is a loop. We need to show that

$$n_M((A\cup e)\setminus K) = n_{M/e}(A\setminus(K\setminus e)) + [e\notin K].$$

We know that e is a loop. Thus, $r_M(e) = 0$ (by Proposition 2.11 **(a)**). Proposition 2.12 **(a)** (applied to $S = A\setminus K$) yields $r_M((A\setminus K)\cup e) = r_M(A\setminus K)$.

We are in one of the following two cases:

Case 1: We have $e\in K$.

Case 2: We have $e\notin K$.

Let us first consider Case 1. In this case, we have $e\in K$. Thus, Lemma 4.12 **(a)** yields

$$\begin{aligned}&n_M((A\cup e)\setminus K)\\ &= n_{M/e}(A\setminus(K\setminus e)) + \underbrace{r_M((A\setminus K)\cup e)}_{=r_M(A\setminus K)} - r_M(A\setminus K) - \underbrace{r_M(e)}_{=0}\\ &= n_{M/e}(A\setminus(K\setminus e)) + r_M(A\setminus K) - r_M(A\setminus K)\\ &= n_{M/e}(A\setminus(K\setminus e)).\end{aligned}$$

Compared with

$$n_{M/e}(A\setminus(K\setminus e)) + \underbrace{[e\notin K]}_{\substack{=0\\ (\text{since } e\in K)}} = n_{M/e}(A\setminus(K\setminus e)),$$

this yields $n_M((A\cup e)\setminus K) = n_{M/e}(A\setminus(K\setminus e)) + [e\notin K]$. Thus, Lemma 4.12 **(c)** is proven in Case 1.

Let us now consider Case 2. In this case, we have $e\notin K$. Hence, Lemma 4.12 **(b)** yields

$$\begin{aligned}n_M((A\cup e)\setminus K) &= n_{M/e}(A\setminus(K\setminus e)) + 1 - \underbrace{r_M(e)}_{=0}\\ &= n_{M/e}(A\setminus(K\setminus e)) + 1.\end{aligned}$$

Compared with

$$n_{M/e}(A\setminus(K\setminus e)) + \underbrace{[e\notin K]}_{\substack{=1\\ (\text{since } e\notin K)}} = n_{M/e}(A\setminus(K\setminus e)) + 1,$$

this yields $n_M((A\cup e)\setminus K) = n_{M/e}(A\setminus(K\setminus e)) + [e\notin K]$. Thus, Lemma 4.12 **(c)** is proven in Case 2.

Hence, Lemma 4.12 **(c)** is proven in each of the two Cases 1 and 2. Thus, Lemma 4.12 **(c)** always holds.

(d) Assume that e is a coloop. We need to show that $n_M((A\cup e)\setminus K) = n_{M/e}(A\setminus(K\setminus e)) + 1$.

We know that e is a coloop. Thus, Proposition 2.12 **(b)** (applied to $S = \varnothing$) yields $r_M(\varnothing \cup e) = \underbrace{r_M(\varnothing)}_{=0} + 1 = 1$. Since $r_M\left(\underbrace{\varnothing \cup e}_{=\{e\}}\right) = r_M(\{e\}) = r_M(e)$, this rewrites as $r_M(e) = 1$. But Proposition 2.12 **(b)** (applied to $S = A \setminus K$) yields $r_M((A \setminus K) \cup e) = r_M(A \setminus K) + 1$ (since $e \notin A \setminus K$ (since $e \notin A$)).

We are in one of the following two cases:

Case 1: We have $e \in K$.

Case 2: We have $e \notin K$.

Let us first consider Case 1. In this case, we have $e \in K$. Thus, Lemma 4.12 **(a)** yields

$$\begin{aligned} & n_M((A \cup e) \setminus K) \\ & = n_{M/e}(A \setminus (K \setminus e)) + \underbrace{r_M((A \setminus K) \cup e)}_{=r_M(A\setminus K)+1} - r_M(A \setminus K) - \underbrace{r_M(e)}_{=1} \\ & = n_{M/e}(A \setminus (K \setminus e)) + r_M(A \setminus K) + 1 - r_M(A \setminus K) - 1 \\ & = n_{M/e}(A \setminus (K \setminus e)). \end{aligned}$$

Thus, Lemma 4.12 **(d)** is proven in Case 1.

Let us now consider Case 2. In this case, we have $e \notin K$. Hence, Lemma 4.12 **(b)** yields

$$\begin{aligned} n_M((A \cup e) \setminus K) &= n_{M/e}(A \setminus (K \setminus e)) + 1 - \underbrace{r_M(e)}_{=1} \\ &= n_{M/e}(A \setminus (K \setminus e)) + 1 - 1 = n_{M/e}(A \setminus (K \setminus e)). \end{aligned}$$

Thus, Lemma 4.12 **(d)** is proven in Case 2.

Hence, Lemma 4.12 **(d)** is proven in each of the two Cases 1 and 2. Thus, Lemma 4.12 **(d)** always holds.

(e) Assume that $K = E$.

From $A \cup e \subseteq E = K$, we obtain $(A \cup e) \setminus K = \varnothing$, so that $n_M\left(\underbrace{(A \cup e) \setminus K}_{=\varnothing}\right) = n_M(\varnothing) = 0$.

On the other hand, $A \subseteq E = K$. Combined with $e \notin A$, this yields $A \subseteq K \setminus e$. Thus, $A \setminus (K \setminus e) = \varnothing$, so that $n_{M/e}\left(\underbrace{A \setminus (K \setminus e)}_{=\varnothing}\right) = n_{M/e}(\varnothing) = 0$. Compared with $n_M((A \cup e) \setminus K) = 0$, this yields $n_M((A \cup e) \setminus K) = n_{M/e}(A \setminus (K \setminus e))$. This proves Lemma 4.12 **(e)**. ☐

Lemma 4.13. Let $M = (E, \mathcal{I})$ be a matroid. Let $e \in E$. Let $A \in \mathcal{P}(E)$ be such that $e \notin A$. Let $K \in \mathcal{P}(E)$.

(a) If e is a loop, then

$$r_M(A \cup K) = r_{M\setminus e}(A \cup (K \setminus e)).$$

(b) If e is a coloop, then

$$r_M(A \cup K) = r_{M\setminus e}(A \cup (K \setminus e)) + [e \in K].$$

(c) If $K = E$ and if e is not a coloop, then

$$r_M(A \cup K) = r_{M\setminus e}(A \cup (K \setminus e)).$$

Proof of Lemma 4.13. We have $e \notin A$, thus $A \subseteq E \setminus e$. Hence, $\underbrace{A}_{\subseteq E\setminus e} \cup \left(\underbrace{K}_{\subseteq E} \setminus e\right) \subseteq (E \setminus e) \cup (E \setminus e) = E \setminus e$. In other words, $A \cup (K \setminus e) \in \mathcal{P}(E \setminus e)$. Hence, Proposition 2.19 **(a)** (applied to $Z = \{e\}$ and $S = A \cup (K \setminus e)$) shows that $r_{M\setminus\{e\}}(A \cup (K \setminus e)) = r_M(A \cup (K \setminus e))$. Since $M \setminus \{e\} = M \setminus e$, this rewrites as $r_{M\setminus e}(A \cup (K \setminus e)) = r_M(A \cup (K \setminus e))$.

(a) Assume that e is a loop.

If $K \setminus e = K$, then

$$r_{M\setminus e}(A \cup (K \setminus e)) = r_M\left(A \cup \underbrace{(K \setminus e)}_{=K}\right) = r_M(A \cup K).$$

Hence, Lemma 4.13 **(a)** holds in the case when $K \setminus e = K$. Thus, for the rest of this proof, we WLOG assume that $K \setminus e \neq K$.

If we had $e \notin K$, then we would have $K \setminus e = K$, which would contradict $K \setminus e \neq K$. Hence, we cannot have $e \notin K$. Thus, $e \in K$. Hence, $(K \setminus e) \cup e = K$.

But Proposition 2.12 **(a)** (applied to $S = A \cup (K \setminus e)$) shows that $r_M(A \cup (K \setminus e) \cup e) = r_M(A \cup (K \setminus e))$. Hence, $r_M(A \cup (K \setminus e)) = r_M\left(A \cup \underbrace{(K \setminus e) \cup e}_{=K}\right) = r_M(A \cup K)$.

Hence,

$$r_{M\setminus e}(A \cup (K \setminus e)) = r_M(A \cup (K \setminus e)) = r_M(A \cup K).$$

This proves Lemma 4.13.

(b) Assume that e is a coloop.

We are in one of the following two cases:

Case 1: We have $e \in K$.

Case 2: We have $e \notin K$.

Let us first consider Case 1. In this case, we have $e \in K$. Thus, $(K \setminus e) \cup e = K$.

But $e \notin A \cup (K \setminus e)$ (since $e \notin A$ and $e \notin K \setminus e$). Hence, Proposition 2.12 **(b)** (applied to $S = A \cup (K \setminus e)$) shows that $r_M(A \cup (K \setminus e) \cup e) = r_M(A \cup (K \setminus e)) + 1$.

Since $A \cup \underbrace{(K \setminus e) \cup e}_{=K} = A \cup K$, this rewrites as $r_M(A \cup K) = r_M(A \cup (K \setminus e)) + 1$. Compared with $\underbrace{r_{M \setminus e}(A \cup (K \setminus e))}_{=r_M(A \cup (K \setminus e))} + \underbrace{[e \in K]}_{\substack{=1 \\ (\text{since } e \in K)}} = r_M(A \cup (K \setminus e)) + 1$, this yields $r_M(A \cup K) = r_{M \setminus e}(A \cup (K \setminus e)) + [e \in K]$. Thus, Lemma 4.13 **(b)** is proven in Case 1.

Let us now consider Case 2. In this case, we have $e \notin K$. Thus, $K \setminus e = K$. Now,

$$r_{M \setminus e}(A \cup (K \setminus e)) + \underbrace{[e \in K]}_{\substack{=0 \\ (\text{since } e \notin K)}} = r_{M \setminus e}(A \cup (K \setminus e)) = r_M\left(A \cup \underbrace{(K \setminus e)}_{=K}\right) = r_M(A \cup K).$$

Thus, Lemma 4.13 **(b)** is proven in Case 2.

Hence, Lemma 4.13 **(b)** is proven in each of the two Cases 1 and 2. Thus, Lemma 4.13 **(b)** always holds.

(c) Assume that $K = E$, and that e is not a coloop.

If we had $r_M(E \setminus e) \neq r_M(E)$, then the element e would be a coloop (by Proposition 2.11 **(c)**), which would contradict the assumption that e is not a coloop. Hence, we cannot have $r_M(E \setminus e) \neq r_M(E)$. Thus, we must have $r_M(E \setminus e) = r_M(E)$.

Now, $A \subseteq E \setminus e$ (since $e \notin A$) and thus $A \cup (E \setminus e) = E \setminus e$. Now,

$$r_{M \setminus e}(A \cup (K \setminus e)) = r_M\left(A \cup \left(\underbrace{K}_{=E} \setminus e\right)\right) = r_M\left(\underbrace{A \cup (E \setminus e)}_{=E \setminus e}\right) = r_M(E \setminus e) = r_M(E).$$

Compared with

$$r_M\left(A \cup \underbrace{K}_{=E}\right) = r_M\left(\underbrace{A \cup E}_{\substack{=E \\ (\text{since } A \subseteq E)}}\right) = r_M(E),$$

this yields $r_M(A \cup K) = r_{M \setminus e}(A \cup (K \setminus e))$. This proves Lemma 4.13 **(c)**. □

Lemma 4.14. Let $M = (E, \mathcal{I})$ be a matroid. Let $e \in E$. Let $A \in \mathcal{P}(E)$ be such that $e \notin A$. Let $K \in \mathcal{P}(E)$. Then,

$$n_M(A \setminus K) = n_{M \setminus e}(A \setminus (K \setminus e)).$$

Proof of Lemma 4.14. We have $e \notin A$, hence $A \subseteq E \setminus e$, thus $A \setminus K \subseteq A \subseteq E \setminus e$. Hence, $A \setminus K \in \mathcal{P}(E \setminus e)$.

For every $S \in \mathcal{P}(E \setminus e)$, we have $n_{M\setminus e}(S) = n_M(S)$ [4]. Applying this to $S = A \setminus K$, we obtain $n_{M\setminus e}(A \setminus K) = n_M(A \setminus K)$.

But $e \notin A$. Hence, $A \setminus (K \setminus e) = A \setminus K$. Thus, $n_{M\setminus e}\left(\underbrace{A \setminus (K \setminus e)}_{=A\setminus K}\right) = n_{M\setminus e}(A \setminus K) = n_M(A \setminus K)$. This proves Lemma 4.14. ☐

Lemma 4.15. Let $\mathbf{M} = (M, \mathbf{E})$ be a filtered matroid, with $M = (E, \mathcal{I})$. Write the list $\mathbf{E}$ as $(E_0, E_1, \ldots, E_m)$. Let $e \in E$. Assume that e is a loop.

(a) Let $k \in \{1, 2, \ldots, m\}$ be such that $e \in E_k \setminus E_{k-1}$. Then,

$$\sum_{\substack{A\subseteq E;\\ e\in A}} \left(\prod_{i=1}^{m} (x_i - 1)^{r_M(A\cup E_i) - r_M(A\cup E_{i-1})}\right) \left(\prod_{i=1}^{m} (y_i - 1)^{n_M(A\setminus E_{i-1}) - n_M(A\setminus E_i)}\right) = (y_k - 1)\, T_{\mathbf{M}/e}.$$

(b) We have

$$\sum_{\substack{A\subseteq E;\\ e\notin A}} \left(\prod_{i=1}^{m} (x_i - 1)^{r_M(A\cup E_i) - r_M(A\cup E_{i-1})}\right) \left(\prod_{i=1}^{m} (y_i - 1)^{n_M(A\setminus E_{i-1}) - n_M(A\setminus E_i)}\right) = T_{\mathbf{M}/e}.$$

Proof of Lemma 4.15. The definition of $\mathbf{E} \setminus e$ shows that $\mathbf{E} \setminus e = (E_0 \setminus e, E_1 \setminus e, \ldots, E_m \setminus e)$. The definition of $\mathbf{M}/e$ shows that $\mathbf{M}/e = (M/e, \mathbf{E} \setminus e)$. The matroid M/e has

[4] *Proof.* Let $S \in \mathcal{P}(E \setminus e)$. Thus, $S \in \mathcal{P}(E \setminus e) = \mathcal{P}(E \setminus \{e\})$. Hence, Proposition 2.19 **(a)** (applied to $Z = \{e\}$) shows that $r_{M\setminus\{e\}}(S) = r_M(S)$. Since $M \setminus \{e\} = M \setminus e$, this rewrites as $r_{M\setminus e}(S) = r_M(S)$. The definition of $n_M(S)$ yields $n_M(S) = |S| - r_M(S)$. But the definition of $n_{M\setminus e}(S)$ yields

$$n_{M\setminus e}(S) = |S| - \underbrace{r_{M\setminus e}(S)}_{=r_M(S)} = |S| - r_M(S) = n_M(S),$$

qed.

ground set $E \setminus e$. Hence, the definition of $T_{\mathbf{M}/e}$ yields

$$T_{\mathbf{M}/e} = \underbrace{\sum_{A \subseteq E \setminus e}}_{= \sum_{A \subseteq E;\ e \notin A}} \left(\prod_{i=1}^{m} (x_i - 1)^{r_{M/e}(A \cup (E_i \setminus e)) - r_{M/e}(A \cup (E_{i-1} \setminus e))} \right) \left(\prod_{i=1}^{m} (y_i - 1)^{n_{M/e}(A \setminus (E_{i-1} \setminus e)) - n_{M/e}(A \setminus (E_i \setminus e))} \right)$$

$$= \sum_{\substack{A \subseteq E; \\ e \notin A}} \left(\prod_{i=1}^{m} (x_i - 1)^{r_{M/e}(A \cup (E_i \setminus e)) - r_{M/e}(A \cup (E_{i-1} \setminus e))} \right) \left(\prod_{i=1}^{m} (y_i - 1)^{n_{M/e}(A \setminus (E_{i-1} \setminus e)) - n_{M/e}(A \setminus (E_i \setminus e))} \right). \tag{17}$$

Since $(M, \mathbf{E})$ is a filtered matroid, we have $\varnothing = E_0 \subseteq E_1 \subseteq \cdots \subseteq E_m = E$. Thus, $E = \bigsqcup_{i=1}^{m} (E_i \setminus E_{i-1})$. Hence, $e \in E = \bigsqcup_{i=1}^{m} (E_i \setminus E_{i-1})$. In other words, there exists exactly one $i \in \{1, 2, \ldots, m\}$ satisfying $e \in E_i \setminus E_{i-1}$. This i must be k (since $e \in E_k \setminus E_{k-1}$). Thus, for any $i \in \{1, 2, \ldots, m\}$, we have $e \in E_i \setminus E_{i-1}$ if and only if $i = k$. In other words, for any $i \in \{1, 2, \ldots, m\}$, we have

$$[e \in E_i \setminus E_{i-1}] = [i = k]. \tag{18}$$

(a) Let $A \in \mathcal{P}(E)$ be such that $e \notin A$.
For every $i \in \{1, 2, \ldots, m\}$, we have

$$r_M((A \cup e) \cup E_i) - r_M((A \cup e) \cup E_{i-1}) = r_{M/e}(A \cup (E_i \setminus e)) - r_{M/e}(A \cup (E_{i-1} \setminus e)) \tag{19}$$

[5].

For every $i \in \{1, 2, \ldots, m\}$, we have

$$n_M((A \cup e) \setminus E_{i-1}) - n_M((A \cup e) \setminus E_i) = n_{M/e}(A \setminus (E_{i-1} \setminus e)) - n_{M/e}(A \setminus (E_i \setminus e)) + [i = k] \tag{21}$$

[5]*Proof of (19):* Let $i \in \{1, 2, \ldots, m\}$. Applying Lemma 4.11 to $K = E_i$, we obtain

$$r_M((A \cup e) \cup E_i) = r_{M/e}(A \cup (E_i \setminus e)) + r_M(e). \tag{20}$$

Applying Lemma 4.11 to $K = E_{i-1}$, we obtain

$$r_M((A \cup e) \cup E_{i-1}) = r_{M/e}(A \cup (E_{i-1} \setminus e)) + r_M(e).$$

Subtracting this equality from (20), we obtain

$$r_M((A \cup e) \cup E_i) - r_M((A \cup e) \cup E_{i-1}) = (r_{M/e}(A \cup (E_i \setminus e)) + r_M(e)) - (r_{M/e}(A \cup (E_{i-1} \setminus e)) + r_M(e)) = r_{M/e}(A \cup (E_i \setminus e)) - r_{M/e}(A \cup (E_{i-1} \setminus e)).$$

This proves (19).

[6]. Thus,

$$\prod_{i=1}^{m} \underbrace{(y_i - 1)^{n_M((A\cup e)\setminus E_{i-1}) - n_M((A\cup e)\setminus E_i)}}_{\substack{=(y_i-1)^{n_{M/e}(A\setminus(E_{i-1}\setminus e)) - n_{M/e}(A\setminus(E_i\setminus e)) + [i=k]} \\ \text{(by (21))}}}$$
$$= \prod_{i=1}^{m} \underbrace{(y_i - 1)^{n_{M/e}(A\setminus(E_{i-1}\setminus e)) - n_{M/e}(A\setminus(E_i\setminus e)) + [i=k]}}_{=(y_i-1)^{n_{M/e}(A\setminus(E_{i-1}\setminus e)) - n_{M/e}(A\setminus(E_i\setminus e))}(y_i-1)^{[i=k]}}$$
$$= \prod_{i=1}^{m} \left((y_i - 1)^{n_{M/e}(A\setminus(E_{i-1}\setminus e)) - n_{M/e}(A\setminus(E_i\setminus e))} (y_i - 1)^{[i=k]} \right)$$
$$= \left(\prod_{i=1}^{m} (y_i - 1)^{n_{M/e}(A\setminus(E_{i-1}\setminus e)) - n_{M/e}(A\setminus(E_i\setminus e))} \right) \underbrace{\left(\prod_{i=1}^{m} (y_i - 1)^{[i=k]} \right)}_{\substack{=y_k - 1 \\ \text{(since all factors of this product are 1} \\ \text{except of the factor for } i=k)}}$$
$$= \left(\prod_{i=1}^{m} (y_i - 1)^{n_{M/e}(A\setminus(E_{i-1}\setminus e)) - n_{M/e}(A\setminus(E_i\setminus e))} \right) (y_k - 1). \tag{23}$$

Now, the map

$$\{A \subseteq E \mid e \notin A\} \to \{A \subseteq E \mid e \in A\}, \qquad A \mapsto A \cup e$$

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[6]*Proof of (21):* Let $i \in \{1, 2, \ldots, m\}$. Thus, $E_{i-1} \subseteq E_i$ (since $E_0 \subseteq E_1 \subseteq \cdots \subseteq E_m$).

Applying Lemma 4.12 **(c)** to $K = E_i$, we obtain

$$n_M((A \cup e) \setminus E_i) = n_{M/e}(A \setminus (E_i \setminus e)) + [e \notin E_i]. \tag{22}$$

Applying Lemma 4.12 **(c)** to $K = E_{i-1}$, we obtain

$$n_M((A \cup e) \setminus E_{i-1}) = n_{M/e}(A \setminus (E_{i-1} \setminus e)) + [e \notin E_{i-1}].$$

Subtracting (20) from this equality, we obtain

$$\begin{aligned} & n_M((A \cup e) \setminus E_{i-1}) - n_M((A \cup e) \setminus E_i) \\ &= (n_{M/e}(A \setminus (E_{i-1} \setminus e)) + [e \notin E_{i-1}]) - (n_{M/e}(A \setminus (E_i \setminus e)) + [e \notin E_i]) \\ &= n_{M/e}(A \setminus (E_{i-1} \setminus e)) - n_{M/e}(A \setminus (E_i \setminus e)) + \underbrace{[e \notin E_{i-1}] - [e \notin E_i]}_{\substack{=[e \in E_i \setminus E_{i-1}] \\ \text{(since } E_{i-1} \subseteq E_i)}} \\ &= n_{M/e}(A \setminus (E_{i-1} \setminus e)) - n_{M/e}(A \setminus (E_i \setminus e)) + \underbrace{[e \in E_i \setminus E_{i-1}]}_{\substack{=[i=k] \\ \text{(by (18))}}} \\ &= n_{M/e}(A \setminus (E_{i-1} \setminus e)) - n_{M/e}(A \setminus (E_i \setminus e)) + [i = k]. \end{aligned}$$

This proves (21).

is a bijection. Hence, we can substitute $A \cup e$ for A in the sum

$$\sum_{\substack{A \subseteq E;\\ e \in A}} \left(\prod_{i=1}^{m} (x_i - 1)^{r_M(A \cup E_i) - r_M(A \cup E_{i-1})} \right) \left(\prod_{i=1}^{m} (y_i - 1)^{n_M(A \setminus E_{i-1}) - n_M(A \setminus E_i)} \right).$$

We thus obtain

$$\begin{aligned}
&\sum_{\substack{A \subseteq E;\\ e \in A}} \left(\prod_{i=1}^{m} (x_i - 1)^{r_M(A \cup E_i) - r_M(A \cup E_{i-1})} \right) \left(\prod_{i=1}^{m} (y_i - 1)^{n_M(A \setminus E_{i-1}) - n_M(A \setminus E_i)} \right) \\
&= \sum_{\substack{A \subseteq E;\\ e \notin A}} \left(\prod_{i=1}^{m} \underbrace{(x_i - 1)^{r_M((A \cup e) \cup E_i) - r_M((A \cup e) \cup E_{i-1})}}_{\substack{= (x_i - 1)^{r_{M/e}(A \cup (E_i \setminus e)) - r_{M/e}(A \cup (E_{i-1} \setminus e))} \\ \text{(by (19))}}} \right) \\
&\qquad \underbrace{\left(\prod_{i=1}^{m} (y_i - 1)^{n_M((A \cup e) \setminus E_{i-1}) - n_M((A \cup e) \setminus E_i)} \right)}_{\substack{= \left(\prod_{i=1}^{m} (y_i - 1)^{n_{M/e}(A \setminus (E_{i-1} \setminus e)) - n_{M/e}(A \setminus (E_i \setminus e))} \right) (y_k - 1) \\ \text{(by (23))}}} \\
&= \sum_{\substack{A \subseteq E;\\ e \notin A}} \left(\prod_{i=1}^{m} (x_i - 1)^{r_{M/e}(A \cup (E_i \setminus e)) - r_{M/e}(A \cup (E_{i-1} \setminus e))} \right) \\
&\qquad \left(\prod_{i=1}^{m} (y_i - 1)^{n_{M/e}(A \setminus (E_{i-1} \setminus e)) - n_{M/e}(A \setminus (E_i \setminus e))} \right) (y_k - 1) \\
&= (y_k - 1) \sum_{\substack{A \subseteq E;\\ e \notin A}} \left(\prod_{i=1}^{m} (x_i - 1)^{r_{M/e}(A \cup (E_i \setminus e)) - r_{M/e}(A \cup (E_{i-1} \setminus e))} \right) \\
&\qquad \left(\prod_{i=1}^{m} (y_i - 1)^{n_{M/e}(A \setminus (E_{i-1} \setminus e)) - n_{M/e}(A \setminus (E_i \setminus e))} \right) \\
&= (y_k - 1)\, T_{\mathbf{M}/e}
\end{aligned}$$

(this follows by multiplying both sides of the equality (17) by $y_k - 1$). This proves Lemma 4.15 **(a)**.

(b) Let $A \in \mathcal{P}(E)$ be such that $e \notin A$.

For every $i \in \{1, 2, \ldots, m\}$, we have

$$\begin{aligned}
&r_M(A \cup E_i) - r_M(A \cup E_{i-1}) \\
&= r_{M \setminus e}(A \cup (E_i \setminus e)) - r_{M \setminus e}(A \cup (E_{i-1} \setminus e))
\end{aligned} \tag{24}$$

[7].

For every $i \in \{1, 2, \ldots, m\}$, we have

$$n_M(A \setminus E_{i-1}) - n_M(A \setminus E_i) = n_{M\setminus e}(A \setminus (E_{i-1} \setminus e)) - n_{M\setminus e}(A \setminus (E_i \setminus e)) \qquad (26)$$

[8].

But Proposition 2.23 **(a)** shows that $M/e = M \setminus e$. Now,

$$\sum_{\substack{A\subseteq E;\\ e\notin A}} \left(\prod_{i=1}^{m} \underbrace{(x_i - 1)^{r_M(A\cup E_i) - r_M(A\cup E_{i-1})}}_{\substack{=(x_i-1)^{r_{M\setminus e}(A\cup(E_i\setminus e)) - r_{M\setminus e}(A\cup(E_{i-1}\setminus e))}\\ \text{(by (24))}}} \right) \left(\prod_{i=1}^{m} \underbrace{(y_i - 1)^{n_M(A\setminus E_{i-1}) - n_M(A\setminus E_i)}}_{\substack{=(y_i-1)^{n_{M\setminus e}(A\setminus(E_{i-1}\setminus e)) - n_{M\setminus e}(A\setminus(E_i\setminus e))}\\ \text{(by (26))}}} \right)$$

$$= \sum_{\substack{A\subseteq E;\\ e\notin A}} \left(\prod_{i=1}^{m} (x_i - 1)^{r_{M\setminus e}(A\cup(E_i\setminus e)) - r_{M\setminus e}(A\cup(E_{i-1}\setminus e))} \right) \left(\prod_{i=1}^{m} (y_i - 1)^{n_{M\setminus e}(A\setminus(E_{i-1}\setminus e)) - n_{M\setminus e}(A\setminus(E_i\setminus e))} \right)$$

$$= \sum_{\substack{A\subseteq E;\\ e\notin A}} \left(\prod_{i=1}^{m} (x_i - 1)^{r_{M/e}(A\cup(E_i\setminus e)) - r_{M/e}(A\cup(E_{i-1}\setminus e))} \right) \left(\prod_{i=1}^{m} (y_i - 1)^{n_{M/e}(A\setminus(E_{i-1}\setminus e)) - n_{M/e}(A\setminus(E_i\setminus e))} \right)$$

$$(\text{since } M \setminus e = M/e)$$

$$= T_{\mathbf{M}/e} \qquad (\text{by (17)}).$$

[7] *Proof of (24):* Let $i \in \{1, 2, \ldots, m\}$. Applying Lemma 4.13 **(a)** to $K = E_i$, we obtain

$$r_M(A \cup E_i) = r_{M\setminus e}(A \cup (E_i \setminus e)). \qquad (25)$$

Applying Lemma 4.13 **(a)** to $K = E_{i-1}$, we obtain

$$r_M(A \cup E_{i-1}) = r_{M\setminus e}(A \cup (E_{i-1} \setminus e)).$$

Subtracting this equality from (25), we obtain

$$r_M(A \cup E_i) - r_M(A \cup E_{i-1}) = r_{M\setminus e}(A \cup (E_i \setminus e)) - r_{M\setminus e}(A \cup (E_{i-1} \setminus e)).$$

This proves (24).

[8] *Proof of (26):* Let $i \in \{1, 2, \ldots, m\}$.

Applying Lemma 4.14 to $K = E_i$, we obtain

$$n_M(A \setminus E_i) = n_{M\setminus e}(A \setminus (E_i \setminus e)). \qquad (27)$$

Applying Lemma 4.14 to $K = E_{i-1}$, we obtain

$$n_M(A \setminus E_{i-1}) = n_{M\setminus e}(A \setminus (E_{i-1} \setminus e)).$$

Subtracting (25) from this equality, we obtain

$$n_M(A \setminus E_{i-1}) - n_M(A \setminus E_i) = n_{M\setminus e}(A \setminus (E_{i-1} \setminus e)) - n_{M\setminus e}(A \setminus (E_i \setminus e)).$$

This proves (26).

This proves Lemma 4.15 **(b)**. □

Lemma 4.16. Let $\mathbf{M} = (M, \mathbf{E})$ be a filtered matroid, with $M = (E, \mathcal{I})$. Write the list $\mathbf{E}$ as $(E_0, E_1, \ldots, E_m)$. Let $e \in E$. Assume that e is a coloop.

(a) We have

$$\sum_{\substack{A \subseteq E;\\ e \in A}} \left(\prod_{i=1}^{m} (x_i - 1)^{r_M(A \cup E_i) - r_M(A \cup E_{i-1})} \right) \left(\prod_{i=1}^{m} (y_i - 1)^{n_M(A \setminus E_{i-1}) - n_M(A \setminus E_i)} \right)$$
$$= T_{\mathbf{M}/e}.$$

(b) Let $k \in \{1, 2, \ldots, m\}$ be such that $e \in E_k \setminus E_{k-1}$. Then,

$$\sum_{\substack{A \subseteq E;\\ e \notin A}} \left(\prod_{i=1}^{m} (x_i - 1)^{r_M(A \cup E_i) - r_M(A \cup E_{i-1})} \right) \left(\prod_{i=1}^{m} (y_i - 1)^{n_M(A \setminus E_{i-1}) - n_M(A \setminus E_i)} \right)$$
$$= (x_k - 1)\, T_{\mathbf{M}/e}.$$

Proof of Lemma 4.16. The definition of $\mathbf{E} \setminus e$ shows that $\mathbf{E} \setminus e = (E_0 \setminus e, E_1 \setminus e, \ldots, E_m \setminus e)$. The definition of $\mathbf{M}/e$ shows that $\mathbf{M}/e = (M/e, \mathbf{E} \setminus e)$. The matroid M/e has ground set $E \setminus e$. Hence, the definition of $T_{\mathbf{M}/e}$ yields

$$T_{\mathbf{M}/e}$$
$$= \underbrace{\sum_{A \subseteq E \setminus e}}_{= \sum_{\substack{A \subseteq E;\\ e \notin A}}} \left(\prod_{i=1}^{m} (x_i - 1)^{r_{M/e}(A \cup (E_i \setminus e)) - r_{M/e}(A \cup (E_{i-1} \setminus e))} \right) \left(\prod_{i=1}^{m} (y_i - 1)^{n_{M/e}(A \setminus (E_{i-1} \setminus e)) - n_{M/e}(A \setminus (E_i \setminus e))} \right)$$
$$= \sum_{\substack{A \subseteq E;\\ e \notin A}} \left(\prod_{i=1}^{m} (x_i - 1)^{r_{M/e}(A \cup (E_i \setminus e)) - r_{M/e}(A \cup (E_{i-1} \setminus e))} \right) \left(\prod_{i=1}^{m} (y_i - 1)^{n_{M/e}(A \setminus (E_{i-1} \setminus e)) - n_{M/e}(A \setminus (E_i \setminus e))} \right). \tag{28}$$

Since $(M, \mathbf{E})$ is a filtered matroid, we have $\varnothing = E_0 \subseteq E_1 \subseteq \cdots \subseteq E_m = E$. Thus, $E = \bigsqcup_{i=1}^{m} (E_i \setminus E_{i-1})$. Hence, $e \in E = \bigsqcup_{i=1}^{m} (E_i \setminus E_{i-1})$. In other words, there exists exactly one $i \in \{1, 2, \ldots, m\}$ satisfying $e \in E_i \setminus E_{i-1}$. This i must be k (since $e \in E_k \setminus E_{k-1}$). Thus, for any $i \in \{1, 2, \ldots, m\}$, we have $e \in E_i \setminus E_{i-1}$ if and only if $i = k$. In other words, for any $i \in \{1, 2, \ldots, m\}$, we have

$$[e \in E_i \setminus E_{i-1}] = [i = k]. \tag{29}$$

(a) Let $A \in \mathcal{P}(E)$ be such that $e \notin A$.

For every $i \in \{1, 2, \ldots, m\}$, we have

$$r_M((A \cup e) \cup E_i) - r_M((A \cup e) \cup E_{i-1}) \\ = r_{M/e}(A \cup (E_i \setminus e)) - r_{M/e}(A \cup (E_{i-1} \setminus e)) \qquad (30)$$

[9].

For every $i \in \{1, 2, \ldots, m\}$, we have

$$n_M((A \cup e) \setminus E_{i-1}) - n_M((A \cup e) \setminus E_i) \\ = n_{M/e}(A \setminus (E_{i-1} \setminus e)) - n_{M/e}(A \setminus (E_i \setminus e)) \qquad (32)$$

[10].

Now, the map

$$\{A \subseteq E \mid e \notin A\} \to \{A \subseteq E \mid e \in A\}, \qquad A \mapsto A \cup e$$

is a bijection. Hence, we can substitute $A \cup e$ for A in the sum

$$\sum_{\substack{A \subseteq E; \\ e \in A}} \left(\prod_{i=1}^{m} (x_i - 1)^{r_M(A \cup E_i) - r_M(A \cup E_{i-1})} \right) \left(\prod_{i=1}^{m} (y_i - 1)^{n_M(A \setminus E_{i-1}) - n_M(A \setminus E_i)} \right).$$

[9] *Proof of (30):* Let $i \in \{1, 2, \ldots, m\}$. Applying Lemma 4.11 to $K = E_i$, we obtain

$$r_M((A \cup e) \cup E_i) = r_{M/e}(A \cup (E_i \setminus e)) + r_M(e). \qquad (31)$$

Applying Lemma 4.11 to $K = E_{i-1}$, we obtain

$$r_M((A \cup e) \cup E_{i-1}) = r_{M/e}(A \cup (E_{i-1} \setminus e)) + r_M(e).$$

Subtracting this equality from (31), we obtain

$$r_M((A \cup e) \cup E_i) - r_M((A \cup e) \cup E_{i-1}) \\ = (r_{M/e}(A \cup (E_i \setminus e)) + r_M(e)) - (r_{M/e}(A \cup (E_{i-1} \setminus e)) + r_M(e)) \\ = r_{M/e}(A \cup (E_i \setminus e)) - r_{M/e}(A \cup (E_{i-1} \setminus e)).$$

This proves (30).

[10] *Proof of (32):* Let $i \in \{1, 2, \ldots, m\}$.

Applying Lemma 4.12 **(d)** to $K = E_i$, we obtain

$$n_M((A \cup e) \setminus E_i) = n_{M/e}(A \setminus (E_i \setminus e)). \qquad (33)$$

Applying Lemma 4.12 **(d)** to $K = E_{i-1}$, we obtain

$$n_M((A \cup e) \setminus E_{i-1}) = n_{M/e}(A \setminus (E_{i-1} \setminus e)).$$

Subtracting (31) from this equality, we obtain

$$n_M((A \cup e) \setminus E_{i-1}) - n_M((A \cup e) \setminus E_i) = n_{M/e}(A \setminus (E_{i-1} \setminus e)) - n_{M/e}(A \setminus (E_i \setminus e)).$$

This proves (32).

We thus obtain

$$\sum_{\substack{A\subseteq E;\\ e\in A}} \left(\prod_{i=1}^{m} (x_i - 1)^{r_M(A\cup E_i) - r_M(A\cup E_{i-1})}\right) \left(\prod_{i=1}^{m} (y_i - 1)^{n_M(A\setminus E_{i-1}) - n_M(A\setminus E_i)}\right)$$

$$= \sum_{\substack{A\subseteq E;\\ e\notin A}} \left(\prod_{i=1}^{m} \underbrace{(x_i - 1)^{r_M((A\cup e)\cup E_i) - r_M((A\cup e)\cup E_{i-1})}}_{\substack{=(x_i-1)^{r_{M/e}(A\cup(E_i\setminus e)) - r_{M/e}(A\cup(E_{i-1}\setminus e))} \\ \text{(by (30))}}}\right)$$

$$\left(\prod_{i=1}^{m} \underbrace{(y_i - 1)^{n_M((A\cup e)\setminus E_{i-1}) - n_M((A\cup e)\setminus E_i)}}_{\substack{=(y_i-1)^{n_{M/e}(A\setminus(E_{i-1}\setminus e)) - n_{M/e}(A\setminus(E_i\setminus e))} \\ \text{(by (32))}}}\right)$$

$$= \sum_{\substack{A\subseteq E;\\ e\notin A}} \left(\prod_{i=1}^{m} (x_i - 1)^{r_{M/e}(A\cup(E_i\setminus e)) - r_{M/e}(A\cup(E_{i-1}\setminus e))}\right)$$

$$\left(\prod_{i=1}^{m} (y_i - 1)^{n_{M/e}(A\setminus(E_{i-1}\setminus e)) - n_{M/e}(A\setminus(E_i\setminus e))}\right)$$

$$= \sum_{\substack{A\subseteq E;\\ e\notin A}} \left(\prod_{i=1}^{m} (x_i - 1)^{r_{M/e}(A\cup(E_i\setminus e)) - r_{M/e}(A\cup(E_{i-1}\setminus e))}\right) \left(\prod_{i=1}^{m} (y_i - 1)^{n_{M/e}(A\setminus(E_{i-1}\setminus e)) - n_{M/e}(A\setminus(E_i\setminus e))}\right)$$

$$= T_{\mathbf{M}/e} \qquad \text{(by (28))}$$

This proves Lemma 4.16 **(a)**.

(b) Let $A \in \mathcal{P}(E)$ be such that $e \notin A$.

For every $i \in \{1, 2, \ldots, m\}$, we have

$$r_M(A \cup E_i) - r_M(A \cup E_{i-1}) \\ = r_{M\setminus e}(A \cup (E_i \setminus e)) - r_{M\setminus e}(A \cup (E_{i-1} \setminus e)) + [i = k] \tag{34}$$

[11]. Thus,

$$\prod_{i=1}^{m} \underbrace{(x_i-1)^{n_M((A\cup e)\setminus E_{i-1})-n_M((A\cup e)\setminus E_i)}}_{\substack{=(x_i-1)^{r_{M\setminus e}(A\cup(E_i\setminus e))-r_{M\setminus e}(A\cup(E_{i-1}\setminus e))+[i=k]} \\ \text{(by (34))}}}$$
$$= \prod_{i=1}^{m} \underbrace{(x_i-1)^{r_{M\setminus e}(A\cup(E_i\setminus e))-r_{M\setminus e}(A\cup(E_{i-1}\setminus e))+[i=k]}}_{=(x_i-1)^{r_{M\setminus e}(A\cup(E_i\setminus e))-r_{M\setminus e}(A\cup(E_{i-1}\setminus e))}(x_i-1)^{[i=k]}}$$
$$= \prod_{i=1}^{m} \left((x_i-1)^{r_{M\setminus e}(A\cup(E_i\setminus e))-r_{M\setminus e}(A\cup(E_{i-1}\setminus e))} (x_i-1)^{[i=k]} \right)$$
$$= \left(\prod_{i=1}^{m} (x_i-1)^{r_{M\setminus e}(A\cup(E_i\setminus e))-r_{M\setminus e}(A\cup(E_{i-1}\setminus e))} \right) \underbrace{\left(\prod_{i=1}^{m} (x_i-1)^{[i=k]} \right)}_{\substack{=x_k-1 \\ \text{(since all factors of this product are 1} \\ \text{except of the factor for } i=k)}}$$
$$= \left(\prod_{i=1}^{m} (x_i-1)^{r_{M\setminus e}(A\cup(E_i\setminus e))-r_{M\setminus e}(A\cup(E_{i-1}\setminus e))} \right) (x_k-1). \tag{36}$$

For every $i \in \{1, 2, \ldots, m\}$, we have

$$n_M(A \setminus E_{i-1}) - n_M(A \setminus E_i) = n_{M\setminus e}(A \setminus (E_{i-1} \setminus e)) - n_{M\setminus e}(A \setminus (E_i \setminus e)) \tag{37}$$

[11] *Proof of (34):* Let $i \in \{1, 2, \ldots, m\}$. Thus, $E_{i-1} \subseteq E_i$ (since $E_0 \subseteq E_1 \subseteq \cdots \subseteq E_m$).

Applying Lemma 4.13 **(b)** to $K = E_i$, we obtain

$$r_M(A \cup E_i) = r_{M\setminus e}(A \cup (E_i \setminus e)) + [e \in E_i]. \tag{35}$$

Applying Lemma 4.13 **(a)** to $K = E_{i-1}$, we obtain

$$r_M(A \cup E_{i-1}) = r_{M\setminus e}(A \cup (E_{i-1} \setminus e)) + [e \in E_{i-1}].$$

Subtracting this equality from (35), we obtain

$$r_M(A \cup E_i) - r_M(A \cup E_{i-1})$$
$$= \left(r_{M\setminus e}(A \cup (E_i \setminus e)) + [e \in E_i] \right) - \left(r_{M\setminus e}(A \cup (E_{i-1} \setminus e)) + [e \in E_{i-1}] \right)$$
$$= r_{M\setminus e}(A \cup (E_i \setminus e)) - r_{M\setminus e}(A \cup (E_{i-1} \setminus e)) + \underbrace{[e \in E_i] - [e \in E_{i-1}]}_{\substack{=[e\in E_i\setminus E_{i-1}] \\ \text{(since } E_{i-1}\subseteq E_i)}}$$
$$= r_{M\setminus e}(A \cup (E_i \setminus e)) - r_{M\setminus e}(A \cup (E_{i-1} \setminus e)) + \underbrace{[e \in E_i \setminus E_{i-1}]}_{\substack{=[i=k] \\ \text{(by (29))}}}$$
$$= r_{M\setminus e}(A \cup (E_i \setminus e)) - r_{M\setminus e}(A \cup (E_{i-1} \setminus e)) + [i = k].$$

This proves (34).

[12].

But Proposition 2.23 **(a)** shows that $M/e = M \setminus e$. Now,

$$\sum_{\substack{A\subseteq E;\\ e\notin A}} \underbrace{\left(\prod_{i=1}^{m}(x_i-1)^{r_M(A\cup E_i)-r_M(A\cup E_{i-1})}\right)}_{\substack{=\left(\prod_{i=1}^{m}(x_i-1)^{r_{M\setminus e}(A\cup(E_i\setminus e))-r_{M\setminus e}(A\cup(E_{i-1}\setminus e))}\right)(x_k-1)\\ \text{(by (36))}}} \left(\prod_{i=1}^{m}\underbrace{(y_i-1)^{n_M(A\setminus E_{i-1})-n_M(A\setminus E_i)}}_{\substack{=(y_i-1)^{n_{M\setminus e}(A\setminus(E_{i-1}\setminus e))-n_{M\setminus e}(A\setminus(E_i\setminus e))}\\ \text{(by (37))}}}\right)$$

$$= \sum_{\substack{A\subseteq E;\\ e\notin A}} \left(\prod_{i=1}^{m}(x_i-1)^{r_{M\setminus e}(A\cup(E_i\setminus e))-r_{M\setminus e}(A\cup(E_{i-1}\setminus e))}\right)$$
$$(x_k-1)\left(\prod_{i=1}^{m}(y_i-1)^{n_{M\setminus e}(A\setminus(E_{i-1}\setminus e))-n_{M\setminus e}(A\setminus(E_i\setminus e))}\right)$$
$$= (x_k-1)\sum_{\substack{A\subseteq E;\\ e\notin A}} \left(\prod_{i=1}^{m}(x_i-1)^{r_{M\setminus e}(A\cup(E_i\setminus e))-r_{M\setminus e}(A\cup(E_{i-1}\setminus e))}\right)$$
$$\left(\prod_{i=1}^{m}(y_i-1)^{n_{M\setminus e}(A\setminus(E_{i-1}\setminus e))-n_{M\setminus e}(A\setminus(E_i\setminus e))}\right)$$
$$= (x_k-1)\sum_{\substack{A\subseteq E;\\ e\notin A}} \left(\prod_{i=1}^{m}(x_i-1)^{r_{M/e}(A\cup(E_i\setminus e))-r_{M/e}(A\cup(E_{i-1}\setminus e))}\right)$$
$$\left(\prod_{i=1}^{m}(y_i-1)^{n_{M/e}(A\setminus(E_{i-1}\setminus e))-n_{M/e}(A\setminus(E_i\setminus e))}\right)$$
$$(\text{since } M\setminus e = M/e)$$
$$= (x_k-1)\,T_{\mathbf{M}/e}$$

(this follows by multiplying both sides of the equality (28) by $x_k - 1$). This proves Lemma 4.16 **(b)**. □

[12]*Proof of (37):* Let $i \in \{1, 2, \ldots, m\}$.

Applying Lemma 4.14 to $K = E_i$, we obtain

$$n_M(A \setminus E_i) = n_{M\setminus e}(A \setminus (E_i \setminus e)). \tag{38}$$

Applying Lemma 4.14 to $K = E_{i-1}$, we obtain

$$n_M(A \setminus E_{i-1}) = n_{M\setminus e}(A \setminus (E_{i-1} \setminus e)).$$

Subtracting (35) from this equality, we obtain

$$n_M(A \setminus E_{i-1}) - n_M(A \setminus E_i) = n_{M\setminus e}(A \setminus (E_{i-1} \setminus e)) - n_{M\setminus e}(A \setminus (E_i \setminus e)).$$

This proves (37).

Lemma 4.17. Let $\mathbf{M} = (M, \mathbf{E})$ be a filtered matroid, with $M = (E, \mathcal{I})$. Write the list $\mathbf{E}$ as $(E_0, E_1, \ldots, E_m)$. Let $e \in E$ be such that $e \in E_m \setminus E_{m-1}$. Assume that e is neither a loop nor a coloop.

(a) We have

$$\sum_{\substack{A\subseteq E;\\ e\in A}} \left(\prod_{i=1}^{m} (x_i - 1)^{r_M(A\cup E_i) - r_M(A\cup E_{i-1})} \right) \left(\prod_{i=1}^{m} (y_i - 1)^{n_M(A\setminus E_{i-1}) - n_M(A\setminus E_i)} \right)$$
$$= T_{\mathbf{M}/e}.$$

(b) We have

$$\sum_{\substack{A\subseteq E;\\ e\notin A}} \left(\prod_{i=1}^{m} (x_i - 1)^{r_M(A\cup E_i) - r_M(A\cup E_{i-1})} \right) \left(\prod_{i=1}^{m} (y_i - 1)^{n_M(A\setminus E_{i-1}) - n_M(A\setminus E_i)} \right)$$
$$= T_{\mathbf{M}\setminus e}.$$

Proof of Lemma 4.17. Since $(M, \mathbf{E})$ is a filtered matroid, we have $\varnothing = E_0 \subseteq E_1 \subseteq \cdots \subseteq E_m = E$.

But from $e \in E_m \setminus E_{m-1} \notin E_{m-1}$, we obtain

$$e \notin E_j \qquad \text{for every } j \in \{0, 1, \ldots, m-1\} \tag{39}$$

[13].

Proposition 2.11 **(a)** shows that the element e is a loop (of M) if and only if $r_M(e) = 0$. Thus, we don't have $r_M(e) = 0$ (since e is not a loop). In other words, we have $r_M(e) \neq 0$. Consequently, $r_M(e) = 1$ [14].

The definition of $\mathbf{E} \setminus e$ shows that $\mathbf{E} \setminus e = (E_0 \setminus e, E_1 \setminus e, \ldots, E_m \setminus e)$. The definition of $\mathbf{M}/e$ shows that $\mathbf{M}/e = (M/e, \mathbf{E} \setminus e)$. The matroid M/e has ground set $E \setminus e$. Hence, the definition of $T_{\mathbf{M}/e}$ yields

[13] *Proof of (39):* Let $j \in \{0, 1, \ldots, m-1\}$. Thus, $j \leq m-1$, so that $E_j \subseteq E_{m-1}$ (since $E_0 \subseteq E_1 \subseteq \cdots \subseteq E_m$). Thus, $e \notin E_j$ (since $e \notin E_{m-1}$). This proves (39).

[14] *Proof.* Applying (2) to $S = \{e\}$, we obtain $r_M(\{e\}) \leq |\{e\}| = 1$. Thus, $r_M(e) = r_M(\{e\}) \leq 1$. Combining this with $r_M(e) \neq 0$, we obtain $r_M(e) = 1$, qed.

$$T_{\mathbf{M}/e} = \underbrace{\sum_{A \subseteq E \setminus e}}_{= \sum_{A \subseteq E;\, e \notin A}} \left(\prod_{i=1}^{m} (x_i - 1)^{r_{M/e}(A \cup (E_i \setminus e)) - r_{M/e}(A \cup (E_{i-1} \setminus e))} \right) \left(\prod_{i=1}^{m} (y_i - 1)^{n_{M/e}(A \setminus (E_{i-1} \setminus e)) - n_{M/e}(A \setminus (E_i \setminus e))} \right)$$
$$= \sum_{\substack{A \subseteq E;\\ e \notin A}} \left(\prod_{i=1}^{m} (x_i - 1)^{r_{M/e}(A \cup (E_i \setminus e)) - r_{M/e}(A \cup (E_{i-1} \setminus e))} \right) \left(\prod_{i=1}^{m} (y_i - 1)^{n_{M/e}(A \setminus (E_{i-1} \setminus e)) - n_{M/e}(A \setminus (E_i \setminus e))} \right). \tag{40}$$

The definition of $\mathbf{E} \setminus e$ shows that $\mathbf{E} \setminus e = (E_0 \setminus e, E_1 \setminus e, \ldots, E_m \setminus e)$. The definition of $\mathbf{M} \setminus e$ shows that $\mathbf{M}/e = (M \setminus e, \mathbf{E} \setminus e)$. The matroid $M \setminus e$ has ground set $E \setminus e$. Hence, the definition of $T_{\mathbf{M} \setminus e}$ yields

$$T_{\mathbf{M} \setminus e} = \underbrace{\sum_{A \subseteq E \setminus e}}_{= \sum_{A \subseteq E;\, e \notin A}} \left(\prod_{i=1}^{m} (x_i - 1)^{r_{M \setminus e}(A \cup (E_i \setminus e)) - r_{M \setminus e}(A \cup (E_{i-1} \setminus e))} \right) \left(\prod_{i=1}^{m} (y_i - 1)^{n_{M \setminus e}(A \setminus (E_{i-1} \setminus e)) - n_{M \setminus e}(A \setminus (E_i \setminus e))} \right)$$
$$= \sum_{\substack{A \subseteq E;\\ e \notin A}} \left(\prod_{i=1}^{m} (x_i - 1)^{r_{M \setminus e}(A \cup (E_i \setminus e)) - r_{M \setminus e}(A \cup (E_{i-1} \setminus e))} \right) \left(\prod_{i=1}^{m} (y_i - 1)^{n_{M \setminus e}(A \setminus (E_{i-1} \setminus e)) - n_{M \setminus e}(A \setminus (E_i \setminus e))} \right). \tag{41}$$

(a) Let $A \in \mathcal{P}(E)$ be such that $e \notin A$.
For every $i \in \{1, 2, \ldots, m\}$, we have

$$r_M((A \cup e) \cup E_i) - r_M((A \cup e) \cup E_{i-1}) = r_{M/e}(A \cup (E_i \setminus e)) - r_{M/e}(A \cup (E_{i-1} \setminus e)) \tag{42}$$

[15].

Every $j \in \{0, 1, \ldots, m\}$ satisfies

$$n_M((A \cup e) \setminus E_j) = n_{M/e}(A \setminus (E_j \setminus e)) \tag{44}$$

[15]*Proof of (42):* Let $i \in \{1, 2, \ldots, m\}$. Applying Lemma 4.11 to $K = E_i$, we obtain

$$r_M((A \cup e) \cup E_i) = r_{M/e}(A \cup (E_i \setminus e)) + r_M(e). \tag{43}$$

Applying Lemma 4.11 to $K = E_{i-1}$, we obtain

$$r_M((A \cup e) \cup E_{i-1}) = r_{M/e}(A \cup (E_{i-1} \setminus e)) + r_M(e).$$

[16].

For every $i \in \{1, 2, \ldots, m\}$, we have

$$n_M((A \cup e) \setminus E_{i-1}) - n_M((A \cup e) \setminus E_i) = n_{M/e}(A \setminus (E_{i-1} \setminus e)) - n_{M/e}(A \setminus (E_i \setminus e)) \tag{45}$$

[17].

Now, the map

$$\{A \subseteq E \mid e \notin A\} \to \{A \subseteq E \mid e \in A\}, \qquad A \mapsto A \cup e$$

is a bijection. Hence, we can substitute $A \cup e$ for A in the sum

$$\sum_{\substack{A \subseteq E; \\ e \in A}} \left(\prod_{i=1}^{m} (x_i - 1)^{r_M(A \cup E_i) - r_M(A \cup E_{i-1})} \right) \left(\prod_{i=1}^{m} (y_i - 1)^{n_M(A \setminus E_{i-1}) - n_M(A \setminus E_i)} \right).$$

Subtracting this equality from (43), we obtain

$$\begin{aligned} & r_M((A \cup e) \cup E_i) - r_M((A \cup e) \cup E_{i-1}) \\ &= (r_{M/e}(A \cup (E_i \setminus e)) + r_M(e)) - (r_{M/e}(A \cup (E_{i-1} \setminus e)) + r_M(e)) \\ &= r_{M/e}(A \cup (E_i \setminus e)) - r_{M/e}(A \cup (E_{i-1} \setminus e)). \end{aligned}$$

This proves (42).

[16] *Proof of (44):* Let $j \in \{0, 1, \ldots, m\}$. We must prove (44). We are in one of the following two cases:

Case 1: We have $j < m$.

Case 2: We have $j = m$.

Let us first consider Case 1. In this case, we have $j < m$. Hence, $j \in \{0, 1, \ldots, m\}$, so that $e \notin E_j$ (by (39)). Hence, Lemma 4.12 **(b)** (applied to $K = E_j$) shows that

$$\begin{aligned} n_M((A \cup e) \setminus E_j) &= n_{M/e}(A \setminus (E_j \setminus e)) + 1 - \underbrace{r_M(e)}_{=1} \\ &= n_{M/e}(A \setminus (E_j \setminus e)) + 1 - 1 = n_{M/e}(A \setminus (E_j \setminus e)). \end{aligned}$$

Thus, (44) is proven in Case 1.

Let us now consider Case 2. In this case, we have $j = m$. Thus, $E_j = E_m = E$. Hence, Lemma 4.12 **(e)** (applied to $K = E_j$) shows that $n_M((A \cup e) \setminus E_j) = n_{M/e}(A \setminus (E_j \setminus e))$. Thus, (44) is proven in Case 2.

We have thus proven (44) in each of the two Cases 1 and 2. Hence, (44) always holds.

[17] *Proof of (45):* Let $i \in \{1, 2, \ldots, m\}$.

Applying (44) to $j = i$, we obtain

$$n_M((A \cup e) \setminus E_i) = n_{M/e}(A \setminus (E_i \setminus e)). \tag{46}$$

Applying (44) to $j = i - 1$, we obtain

$$n_M((A \cup e) \setminus E_{i-1}) = n_{M/e}(A \setminus (E_{i-1} \setminus e)).$$

Subtracting (43) from this equality, we obtain

$$n_M((A \cup e) \setminus E_{i-1}) - n_M((A \cup e) \setminus E_i) = n_{M/e}(A \setminus (E_{i-1} \setminus e)) - n_{M/e}(A \setminus (E_i \setminus e)).$$

This proves (45).

We thus obtain

$$\sum_{\substack{A\subseteq E;\\ e\in A}} \left(\prod_{i=1}^{m} (x_i-1)^{r_M(A\cup E_i)-r_M(A\cup E_{i-1})}\right)\left(\prod_{i=1}^{m} (y_i-1)^{n_M(A\setminus E_{i-1})-n_M(A\setminus E_i)}\right)$$

$$= \sum_{\substack{A\subseteq E;\\ e\notin A}} \left(\prod_{i=1}^{m} \underbrace{(x_i-1)^{r_M((A\cup e)\cup E_i)-r_M((A\cup e)\cup E_{i-1})}}_{\substack{=(x_i-1)^{r_{M/e}(A\cup(E_i\setminus e))-r_{M/e}(A\cup(E_{i-1}\setminus e))}\\ \text{(by (42))}}}\right)$$

$$\left(\prod_{i=1}^{m} \underbrace{(y_i-1)^{n_M((A\cup e)\setminus E_{i-1})-n_M((A\cup e)\setminus E_i)}}_{\substack{=(y_i-1)^{n_{M/e}(A\setminus(E_{i-1}\setminus e))-n_{M/e}(A\setminus(E_i\setminus e))}\\ \text{(by (45))}}}\right)$$

$$= \sum_{\substack{A\subseteq E;\\ e\notin A}} \left(\prod_{i=1}^{m} (x_i-1)^{r_{M/e}(A\cup(E_i\setminus e))-r_{M/e}(A\cup(E_{i-1}\setminus e))}\right)$$

$$\left(\prod_{i=1}^{m} (y_i-1)^{n_{M/e}(A\setminus(E_{i-1}\setminus e))-n_{M/e}(A\setminus(E_i\setminus e))}\right)$$

$$= \sum_{\substack{A\subseteq E;\\ e\notin A}} \left(\prod_{i=1}^{m} (x_i-1)^{r_{M/e}(A\cup(E_i\setminus e))-r_{M/e}(A\cup(E_{i-1}\setminus e))}\right)\left(\prod_{i=1}^{m} (y_i-1)^{n_{M/e}(A\setminus(E_{i-1}\setminus e))-n_{M/e}(A\setminus(E_i\setminus e))}\right)$$

$$= T_{\mathbf{M}/e} \qquad \text{(by (40))}$$

This proves Lemma 4.17 **(a)**.

(b) Let $A \in \mathcal{P}(E)$ be such that $e \notin A$.

Every $j \in \{0, 1, \ldots, m\}$ satisfies

$$r_M(A \cup E_j) = r_{M\setminus e}(A \cup (E_j \setminus e)) \tag{47}$$

[18].

[18] *Proof of (47):* Let $j \in \{0, 1, \ldots, m\}$. We must prove (47). We are in one of the following two cases:

Case 1: We have $j < m$.

Case 2: We have $j = m$.

Let us first consider Case 1. In this case, we have $j < m$. Hence, $j \in \{0, 1, \ldots, m\}$, so that $e \notin E_j$ (by (39)). Hence, $E_j \setminus e = E_j$.

But combining $A \subseteq E \setminus e$ (since $e \notin A$) with $E_j \subseteq E \setminus e$ (since $e \notin E_j$), we obtain $A \cup E_j \subseteq E \setminus e$, thus $A \cup E_j \in \mathcal{P}(E \setminus e)$. Hence, Proposition 2.19 **(a)** (applied to $Z = \{e\}$ and $S = A \cup E_j$) shows that $r_{M\setminus\{e\}}(A \cup E_j) = r_M(A \cup E_j)$. Since $M \setminus \{e\} = M \setminus e$, this rewrites as

For every $i \in \{1, 2, \ldots, m\}$, we have

$$r_M(A \cup E_i) - r_M(A \cup E_{i-1}) \\ = r_{M \setminus e}(A \cup (E_i \setminus e)) - r_{M \setminus e}(A \cup (E_{i-1} \setminus e)) \quad (48)$$

[19].

For every $i \in \{1, 2, \ldots, m\}$, we have

$$n_M(A \setminus E_{i-1}) - n_M(A \setminus E_i) \\ = n_{M \setminus e}(A \setminus (E_{i-1} \setminus e)) - n_{M \setminus e}(A \setminus (E_i \setminus e)) \quad (50)$$

[20].

$r_{M\setminus e}(A \cup E_j) = r_M(A \cup E_j)$. Thus, $r_M(A \cup E_j) = r_{M\setminus e}\left(A \cup \underbrace{E_j}_{=E_j \setminus e}\right) = r_{M\setminus e}(A \cup (E_j \setminus e))$.

Hence, (47) is proven in Case 1.

Let us now consider Case 2. In this case, we have $j = m$. Thus, $E_j = E_m = E$. Hence, Lemma 4.13 **(c)** (applied to $K = E_j$) shows that $r_M(A \cup E_j) = r_{M\setminus e}(A \cup (E_j \setminus e))$ (since e is not a coloop). Thus, (47) is proven in Case 2.

We have thus proven (47) in each of the two Cases 1 and 2. Hence, (47) always holds.

[19] *Proof of (48):* Let $i \in \{1, 2, \ldots, m\}$.

Applying (47) to $j = i$, we obtain

$$r_M(A \cup E_i) = r_{M\setminus e}(A \cup (E_i \setminus e)). \quad (49)$$

Applying (47) to $j = i - 1$, we obtain

$$r_M(A \cup E_{i-1}) = r_{M\setminus e}(A \cup (E_{i-1} \setminus e)).$$

Subtracting this equality from (49), we obtain

$$r_M(A \cup E_i) - r_M(A \cup E_{i-1}) = r_{M\setminus e}(A \cup (E_i \setminus e)) - r_{M\setminus e}(A \cup (E_{i-1} \setminus e)).$$

This proves (48).

[20] *Proof of (50):* Let $i \in \{1, 2, \ldots, m\}$.

Applying Lemma 4.14 to $K = E_i$, we obtain

$$n_M(A \setminus E_i) = n_{M\setminus e}(A \setminus (E_i \setminus e)). \quad (51)$$

Applying Lemma 4.14 to $K = E_{i-1}$, we obtain

$$n_M(A \setminus E_{i-1}) = n_{M\setminus e}(A \setminus (E_{i-1} \setminus e)).$$

Subtracting (49) from this equality, we obtain

$$n_M(A \setminus E_{i-1}) - n_M(A \setminus E_i) = n_{M\setminus e}(A \setminus (E_{i-1} \setminus e)) - n_{M\setminus e}(A \setminus (E_i \setminus e)).$$

This proves (50).

Now,

$$\sum_{\substack{A\subseteq E;\\ e\notin A}} \left(\prod_{i=1}^{m} \underbrace{(x_i-1)^{r_M(A\cup E_i)-r_M(A\cup E_{i-1})}}_{\substack{=(x_i-1)^{r_{M\setminus e}(A\cup(E_i\setminus e))-r_{M\setminus e}(A\cup(E_{i-1}\setminus e))} \\ \text{(by (48))}}} \right) \left(\prod_{i=1}^{m} \underbrace{(y_i-1)^{n_M(A\setminus E_{i-1})-n_M(A\setminus E_i)}}_{\substack{=(y_i-1)^{n_{M\setminus e}(A\setminus(E_{i-1}\setminus e))-n_{M\setminus e}(A\setminus(E_i\setminus e))} \\ \text{(by (50))}}} \right)$$
$$= \sum_{\substack{A\subseteq E;\\ e\notin A}} \left(\prod_{i=1}^{m} (x_i-1)^{r_{M\setminus e}(A\cup(E_i\setminus e))-r_{M\setminus e}(A\cup(E_{i-1}\setminus e))} \right) \left(\prod_{i=1}^{m} (y_i-1)^{n_{M\setminus e}(A\setminus(E_{i-1}\setminus e))-n_{M\setminus e}(A\setminus(E_i\setminus e))} \right)$$
$$= T_{\mathbf{M}\setminus e} \qquad \text{(by (41))}.$$

This proves Lemma 4.17 **(b)**. □

Proof of Proposition 4.8. **(a)** Assume that e is a loop. Let $k \in \{1, 2, \ldots, m\}$ be such that $e \in E_k \setminus E_{k-1}$. We need to show that $T_{\mathbf{M}} = y_k T_{\mathbf{M}\setminus e}$.

Proposition 2.23 **(a)** shows that $M/e = M \setminus e$. The definition of $\mathbf{M}/e$ shows that $\mathbf{M}/e = (M/e, \mathbf{E} \setminus e)$. Similarly, $\mathbf{M} \setminus e = (M \setminus e, \mathbf{E} \setminus e)$. Hence, $\mathbf{M}/e = \left(\underbrace{M/e}_{=M\setminus e}, \mathbf{E} \setminus e \right) = (M \setminus e, \mathbf{E} \setminus e) = \mathbf{M} \setminus e$.

The definition of $T_{\mathbf{M}}$ shows that

$$T_{\mathbf{M}} = \sum_{A\subseteq E} \left(\prod_{i=1}^{m} (x_i-1)^{r_M(A\cup E_i)-r_M(A\cup E_{i-1})} \right) \left(\prod_{i=1}^{m} (y_i-1)^{n_M(A\setminus E_{i-1})-n_M(A\setminus E_i)} \right)$$
$$= \underbrace{\sum_{\substack{A\subseteq E;\\ e\in A}} \left(\prod_{i=1}^{m} (x_i-1)^{r_M(A\cup E_i)-r_M(A\cup E_{i-1})} \right) \left(\prod_{i=1}^{m} (y_i-1)^{n_M(A\setminus E_{i-1})-n_M(A\setminus E_i)} \right)}_{\substack{=(y_k-1)T_{\mathbf{M}/e} \\ \text{(by Lemma 4.15 (a))}}}$$
$$+ \underbrace{\sum_{\substack{A\subseteq E;\\ e\notin A}} \left(\prod_{i=1}^{m} (x_i-1)^{r_M(A\cup E_i)-r_M(A\cup E_{i-1})} \right) \left(\prod_{i=1}^{m} (y_i-1)^{n_M(A\setminus E_{i-1})-n_M(A\setminus E_i)} \right)}_{\substack{=T_{\mathbf{M}/e} \\ \text{(by Lemma 4.15 (b))}}}$$
$$= (y_k-1)\, T_{\mathbf{M}/e} + T_{\mathbf{M}/e} = y_k T_{\mathbf{M}/e} = y_k T_{\mathbf{M}\setminus e} \qquad (\text{since } \mathbf{M}/e = \mathbf{M}\setminus e).$$

This proves Proposition 4.8 **(a)**.

(b) Assume that e is a coloop. Let $k \in \{1, 2, \ldots, m\}$ be such that $e \in E_k \setminus E_{k-1}$. We need to show that $T_{\mathbf{M}} = x_k T_{\mathbf{M}/e}$.

The definition of $T_{\mathbf{M}}$ shows that

$$T_{\mathbf{M}} = \sum_{A \subseteq E} \left(\prod_{i=1}^{m} (x_i - 1)^{r_M(A \cup E_i) - r_M(A \cup E_{i-1})} \right) \left(\prod_{i=1}^{m} (y_i - 1)^{n_M(A \setminus E_{i-1}) - n_M(A \setminus E_i)} \right)$$

$$= \underbrace{\sum_{\substack{A \subseteq E; \\ e \in A}} \left(\prod_{i=1}^{m} (x_i - 1)^{r_M(A \cup E_i) - r_M(A \cup E_{i-1})} \right) \left(\prod_{i=1}^{m} (y_i - 1)^{n_M(A \setminus E_{i-1}) - n_M(A \setminus E_i)} \right)}_{\substack{= T_{\mathbf{M}/e} \\ \text{(by Lemma 4.16 (a))}}}$$

$$+ \underbrace{\sum_{\substack{A \subseteq E; \\ e \notin A}} \left(\prod_{i=1}^{m} (x_i - 1)^{r_M(A \cup E_i) - r_M(A \cup E_{i-1})} \right) \left(\prod_{i=1}^{m} (y_i - 1)^{n_M(A \setminus E_{i-1}) - n_M(A \setminus E_i)} \right)}_{\substack{= (x_k - 1) T_{\mathbf{M}/e} \\ \text{(by Lemma 4.16 (b))}}}$$

$$= T_{\mathbf{M}/e} + (x_k - 1) T_{\mathbf{M}/e} = x_k T_{\mathbf{M}/e}.$$

This proves Proposition 4.8 **(b)**.

(c) Assume that e belongs to $E_m \setminus E_{m-1}$ and is neither a loop nor a coloop. The definition of $T_{\mathbf{M}}$ shows that

$$T_{\mathbf{M}} = \sum_{A \subseteq E} \left(\prod_{i=1}^{m} (x_i - 1)^{r_M(A \cup E_i) - r_M(A \cup E_{i-1})} \right) \left(\prod_{i=1}^{m} (y_i - 1)^{n_M(A \setminus E_{i-1}) - n_M(A \setminus E_i)} \right)$$

$$= \underbrace{\sum_{\substack{A \subseteq E; \\ e \in A}} \left(\prod_{i=1}^{m} (x_i - 1)^{r_M(A \cup E_i) - r_M(A \cup E_{i-1})} \right) \left(\prod_{i=1}^{m} (y_i - 1)^{n_M(A \setminus E_{i-1}) - n_M(A \setminus E_i)} \right)}_{\substack{= T_{\mathbf{M}/e} \\ \text{(by Lemma 4.17 (a))}}}$$

$$+ \underbrace{\sum_{\substack{A \subseteq E; \\ e \notin A}} \left(\prod_{i=1}^{m} (x_i - 1)^{r_M(A \cup E_i) - r_M(A \cup E_{i-1})} \right) \left(\prod_{i=1}^{m} (y_i - 1)^{n_M(A \setminus E_{i-1}) - n_M(A \setminus E_i)} \right)}_{\substack{= T_{\mathbf{M} \setminus e} \\ \text{(by Lemma 4.17 (b))}}}$$

$$= T_{\mathbf{M}/e} + T_{\mathbf{M} \setminus e} = T_{\mathbf{M} \setminus e} + T_{\mathbf{M}/e}.$$

This proves Proposition 4.8 **(c)**. □

Proof of Proposition 4.9. Let E denote the ground set of M (so that $M = (E, \mathcal{I})$ for some $\mathcal{I} \subseteq \mathcal{P}(E)$). We shall prove Proposition 4.9 by induction on $m + |E|$. The *induction base* (i.e., the case when $m + |E| = 0$) is trivial and left to the reader.

Induction step: Let N be a positive integer. Assume (as the induction hypothesis) that Proposition 4.9 holds whenever $m + |E| = N - 1$. We need to show that Proposition 4.9 holds whenever $m + |E| = N$.

So let us assume that we are in the situation of Proposition 4.9, and that we have $m + |E| = N$. We need to show that

$$T_{\mathbf{M}} \in \mathbb{N}[x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_m]. \quad (52)$$

Since $(M, \mathbf{E})$ is a filtered matroid, we have $\varnothing = E_0 \subseteq E_1 \subseteq \cdots \subseteq E_m = E$.

We have $m > 0$ [21]. Hence, $E_m \setminus E_{m-1}$ is well-defined. We thus are in one of the following two cases:

Case 1: We have $E_m \setminus E_{m-1} = \varnothing$.

Case 2: We have $E_m \setminus E_{m-1} \neq \varnothing$.

Let us first consider Case 1. In this case, we have $E_m \setminus E_{m-1} = \varnothing$. Combined with $E_{m-1} \subseteq E_m$ (since $E_0 \subseteq E_1 \subseteq \cdots \subseteq E_m$), this entails that $E_{m-1} = E_m$. Let $\mathbf{E}'$ be the list $(E_0, E_1, \ldots, E_{m-1})$. Proposition 4.7 thus shows that $(M, \mathbf{E}')$ is a filtered matroid, and satisfies $T_{(M,\mathbf{E})} = T_{(M,\mathbf{E}')}$. Now, $(m-1) + |E| = \underbrace{m + |E|}_{=N} - 1 = N - 1$. Hence, the induction hypothesis shows that we can apply Proposition 4.9 to $(M, \mathbf{E}')$, $(E_0, E_1, \ldots, E_{m-1})$ and $\mathbf{E}'$ instead of $\mathbf{M}$, $(E_0, E_1, \ldots, E_{m-1})$ and $\mathbf{E}$. As a consequence, we conclude that $T_{(M,\mathbf{E}')} \in \mathbb{N}[x_1, x_2, \ldots, x_{m-1}, y_1, y_2, \ldots, y_{m-1}]$. Now, from $\mathbf{M} = (M, \mathbf{E})$, we obtain

$$\begin{aligned} T_{\mathbf{M}} &= T_{(M,\mathbf{E})} = T_{(M,\mathbf{E}')} \in \mathbb{N}[x_1, x_2, \ldots, x_{m-1}, y_1, y_2, \ldots, y_{m-1}] \\ &\subseteq \mathbb{N}[x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_m]. \end{aligned}$$

Thus, (52) is proven in Case 1.

Let us now consider Case 2. In this case, we have $E_m \setminus E_{m-1} \neq \varnothing$. Hence, there exists some $e \in E_m \setminus E_{m-1}$. Pick such an e. Clearly, $e \in E_m \setminus E_{m-1} \subseteq E_m = E$.

The matroid $M \setminus e$ has ground set $E \setminus e$. The filtered matroid $\mathbf{M} \setminus e = (M \setminus e, \mathbf{E} \setminus e)$ has $\mathbf{E} \setminus e = (E_0 \setminus e, E_1 \setminus e, \ldots, E_m \setminus e)$. Now, $m + \underbrace{|E \setminus e|}_{\substack{=|E|-1 \\ (\text{since } e \in E)}} = \underbrace{m + |E|}_{=N} - 1 = N - 1$. Hence, the induction hypothesis shows that we can apply Proposition 4.9 to $\mathbf{M} \setminus e$, $M \setminus e$, $\mathbf{E} \setminus e$ and $(E_0 \setminus e, E_1 \setminus e, \ldots, E_m \setminus e)$ instead of $\mathbf{M}$, M, $\mathbf{E}$ and $(E_0, E_1, \ldots, E_m)$. As a result, we obtain

$$T_{\mathbf{M} \setminus e} \in \mathbb{N}[x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_m].$$

The matroid M/e has ground set $E \setminus e$. The filtered matroid $\mathbf{M}/e = (M/e, \mathbf{E} \setminus e)$ has $\mathbf{E} \setminus e = (E_0 \setminus e, E_1 \setminus e, \ldots, E_m \setminus e)$. Now, recall that $m + |E \setminus e| = N - 1$. Hence, the induction hypothesis shows that we can apply Proposition 4.9 to

[21] *Proof.* We have $m + |E| = N > 0$. If we had $m = 0$, then we would have $E_m = E_0$, which would yield $\underbrace{m}_{=0} + \left| \underbrace{E}_{=E_m = E_0 = \varnothing} \right| = |\varnothing| = 0$, which would contradict $m + |E| > 0$. Hence, we cannot have $m = 0$. We thus have $m > 0$, qed.

$\mathbf{M}/e$, M/e, $\mathbf{E}\setminus e$ and $(E_0 \setminus e, E_1 \setminus e, \ldots, E_m \setminus e)$ instead of $\mathbf{M}$, M, $\mathbf{E}$ and $(E_0, E_1, \ldots, E_m)$. As a result, we obtain

$$T_{\mathbf{M}/e} \in \mathbb{N}\left[x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_m\right].$$

We are in one of the following three subcases:

Subcase 2.1: The element e is a loop (of M).

Subcase 2.2: The element e is a coloop (of M).

Subcase 2.3: The element e is neither a loop nor a coloop.

Let us first consider Subcase 2.1. In this Subcase, the element e is a loop (of M). Hence, Proposition 4.8 **(a)** (applied to $k = m$) yields

$$\begin{aligned} T_{\mathbf{M}} = y_m \underbrace{T_{\mathbf{M}\setminus e}}_{\in \mathbb{N}[x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_m]} &\in y_m \mathbb{N}\left[x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_m\right] \\ &\subseteq \mathbb{N}\left[x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_m\right]. \end{aligned}$$

Thus, (52) is proven in Subcase 2.1.

Let us now consider Subcase 2.2. In this Subcase, the element e is a coloop (of M). Hence, Proposition 4.8 **(b)** (applied to $k = m$) yields

$$\begin{aligned} T_{\mathbf{M}} = x_m \underbrace{T_{\mathbf{M}/e}}_{\in \mathbb{N}[x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_m]} &\in x_m \mathbb{N}\left[x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_m\right] \\ &\subseteq \mathbb{N}\left[x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_m\right]. \end{aligned}$$

Thus, (52) is proven in Subcase 2.2.

Let us now consider Subcase 2.3. In this Subcase, the element e is neither a loop nor a coloop. Hence, Proposition 4.8 **(c)** yields

$$\begin{aligned} T_{\mathbf{M}} &= \underbrace{T_{\mathbf{M}\setminus e}}_{\in \mathbb{N}[x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_m]} + \underbrace{T_{\mathbf{M}/e}}_{\in \mathbb{N}[x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_m]} \\ &\in \mathbb{N}\left[x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_m\right] + \mathbb{N}\left[x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_m\right] \\ &\subseteq \mathbb{N}\left[x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_m\right]. \end{aligned}$$

Thus, (52) is proven in Subcase 2.3.

We thus have proven (52) in each of the three Subcases 2.1, 2.2 and 2.3. Since these three Subcases cover all of Case 2, this yields that (52) always holds in Case 2.

We thus have proven (52) in each of the two Cases 1 and 2. Thus, (52) always holds. In other words, Proposition 4.9 holds whenever $m + |E| = N$. This completes the induction step. The inductive proof of Proposition 4.9 is thus complete. □

Proof of Proposition 4.10. Since $(M, \mathbf{E})$ is a filtered matroid, we have $\varnothing = E_0 \subseteq E_1 \subseteq \cdots \subseteq E_m = E$. Thus, $(M^*, \mathbf{E})$ is a filtered matroid (since E is the ground set of M^*). It remains to prove that $T_{(M^*, \mathbf{E})} = T_{\mathbf{M}}\left(y_1, y_2, \ldots, y_m, x_1, x_2, \ldots, x_m\right)$.

For every $S \in \mathcal{P}(E)$, let us write $\overline{S}$ for the set $E \setminus S \in \mathcal{P}(E)$. We have

$$n_{M^*}(S) = r_M(E) - r_M\left(\overline{S}\right) \qquad \text{for every } S \in \mathcal{P}(E) \tag{53}$$

[22]. Furthermore,

$$r_{M^*}(S) = n_M(E) - n_M\left(\overline{S}\right) \qquad \text{for every } S \in \mathcal{P}(E) \tag{54}$$

[23].

The definition of $T_{\mathbf{M}}$ shows that

$$T_{\mathbf{M}} = \sum_{A \subseteq E} \left(\prod_{i=1}^{m} (x_i - 1)^{r_M(A \cup E_i) - r_M(A \cup E_{i-1})} \right) \left(\prod_{i=1}^{m} (y_i - 1)^{n_M(A \setminus E_{i-1}) - n_M(A \setminus E_i)} \right).$$

Substituting $(y_1, y_2, \ldots, y_m, x_1, x_2, \ldots, x_m)$ for $(x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_m)$ in this

[22] *Proof of (53):* Let $S \in \mathcal{P}(E)$. The definition of n_{M^*} yields

$$\begin{aligned} n_{M^*}(S) &= |S| - \underbrace{r_{M^*}(S)}_{\substack{= |S| + r_M(E \setminus S) - r_M(E) \\ \text{(by (10))}}} = |S| - (|S| + r_M(E \setminus S) - r_M(E)) \\ &= r_M(E) - r_M\Bigg(\underbrace{E \setminus S}_{\substack{= \overline{S} \\ (\text{since } \overline{S} = E \setminus S)}} \Bigg) = r_M(E) - r_M\left(\overline{S}\right). \end{aligned}$$

This proves (53).

[23] *Proof of (54):* Let $S \in \mathcal{P}(E)$. The definition of n_M yields $n_M(E) = |E| - r_M(E)$. The definition of n_M yields $n_M\left(\overline{S}\right) = \left|\overline{S}\right| - r_M\left(\overline{S}\right)$. Since $S \subseteq E$, we have $|E| = |S| + |E \setminus S|$, so that $|S| = |E| - \Bigg| \underbrace{E \setminus S}_{\substack{= \overline{S} \\ (\text{since } \overline{S} = E \setminus S)}} \Bigg| = |E| - \left|\overline{S}\right|$. Now,

$$\begin{aligned} &\underbrace{n_M(E)}_{= |E| - r_M(E)} - \underbrace{n_M\left(\overline{S}\right)}_{= \left|\overline{S}\right| - r_M\left(\overline{S}\right)} \\ &= (|E| - r_M(E)) - \left(\left|\overline{S}\right| - r_M\left(\overline{S}\right)\right) = \underbrace{|E| - \left|\overline{S}\right|}_{= |S|} + r_M\Bigg(\underbrace{\overline{S}}_{= E \setminus S} \Bigg) - r_M(E) \\ &= |S| + r_M(E \setminus S) - r_M(E) = r_{M^*}(S) \qquad (\text{by (10)}). \end{aligned}$$

This proves (54).

identity, we obtain

$$\begin{aligned}
& T_{\mathbf{M}}\left(y_1, y_2, \ldots, y_m, x_1, x_2, \ldots, x_m\right) \\
& = \sum_{A \subseteq E} \left(\prod_{i=1}^m \left(y_i - 1\right)^{r_M(A \cup E_i) - r_M(A \cup E_{i-1})} \right) \left(\prod_{i=1}^m \left(x_i - 1\right)^{n_M(A \setminus E_{i-1}) - n_M(A \setminus E_i)} \right) \\
& = \sum_{A \subseteq E} \left(\prod_{i=1}^m \left(y_i - 1\right)^{r_M\left(\overline{A} \cup E_i\right) - r_M\left(\overline{A} \cup E_{i-1}\right)} \right) \left(\prod_{i=1}^m \left(x_i - 1\right)^{n_M\left(\overline{A} \setminus E_{i-1}\right) - n_M\left(\overline{A} \setminus E_i\right)} \right) \\
& \qquad \left(\begin{array}{c} \text{here, we have substituted } \overline{A} \text{ for } A \text{ in the sum, since the map} \\ \mathcal{P}(E) \to \mathcal{P}(E), \ A \mapsto \overline{A} \text{ is a bijection} \end{array} \right) \\
& = \sum_{A \subseteq E} \left(\prod_{i=1}^m \left(x_i - 1\right)^{n_M\left(\overline{A} \setminus E_{i-1}\right) - n_M\left(\overline{A} \setminus E_i\right)} \right) \left(\prod_{i=1}^m \left(y_i - 1\right)^{r_M\left(\overline{A} \cup E_i\right) - r_M\left(\overline{A} \cup E_{i-1}\right)} \right).
\end{aligned} \tag{55}$$

On the other hand, every $i \in \{1, 2, \ldots, m\}$ satisfies

$$r_{M^*}\left(A \cup E_i\right) - r_{M^*}\left(A \cup E_{i-1}\right) = n_M\left(\overline{A} \setminus E_{i-1}\right) - n_M\left(\overline{A} \setminus E_i\right) \tag{56}$$

[24]. Furthermore, every $i \in \{1, 2, \ldots, m\}$ satisfies

$$n_{M^*}\left(A \setminus E_{i-1}\right) - n_{M^*}\left(A \setminus E_i\right) = r_M\left(\overline{A} \cup E_i\right) - r_M\left(\overline{A} \cup E_{i-1}\right) \tag{58}$$

[25]. Now, recall that the matroid M^* has ground set E. Hence, the definition of

[24] *Proof of (56):* Let $i \in \{1, 2, \ldots, m\}$.

We have $\overline{A \cup E_i} = E \setminus (A \cup E_i) = \underbrace{(E \setminus A)}_{=\overline{A}} \setminus E_i = \overline{A} \setminus E_i$. Applying (54) to $S = A \cup E_i$, we obtain

$$r_{M^*}\left(A \cup E_i\right) = n_M(E) - n_M\left(\underbrace{\overline{A \cup E_i}}_{=\overline{A} \setminus E_i}\right) = n_M(E) - n_M\left(\overline{A} \setminus E_i\right). \tag{57}$$

The same argument (but applied to E_{i-1} instead of E_i) yields

$$r_{M^*}\left(A \cup E_{i-1}\right) = n_M(E) - n_M\left(\overline{A} \setminus E_{i-1}\right).$$

Subtracting this equality from (57), we obtain

$$\begin{aligned}
& r_{M^*}\left(A \cup E_i\right) - r_{M^*}\left(A \cup E_{i-1}\right) \\
& = \left(n_M(E) - n_M\left(\overline{A} \setminus E_i\right)\right) - \left(n_M(E) - n_M\left(\overline{A} \setminus E_{i-1}\right)\right) \\
& = n_M\left(\overline{A} \setminus E_{i-1}\right) - n_M\left(\overline{A} \setminus E_i\right).
\end{aligned}$$

This proves (56).

[25] *Proof of (58):* Let $i \in \{1, 2, \ldots, m\}$.

We have $\overline{A \setminus E_i} = E \setminus (A \setminus E_i) = (E \setminus A) \cup E_i$ (since $E_i \subseteq E$), hence $\overline{A \setminus E_i} = \underbrace{(E \setminus A)}_{=\overline{A}} \cup E_i =$

$T_{(M^*,\mathbf{E})}$ yields

$$\begin{aligned}
&T_{(M^*,\mathbf{E})} \\
&= \sum_{A \subseteq E} \left(\prod_{i=1}^{m} \underbrace{(x_i - 1)^{r_{M^*}(A \cup E_i) - r_{M^*}(A \cup E_{i-1})}}_{\substack{=(x_i-1)^{n_M\left(\overline{A} \setminus E_{i-1}\right) - n_M\left(\overline{A} \setminus E_i\right)} \\ \text{(by (56))}}} \right) \left(\prod_{i=1}^{m} \underbrace{(y_i - 1)^{n_{M^*}(A \setminus E_{i-1}) - n_{M^*}(A \setminus E_i)}}_{\substack{=(y_i-1)^{r_M\left(\overline{A} \cup E_i\right) - r_M\left(\overline{A} \cup E_{i-1}\right)} \\ \text{(by (56))}}} \right) \\
&= \sum_{A \subseteq E} \left(\prod_{i=1}^{m} (x_i - 1)^{n_M\left(\overline{A} \setminus E_{i-1}\right) - n_M\left(\overline{A} \setminus E_i\right)} \right) \left(\prod_{i=1}^{m} (y_i - 1)^{r_M\left(\overline{A} \cup E_i\right) - r_M\left(\overline{A} \cup E_{i-1}\right)} \right) \\
&= T_{\mathbf{M}} \left(y_1, y_2, \ldots, y_m, x_1, x_2, \ldots, x_m \right) \qquad \text{(by (55))}.
\end{aligned}$$

This proves Proposition 4.10. □

As promised, we can now derive Proposition 3.2 and Corollary 3.3:

Proof of Proposition 3.2. Let $\mathbf{E}$ be the list $(\varnothing, E)$. Then, Proposition 4.5 shows that $(M, \mathbf{E})$ is a filtered matroid, and satisfies $T_{(M,\mathbf{E})} = T_M(x_1, y_1)$. Clearly, $e \in E = E \setminus \varnothing$.

Let $\mathbf{M}$ be the filtered matroid $(M, \mathbf{E})$. Thus, $T_{\mathbf{M}} = T_{(M,\mathbf{E})} = T_M(x_1, y_1)$.

It is easy to see that $\mathbf{M}/e = (M/e, (\varnothing, E \setminus e))$. Write the matroid M/e in the form $(E \setminus e, \mathcal{J})$. Then, Proposition 4.5 (applied to M/e, $E \setminus e$ and $\mathcal{J}$ instead of M, E and $\mathcal{I}$) shows that $(M/e, (\varnothing, E \setminus e))$ is a filtered matroid, and satisfies $T_{M/e,(\varnothing,E\setminus e)} = T_{M/e}(x_1, y_1)$. This latter equality rewrites as $T_{\mathbf{M}/e} = T_{M/e}(x_1, y_1)$ (since $\mathbf{M}/e = (M/e, (\varnothing, E \setminus e))$).

It is easy to see that $\mathbf{M} \setminus e = (M \setminus e, (\varnothing, E \setminus e))$. Write the matroid $M \setminus e$ in the form $(E \setminus e, \mathcal{K})$. Then, Proposition 4.5 (applied to $M \setminus e$, $E \setminus e$ and $\mathcal{K}$ instead of M, E and $\mathcal{I}$) shows that $(M \setminus e, (\varnothing, E \setminus e))$ is a filtered matroid, and satisfies

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$\overline{A} \cup E_i$. Applying (53) to $S = A \setminus E_i$, we obtain

$$n_{M^*}(A \setminus E_i) = r_M(E) - r_M\left(\underbrace{\overline{A \setminus E_i}}_{=\overline{A} \cup E_i} \right) = r_M(E) - r_M\left(\overline{A} \cup E_i\right). \tag{59}$$

The same argument (but applied to E_{i-1} instead of E_i) yields

$$n_{M^*}(A \setminus E_{i-1}) = r_M(E) - r_M\left(\overline{A} \cup E_{i-1}\right).$$

Subtracting this equality from (59), we obtain

$$\begin{aligned}
&n_{M^*}(A \cup E_i) - n_{M^*}(A \cup E_{i-1}) \\
&= \left(r_M(E) - r_M\left(\overline{A} \setminus E_i\right)\right) - \left(r_M(E) - r_M\left(\overline{A} \setminus E_{i-1}\right)\right) \\
&= r_M\left(\overline{A} \setminus E_{i-1}\right) - r_M\left(\overline{A} \setminus E_i\right).
\end{aligned}$$

This proves (58).

$T_{M\setminus e,(\varnothing,E\setminus e)} = T_{M\setminus e}(x_1, y_1)$. This latter equality rewrites as $T_{\mathbf{M}\setminus e} = T_{M\setminus e}(x_1, y_1)$ (since $\mathbf{M} \setminus e = (M \setminus e, (\varnothing, E \setminus e))$).

(a) Assume that e is a loop. Proposition 4.8 **(a)** (applied to $(M, \mathbf{E})$, $(\varnothing, E)$ and 1 instead of $\mathbf{M}$, $(E_0, E_1, \ldots, E_m)$ and k) shows that $T_{\mathbf{M}} = y_1 T_{\mathbf{M}\setminus e}$. Since $T_{\mathbf{M}} = T_M(x_1, y_1)$ and $T_{\mathbf{M}\setminus e} = T_{M\setminus e}(x_1, y_1)$, this rewrites as $T_M(x_1, y_1) = y_1 T_{M\setminus e}(x_1, y_1)$. Substituting x and y for x_1 and y_1 in this equality, we obtain $T_M(x, y) = yT_{M\setminus e}(x, y)$. In other words, $T_M = yT_{M\setminus e}$. This proves Proposition 3.2 **(a)**.

(b) Assume that e is a coloop. Proposition 4.8 **(b)** (applied to $(M, \mathbf{E})$, $(\varnothing, E)$ and 1 instead of $\mathbf{M}$, $(E_0, E_1, \ldots, E_m)$ and k) shows that $T_{\mathbf{M}} = x_1 T_{\mathbf{M}/e}$. Since $T_{\mathbf{M}} = T_M(x_1, y_1)$ and $T_{\mathbf{M}/e} = T_{M/e}(x_1, y_1)$, this rewrites as $T_M(x_1, y_1) = x_1 T_{M/e}(x_1, y_1)$. Substituting x and y for x_1 and y_1 in this equality, we obtain $T_M(x, y) = xT_{M/e}(x, y)$. In other words, $T_M = xT_{M/e}$. This proves Proposition 3.2 **(a)**.

(c) Assume that e is neither a loop nor a coloop. Proposition 4.8 **(c)** (applied to $(M, \mathbf{E})$ and $(\varnothing, E)$ instead of $\mathbf{M}$ and $(E_0, E_1, \ldots, E_m)$) shows that $T_{\mathbf{M}} = T_{\mathbf{M}\setminus e} + T_{\mathbf{M}/e}$. Since $T_{\mathbf{M}} = T_M(x_1, y_1)$, $T_{\mathbf{M}/e} = T_{M/e}(x_1, y_1)$ and $T_{\mathbf{M}\setminus e} = T_{M\setminus e}(x_1, y_1)$, this rewrites as $T_M(x_1, y_1) = T_{M\setminus e}(x_1, y_1) + T_{M/e}(x_1, y_1)$. Substituting x and y for x_1 and y_1 in this equality, we obtain $T_M(x, y) = T_{M\setminus e}(x, y) + T_{M/e}(x, y)$. In other words, $T_M = T_{M\setminus e} + T_{M/e}$. This proves Proposition 3.2 **(c)**. □

Proof of Corollary 3.3. Let $\mathbf{E}$ be the list $(\varnothing, E)$. Then, Proposition 4.5 shows that $(M, \mathbf{E})$ is a filtered matroid, and satisfies $T_{(M,\mathbf{E})} = T_M(x_1, y_1)$. Proposition 4.9 (applied to $(M, \mathbf{E})$ and $(\varnothing, E)$ instead of $\mathbf{M}$ and $(E_0, E_1, \ldots, E_m)$) shows that $T_{(M,\mathbf{E})} \in \mathbb{N}[x_1, y_1]$. Since $T_{(M,\mathbf{E})} = T_M(x_1, y_1)$, this rewrites as $T_M(x_1, y_1) \in \mathbb{N}[x_1, y_1]$. Substituting x and y for x_1 and y_1 in this relation, we obtain $T_M(x, y) \in \mathbb{N}[x, y]$. In other words, $T_M \in \mathbb{N}[x, y]$. This proves Corollary 3.3. □

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