The enriched q-monomial basis of the quasisymmetric functions

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Abstract. We construct a new family $\left(\eta_{\alpha}^{(q)}\right)_{\alpha\in\mathrm{Comp}}$ of quasisymmetric functions for each element q of the base ring. We call them the "enriched q-monomial quasisymmetric functions". When $r := q + 1$ is invertible, this family is a basis of QSym. It generalizes Hoffman's "essential quasisymmetric functions" (obtained for $q = 0$) and Hsiao's "monomial peak functions" (obtained for $q = 1$), but also includes the monomial quasisymmetric functions as a limiting case.

We describe these functions $\eta_{\alpha}^{(q)}$ by several formulas, and compute their products, coproducts and antipodes. The product expansion is given by an exotic variant of the shuffle product which we call the "stufufuffle product" due to its ability to pick several consecutive entries from each composition. This "stufufuffle product" has previously appeared in recent work by Bouillot, Novelli and Thibon, generalizing the "block shuffle product" from the theory of multizeta values.

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Contents

1. Introduction

Among the combinatorial Hopf algebras that consist of power series in commuting indeterminates, one of the largest and most all-embracing is that of the *quasisymmetric functions*, called QSym. Originally introduced by Gessel in 1984 [Gessel84], it has since found applications (e.g.) to enumerative combinatorics ([Sagan20, Chapter 8], [Stanle24, §7.19–7.23], [GesZhu18]), multizeta values (e.g., [Hoffma15]), algebraic geometry ([Oesing18]) and the representation theory of 0-Hecke algebras ([Meliot17, §6.2]).

It was observed by Ehrenborg ([Ehrenb96, Lemma 4.2]; see [Biller10, §3.3] for a survey) that quasisymmetric functions can also be used to encode the "flag f-vector" of a finite graded poset – i.e., essentially, the number of chains over a given sequence of ranks, for each possible sequence of ranks. Soon after, work of Bergeron, Mykytiuk, Sottile and van Willigenburg ([BMSW00, Example 5.3], but see [Biller10, §3.4] for an explicit statement) showed that if the graded poset is Eulerian (a property shared by face posets of polytopes and simplicial spheres), then the resulting quasisymmetric function is not arbitrary but rather belongs to a certain subalgebra of QSym called *Stembridge's Hopf algebra* or the *peak algebra* or the *odd subalgebra* Π_- *of* QSym. It was initially defined by Stembridge [Stembr97, §3] in order to find a fundamental expansion of the Schur P- and Q-functions, and has since been studied by others for related and unrelated reasons ([AgBeSo14, §6, particularly Proposition 6.5], [BMSW99], [BMSW00, §5], [Hsiao07] etc.); among other properties, it is a Hopf subalgebra of QSym.

Almost all bases of QSym constructed so far are indexed by *compositions* (i.e., tuples of positive integers), and their structure constants are often governed by versions of shuffle products and deconcatenation coproducts. The peak algebra is smaller, and its bases are often indexed by *odd compositions*, i.e., compositions whose entries are all odd. One of its simplest bases is defined as follows (for the sake of simplicity, we use $\mathbb{Q}$ as a base ring here): If $n \in \mathbb{N}$ and if $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ is a composition of n (that is, a tuple of positive integers with $\alpha_1 + \alpha_2 + \cdots + \alpha_\ell = n$), then we define the formal power series

$$\begin{aligned} \eta_\alpha &= \sum_{\substack{1 \leq g_1 \leq g_2 \leq \cdots \leq g_n; \\ g_i = g_{i+1} \text{ for each } i \in E(\alpha)}} 2^{|\{g_1, g_2, \ldots, g_n\}|} x_{g_1} x_{g_2} \cdots x_{g_n} \\ &\in \mathbb{Q}\left[\left[x_1, x_2, x_3, \ldots\right]\right], \end{aligned} \tag{1}$$

where $E(\alpha)$ denotes the set $\{1, 2, \ldots, n-1\} \setminus \{\alpha_1 + \alpha_2 + \cdots + \alpha_i \mid 0 < i < \ell\}$. This η_α belongs to the $\mathbb{Q}$-algebra QSym of quasisymmetric functions over $\mathbb{Q}$. If we let α range over all *odd compositions* (i.e., compositions $(\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ whose entries α_i are all odd), then the η_α form a basis of the peak algebra over $\mathbb{Q}$.

In this form, the power series η_α have been introduced by Hsiao ([Hsiao07, Proposition 2.1], although his η_α differ from ours by a sign), who called them the *monomial peak functions*. Hsiao computed their products, coproducts (in the sense of Hopf algebra) and antipodes, and obtained some structural results for the peak algebra.

In this paper, we generalize the η_α by replacing the power of 2 in (1) by a power of an arbitrary element r of the base ring. We furthermore study the resulting quasisymmetric functions for all compositions α (not only for the odd ones). Thus we obtain a new family $\left(\eta_\alpha^{(q)}\right)_{\alpha \in \mathrm{Comp}}$ of quasisymmetric functions for each element q of the base ring. When $r := q + 1$ is invertible, this family is a basis of QSym. It generalizes Hoffman's "essential quasi-symmetric functions" (obtained for $q = 0$) and Hsiao's monomial peak functions (obtained for $q = 1$), but also includes the monomial quasisymmetric functions as a limiting case.

We call our functions $\eta_\alpha^{(q)}$ the *enriched q-monomial quasisymmetric functions*. We describe them by several formulas, and compute their products, coproducts and antipodes (generalizing Hsiao's results). The product expansion is the most interesting one, as it is given by an exotic variant of the shuffle product which we call the "stufufuffle product" due to its ability to pick several consecutive entries from each composition. This "stufufuffle product" has previously appeared in recent work by Bouillot, Novelli and Thibon [BoNoTh22, (1)], where it was proposed as a generalization of the "block shuffle product" from the theory of multizeta values ([HirSat22]). While the authors of [BoNoTh22] have already found a basis of QSym that multiplies according to this product, ours is simpler and more natural. The coproduct and antipode formulas for $\eta_\alpha^{(q)}$ are fairly simple (the coproduct is given by deconcatenation, whereas the antipode involves the parameter q being replaced by its reciprocal $1/q$ and the composition α being reversed). We also express the functions $\eta_\alpha^{(q)}$ in terms of the monomial and fundamental bases of QSym and vice versa. Finally, we discuss how Hopf subalgebras of QSym can be constructed by picking a subset of the set of all compositions. (This generalizes the peak subalgebra.)

This paper is the first of (at least) two. The next shall extend the theory of extended P-partitions to incorporate the parameter q, which will shed a new light onto the enriched q-monomial quasisymmetric functions $\eta_\alpha^{(q)}$ while also leading to a new basis of QSym.

Several results in this paper have appeared (mostly without proof) in the extended abstracts [GriVas21] and [GriVas22].

1.1. Structure of the paper

This paper is organized as follows:

We begin by recalling the definition of quasisymmetric functions (and some concomitant notions) in Section 2.

Then, in Section 3, we define the quasisymmetric functions $\eta_\alpha^{(q)}$ and prove their simplest properties (conversion formulas to the M- and L-bases, formulas for antipode and coproduct). In particular, we show that the family of these functions $\eta_\alpha^{(q)}$ (where α ranges over all compositions) forms a basis of QSym if and only if $r := q + 1$ is invertible in the base ring.

Consequently, in Section 4, we introduce and study the basis of NSym dual to this basis of QSym.

In Section 5, we use this to express the product $\eta_\delta^{(q)} \eta_\varepsilon^{(q)}$ in three equivalent ways.

Finally, we discuss some applications in Section 6, and establish one last formula for $\eta_\alpha^{(q)}$ in Section 7.

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2. Quasisymmetric functions

2.1. Formal power series and quasisymmetry

We will use some of the standard notations from [GriRei20, Chapter 5]. Namely:

- We let $\mathbb{N} = \{0, 1, 2, \ldots\}$.
- We fix a commutative ring $\mathbf{k}$.
- We consider the ring $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ of formal power series in countably many commuting variables $x_1, x_2, x_3, \ldots$. A *monomial* shall mean a formal expression of the form $x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \cdots$, where $\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots) \in \mathbb{N}^\infty$ is a sequence of nonnegative integers such that only finitely many α_i are positive. Formal power series are formal infinite $\mathbf{k}$-linear combinations of such monomials.
- Each monomial $x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \cdots$ has degree $\alpha_1 + \alpha_2 + \alpha_3 + \cdots$.
- A formal power series $f \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ is said to be *of bounded degree* if there exists some $d \in \mathbb{N}$ such that each monomial in f has degree $\leq d$ (that is, each monomial of degree $> d$ has coefficient 0 in f).

For example, the formal power series $(x_1 + x_2 + x_3 + \cdots)^3$ is of bounded degree, but the formal power series $\dfrac{1}{1 - x_1} = 1 + x_1 + x_1^2 + x_1^3 + \cdots$ is not.

We shall now introduce the notion of *pack-equivalent monomials*. Let us first illustrate it by an example:

Example 2.1. Question: What do the monomials $x_1^4 x_3^7 x_4 x_9^2$ and $x_3^4 x_4^7 x_{10} x_{16}^2$ and $x_5^4 x_6^7 x_8 x_9^2$ have in common (but not in common with $x_1^7 x_3^4 x_4 x_9^2$) ?

Answer: They have the same sequence of nonzero exponents (when the variables are ordered in increasing order – i.e., if $i < j$, then x_i goes before x_j). Or, to put it differently, they all have the form $x_a^4 x_b^7 x_c x_d^2$ for some $a < b < c < d$. We shall call such pairs of monomials *pack-equivalent*.

Let us define this concept more rigorously:

Definition 2.2. Two monomials $\mathfrak{m}$ and $\mathfrak{n}$ are said to be *pack-equivalent* if they can be written in the forms

$$\mathfrak{m} = x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_\ell}^{a_\ell} \qquad \text{and} \qquad \mathfrak{n} = x_{j_1}^{a_1} x_{j_2}^{a_2} \cdots x_{j_\ell}^{a_\ell}$$

for some $\ell \in \mathbb{N}$, some positive integers $a_1, a_2, \ldots, a_\ell$ and two strictly increasing ℓ-tuples $(i_1 < i_2 < \cdots < i_\ell)$ and $(j_1 < j_2 < \cdots < j_\ell)$ of positive integers.

For example, the monomials $x_1^4 x_3^7 x_4 x_9^2$ and $x_3^4 x_4^7 x_{10} x_{16}^2$ are pack-equivalent, since they can be written as $x_1^4 x_3^7 x_4 x_9^2 = x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_\ell}^{a_\ell}$ and $x_3^4 x_4^7 x_{10} x_{16}^2 = x_{j_1}^{a_1} x_{j_2}^{a_2} \cdots x_{j_\ell}^{a_\ell}$ for $\ell = 4$ and $(a_1, a_2, \ldots, a_\ell) = (4, 7, 1, 2)$ and $(i_1 < i_2 < \cdots < i_\ell) = (1, 3, 4, 9)$ and $(j_1 < j_2 < \cdots < j_\ell) = (3, 4, 10, 16)$.

We are now ready to define the quasisymmetric functions:

Definition 2.3. **(a)** A formal power series $f \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ is said to be *quasisymmetric* if it has the property that any two pack-equivalent monomials have the same coefficient in f (that is: if $\mathfrak{m}$ and $\mathfrak{n}$ are two pack-equivalent monomials, then the coefficient of $\mathfrak{m}$ in f equals the coefficient of $\mathfrak{n}$ in f).

(b) A *quasisymmetric function* means a formal power series $f \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ that is quasisymmetric and of bounded degree.

Quasisymmetric functions have been introduced by Gessel in [Gessel84] (for $\mathbf{k} = \mathbb{Z}$ at least, but the general case is not much different). Introductions to their theory can be found in [GriRei20, Chapters 5–6], [Stanle24, §7.19], [Sagan20, Chapter 8], [Malven93, §4] and various other texts.

It is known (see [Malven93, Corollaire 4.7] or [GriRei20, Proposition 5.1.3]) that the set of all quasisymmetric functions is a $\mathbf{k}$-subalgebra of $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$. It is denoted by QSym and called the *ring of quasisymmetric functions*. It has several bases (as a $\mathbf{k}$-module), most of which are indexed by compositions.

2.2. Compositions

A *composition* means a finite list $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ of positive integers. The set of all compositions is denoted by Comp. The *empty composition* $\varnothing$ is the composition $()$, which is a 0-tuple.

The *length* $\ell(\alpha)$ of a composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ is defined to be the number k.

If $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ is a composition, then the nonnegative integer $\alpha_1 + \alpha_2 + \cdots + \alpha_k$ is called the *size* of α and is denoted by $|\alpha|$. For any $n \in \mathbb{N}$, we define a *composition of n* to be a composition that has size n. We let Comp_n be the set of all compositions of n (for given $n \in \mathbb{N}$). For example, $(1, 5, 2, 1)$ is a composition with size 9 (since $|(1, 5, 2, 1)| = 1 + 5 + 2 + 1 = 9$), so that $(1, 5, 2, 1) \in \mathrm{Comp}_9$.

For any $n \in \mathbb{Z}$, we let $[n]$ denote the set $\{1, 2, \ldots, n\}$. This set is empty whenever $n \leq 0$, and otherwise has size n.

It is well-known that any positive integer n has exactly 2^{n-1} compositions. This has a standard bijective proof ("stars and bars") which is worth recalling, as the bijection itself will be used a lot:

Definition 2.4. Let $n \in \mathbb{N}$. Let $\mathcal{P}([n-1])$ be the powerset of $[n-1]$ (that is, the set of all subsets of $[n-1]$).

(a) We define a map $D : \operatorname{Comp}_n \to \mathcal{P}([n-1])$ by

$$\begin{aligned} D(\alpha_1, \alpha_2, \ldots, \alpha_k) &= \{\alpha_1 + \alpha_2 + \cdots + \alpha_i \mid i \in [k-1]\} \\ &= \{\alpha_1,\ \alpha_1 + \alpha_2,\ \alpha_1 + \alpha_2 + \alpha_3,\ \ldots,\ \alpha_1 + \alpha_2 + \cdots + \alpha_{k-1}\}. \end{aligned}$$

(b) We define a map $\operatorname{comp} : \mathcal{P}([n-1]) \to \operatorname{Comp}_n$ as follows: For any $I \in \mathcal{P}([n-1])$, we set

$$\operatorname{comp}(I) = (i_1 - i_0, i_2 - i_1, \ldots, i_m - i_{m-1}),$$

where $i_0, i_1, \ldots, i_m$ are the elements of the set $I \cup \{0, n\}$ listed in increasing order (so that $i_0 < i_1 < \cdots < i_m$, therefore $i_0 = 0$ and $i_m = n$ and $\{i_1 < i_2 < \cdots < i_{m-1}\} = I$).

The maps D and comp are mutually inverse bijections. (See [Grinbe15, detailed version, Proposition 10.17] for a detailed proof of this.)

For example, for $n = 8$, we have $D(2,1,3,2) = \{2,\ 2+1,\ 2+1+3\} = \{2,3,6\}$ and $\operatorname{comp}\{2,3,6\} = (2-0,\ 3-2,\ 6-3,\ 8-6) = (2,1,3,2)$. Note that the meaning of $\operatorname{comp}(I)$ for a given set I depends on n, and thus the notation is ambiguous unless n is specified. In contrast, the notation $D(\alpha)$ is unambiguous, since $\alpha \in \operatorname{Comp}_n$ uniquely determines n to be $|\alpha|$.

The notation D in Definition 2.4 presumably originates in the word "**d**escent", but the connection between D and actual descents is indirect and rather misleading. We prefer to call D the "partial sum map" (as $D(\alpha)$ consists of the partial sums of the composition α) and its inverse comp the "interstitial map" (as $\operatorname{comp}(I)$ consists of the lengths of the intervals into which the elements of I split the interval $[n]$). Note that Stanley, in [Stanle24, §7.19], writes S_α for $D(\alpha)$ and writes $\operatorname{co}(I)$ for $\operatorname{comp}(I)$.

Note that every composition α of size $|\alpha| > 0$ satisfies $|D(\alpha)| = \ell(\alpha) - 1$, so that $|D(\alpha)| + 1 = \ell(\alpha)$. But this fails if α is the empty composition $\varnothing = ()$ (since $D() = \varnothing$ and $\ell() = 0$).

2.3. The monomial and fundamental bases of QSym

We will only need two bases of QSym: the monomial basis and the fundamental basis.

If $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ is a composition, then we define the *monomial quasisymmetric function* $M_\alpha \in \operatorname{QSym}$ by

$$M_\alpha = \sum_{i_1 < i_2 < \cdots < i_\ell} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell} = \sum_{\substack{\mathfrak{m} \text{ is a monomial} \\ \text{pack-equivalent} \\ \text{to } x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_\ell^{\alpha_\ell}}} \mathfrak{m}. \tag{2}$$

For example,

$$M_{(2,1)} = \sum_{i<j} x_i^2 x_j = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1^2 x_4 + x_2^2 x_4 + x_3^2 x_4 + \cdots .$$

The family $(M_\alpha)_{\alpha \in \operatorname{Comp}}$ is a basis of the $\mathbf{k}$-module QSym, and is known as the *monomial basis* of QSym.

For any composition α, we define the *fundamental quasisymmetric function* $L_\alpha \in \operatorname{QSym}$ by

$$L_\alpha = \sum_{\substack{\beta \in \operatorname{Comp}_n; \\ D(\beta) \supseteq D(\alpha)}} M_\beta, \tag{3}$$

where $n = |\alpha|$ (so that $\alpha \in \operatorname{Comp}_n$). It is not hard to rewrite this as[1]

$$L_\alpha = \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_n; \\ i_j < i_{j+1} \text{ whenever } j \in D(\alpha)}} x_{i_1} x_{i_2} \cdots x_{i_n} \tag{4}$$

(again with $n = |\alpha|$). This quasisymmetric function L_α was originally called F_α in Gessel's paper [Gessel84] (and in some later work such as [Malven93]), but the notation L_α has since spread more widely.

The family $(L_\alpha)_{\alpha \in \operatorname{Comp}}$ is a basis of the $\mathbf{k}$-module QSym, and is known as the *fundamental basis* of QSym.

3. The enriched q-monomial functions

3.1. Definition and restatements

Convention 3.1. From now on, we fix an element q of the base ring $\mathbf{k}$. We set

$$r := q + 1.$$

We shall now introduce a new family of quasisymmetric functions depending on q:

[1] See [Grinbe15, detailed version, Corollary 10.18] for a proof that the right hand side of (4) equals the right hand side of (3).

Definition 3.2. For any $n \in \mathbb{N}$ and any composition $\alpha \in \operatorname{Comp}_n$, we define a quasisymmetric function $\eta_\alpha^{(q)} \in \mathrm{QSym}$ by

$$\eta_\alpha^{(q)} = \sum_{\substack{\beta \in \operatorname{Comp}_n; \\ D(\beta) \subseteq D(\alpha)}} r^{\ell(\beta)} M_\beta. \tag{5}$$

We shall refer to $\eta_\alpha^{(q)}$ as the *enriched q-monomial function* corresponding to α.

Example 3.3.

(a) Setting $n = 5$ and $\alpha = (1,3,1)$ in this definition, we obtain

$$\begin{aligned} \eta_{(1,3,1)}^{(q)} &= \sum_{\substack{\beta \in \operatorname{Comp}_5; \\ D(\beta) \subseteq D(1,3,1)}} r^{\ell(\beta)} M_\beta \\ &= \sum_{\substack{\beta \in \operatorname{Comp}_5; \\ D(\beta) \subseteq \{1,4\}}} r^{\ell(\beta)} M_\beta \qquad (\text{since } D(1,3,1) = \{1,4\}) \\ &= r^{\ell(5)} M_{(5)} + r^{\ell(1,4)} M_{(1,4)} + r^{\ell(4,1)} M_{(4,1)} + r^{\ell(1,3,1)} M_{(1,3,1)} \end{aligned}$$

(since the compositions $\beta \in \operatorname{Comp}_5$ satisfying $D(\beta) \subseteq \{1,4\}$ are (5), $(1,4)$, $(4,1)$ and $(1,3,1)$). This simplifies to

$$\eta_{(1,3,1)}^{(q)} = rM_{(5)} + r^2 M_{(1,4)} + r^2 M_{(4,1)} + r^3 M_{(1,3,1)}.$$

(b) For any positive integer n, we have

$$\eta_{(n)}^{(q)} = rM_{(n)},$$

because the only composition $\beta \in \operatorname{Comp}_n$ satisfying $D(\beta) \subseteq D(n)$ is the composition (n) itself (since $D(n)$ is the empty set $\varnothing$) and has length $\ell(n) = 1$. Likewise, the empty composition $\varnothing = ()$ satisfies

$$\eta_\varnothing^{(q)} = M_\varnothing = 1.$$

The quasisymmetric function $\eta_\alpha^{(q)}$ generalizes several known power series. For $q = 0$, the series $\eta_\alpha^{(q)} = \eta_\alpha^{(0)}$ is the "essential quasi-symmetric function" E_I (for $I = D(\alpha)$) defined in [Hoffma15, (8)]. When α is an odd composition (i.e., all entries of α are odd) and $q = 1$, the series $\eta_\alpha^{(q)} = \eta_\alpha^{(1)}$ is precisely the η_α defined in [AgBeSo14, (6.1)], and differs only in sign from the η_α given in [Hsiao07, (2.1)] (because of [Hsiao07, Proposition 2.1]). (This is the reason for the notation $\eta_\alpha^{(q)}$.)

Finally, in an appropriate sense, we can view M_α as the "$q \to \infty$ limit" of $\eta_\alpha^{(q)}$; to be precise, this is saying that when $\eta_\alpha^{(q)}$ is considered as a polynomial in q (over QSym), its leading term is $q^{\ell(\alpha)} M_\alpha$ (which is obvious from (5) and $r = q + 1$).

The following two propositions are essentially restatements of (5):

Proposition 3.4. Let $n \in \mathbb{N}$ and $\alpha \in \operatorname{Comp}_n$. Then,

$$\eta_\alpha^{(q)} = \sum_{\substack{g_1 \le g_2 \le \cdots \le g_n; \\ g_i = g_{i+1} \text{ for each } i \in [n-1] \setminus D(\alpha)}} r^{|\{g_1, g_2, \ldots, g_n\}|} x_{g_1} x_{g_2} \cdots x_{g_n}, \tag{6}$$

where the sum is over all weakly increasing n-tuples $(g_1 \le g_2 \le \cdots \le g_n)$ of positive integers that satisfy $(g_i = g_{i+1}$ for each $i \in [n-1] \setminus D(\alpha))$.

Proof. If $\mathbf{g} = (g_1 \le g_2 \le \cdots \le g_n)$ is a weakly increasing n-tuple of positive integers, then we let $\operatorname{Asc} \mathbf{g}$ denote the set of all $i \in [n-1]$ satisfying $g_i < g_{i+1}$ (or, equivalently, $g_i \ne g_{i+1}$). Thus, $\operatorname{Asc} \mathbf{g}$ is the set of all positions at which the entries of $\mathbf{g}$ get larger (or, equivalently, change values). For instance, $\operatorname{Asc}(3,3,4,6,8,8,8) = \{2,3,4\}$.

Let $\beta \in \operatorname{Comp}_n$. Write β as $\beta = (\beta_1, \beta_2, \ldots, \beta_\ell)$; thus, $\ell(\beta) = \ell$ and $\beta_1 + \beta_2 + \cdots + \beta_\ell = n$ and

$$M_\beta = \sum_{i_1 < i_2 < \cdots < i_\ell} x_{i_1}^{\beta_1} x_{i_2}^{\beta_2} \cdots x_{i_\ell}^{\beta_\ell} \tag{7}$$

(by the definition of M_β). Moreover, the definition of $D(\beta)$ yields that

$$\begin{aligned} D(\beta) &= \{\beta_1,\ \beta_1 + \beta_2,\ \beta_1 + \beta_2 + \beta_3,\ \ldots,\ \beta_1 + \beta_2 + \cdots + \beta_{\ell-1}\} \\ &= \{\beta_1 < \beta_1 + \beta_2 < \beta_1 + \beta_2 + \beta_3 < \cdots < \beta_1 + \beta_2 + \cdots + \beta_{\ell-1}\}; \end{aligned}$$

thus, the elements of $D(\beta)$ subdivide the interval $\{1, 2, \ldots, n\}$ into ℓ subintervals of sizes $\beta_1, \beta_2, \ldots, \beta_\ell$ (from left to right).

A weakly increasing n-tuple $\mathbf{g} = (g_1 \le g_2 \le \cdots \le g_n)$ satisfies $\operatorname{Asc} \mathbf{g} = D(\beta)$ if and only if its entries are constant on each of these ℓ subintervals (i.e., we must have $g_i = g_j$ whenever i and j belong to the same subinterval) but strictly increase as we pass from one subinterval to the next (since $\operatorname{Asc} \mathbf{g}$ is the set of all positions at which the entries of $\mathbf{g}$ change values). Hence, a weakly increasing n-tuple $\mathbf{g} = (g_1 \le g_2 \le \cdots \le g_n)$ satisfies $\operatorname{Asc} \mathbf{g} = D(\beta)$ if and only if it begins with β_1 copies of a positive integer i_1, then continues with β_2 copies of a larger positive integer i_2, then continues further with β_3 copies of a yet-larger positive integer i_3, and so on. Consequently, for any weakly increasing n-tuple $\mathbf{g} = (g_1 \le g_2 \le \cdots \le g_n)$ that satisfies $\operatorname{Asc} \mathbf{g} = D(\beta)$, we can rewrite the monomial $x_{g_1} x_{g_2} \cdots x_{g_n}$ as $x_{i_1}^{\beta_1} x_{i_2}^{\beta_2} \cdots x_{i_\ell}^{\beta_\ell}$ using these positive integers $i_1 < i_2 < \cdots < i_\ell$.

Conversely, any monomial of the form $x_{i_1}^{\beta_1} x_{i_2}^{\beta_2} \cdots x_{i_\ell}^{\beta_\ell}$ with $i_1 < i_2 < \cdots < i_\ell$ can be rewritten in the form $x_{g_1} x_{g_2} \cdots x_{g_n}$ for a unique weakly increasing n-tuple $\mathbf{g} = (g_1 \le g_2 \le \cdots \le g_n)$ that satisfies $\operatorname{Asc} \mathbf{g} = D(\beta)$ (indeed, it has degree $\beta_1 + \beta_2 +$

$\cdots + \beta_\ell = n$, and so can be written in the form $x_{g_1} x_{g_2} \cdots x_{g_n}$ for a unique weakly increasing n-tuple $\mathbf{g} = (g_1 \leq g_2 \leq \cdots \leq g_n)$; but then, the property $\operatorname{Asc} \mathbf{g} = D(\beta)$ follows from the arguments in the previous paragraph).

Combining the results of the previous two paragraphs, we conclude that the monomials $x_{i_1}^{\beta_1} x_{i_2}^{\beta_2} \cdots x_{i_\ell}^{\beta_\ell}$ that appear on the right hand side of (7) are precisely the monomials $x_{g_1} x_{g_2} \cdots x_{g_n}$ for all weakly increasing n-tuples $\mathbf{g} = (g_1 \leq g_2 \leq \cdots \leq g_n)$ satisfying $\operatorname{Asc} \mathbf{g} = D(\beta)$. Therefore, we can rewrite (7) as

$$M_\beta = \sum_{\substack{\mathbf{g}=(g_1 \leq g_2 \leq \cdots \leq g_n);\\ \operatorname{Asc} \mathbf{g} = D(\beta)}} x_{g_1} x_{g_2} \cdots x_{g_n}. \tag{8}$$

Let us furthermore observe the following: If a weakly increasing n-tuple $\mathbf{g} = (g_1 \leq g_2 \leq \cdots \leq g_n)$ satisfies $\operatorname{Asc} \mathbf{g} = D(\beta)$, then

$$|\{g_1, g_2, \ldots, g_n\}| = \ell$$

(since we have proved above that the monomial $x_{g_1} x_{g_2} \cdots x_{g_n}$ can be written as $x_{i_1}^{\beta_1} x_{i_2}^{\beta_2} \cdots x_{i_\ell}^{\beta_\ell}$ for some integers $i_1 < i_2 < \cdots < i_\ell$, and thus contains exactly ℓ distinct indeterminates, but this is saying precisely that the set $\{g_1, g_2, \ldots, g_n\}$ has exactly ℓ distinct elements) and thus

$$\ell(\beta) = \ell = |\{g_1, g_2, \ldots, g_n\}|. \tag{9}$$

Now, multiplying the equality (8) by $r^{\ell(\beta)}$, we find

$$\begin{aligned} r^{\ell(\beta)} M_\beta &= \sum_{\substack{\mathbf{g}=(g_1 \leq g_2 \leq \cdots \leq g_n);\\ \operatorname{Asc} \mathbf{g} = D(\beta)}} \underbrace{r^{\ell(\beta)}}_{\substack{= r^{|\{g_1, g_2, \ldots, g_n\}|} \\ \text{(by (9))}}} x_{g_1} x_{g_2} \cdots x_{g_n} \\ &= \sum_{\substack{\mathbf{g}=(g_1 \leq g_2 \leq \cdots \leq g_n);\\ \operatorname{Asc} \mathbf{g} = D(\beta)}} r^{|\{g_1, g_2, \ldots, g_n\}|} x_{g_1} x_{g_2} \cdots x_{g_n}. \end{aligned} \tag{10}$$

Forget that we fixed β. We thus have proved (10) for each $\beta \in \operatorname{Comp}_n$. Now, (5) becomes

$$\begin{aligned}
\eta_\alpha^{(q)} &= \sum_{\substack{\beta\in\operatorname{Comp}_n;\\ D(\beta)\subseteq D(\alpha)}} r^{\ell(\beta)} M_\beta \\
&= \sum_{\substack{\beta\in\operatorname{Comp}_n;\\ D(\beta)\subseteq D(\alpha)}} \sum_{\substack{\mathbf{g}=(g_1\leq g_2\leq\cdots\leq g_n);\\ \operatorname{Asc}\mathbf{g}=D(\beta)}} r^{\left|\{g_1,g_2,\ldots,g_n\}\right|} x_{g_1}x_{g_2}\cdots x_{g_n} \qquad \text{(by (10))} \\
&= \sum_{\substack{I\subseteq[n-1];\\ I\subseteq D(\alpha)}} \sum_{\substack{\mathbf{g}=(g_1\leq g_2\leq\cdots\leq g_n);\\ \operatorname{Asc}\mathbf{g}=I}} r^{\left|\{g_1,g_2,\ldots,g_n\}\right|} x_{g_1}x_{g_2}\cdots x_{g_n} \\
&\qquad \left(\begin{array}{c}\text{here, we have substituted } I \text{ for } D(\beta) \text{ in the first sum,}\\ \text{since the map } D:\operatorname{Comp}_n\to\mathcal{P}([n-1]) \text{ is a bijection}\end{array}\right) \\
&= \sum_{\substack{\mathbf{g}=(g_1\leq g_2\leq\cdots\leq g_n);\\ \operatorname{Asc}\mathbf{g}\subseteq D(\alpha)}} r^{\left|\{g_1,g_2,\ldots,g_n\}\right|} x_{g_1}x_{g_2}\cdots x_{g_n} \\
&= \sum_{\substack{\mathbf{g}=(g_1\leq g_2\leq\cdots\leq g_n);\\ g_i=g_{i+1} \text{ for each } i\in[n-1]\setminus D(\alpha)}} r^{\left|\{g_1,g_2,\ldots,g_n\}\right|} x_{g_1}x_{g_2}\cdots x_{g_n}
\end{aligned}$$

(since the condition "$\operatorname{Asc}\mathbf{g}\subseteq D(\alpha)$" imposed on a weakly increasing n-tuple $\mathbf{g} = (g_1\leq g_2\leq\cdots\leq g_n)$ is equivalent to the condition "$g_i = g_{i+1}$ for each $i\in[n-1]\setminus D(\alpha)$"). This proves Proposition 3.4. $\square$

Proposition 3.5. Let $\alpha = (\alpha_1,\alpha_2,\ldots,\alpha_\ell)\in\operatorname{Comp}$. Then,

$$\eta_\alpha^{(q)} = \sum_{i_1\leq i_2\leq\cdots\leq i_\ell} r^{\left|\{i_1,i_2,\ldots,i_\ell\}\right|} x_{i_1}^{\alpha_1}x_{i_2}^{\alpha_2}\cdots x_{i_\ell}^{\alpha_\ell}, \tag{11}$$

where the sum is over all weakly increasing ℓ-tuples $(i_1\leq i_2\leq\cdots\leq i_\ell)$ of positive integers.

Proof. Let $n=|\alpha|$, so that $\alpha\in\operatorname{Comp}_n$. Clearly, it suffices to show that the right hand sides of (6) and (11) are identical.

We claim that the monomials $x_{g_1}x_{g_2}\cdots x_{g_n}$ for all n-tuples[2] $(g_1\leq g_2\leq\cdots\leq g_n)$ satisfying $(g_i=g_{i+1}$ for each $i\in[n-1]\setminus D(\alpha))$ are precisely the monomials of the form $x_{i_1}^{\alpha_1}x_{i_2}^{\alpha_2}\cdots x_{i_\ell}^{\alpha_\ell}$ for all ℓ-tuples $(i_1\leq i_2\leq\cdots\leq i_\ell)$.

Indeed, let us set

$$s_k := \alpha_1+\alpha_2+\cdots+\alpha_k \qquad \text{for each } k\in[\ell].$$

Thus, $D(\alpha)=\{s_1,s_2,\ldots,s_{\ell-1}\}$ and $s_\ell = |\alpha| = n$. Hence, an n-tuple $(g_1\leq g_2\leq\cdots\leq g_n)$ satisfies

$$(g_i = g_{i+1} \text{ for each } i\in[n-1]\setminus D(\alpha))$$

[2] We are no longer saying "weakly increasing", since this is automatically implied by the notation $(g_1\leq g_2\leq\cdots\leq g_n)$.

if and only if it satisfies

$$\begin{aligned} & g_1 = g_2 = \cdots = g_{s_1} \\ \leq\ & g_{s_1+1} = g_{s_1+2} = \cdots = g_{s_2} \\ \leq\ & g_{s_2+1} = g_{s_2+2} = \cdots = g_{s_3} \\ \leq\ & \cdots \\ \leq\ & g_{s_{\ell-1}+1} = g_{s_{\ell-1}+2} = \cdots = g_{s_\ell} \end{aligned}$$

(that is, if its entries can grow only at the positions $s_1, s_2, \ldots, s_{\ell-1}$). Therefore, such an n-tuple is uniquely determined by its "essential" entries $g_{s_1}, g_{s_2}, \ldots, g_{s_\ell}$. More precisely, there is a bijection from the set

$$\{n\text{-tuples } (g_1 \leq g_2 \leq \cdots \leq g_n) \text{ satisfying } (g_i = g_{i+1} \text{ for each } i \in [n-1] \setminus D(\alpha))\}$$

to the set

$$\{\ell\text{-tuples } (i_1 \leq i_2 \leq \cdots \leq i_\ell)\},$$

which sends each n-tuple $(g_1 \leq g_2 \leq \cdots \leq g_n)$ from the former set to its "subword" $(g_{s_1} \leq g_{s_2} \leq \cdots \leq g_{s_\ell})$ [3]. This bijection has the property that

$$x_{g_1} x_{g_2} \cdots x_{g_n} = x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell} \qquad \text{and} \qquad \{g_1, g_2, \ldots, g_n\} = \{i_1, i_2, \ldots, i_\ell\}$$

whenever it sends an n-tuple $(g_1 \leq g_2 \leq \cdots \leq g_n)$ to the ℓ-tuple $(i_1 \leq i_2 \leq \cdots \leq i_\ell) = (g_{s_1} \leq g_{s_2} \leq \cdots \leq g_{s_\ell})$ (because the n-tuple $(g_1 \leq g_2 \leq \cdots \leq g_n)$ consists of α_1 copies of i_1, followed by α_2 copies of i_2, followed by α_3 copies of i_3, and so on). This shows that the right hand sides of the equalities (6) and (11) are identical. Hence, (11) follows from (6). This proves Proposition 3.5. □

3.2. The $\eta_\alpha^{(q)}$ as a basis

The equality (5) writes each enriched q-monomial function $\eta_\alpha^{(q)}$ as a $\mathbf{k}$-linear combination of M_β's. Conversely, we can expand each monomial quasisymmetric function M_β as a $\mathbf{k}$-linear combination of $\eta_\alpha^{(q)}$'s, at least after multiplying it by $r^{\ell(\beta)}$:

[3] This bijection is precisely the bijection Φ from [Grinbe15, detailed version, proof of Proposition 10.69]. Indeed, our set

$$\{n\text{-tuples } (g_1 \leq g_2 \leq \cdots \leq g_n) \text{ satisfying } (g_i = g_{i+1} \text{ for each } i \in [n-1] \setminus D(\alpha))\}$$

is the set $\mathcal{I}$ from [Grinbe15, detailed version, proof of Proposition 10.69] (since the condition "$g_i = g_{i+1}$ for each $i \in [n-1] \setminus D(\alpha)$" can be rewritten as "$\{j \in [n-1] \mid g_j < g_{j+1}\} \subseteq D(\alpha)$"), whereas our set

$$\{\ell\text{-tuples } (i_1 \leq i_2 \leq \cdots \leq i_\ell)\}$$

is the set $\mathcal{J}$ from [Grinbe15, detailed version, proof of Proposition 10.69].

Proposition 3.6. Let $n \in \mathbb{N}$. Let $\beta \in \operatorname{Comp}_n$ be a composition. Then,

$$r^{\ell(\beta)} M_\beta = \sum_{\substack{\alpha \in \operatorname{Comp}_n; \\ D(\alpha) \subseteq D(\beta)}} (-1)^{\ell(\beta)-\ell(\alpha)} \eta_\alpha^{(q)}.$$

For the proof of this proposition (and some later ones as well), we will need the *Iverson bracket notation*:

Convention 3.7. If $\mathcal{A}$ is a logical statement, then $[\mathcal{A}]$ shall denote the truth value of $\mathcal{A}$ (that is, the number 1 if $\mathcal{A}$ is true, and the number 0 if $\mathcal{A}$ is false).

For example, $[2+2=4] = 1$ and $[2+2=5] = 0$.

The following lemma is a classical elementary property of finite sets, but we recall its proof for the sake of completeness:

Lemma 3.8. Let S and T be two finite sets. Then,

$$\sum_{\substack{I \subseteq S; \\ T \subseteq I}} (-1)^{|S|-|I|} = [S = T].$$

Proof. If $T \not\subseteq S$, then the left hand side is an empty sum and thus equals 0, as does the right hand side. Hence, for the rest of this proof, we WLOG assume that $T \subseteq S$. Hence, $S \setminus T = \varnothing$ holds if and only if $S = T$. Therefore, $[S \setminus T = \varnothing] = [S = T]$.

Each subset I of S can be uniquely written as $S \setminus J$ for a unique subset J of S (namely, this J is the complement of I in S). Thus, we can reindex the sum $\sum_{\substack{I \subseteq S; \\ T \subseteq I}} (-1)^{|S|-|I|}$ by substituting $S \setminus J$ for I. Thus, we obtain

$$\begin{aligned}
\sum_{\substack{I \subseteq S; \\ T \subseteq I}} (-1)^{|S|-|I|} &= \sum_{\substack{J \subseteq S; \\ T \subseteq S \setminus J}} \underbrace{(-1)^{|S|-|S \setminus J|}}_{\substack{=(-1)^{|J|} \\ (\text{since } |S|-|S\setminus J|=|J|)}} = \sum_{\substack{J \subseteq S; \\ T \subseteq S \setminus J}} (-1)^{|J|} = \sum_{\substack{J \subseteq S; \\ J \subseteq S \setminus T}} (-1)^{|J|} \\
&\qquad \left(\begin{array}{c} \text{since the condition “} T \subseteq S \setminus J \text{”} \\ \text{is equivalent to “} J \subseteq S \setminus T \text{” when} \\ \text{both } J \text{ and } T \text{ are subsets of } S \end{array}\right) \\
&= \sum_{J \subseteq S \setminus T} (-1)^{|J|}
\end{aligned} \tag{12}$$

(since each subset $J \subseteq S \setminus T$ satisfies $J \subseteq S$).

However, a known fact about finite sets (see, e.g., [Grinbe20, Proposition 2.9.10]) says that $\sum_{I \subseteq S} (-1)^{|I|} = [S = \varnothing]$. Applying this fact to $S \setminus T$ instead of S, we obtain

$\sum_{I\subseteq S\setminus T} (-1)^{|I|} = [S \setminus T = \varnothing] = [S = T]$. Thus, (12) becomes

$$\sum_{\substack{I\subseteq S;\\ T\subseteq I}} (-1)^{|S|-|I|} = \sum_{J\subseteq S\setminus T} (-1)^{|J|} = \sum_{I\subseteq S\setminus T} (-1)^{|I|} = [S = T].$$

This proves Lemma 3.8. □

We will also use the following property of compositions:

Lemma 3.9. Let $n \in \mathbb{N}$. Then:

(a) We have $\ell(\delta) = |D(\delta)| + [n \neq 0]$ for each $\delta \in \operatorname{Comp}_n$.

(b) We have $\ell(\beta) - \ell(\alpha) = |D(\beta)| - |D(\alpha)|$ for any $\alpha \in \operatorname{Comp}_n$ and $\beta \in \operatorname{Comp}_n$.

Proof. **(a)** Part **(a)** follows easily from the definition of $D(\delta)$. For a detailed proof, see [GriVas23, Corollary 2.6].

(b) Use part **(a)**. For a detailed proof, see [GriVas23, Corollary 2.7]. □

Proof of Proposition 3.6. For each $\alpha \in \operatorname{Comp}_n$, we have

$$\eta_\alpha^{(q)} = \sum_{\substack{\gamma\in\operatorname{Comp}_n;\\ D(\gamma)\subseteq D(\alpha)}} r^{\ell(\gamma)} M_\gamma$$

(by (5), with β renamed as γ). Hence,

$$\begin{aligned}
&\sum_{\substack{\alpha\in\operatorname{Comp}_n;\\ D(\alpha)\subseteq D(\beta)}} (-1)^{\ell(\beta)-\ell(\alpha)} \eta_\alpha^{(q)} \\
&= \sum_{\substack{\alpha\in\operatorname{Comp}_n;\\ D(\alpha)\subseteq D(\beta)}} (-1)^{\ell(\beta)-\ell(\alpha)} \sum_{\substack{\gamma\in\operatorname{Comp}_n;\\ D(\gamma)\subseteq D(\alpha)}} r^{\ell(\gamma)} M_\gamma \\
&= \underbrace{\sum_{\substack{\alpha\in\operatorname{Comp}_n;\\ D(\alpha)\subseteq D(\beta)}} \sum_{\substack{\gamma\in\operatorname{Comp}_n;\\ D(\gamma)\subseteq D(\alpha)}}}_{=\sum_{\gamma\in\operatorname{Comp}_n} \sum_{\substack{\alpha\in\operatorname{Comp}_n;\\ D(\gamma)\subseteq D(\alpha)\subseteq D(\beta)}}} (-1)^{\ell(\beta)-\ell(\alpha)} r^{\ell(\gamma)} M_\gamma \\
&= \sum_{\gamma\in\operatorname{Comp}_n} r^{\ell(\gamma)} M_\gamma \sum_{\substack{\alpha\in\operatorname{Comp}_n;\\ D(\gamma)\subseteq D(\alpha)\subseteq D(\beta)}} (-1)^{\ell(\beta)-\ell(\alpha)}.
\end{aligned} \tag{13}$$

However, for each $\gamma \in \operatorname{Comp}_n$, we have

$$\sum_{\substack{\alpha\in\operatorname{Comp}_n;\\ D(\gamma)\subseteq D(\alpha)\subseteq D(\beta)}} \underbrace{(-1)^{\ell(\beta)-\ell(\alpha)}}_{\substack{=(-1)^{|D(\beta)|-|D(\alpha)|}\\ \text{(by Lemma 3.9 (b))}}}$$
$$= \sum_{\substack{\alpha\in\operatorname{Comp}_n;\\ D(\gamma)\subseteq D(\alpha)\subseteq D(\beta)}} (-1)^{|D(\beta)|-|D(\alpha)|} = \sum_{\substack{I\subseteq[n-1];\\ D(\gamma)\subseteq I\subseteq D(\beta)}} (-1)^{|D(\beta)|-|I|}$$
$$\left(\begin{array}{c}\text{here, we have substituted } I \text{ for } D(\alpha) \text{ in the sum,}\\ \text{since the map } D:\operatorname{Comp}_n\to\mathcal{P}([n-1]) \text{ is a bijection}\end{array}\right)$$
$$= \sum_{\substack{I\subseteq D(\beta);\\ D(\gamma)\subseteq I}} (-1)^{|D(\beta)|-|I|} \qquad \left(\begin{array}{c}\text{since } D(\beta)\subseteq[n-1], \text{ so that each subset } I\\ \text{of } D(\beta) \text{ is also a subset of } [n-1]\end{array}\right)$$
$$= [D(\beta)=D(\gamma)] \qquad (\text{by Lemma 3.8, applied to } S=D(\beta) \text{ and } T=D(\gamma))$$
$$= [\beta=\gamma] \qquad (\text{since the map } D \text{ is a bijection}).$$

Hence, (13) becomes

$$\sum_{\substack{\alpha\in\operatorname{Comp}_n;\\ D(\alpha)\subseteq D(\beta)}} (-1)^{\ell(\beta)-\ell(\alpha)}\,\eta_\alpha^{(q)} = \sum_{\gamma\in\operatorname{Comp}_n} r^{\ell(\gamma)} M_\gamma \underbrace{\sum_{\substack{\alpha\in\operatorname{Comp}_n;\\ D(\gamma)\subseteq D(\alpha)\subseteq D(\beta)}} (-1)^{\ell(\beta)-\ell(\alpha)}}_{=[\beta=\gamma]}$$
$$= \sum_{\gamma\in\operatorname{Comp}_n} r^{\ell(\gamma)} M_\gamma\,[\beta=\gamma] = r^{\ell(\beta)} M_\beta$$

(since the factor $[\beta=\gamma]$ in the sum ensures that the only nonzero addend in the sum is the addend for $\gamma=\beta$). This proves Proposition 3.6. □

Proposition 3.6 shows that the quasisymmetric functions $r^{\ell(\beta)} M_\beta$ for all $\beta \in \operatorname{Comp}$ are $\mathbf{k}$-linear combinations of the enriched q-monomial quasisymmetric functions $\eta_\alpha^{(q)}$. If r is invertible in $\mathbf{k}$, then it follows that the monomial quasisymmetric functions M_β are such combinations as well, and thus the family $\left(\eta_\alpha^{(q)}\right)_{\alpha\in\operatorname{Comp}}$ spans the $\mathbf{k}$-module QSym in this case. But we can actually say more:

Theorem 3.10. Assume that r is invertible in $\mathbf{k}$. Then:

(a) The family $\left(\eta_\alpha^{(q)}\right)_{\alpha\in\operatorname{Comp}}$ is a basis of the $\mathbf{k}$-module QSym.

(b) Let $n \in \mathbb{N}$. Consider the n-th graded component QSym_n of the graded $\mathbf{k}$-module QSym. Then, the family $\left(\eta_\alpha^{(q)}\right)_{\alpha\in\operatorname{Comp}_n}$ is a basis of the $\mathbf{k}$-module QSym_n.

Proof. **(b)** Define a partial order $\prec$ on the finite set Comp_n by setting

$$\beta \prec \alpha \qquad \text{if and only if} \qquad \ell(\beta) < \ell(\alpha).$$

If two compositions $\alpha, \beta \in \operatorname{Comp}_n$ satisfy $D(\beta) \subseteq D(\alpha)$ and $\beta \neq \alpha$, then $D(\beta)$ is a proper subset of $D(\alpha)$ (since D is a bijection, so that $\beta \neq \alpha$ entails $D(\beta) \neq D(\alpha)$), and thus we have $|D(\beta)| < |D(\alpha)|$ and therefore $\ell(\beta) < \ell(\alpha)$ (by Lemma 3.9 **(b)**) and thus

$$\beta \prec \alpha \tag{14}$$

(by the definition of the partial order $\prec$).

For each $\alpha \in \operatorname{Comp}_n$, we have

$$\begin{aligned}
\eta_\alpha^{(q)} &= \sum_{\substack{\beta \in \operatorname{Comp}_n; \\ D(\beta) \subseteq D(\alpha)}} r^{\ell(\beta)} M_\beta \qquad \left(\text{by the definition of } \eta_\alpha^{(q)}\right) \\
&= r^{\ell(\alpha)} M_\alpha + \sum_{\substack{\beta \in \operatorname{Comp}_n; \\ D(\beta) \subseteq D(\alpha); \\ \beta \neq \alpha}} r^{\ell(\beta)} M_\beta \\
&= r^{\ell(\alpha)} M_\alpha + \left(\text{a linear combination of } M_\beta \text{ with } \beta \in \operatorname{Comp}_n \text{ satisfying } \beta \prec \alpha\right)
\end{aligned}$$

(by (14)). Since r is invertible, this shows that the family $\left(\eta_\alpha^{(q)}\right)_{\alpha \in \operatorname{Comp}_n}$ expands invertibly triangularly in the family $(M_\alpha)_{\alpha \in \operatorname{Comp}_n}$ with respect to the partial order $\prec$ (where we are using the terminology from [GriRei20, §11.1]). Hence, a classical fact about triangular expansions ([GriRei20, Corollary 11.1.19(e)]) shows that the family $\left(\eta_\alpha^{(q)}\right)_{\alpha \in \operatorname{Comp}_n}$ is a basis of the $\mathbf{k}$-module QSym_n (since the family $(M_\alpha)_{\alpha \in \operatorname{Comp}_n}$ is a basis of QSym_n). This proves Theorem 3.10 **(b)**.

(a) Theorem 3.10 **(b)** says that the family $\left(\eta_\alpha^{(q)}\right)_{\alpha \in \operatorname{Comp}_n}$ is a basis of the $\mathbf{k}$-module QSym_n for each $n \in \mathbb{N}$. Hence, the family $\left(\eta_\alpha^{(q)}\right)_{\alpha \in \operatorname{Comp}}$ is a basis of the $\mathbf{k}$-module $\bigoplus_{n \in \mathbb{N}} \operatorname{QSym}_n = \operatorname{QSym}$. This proves Theorem 3.10 **(a)**. □

Theorem 3.10 **(a)** has a converse: If the family $\left(\eta_\alpha^{(q)}\right)_{\alpha \in \operatorname{Comp}}$ is a basis of QSym, then r is invertible. (This is already clear from considering its unique degree-1 entry $\eta_{(1)}^{(q)} = rM_{(1)}$.)

3.3. Relation to the fundamental basis

We can also expand the $\eta_\alpha^{(q)}$ in the fundamental basis and vice versa:

Proposition 3.11. Let n be a positive integer. Let $\alpha \in \operatorname{Comp}_n$. Then,

$$\eta_\alpha^{(q)} = r \sum_{\gamma \in \operatorname{Comp}_n} (-1)^{|D(\gamma) \setminus D(\alpha)|} q^{|D(\gamma) \cap D(\alpha)|} L_\gamma.$$

Proposition 3.12. Let n be a positive integer. Let $\gamma \in \operatorname{Comp}_n$. Then,

$$r^n L_\gamma = \sum_{\alpha \in \operatorname{Comp}_n} (-1)^{|D(\gamma) \setminus D(\alpha)|} q^{|[n-1] \setminus (D(\gamma) \cup D(\alpha))|} \eta_\alpha^{(q)}.$$

Note that Proposition 3.11 generalizes [Hsiao07, Proposition 2.2].
Both propositions can be proved by the help of a rather simple identity:[4]

Lemma 3.13. Let S and T be two finite sets. Then,

$$\sum_{I \subseteq S} (-1)^{|I \setminus T|} q^{|I \cap T|} = [S \subseteq T] \cdot r^{|S|}.$$

Proof of Lemma 3.13. We are in one of the following two cases:
Case 1: We have $S \subseteq T$.
Case 2: We have $S \not\subseteq T$.
Let us first consider Case 1. In this case, we have $S \subseteq T$. Hence, $[S \subseteq T] = 1$, so that $[S \subseteq T] \cdot r^{|S|} = 1 \cdot r^{|S|} = r^{|S|}$.
Also, each subset I of S satisfies $I \subseteq S \subseteq T$ and therefore satisfies $I \setminus T = \varnothing$ and $I \cap T = I$. Hence,

$$\begin{aligned} \sum_{I \subseteq S} \underbrace{(-1)^{|I \setminus T|} q^{|I \cap T|}}_{\substack{=(-1)^{|\varnothing|} q^{|I|} \\ (\text{since } I \setminus T = \varnothing \text{ and } I \cap T = I)}} &= \sum_{I \subseteq S} \underbrace{(-1)^{|\varnothing|}}_{=(-1)^0 = 1} q^{|I|} \\ &= \sum_{I \subseteq S} q^{|I|}. \end{aligned} \tag{15}$$

Now, consider the sum $\sum\limits_{I \subseteq S} q^{|I|}$ on the right hand side. This sum contains $\dbinom{|S|}{k}$ copies of each possible power q^k (since the number of k-element subsets of S is

[4] We will use Convention 3.7.

$\dbinom{|S|}{k}$), and thus can be rewritten as $\sum\limits_{k=0}^{|S|} \dbinom{|S|}{k} q^k$. Hence, (15) rewrites as

$$\begin{aligned}
\sum_{I \subseteq S} (-1)^{|I \setminus T|} q^{|I \cap T|} &= \sum_{k=0}^{|S|} \binom{|S|}{k} q^k \\
&= (1+q)^{|S|} \qquad \text{(by the binomial formula)} \\
&= r^{|S|} \qquad (\text{since } 1+q = q+1 = r) \\
&= [S \subseteq T] \cdot r^{|S|} \qquad \left(\text{since } [S \subseteq T] \cdot r^{|S|} = r^{|S|}\right).
\end{aligned}$$

Thus, Lemma 3.13 is proved in Case 1.

Let us now consider Case 2. In this case, we have $S \not\subseteq T$. Hence, $[S \subseteq T] = 0$, so that $[S \subseteq T] \cdot r^{|S|} = 0 \cdot r^{|S|} = 0$.

There exists some $s \in S$ such that $s \notin T$ (since $S \not\subseteq T$). Consider this s. Now, each subset I of S satisfies either $s \in I$ or $s \notin I$ (but not both). Hence,

$$\begin{aligned}
&\sum_{I \subseteq S} (-1)^{|I \setminus T|} q^{|I \cap T|} \\
&= \sum_{\substack{I \subseteq S; \\ s \in I}} (-1)^{|I \setminus T|} q^{|I \cap T|} + \sum_{\substack{I \subseteq S; \\ s \notin I}} (-1)^{|I \setminus T|} q^{|I \cap T|} \\
&= \sum_{\substack{I \subseteq S; \\ s \notin I}} \underbrace{(-1)^{|(I \cup \{s\}) \setminus T|}}_{\substack{= (-1)^{|I \setminus T|+1} \\ (\text{since } s \notin T \text{ and } s \notin I, \text{ so} \\ \text{that } |(I \cup \{s\}) \setminus T| = |I \setminus T|+1)}} \underbrace{q^{|(I \cup \{s\}) \cap T|}}_{\substack{= q^{|I \cap T|} \\ (\text{since } s \notin T \text{ and} \\ \text{thus } (I \cup \{s\}) \cap T = I \cap T)}} + \sum_{\substack{I \subseteq S; \\ s \notin I}} (-1)^{|I \setminus T|} q^{|I \cap T|} \\
&\qquad \left(\begin{array}{c} \text{here, we have substituted } I \cup \{s\} \text{ for } I \text{ in the first sum,} \\ \text{since the map } \{I \subseteq S \mid s \notin I\} \to \{I \subseteq S \mid s \in I\} \\ \text{that sends each } I \text{ to } I \cup \{s\} \text{ is a bijection} \end{array}\right) \\
&= \sum_{\substack{I \subseteq S; \\ s \notin I}} \underbrace{(-1)^{|I \setminus T|+1}}_{= -(-1)^{|I \setminus T|}} q^{|I \cap T|} + \sum_{\substack{I \subseteq S; \\ s \notin I}} (-1)^{|I \setminus T|} q^{|I \cap T|} \\
&= - \sum_{\substack{I \subseteq S; \\ s \notin I}} (-1)^{|I \setminus T|} q^{|I \cap T|} + \sum_{\substack{I \subseteq S; \\ s \notin I}} (-1)^{|I \setminus T|} q^{|I \cap T|} = 0 = [S \subseteq T] \cdot r^{|S|}
\end{aligned}$$

(since $[S \subseteq T] \cdot r^{|S|} = 0$). Thus, Lemma 3.13 is proved in Case 2. The proof of Lemma 3.13 is thus complete. □

Proof of Proposition 3.11. We begin by observing that

$$|D(\beta)| + 1 = \ell(\beta) \tag{16}$$

for every $\beta \in \operatorname{Comp}_n$ [5].

[5] *Proof.* Let $\beta \in \operatorname{Comp}_n$. From $n \neq 0$ (since n is positive), we obtain $[n \neq 0] = 1$. However, Lemma 3.9 **(a)** (applied to $\delta = \beta$) yields $\ell(\beta) = |D(\beta)| + \underbrace{[n \neq 0]}_{=1} = |D(\beta)| + 1$. This proves (16).

Let $T := D(\alpha)$. Thus, $D(\alpha) = T$, so that

$$\begin{aligned}
& r \sum_{\gamma \in \operatorname{Comp}_n} (-1)^{|D(\gamma) \setminus D(\alpha)|} q^{|D(\gamma) \cap D(\alpha)|} L_\gamma \\
&= r \sum_{\gamma \in \operatorname{Comp}_n} (-1)^{|D(\gamma) \setminus T|} q^{|D(\gamma) \cap T|} \underbrace{L_\gamma}_{\substack{= \sum\limits_{\substack{\beta \in \operatorname{Comp}_n; \\ D(\beta) \supseteq D(\gamma)}} M_\beta \\ \text{(by the definition of } L_\gamma)}} \\
&= r \sum_{\gamma \in \operatorname{Comp}_n} (-1)^{|D(\gamma) \setminus T|} q^{|D(\gamma) \cap T|} \sum_{\substack{\beta \in \operatorname{Comp}_n; \\ D(\beta) \supseteq D(\gamma)}} M_\beta \\
&= r \sum_{\beta \in \operatorname{Comp}_n} \sum_{\substack{\gamma \in \operatorname{Comp}_n; \\ D(\beta) \supseteq D(\gamma)}} (-1)^{|D(\gamma) \setminus T|} q^{|D(\gamma) \cap T|} M_\beta.
\end{aligned}$$

However, every $\beta \in \operatorname{Comp}_n$ satisfies

$$\begin{aligned}
& \sum_{\substack{\gamma \in \operatorname{Comp}_n; \\ D(\beta) \supseteq D(\gamma)}} (-1)^{|D(\gamma) \setminus T|} q^{|D(\gamma) \cap T|} \\
&= \sum_{\substack{I \subseteq [n-1]; \\ D(\beta) \supseteq I}} (-1)^{|I \setminus T|} q^{|I \cap T|} \\
& \qquad \left(\begin{array}{c} \text{here, we have substituted } I \text{ for } D(\gamma) \text{ in the sum,} \\ \text{since the map } D : \operatorname{Comp}_n \to \mathcal{P}([n-1]) \text{ is a bijection} \end{array} \right) \\
&= \sum_{I \subseteq D(\beta)} (-1)^{|I \setminus T|} q^{|I \cap T|} \qquad (\text{since } D(\beta) \subseteq [n-1]) \\
&= [D(\beta) \subseteq T] \cdot r^{|D(\beta)|}
\end{aligned}$$

(by Lemma 3.13, applied to $S = D(\beta)$). Hence, this becomes

$$\begin{aligned}
& r \sum_{\gamma \in \operatorname{Comp}_n} (-1)^{|D(\gamma) \setminus D(\alpha)|} q^{|D(\gamma) \cap D(\alpha)|} L_\gamma \\
&= r \sum_{\beta \in \operatorname{Comp}_n} \underbrace{\sum_{\substack{\gamma \in \operatorname{Comp}_n; \\ D(\beta) \supseteq D(\gamma)}} (-1)^{|D(\gamma) \setminus T|} q^{|D(\gamma) \cap T|}}_{= [D(\beta) \subseteq T] \cdot r^{|D(\beta)|}} M_\beta \\
&= r \sum_{\beta \in \operatorname{Comp}_n} [D(\beta) \subseteq T] \cdot r^{|D(\beta)|} M_\beta = r \sum_{\substack{\beta \in \operatorname{Comp}_n; \\ D(\beta) \subseteq T}} r^{|D(\beta)|} M_\beta \\
&= \sum_{\substack{\beta \in \operatorname{Comp}_n; \\ D(\beta) \subseteq T}} \underbrace{r^{|D(\beta)|+1}}_{\substack{= r^{\ell(\beta)} \\ \text{(by (16))}}} M_\beta = \sum_{\substack{\beta \in \operatorname{Comp}_n; \\ D(\beta) \subseteq T}} r^{\ell(\beta)} M_\beta \\
&= \sum_{\substack{\beta \in \operatorname{Comp}_n; \\ D(\beta) \subseteq D(\alpha)}} r^{\ell(\beta)} M_\beta \qquad (\text{since } T = D(\alpha)) \\
&= \eta_\alpha^{(q)} \qquad \left(\text{by the definition of } \eta_\alpha^{(q)} \right).
\end{aligned}$$

This proves Proposition 3.11. ☐

Proof of Proposition 3.12. For each subset J of $[n-1]$, we let $\overline{J}$ denote its complement $[n-1] \setminus J$. It is easy to see that

$$\left| \overline{D(\beta)} \right| = n - \ell(\beta) \tag{17}$$

for every $\beta \in \operatorname{Comp}_n$ [6].

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[6] *Proof.* Let $\beta \in \operatorname{Comp}_n$. From $n \neq 0$ (since n is positive), we obtain $[n \neq 0] = 1$. However, Lemma 3.9 **(a)** (applied to $\delta = \beta$) yields $\ell(\beta) = |D(\beta)| + \underbrace{[n \neq 0]}_{=1} = |D(\beta)| + 1$, so that $|D(\beta)| = \ell(\beta) - 1$. However, the definition of $\overline{D(\beta)}$ yields $\overline{D(\beta)} = [n-1] \setminus D(\beta)$. Hence,

$$\begin{aligned}
\left| \overline{D(\beta)} \right| &= |[n-1] \setminus D(\beta)| \\
&= \underbrace{|[n-1]|}_{\substack{= n-1 \\ (\text{since } n-1 \in \mathbb{N})}} - \underbrace{|D(\beta)|}_{= \ell(\beta) - 1} \qquad (\text{since } D(\beta) \subseteq [n-1]) \\
&= (n-1) - (\ell(\beta) - 1) = n - \ell(\beta).
\end{aligned}$$

This proves (17).

Let $T := \overline{D(\gamma)}$. Thus, $D(\gamma) = \overline{T}$, so that

$$\begin{aligned}
&\sum_{\alpha \in \operatorname{Comp}_n} (-1)^{|D(\gamma) \setminus D(\alpha)|} q^{|[n-1] \setminus (D(\gamma) \cup D(\alpha))|} \eta_\alpha^{(q)} \\
&= \sum_{\alpha \in \operatorname{Comp}_n} (-1)^{\left|\overline{T} \setminus D(\alpha)\right|} q^{\left|[n-1] \setminus \left(\overline{T} \cup D(\alpha)\right)\right|} \underbrace{\eta_\alpha^{(q)}}_{\substack{= \sum\limits_{\substack{\beta \in \operatorname{Comp}_n; \\ D(\beta) \subseteq D(\alpha)}} r^{\ell(\beta)} M_\beta \\ \text{(by the definition of } \eta_\alpha^{(q)})}} \\
&= \sum_{\alpha \in \operatorname{Comp}_n} (-1)^{\left|\overline{T} \setminus D(\alpha)\right|} q^{\left|[n-1] \setminus \left(\overline{T} \cup D(\alpha)\right)\right|} \sum_{\substack{\beta \in \operatorname{Comp}_n; \\ D(\beta) \subseteq D(\alpha)}} r^{\ell(\beta)} M_\beta \\
&= \sum_{\beta \in \operatorname{Comp}_n} \sum_{\substack{\alpha \in \operatorname{Comp}_n; \\ D(\beta) \subseteq D(\alpha)}} (-1)^{\left|\overline{T} \setminus D(\alpha)\right|} q^{\left|[n-1] \setminus \left(\overline{T} \cup D(\alpha)\right)\right|} r^{\ell(\beta)} M_\beta.
\end{aligned}$$

However, every $\beta \in \operatorname{Comp}_n$ satisfies

$$\begin{aligned}
&\sum_{\substack{\alpha \in \operatorname{Comp}_n; \\ D(\beta) \subseteq D(\alpha)}} (-1)^{\left|\overline{T} \setminus D(\alpha)\right|} q^{\left|[n-1] \setminus \left(\overline{T} \cup D(\alpha)\right)\right|} \\
&= \sum_{\substack{K \subseteq [n-1]; \\ D(\beta) \subseteq K}} (-1)^{\left|\overline{T} \setminus K\right|} q^{\left|[n-1] \setminus \left(\overline{T} \cup K\right)\right|} \\
&\qquad \left(\begin{array}{c} \text{here, we have substituted } K \text{ for } D(\alpha) \text{ in the sum,} \\ \text{since the map } D : \operatorname{Comp}_n \to \mathcal{P}([n-1]) \text{ is a bijection} \end{array}\right) \\
&= \underbrace{\sum_{\substack{I \subseteq [n-1]; \\ D(\beta) \subseteq \overline{I}}}}_{\substack{= \sum\limits_{I \subseteq \overline{D(\beta)}} \\ \text{(since the subsets } I \text{ of } [n-1] \\ \text{satisfying } D(\beta) \subseteq \overline{I} \text{ are precisely} \\ \text{the subsets of } \overline{D(\beta)})}} \underbrace{(-1)^{\left|\overline{T} \setminus \overline{I}\right|}}_{\substack{= (-1)^{|I \setminus T|} \\ \text{(since } \overline{T} \setminus \overline{I} = I \setminus T)}} \underbrace{q^{\left|[n-1] \setminus \left(\overline{T} \cup \overline{I}\right)\right|}}_{\substack{= q^{|I \cap T|} \\ \text{(since we have} \\ [n-1] \setminus \left(\overline{T} \cup \overline{I}\right) = \overline{\overline{T} \cup \overline{I}} = \overline{\overline{I}} \cup \overline{\overline{T}} = I \cap T \\ \text{by de Morgan's laws)}}} \\
&\qquad \left(\begin{array}{c} \text{here, we have substituted } \overline{I} \text{ for } K \text{ in the sum, since} \\ \text{the map } \mathcal{P}([n-1]) \to \mathcal{P}([n-1]) \text{ that sends each} \\ \text{subset } I \text{ to its complement } \overline{I} \text{ is a bijection} \end{array}\right) \\
&= \sum_{I \subseteq \overline{D(\beta)}} (-1)^{|I \setminus T|} q^{|I \cap T|} = \left[\overline{D(\beta)} \subseteq T\right] \cdot r^{\left|\overline{D(\beta)}\right|}
\end{aligned}$$

(by Lemma 3.13, applied to $S = \overline{D(\beta)}$). Hence, this becomes

$$\begin{aligned}
&\sum_{\alpha\in\operatorname{Comp}_n} (-1)^{|D(\gamma)\setminus D(\alpha)|} q^{|[n-1]\setminus(D(\gamma)\cup D(\alpha))|} \eta_\alpha^{(q)} \\
&= \sum_{\beta\in\operatorname{Comp}_n} \underbrace{\sum_{\substack{\alpha\in\operatorname{Comp}_n;\\ D(\beta)\subseteq D(\alpha)}} (-1)^{\left|\overline{T}\setminus D(\alpha)\right|} q^{\left|[n-1]\setminus\left(\overline{T}\cup D(\alpha)\right)\right|}}_{=\left[\overline{D(\beta)}\subseteq T\right]\cdot r^{\left|\overline{D(\beta)}\right|}} r^{\ell(\beta)} M_\beta \\
&= \sum_{\beta\in\operatorname{Comp}_n} \left[\overline{D(\beta)} \subseteq T\right] \cdot r^{\left|\overline{D(\beta)}\right|} r^{\ell(\beta)} M_\beta = \sum_{\substack{\beta\in\operatorname{Comp}_n;\\ \overline{D(\beta)}\subseteq T}} \underbrace{r^{\left|\overline{D(\beta)}\right|}}_{\substack{=r^{n-\ell(\beta)}\\ \text{(by (17))}}} r^{\ell(\beta)} M_\beta \\
&= \sum_{\substack{\beta\in\operatorname{Comp}_n;\\ \overline{D(\beta)}\subseteq T}} \underbrace{r^{n-\ell(\beta)} r^{\ell(\beta)}}_{=r^n} M_\beta = r^n \sum_{\substack{\beta\in\operatorname{Comp}_n;\\ \overline{D(\beta)}\subseteq T}} M_\beta \\
&= r^n \sum_{\substack{\beta\in\operatorname{Comp}_n;\\ \overline{D(\beta)}\subseteq \overline{D(\gamma)}}} M_\beta \qquad \left(\text{since } T = \overline{D(\gamma)}\right) \\
&= r^n \underbrace{\sum_{\substack{\beta\in\operatorname{Comp}_n;\\ D(\beta)\supseteq D(\gamma)}} M_\beta}_{\substack{=L_\gamma\\ \text{(by (3), applied to } \alpha=\gamma)}} \qquad \left(\begin{array}{c}\text{since the condition ``}\overline{D(\beta)} \subseteq \overline{D(\gamma)}\text{'' on a}\\ \text{composition } \beta\in\operatorname{Comp}_n \text{ is equivalent}\\ \text{to the condition ``}D(\beta)\supseteq D(\gamma)\text{''}\end{array}\right) \\
&= r^n L_\gamma .
\end{aligned}$$

This proves Proposition 3.12. □

3.4. The antipode of $\eta_\alpha^{(q)}$

The *antipode* of QSym is a certain $\mathbf{k}$-linear map $S : \operatorname{QSym} \to \operatorname{QSym}$ that can be defined in terms of the Hopf algebra structure of QSym, which we have not defined so far. But there are various formulas for its values on certain quasisymmetric functions that can be used as alternative definitions. For example, for any $n \in \mathbb{N}$ and any $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell) \in \operatorname{Comp}_n$, we have

$$S(M_\alpha) = (-1)^\ell \sum_{\substack{\gamma\in\operatorname{Comp}_n;\\ D(\gamma)\subseteq D(\alpha_\ell,\alpha_{\ell-1},\ldots,\alpha_1)}} M_\gamma. \tag{18}$$

This formula (which appears, e.g., in [Malven93, (4.26)][7] and in [GriRei20, Theorem 5.1.11][8] or in [Grinbe15, detailed version, Proposition 10.70]) can be used to define S (since S is to be $\mathbf{k}$-linear). Also, for each composition α, we have $S(L_\alpha) = (-1)^{|\alpha|} L_{\omega(\alpha)}$, where $\omega(\alpha)$ is a certain composition known as the *conjugate* of α. See [Malven93, Corollaire 4.20] or [GriRei20, Theorem 5.1.11 and Proposition 5.2.15] for details and proofs. Note that S is a $\mathbf{k}$-algebra homomorphism and an involution (that is, $S^2 = \mathrm{id}$). (Again, this is derived from abstract algebraic properties of antipodes in [GriRei20], but can also be showed more directly.)

We will prove two formulas for the antipode of $\eta_\alpha^{(q)}$. Both rely on the following notation:

Definition 3.14. If $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ is a composition, then the *reversal* of α is defined to be the composition $(\alpha_\ell, \alpha_{\ell-1}, \ldots, \alpha_1)$. It is denoted by $\operatorname{rev}\alpha$.

We are now ready to state our first formula for the antipode of $\eta_\alpha^{(q)}$ in the case when q is invertible:

Theorem 3.15. Let $p \in \mathbf{k}$ be such that $pq = 1$. Let $\alpha \in \mathrm{Comp}$, and let $n = |\alpha|$. Then, the antipode S of QSym satisfies

$$S\left(\eta_\alpha^{(q)}\right) = (-q)^{\ell(\alpha)}\, \eta_{\operatorname{rev}\alpha}^{(p)}.$$

Proof. From $pq = 1$, we obtain $p = q^{-1}$. Furthermore, $(p+1)q = \underbrace{pq}_{=1} + q = 1 + q = q + 1 = r$. Solving this for $p+1$, we obtain

$$p + 1 = rq^{-1}. \tag{19}$$

We shall need a few more features of compositions. For any composition $\gamma \in \mathrm{Comp}_n$, we let $\omega(\gamma)$ denote the unique composition of n satisfying

$$D(\omega(\gamma)) = [n-1] \setminus D(\operatorname{rev}\gamma). \tag{20}$$

(This $\omega(\gamma)$ is indeed unique, since the map D is a bijection.) Then, a classical formula ([Malven93, (4.27)] or [GriRei20, (5.2.7)]) says that each $\gamma \in \mathrm{Comp}_n$ satisfies

$$S(L_\gamma) = (-1)^n L_{\omega(\gamma)}. \tag{21}$$

It is also easy to prove (see, e.g., [GriVas23, Proposition 4.3 **(d)**]) that

$$\omega(\omega(\gamma)) = \gamma \qquad \text{for any } \gamma \in \mathrm{Comp}_n. \tag{22}$$

[7] The proof given in [Malven93] requires $\mathbf{k}$ to be a $\mathbb{Q}$-algebra, but it is easy to see that the truth of (18) for $\mathbf{k} = \mathbb{Q}$ implies the truth of (18) for every commutative ring $\mathbf{k}$.

[8] Note that [GriRei20, Theorem 5.1.11] uses the notation $\operatorname{rev}\alpha$ for the composition $(\alpha_\ell, \alpha_{\ell-1}, \ldots, \alpha_1)$, and writes "$\gamma$ coarsens $\operatorname{rev}\alpha$" for what we call "$\gamma \in \mathrm{Comp}_n$ and $D(\gamma) \subseteq D(\operatorname{rev}\alpha)$".

Thus, the map $\omega : \operatorname{Comp}_n \to \operatorname{Comp}_n$ (which sends each $\gamma \in \operatorname{Comp}_n$ to $\omega(\gamma)$) is a bijection.

We WLOG assume that $n \neq 0$ (since the claim of Theorem 3.15 is easily checked by hand in the case when $n = 0$).

From $n = |\alpha|$, we obtain $\alpha \in \operatorname{Comp}_n$.

Now, we make the following combinatorial observation:

Observation 1: Let $\gamma \in \operatorname{Comp}_n$. Then,

$$|D(\omega(\gamma)) \cap D(\alpha)| = \ell(\alpha) - 1 - |D(\gamma) \cap D(\operatorname{rev}\alpha)| \tag{23}$$

and

$$|D(\omega(\gamma)) \setminus D(\alpha)| = n - \ell(\alpha) - |D(\gamma) \setminus D(\operatorname{rev}\alpha)|. \tag{24}$$

The proof of Observation 1 is laborious but fairly straightforward, and can be found in [GriVas23, Proposition 4.4].

Now, Proposition 3.11 (applied to $\operatorname{rev}\alpha$, p and $p+1$ instead of α, q and r) yields

$$\eta^{(p)}_{\operatorname{rev}\alpha} = (p+1) \sum_{\gamma \in \operatorname{Comp}_n} (-1)^{|D(\gamma) \setminus D(\operatorname{rev}\alpha)|} p^{|D(\gamma) \cap D(\operatorname{rev}\alpha)|} L_\gamma. \tag{25}$$

On the other hand, Proposition 3.11 yields

$$\eta^{(q)}_{\alpha} = r \sum_{\gamma \in \operatorname{Comp}_n} (-1)^{|D(\gamma) \setminus D(\alpha)|} q^{|D(\gamma) \cap D(\alpha)|} L_\gamma.$$

Applying the map S to both sides of this equality, we obtain

$$\begin{aligned}
S\left(\eta_\alpha^{(q)}\right) &= S\left(r\sum_{\gamma\in\operatorname{Comp}_n}(-1)^{|D(\gamma)\setminus D(\alpha)|}\,q^{|D(\gamma)\cap D(\alpha)|}L_\gamma\right)\\
&= r\sum_{\gamma\in\operatorname{Comp}_n}(-1)^{|D(\gamma)\setminus D(\alpha)|}\,q^{|D(\gamma)\cap D(\alpha)|}\underbrace{S\left(L_\gamma\right)}_{\substack{=(-1)^nL_{\omega(\gamma)}\\ \text{(by (21))}}} \qquad \left(\begin{array}{c}\text{since the map } S\\ \text{is } \mathbf{k}\text{-linear}\end{array}\right)\\
&= r\sum_{\gamma\in\operatorname{Comp}_n}(-1)^{|D(\gamma)\setminus D(\alpha)|}\,q^{|D(\gamma)\cap D(\alpha)|}\,(-1)^n\,L_{\omega(\gamma)}\\
&= r\sum_{\gamma\in\operatorname{Comp}_n}\underbrace{(-1)^{|D(\omega(\gamma))\setminus D(\alpha)|}\,q^{|D(\omega(\gamma))\cap D(\alpha)|}\,(-1)^n}_{=(-1)^n(-1)^{|D(\omega(\gamma))\setminus D(\alpha)|}q^{|D(\omega(\gamma))\cap D(\alpha)|}}\underbrace{L_{\omega(\omega(\gamma))}}_{\substack{=L_\gamma\\ \text{(by (22))}}}\\
&\qquad \left(\begin{array}{c}\text{here, we have substituted } \omega(\gamma) \text{ for } \gamma \text{ in the sum,}\\ \text{since the map } \omega:\operatorname{Comp}_n\to\operatorname{Comp}_n \text{ is a bijection}\end{array}\right)\\
&= r\sum_{\gamma\in\operatorname{Comp}_n}(-1)^n\underbrace{(-1)^{|D(\omega(\gamma))\setminus D(\alpha)|}}_{\substack{=(-1)^{n-\ell(\alpha)-|D(\gamma)\setminus D(\operatorname{rev}\alpha)|}\\ \text{(by (24))}}}\underbrace{q^{|D(\omega(\gamma))\cap D(\alpha)|}}_{\substack{=q^{\ell(\alpha)-1-|D(\gamma)\cap D(\operatorname{rev}\alpha)|}\\ \text{(by (23))}}}L_\gamma\\
&= r\sum_{\gamma\in\operatorname{Comp}_n}\underbrace{(-1)^n\,(-1)^{n-\ell(\alpha)-|D(\gamma)\setminus D(\operatorname{rev}\alpha)|}}_{=(-1)^{\ell(\alpha)}(-1)^{|D(\gamma)\setminus D(\operatorname{rev}\alpha)|}}\underbrace{q^{\ell(\alpha)-1-|D(\gamma)\cap D(\operatorname{rev}\alpha)|}}_{=q^{\ell(\alpha)}q^{-1}q^{-|D(\gamma)\cap D(\operatorname{rev}\alpha)|}}L_\gamma\\
&= r\sum_{\gamma\in\operatorname{Comp}_n}(-1)^{\ell(\alpha)}\,(-1)^{|D(\gamma)\setminus D(\operatorname{rev}\alpha)|}\,q^{\ell(\alpha)}q^{-1}q^{-|D(\gamma)\cap D(\operatorname{rev}\alpha)|}L_\gamma\\
&= \underbrace{rq^{-1}}_{\substack{=p+1\\ \text{(by (19))}}}\underbrace{(-1)^{\ell(\alpha)}\,q^{\ell(\alpha)}}_{=(-q)^{\ell(\alpha)}}\sum_{\gamma\in\operatorname{Comp}_n}(-1)^{|D(\gamma)\setminus D(\operatorname{rev}\alpha)|}\underbrace{q^{-|D(\gamma)\cap D(\operatorname{rev}\alpha)|}}_{\substack{=\left(q^{-1}\right)^{|D(\gamma)\cap D(\operatorname{rev}\alpha)|}\\ =p^{|D(\gamma)\cap D(\operatorname{rev}\alpha)|}\\ \text{(since } q^{-1}=p)}}L_\gamma\\
&= (p+1)\,(-q)^{\ell(\alpha)}\sum_{\gamma\in\operatorname{Comp}_n}(-1)^{|D(\gamma)\setminus D(\operatorname{rev}\alpha)|}\,p^{|D(\gamma)\cap D(\operatorname{rev}\alpha)|}L_\gamma\\
&= (-q)^{\ell(\alpha)}\underbrace{(p+1)\sum_{\gamma\in\operatorname{Comp}_n}(-1)^{|D(\gamma)\setminus D(\operatorname{rev}\alpha)|}\,p^{|D(\gamma)\cap D(\operatorname{rev}\alpha)|}L_\gamma}_{\substack{=\eta_{\operatorname{rev}\alpha}^{(p)}\\ \text{(by (25))}}}\\
&= (-q)^{\ell(\alpha)}\,\eta_{\operatorname{rev}\alpha}^{(p)}.
\end{aligned}$$

This proves Theorem 3.15. □

Theorem 3.15 generalizes [Hsiao07, Proposition 2.9].

Our second formula for the antipode of $\eta_\alpha^{(q)}$ comes with no requirement on q, but is somewhat more complicated:

Theorem 3.16. Let $n \in \mathbb{N}$. Let $\alpha \in \operatorname{Comp}_n$. Then, the antipode S of QSym satisfies

$$S\left(\eta_\alpha^{(q)}\right) = (-1)^{\ell(\alpha)} \sum_{\substack{\beta \in \operatorname{Comp}_n; \\ D(\beta) \subseteq D(\operatorname{rev} \alpha)}} (q-1)^{\ell(\alpha)-\ell(\beta)} \eta_\beta^{(q)}.$$

We will prove this theorem using the following summation lemma:

Lemma 3.17. Let $n \in \mathbb{N}$. Let $\alpha \in \operatorname{Comp}_n$ and $\gamma \in \operatorname{Comp}_n$ be such that $D(\gamma) \subseteq D(\alpha)$. Then:

(a) For any $u, v \in \mathbf{k}$, we have

$$\sum_{\substack{\beta \in \operatorname{Comp}_n; \\ D(\beta) \subseteq D(\alpha) \text{ and } D(\gamma) \subseteq D(\beta)}} u^{\ell(\beta)-\ell(\gamma)} v^{\ell(\alpha)-\ell(\beta)} = (u+v)^{\ell(\alpha)-\ell(\gamma)}.$$

(b) For any $u \in \mathbf{k}$, we have

$$\sum_{\substack{\beta \in \operatorname{Comp}_n; \\ D(\beta) \subseteq D(\alpha) \text{ and } D(\gamma) \subseteq D(\beta)}} u^{\ell(\beta)} = (u+1)^{\ell(\alpha)-\ell(\gamma)} u^{\ell(\gamma)}.$$

(c) For any $v \in \mathbf{k}$, we have

$$\sum_{\substack{\beta \in \operatorname{Comp}_n; \\ D(\beta) \subseteq D(\alpha) \text{ and } D(\gamma) \subseteq D(\beta)}} v^{\ell(\alpha)-\ell(\beta)} = (1+v)^{\ell(\alpha)-\ell(\gamma)}.$$

Proof of Lemma 3.17. **(a)** Let $u, v \in \mathbf{k}$. Set $A := D(\alpha)$ and $C := D(\gamma)$. Then, $A = D(\alpha) \in \mathcal{P}([n-1])$ (since $\alpha \in \operatorname{Comp}_n$, but $D : \operatorname{Comp}_n \to \mathcal{P}([n-1])$ is a map). In other words, $A \subseteq [n-1]$. Furthermore, $C = D(\gamma) \subseteq D(\alpha) = A \subseteq [n-1]$.

From $C \subseteq A$, we obtain

$$\begin{aligned} |A \setminus C| &= |A| - |C| = |D(\alpha)| - |D(\gamma)| \qquad (\text{since } A = D(\alpha) \text{ and } C = D(\gamma)) \\ &= \ell(\alpha) - \ell(\gamma) \end{aligned}$$

(since Lemma 3.9 **(b)** yields $\ell(\alpha) - \ell(\gamma) = |D(\alpha)| - |D(\gamma)|$).

Now,

$$\begin{aligned}
&\sum_{\substack{\beta\in\operatorname{Comp}_n;\\ D(\beta)\subseteq D(\alpha)\text{ and }D(\gamma)\subseteq D(\beta)}} \underbrace{u^{\ell(\beta)-\ell(\gamma)}}_{\substack{=u^{|D(\beta)|-|D(\gamma)|}\\ \text{(since Lemma 3.9 \textbf{(b)}}\\ \text{yields }\ell(\beta)-\ell(\gamma)=|D(\beta)|-|D(\gamma)|)}} \underbrace{v^{\ell(\alpha)-\ell(\beta)}}_{\substack{=v^{|D(\alpha)|-|D(\beta)|}\\ \text{(since Lemma 3.9 \textbf{(b)}}\\ \text{yields }\ell(\alpha)-\ell(\beta)=|D(\alpha)|-|D(\beta)|)}}\\
&= \sum_{\substack{\beta\in\operatorname{Comp}_n;\\ D(\beta)\subseteq D(\alpha)\text{ and }D(\gamma)\subseteq D(\beta)}} u^{|D(\beta)|-|D(\gamma)|}v^{|D(\alpha)|-|D(\beta)|}\\
&= \sum_{\substack{\beta\in\operatorname{Comp}_n;\\ D(\beta)\subseteq A\text{ and }C\subseteq D(\beta)}} u^{|D(\beta)|-|C|}v^{|A|-|D(\beta)|} \qquad (\text{since } D(\gamma)=C \text{ and } D(\alpha)=A)\\
&= \sum_{\substack{I\subseteq[n-1];\\ I\subseteq A\text{ and }C\subseteq I}} u^{|I|-|C|}v^{|A|-|I|} \qquad \left(\begin{array}{c}\text{here, we have substituted } I \text{ for } D(\beta)\\ \text{in the sum, since the}\\ \text{map } D:\operatorname{Comp}_n\to\mathcal{P}([n-1])\\ \text{is a bijection}\end{array}\right)\\
&= \sum_{\substack{I\subseteq A;\\ C\subseteq I}} u^{|I|-|C|}v^{|A|-|I|} \qquad (26)
\end{aligned}$$

(since the condition "$I \subseteq [n-1]$" under the sum is redundant (because the condition "$I \subseteq A$" already yields $I \subseteq A \subseteq [n-1]$)).

Now, each subset I of A that satisfies $C \subseteq I$ can be written as $C \cup Z$ for some unique subset $Z \subseteq A \setminus C$. Hence, we can substitute $C \cup Z$ for I in the sum $\sum_{\substack{I\subseteq A;\\ C\subseteq I}} u^{|I|-|C|}v^{|A|-|I|}$. As a consequence, we obtain

$$\begin{aligned}
\sum_{\substack{I\subseteq A;\\ C\subseteq I}} u^{|I|-|C|}v^{|A|-|I|} &= \sum_{Z\subseteq A\setminus C} \underbrace{u^{|C\cup Z|-|C|}v^{|A|-|C\cup Z|}}_{\substack{=u^{(|C|+|Z|)-|C|}v^{|A|-(|C|+|Z|)}\\ \text{(since }|C\cup Z|=|C|+|Z|\\ \text{(because }Z\subseteq A\setminus C\text{ shows that}\\ \text{the sets }C\text{ and }Z\text{ are disjoint))}}}\\
&= \sum_{Z\subseteq A\setminus C} \underbrace{u^{(|C|+|Z|)-|C|}}_{=u^{|Z|}} \underbrace{v^{|A|-(|C|+|Z|)}}_{\substack{=v^{|A|-|C|-|Z|}=v^{|A\setminus C|-|Z|}\\ \text{(since }|A|-|C|=|A\setminus C|)}}\\
&= \sum_{Z\subseteq A\setminus C} u^{|Z|}v^{|A\setminus C|-|Z|} = \sum_{k=0}^{|A\setminus C|} \binom{|A\setminus C|}{k} u^k v^{|A\setminus C|-k}
\end{aligned}$$

(here, we have split the sum according to the size $|Z|$ of the subset Z, and observed that for each given $k \in \mathbb{N}$, there are exactly $\binom{|A\setminus C|}{k}$ many subsets $Z \subseteq A \setminus C$ that have size k). Comparing this with

$$(u+v)^{|A\setminus C|} = \sum_{k=0}^{|A\setminus C|} \binom{|A\setminus C|}{k} u^k v^{|A\setminus C|-k} \qquad (\text{by the binomial formula}),$$

we obtain

$$\sum_{\substack{I\subseteq A;\\ C\subseteq I}} u^{|I|-|C|} v^{|A|-|I|} = (u+v)^{|A\setminus C|} = (u+v)^{\ell(\alpha)-\ell(\gamma)}$$

(since $|A \setminus C| = \ell(\alpha) - \ell(\gamma)$). Hence, (26) becomes

$$\sum_{\substack{\beta\in\mathrm{Comp}_n;\\ D(\beta)\subseteq D(\alpha) \text{ and } D(\gamma)\subseteq D(\beta)}} u^{\ell(\beta)-\ell(\gamma)} v^{\ell(\alpha)-\ell(\beta)} = \sum_{\substack{I\subseteq A;\\ C\subseteq I}} u^{|I|-|C|} v^{|A|-|I|} = (u+v)^{\ell(\alpha)-\ell(\gamma)}.$$

This proves Lemma 3.17 **(a)**.

(b) This follows by applying Lemma 3.17 **(a)** to $v = 1$ and then multiplying both sides by $u^{\ell(\gamma)}$.

(c) This follows by applying Lemma 3.17 **(a)** to $u = 1$. $\square$

Proof of Theorem 3.16. We replace α by $\operatorname{rev}\alpha$. Thus, α and $\operatorname{rev}\alpha$ become $\operatorname{rev}\alpha$ and α, respectively, while the length $\ell(\alpha)$ stays unchanged. Hence, the claim we must prove becomes

$$S\left(\eta^{(q)}_{\operatorname{rev}\alpha}\right) = (-1)^{\ell(\alpha)} \sum_{\substack{\beta\in\mathrm{Comp}_n;\\ D(\beta)\subseteq D(\alpha)}} (q-1)^{\ell(\alpha)-\ell(\beta)} \eta^{(q)}_{\beta}. \tag{27}$$

It is this equality that we will be proving.

First, we observe that every $\beta \in \mathrm{Comp}_n$ satisfies

$$S(M_\beta) = (-1)^{\ell(\beta)} \sum_{\substack{\gamma\in\mathrm{Comp}_n;\\ D(\gamma)\subseteq D(\operatorname{rev}\beta)}} M_\gamma. \tag{28}$$

(Indeed, this is just the formula (18), applied to β instead of α and restated using Definition 3.14.) Substituting $\operatorname{rev}\beta$ for β in (28), we obtain the following: Every $\beta \in \mathrm{Comp}_n$ satisfies

$$\begin{aligned} S(M_{\operatorname{rev}\beta}) &= (-1)^{\ell(\operatorname{rev}\beta)} \sum_{\substack{\gamma\in\mathrm{Comp}_n;\\ D(\gamma)\subseteq D(\operatorname{rev}(\operatorname{rev}\beta))}} M_\gamma \qquad (\text{since } \operatorname{rev}\beta \in \mathrm{Comp}_n) \\ &= (-1)^{\ell(\beta)} \sum_{\substack{\gamma\in\mathrm{Comp}_n;\\ D(\gamma)\subseteq D(\beta)}} M_\gamma \end{aligned} \tag{29}$$

(since $\operatorname{rev}(\operatorname{rev}\beta) = \beta$ and $\ell(\operatorname{rev}\beta) = \ell(\beta)$).

Next, we recall two simple facts from [GriVas23]. First, [GriVas23, Corollary 3.10] says that the map

$$\mathrm{Comp}_n \to \mathrm{Comp}_n, \quad \delta \mapsto \operatorname{rev}\delta$$

is a bijection. Furthermore, [GriVas23, Proposition 3.11] shows that if $\beta \in \operatorname{Comp}_n$ is arbitrary, then we have the logical equivalence

$$(D(\operatorname{rev}\beta) \subseteq D(\operatorname{rev}\alpha)) \iff (D(\beta) \subseteq D(\alpha)). \tag{30}$$

The definition of $\eta^{(q)}_{\operatorname{rev}\alpha}$ yields

$$\begin{aligned} \eta^{(q)}_{\operatorname{rev}\alpha} &= \sum_{\substack{\beta\in\operatorname{Comp}_n;\\ D(\beta)\subseteq D(\operatorname{rev}\alpha)}} r^{\ell(\beta)} M_\beta = \underbrace{\sum_{\substack{\beta\in\operatorname{Comp}_n;\\ D(\operatorname{rev}\beta)\subseteq D(\operatorname{rev}\alpha)}}}_{\substack{=\sum_{\substack{\beta\in\operatorname{Comp}_n;\\ D(\beta)\subseteq D(\alpha)}}\\ \text{(by the equivalence (30))}}} \underbrace{r^{\ell(\operatorname{rev}\beta)}}_{\substack{=r^{\ell(\beta)}\\ \text{(since } \ell(\operatorname{rev}\beta)=\ell(\beta))}} M_{\operatorname{rev}\beta} \\ &\qquad \left(\begin{array}{c}\text{here, we have substituted } \operatorname{rev}\beta \text{ for } \beta \text{ in the sum,}\\ \text{since the map } \operatorname{Comp}_n \to \operatorname{Comp}_n,\ \delta\mapsto\operatorname{rev}\delta\\ \text{is a bijection}\end{array}\right) \\ &= \sum_{\substack{\beta\in\operatorname{Comp}_n;\\ D(\beta)\subseteq D(\alpha)}} r^{\ell(\beta)} M_{\operatorname{rev}\beta}. \end{aligned}$$

Applying the $\mathbf{k}$-linear map S to both sides of this equality, we obtain

$$\begin{aligned}
S\left(\eta^{(q)}_{\operatorname{rev}\alpha}\right) &= \sum_{\substack{\beta\in\operatorname{Comp}_n;\\ D(\beta)\subseteq D(\alpha)}} r^{\ell(\beta)} S\left(M_{\operatorname{rev}\beta}\right) \\
&= \sum_{\substack{\beta\in\operatorname{Comp}_n;\\ D(\beta)\subseteq D(\alpha)}} r^{\ell(\beta)} \left(-1\right)^{\ell(\beta)} \sum_{\substack{\gamma\in\operatorname{Comp}_n;\\ D(\gamma)\subseteq D(\beta)}} M_\gamma \qquad \text{(by (29))} \\
&= \underbrace{\sum_{\substack{\beta\in\operatorname{Comp}_n;\\ D(\beta)\subseteq D(\alpha)}} \sum_{\substack{\gamma\in\operatorname{Comp}_n;\\ D(\gamma)\subseteq D(\beta)}}}_{=\sum_{\substack{\gamma\in\operatorname{Comp}_n;\\ D(\gamma)\subseteq D(\alpha)}} \sum_{\substack{\beta\in\operatorname{Comp}_n;\\ D(\beta)\subseteq D(\alpha)\text{ and }D(\gamma)\subseteq D(\beta)}}} \underbrace{r^{\ell(\beta)} \left(-1\right)^{\ell(\beta)}}_{=(-r)^{\ell(\beta)}} M_\gamma \\
&= \sum_{\substack{\gamma\in\operatorname{Comp}_n;\\ D(\gamma)\subseteq D(\alpha)}} \underbrace{\sum_{\substack{\beta\in\operatorname{Comp}_n;\\ D(\beta)\subseteq D(\alpha)\text{ and }D(\gamma)\subseteq D(\beta)}} \left(-r\right)^{\ell(\beta)}}_{\substack{=(-r+1)^{\ell(\alpha)-\ell(\gamma)}(-r)^{\ell(\gamma)}\\ \text{(by Lemma 3.17 \textbf{(b)})}}} M_\gamma \\
&= \sum_{\substack{\gamma\in\operatorname{Comp}_n;\\ D(\gamma)\subseteq D(\alpha)}} \left(\underbrace{-r+1}_{\substack{=-q\\ \text{(since } r=q+1)}}\right)^{\ell(\alpha)-\ell(\gamma)} \left(-r\right)^{\ell(\gamma)} M_\gamma \\
&= \sum_{\substack{\gamma\in\operatorname{Comp}_n;\\ D(\gamma)\subseteq D(\alpha)}} \underbrace{\left(-q\right)^{\ell(\alpha)-\ell(\gamma)} \left(-r\right)^{\ell(\gamma)}}_{=(-1)^{\ell(\alpha)} q^{\ell(\alpha)-\ell(\gamma)} r^{\ell(\gamma)}} M_\gamma \\
&= \left(-1\right)^{\ell(\alpha)} \sum_{\substack{\gamma\in\operatorname{Comp}_n;\\ D(\gamma)\subseteq D(\alpha)}} q^{\ell(\alpha)-\ell(\gamma)} r^{\ell(\gamma)} M_\gamma .
\end{aligned}$$

In view of

$$\begin{aligned}
&\sum_{\substack{\beta\in\operatorname{Comp}_n;\\ D(\beta)\subseteq D(\alpha)}} (q-1)^{\ell(\alpha)-\ell(\beta)} \underbrace{\eta_\beta^{(q)}}_{\substack{=\sum_{\substack{\gamma\in\operatorname{Comp}_n;\\ D(\gamma)\subseteq D(\beta)}} r^{\ell(\gamma)} M_\gamma \\ \text{(by (5))}}} \\
&= \sum_{\substack{\beta\in\operatorname{Comp}_n;\\ D(\beta)\subseteq D(\alpha)}} (q-1)^{\ell(\alpha)-\ell(\beta)} \sum_{\substack{\gamma\in\operatorname{Comp}_n;\\ D(\gamma)\subseteq D(\beta)}} r^{\ell(\gamma)} M_\gamma \\
&= \underbrace{\sum_{\substack{\beta\in\operatorname{Comp}_n;\\ D(\beta)\subseteq D(\alpha)}} \sum_{\substack{\gamma\in\operatorname{Comp}_n;\\ D(\gamma)\subseteq D(\beta)}}}_{=\sum_{\substack{\gamma\in\operatorname{Comp}_n;\\ D(\gamma)\subseteq D(\alpha)}} \sum_{\substack{\beta\in\operatorname{Comp}_n;\\ D(\beta)\subseteq D(\alpha) \text{ and } D(\gamma)\subseteq D(\beta)}}} (q-1)^{\ell(\alpha)-\ell(\beta)} r^{\ell(\gamma)} M_\gamma \\
&= \sum_{\substack{\gamma\in\operatorname{Comp}_n;\\ D(\gamma)\subseteq D(\alpha)}} \underbrace{\sum_{\substack{\beta\in\operatorname{Comp}_n;\\ D(\beta)\subseteq D(\alpha) \text{ and } D(\gamma)\subseteq D(\beta)}} (q-1)^{\ell(\alpha)-\ell(\beta)}}_{\substack{=(1+(q-1))^{\ell(\alpha)-\ell(\gamma)} \\ \text{(by Lemma 3.17 } \textbf{(c)}\text{)}}} r^{\ell(\gamma)} M_\gamma \\
&= \sum_{\substack{\gamma\in\operatorname{Comp}_n;\\ D(\gamma)\subseteq D(\alpha)}} \Bigg(\underbrace{1+(q-1)}_{=q}\Bigg)^{\ell(\alpha)-\ell(\gamma)} r^{\ell(\gamma)} M_\gamma = \sum_{\substack{\gamma\in\operatorname{Comp}_n;\\ D(\gamma)\subseteq D(\alpha)}} q^{\ell(\alpha)-\ell(\gamma)} r^{\ell(\gamma)} M_\gamma,
\end{aligned}$$

we can rewrite this as

$$S\left(\eta_{\operatorname{rev}\alpha}^{(q)}\right) = (-1)^{\ell(\alpha)} \sum_{\substack{\beta\in\operatorname{Comp}_n;\\ D(\beta)\subseteq D(\alpha)}} (q-1)^{\ell(\alpha)-\ell(\beta)} \eta_\beta^{(q)}.$$

Thus, (27) is proved. As we explained, this proves Theorem 3.16. □

3.5. The coproduct of $\eta_\alpha^{(q)}$

The *concatenation* of two compositions $\beta = (\beta_1, \beta_2, \ldots, \beta_i)$ and $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_j)$ is defined to be the composition $(\beta_1, \beta_2, \ldots, \beta_i, \gamma_1, \gamma_2, \ldots, \gamma_j)$. It is denoted by $\beta\gamma$.

The coproduct of the Hopf algebra QSym is a $\mathbf{k}$-linear map $\Delta : \operatorname{QSym} \to \operatorname{QSym} \otimes \operatorname{QSym}$ that can be described by the formula

$$\Delta(M_\alpha) = \sum_{\substack{\beta,\gamma\in\operatorname{Comp};\\ \alpha=\beta\gamma}} M_\beta \otimes M_\gamma, \qquad (31)$$

which holds for all $\alpha \in \operatorname{Comp}$. (See [GriRei20, §5.1] for the definition of Δ, and see [GriRei20, Proposition 5.1.7] for a proof of (31).)

We claim the following simple formula for $\Delta\left(\eta_\alpha^{(q)}\right)$ (analogous to (31)):

Theorem 3.18. Let $\alpha \in \operatorname{Comp}$. Then,

$$\Delta\left(\eta_\alpha^{(q)}\right) = \sum_{\substack{\beta,\gamma \in \operatorname{Comp}; \\ \alpha = \beta\gamma}} \eta_\beta^{(q)} \otimes \eta_\gamma^{(q)}.$$

This generalizes [Hsiao07, Corollary 2.7].

We shall first give a direct proof of Theorem 3.18; later we will outline another one, which is more circuitous.

The direct proof uses the following notion:

Definition 3.19. Let γ be a composition. Then, $C(\gamma)$ shall denote the set of all compositions $\beta \in \operatorname{Comp}_{|\gamma|}$ satisfying $D(\beta) \subseteq D(\gamma)$. (The compositions belonging to $C(\gamma)$ are often called the *coarsenings* of γ.)

For instance, $C(2,1,3) = \{(2,1,3),\ (3,3),\ (2,4),\ (6)\}$.

Using the notion of $C(\gamma)$, we can restate the definition of $\eta_\gamma^{(q)}$:

Proposition 3.20. For any $\gamma \in \operatorname{Comp}$, we have

$$\eta_\gamma^{(q)} = \sum_{\nu \in C(\gamma)} r^{\ell(\nu)} M_\nu.$$

Proof of Proposition 3.20. Let $\gamma \in \operatorname{Comp}$. Then, $\gamma \in \operatorname{Comp}_{|\gamma|}$. Thus, (5) (applied to $|\gamma|$ and γ instead of n and α) yields

$$\eta_\gamma^{(q)} = \sum_{\substack{\beta \in \operatorname{Comp}_{|\gamma|}; \\ D(\beta) \subseteq D(\gamma)}} r^{\ell(\beta)} M_\beta = \sum_{\beta \in C(\gamma)} r^{\ell(\beta)} M_\beta$$

(since the compositions β that the previous sum was ranging over are precisely the elements of $C(\gamma)$). Renaming the summation index β as ν on the right hand side of this equality, we obtain

$$\eta_\gamma^{(q)} = \sum_{\nu \in C(\gamma)} r^{\ell(\nu)} M_\nu.$$

This proves Proposition 3.20. $\square$

We shall also use a simple summation formula ([GriVas23, Proposition 5.17]):

Proposition 3.21. Let $(A, +, 0)$ be an abelian group. Let $u_{\mu,\nu}$ be an element of A for each pair $(\mu, \nu) \in \operatorname{Comp} \times \operatorname{Comp}$ of two compositions. Let $\alpha \in \operatorname{Comp}$. Then,

$$\sum_{\substack{\mu,\nu \in \operatorname{Comp};\\ \mu\nu \in C(\alpha)}} u_{\mu,\nu} = \sum_{\substack{\beta,\gamma \in \operatorname{Comp};\\ \beta\gamma = \alpha}} \sum_{\mu \in C(\beta)} \sum_{\nu \in C(\gamma)} u_{\mu,\nu}.$$

We are now ready to prove Theorem 3.18:

Proof of Theorem 3.18. Proposition 3.20 (applied to $\gamma = \alpha$) yields

$$\eta_\alpha^{(q)} = \sum_{\nu \in C(\alpha)} r^{\ell(\nu)} M_\nu = \sum_{\lambda \in C(\alpha)} r^{\ell(\lambda)} M_\lambda.$$

Applying the map Δ to both sides of this equality, we find

$$\begin{aligned}
\Delta\left(\eta_\alpha^{(q)}\right) &= \Delta\left(\sum_{\lambda \in C(\alpha)} r^{\ell(\lambda)} M_\lambda\right) = \sum_{\lambda \in C(\alpha)} r^{\ell(\lambda)} \underbrace{\Delta(M_\lambda)}_{\substack{= \sum\limits_{\substack{\mu,\nu \in \operatorname{Comp};\\ \lambda = \mu\nu}} M_\mu \otimes M_\nu \\ \text{(by (31), with the} \\ \text{letters } \alpha,\beta,\gamma \text{ renamed as } \lambda,\mu,\nu)}} \\
&\qquad \text{(since the map } \Delta \text{ is } \mathbf{k}\text{-linear)} \\
&= \sum_{\lambda \in C(\alpha)} r^{\ell(\lambda)} \sum_{\substack{\mu,\nu \in \operatorname{Comp};\\ \lambda = \mu\nu}} M_\mu \otimes M_\nu = \underbrace{\sum_{\lambda \in C(\alpha)} \sum_{\substack{\mu,\nu \in \operatorname{Comp};\\ \lambda = \mu\nu}}}_{= \sum\limits_{\substack{\mu,\nu \in \operatorname{Comp};\\ \mu\nu \in C(\alpha)}}} \underbrace{r^{\ell(\lambda)}}_{\substack{= r^{\ell(\mu\nu)} \\ \text{(since } \lambda = \mu\nu)}} M_\mu \otimes M_\nu \\
&= \sum_{\substack{\mu,\nu \in \operatorname{Comp};\\ \mu\nu \in C(\alpha)}} r^{\ell(\mu\nu)} M_\mu \otimes M_\nu \\
&= \sum_{\substack{\beta,\gamma \in \operatorname{Comp};\\ \beta\gamma = \alpha}} \sum_{\mu \in C(\beta)} \sum_{\nu \in C(\gamma)} r^{\ell(\mu\nu)} M_\mu \otimes M_\nu
\end{aligned} \tag{32}$$

(by Proposition 3.21, applied to $A = \operatorname{QSym} \otimes \operatorname{QSym}$ and $u_{\mu,\nu} = r^{\ell(\mu\nu)} M_\mu \otimes M_\nu$).

On the other hand, if $\mu, \nu \in \operatorname{Comp}$ are any two compositions, then

$$\ell(\mu\nu) = \ell(\mu) + \ell(\nu) \qquad \text{(by [GriVas23, Proposition 5.2 **(a)**])}$$

and thus

$$r^{\ell(\mu\nu)} = r^{\ell(\mu)+\ell(\nu)} = r^{\ell(\mu)} r^{\ell(\nu)}. \tag{33}$$

Now,

$$\begin{aligned}
&\sum_{\substack{\beta,\gamma\in\operatorname{Comp};\\ \alpha=\beta\gamma}} \underbrace{\eta_\beta^{(q)}}_{\substack{=\sum_{\mu\in C(\beta)} r^{\ell(\mu)}M_\mu \\ \text{(by Proposition 3.20)}}} \otimes \underbrace{\eta_\gamma^{(q)}}_{\substack{=\sum_{\nu\in C(\gamma)} r^{\ell(\nu)}M_\nu \\ \text{(by Proposition 3.20)}}} \\
&= \sum_{\substack{\beta,\gamma\in\operatorname{Comp};\\ \alpha=\beta\gamma}} \underbrace{\left(\sum_{\mu\in C(\beta)} r^{\ell(\mu)}M_\mu\right)\otimes\left(\sum_{\nu\in C(\gamma)} r^{\ell(\nu)}M_\nu\right)}_{=\sum_{\mu\in C(\beta)}\sum_{\nu\in C(\gamma)} r^{\ell(\mu)}r^{\ell(\nu)}M_\mu\otimes M_\nu} \\
&= \underbrace{\sum_{\substack{\beta,\gamma\in\operatorname{Comp};\\ \alpha=\beta\gamma}}}_{=\sum_{\substack{\beta,\gamma\in\operatorname{Comp};\\ \beta\gamma=\alpha}}} \sum_{\mu\in C(\beta)} \sum_{\nu\in C(\gamma)} \underbrace{r^{\ell(\mu)}r^{\ell(\nu)}}_{\substack{=r^{\ell(\mu\nu)} \\ \text{(by (33))}}} M_\mu\otimes M_\nu \\
&= \sum_{\substack{\beta,\gamma\in\operatorname{Comp};\\ \beta\gamma=\alpha}} \sum_{\mu\in C(\beta)} \sum_{\nu\in C(\gamma)} r^{\ell(\mu\nu)} M_\mu\otimes M_\nu.
\end{aligned}$$

Comparing this with (32), we obtain

$$\Delta\left(\eta_\alpha^{(q)}\right) = \sum_{\substack{\beta,\gamma\in\operatorname{Comp};\\ \alpha=\beta\gamma}} \eta_\beta^{(q)}\otimes\eta_\gamma^{(q)}.$$

This proves Theorem 3.18. □

3.6. The coalgebra morphism T_r

We define a **k**-linear map $T_r : \operatorname{QSym}\to\operatorname{QSym}$ by setting

$$T_r\left(M_\alpha\right) = r^{\ell(\alpha)}M_\alpha \qquad \text{for each } \alpha\in\operatorname{Comp}.$$

This definition is legitimate, since $(M_\alpha)_{\alpha\in\operatorname{Comp}}$ is a basis of the **k**-module QSym (and since a **k**-linear map on a free **k**-module can be defined by specifying its values on a basis). The map T_r is usually not a **k**-algebra homomorphism, but it is always a **k**-coalgebra homomorphism. This chiefly relies on the following lemma:

Lemma 3.22. We have $\Delta\circ T_r = (T_r\otimes T_r)\circ\Delta$ as maps from QSym to $\operatorname{QSym}\otimes\operatorname{QSym}$.

Proof of Lemma 3.22. Let $\alpha\in\operatorname{Comp}$ be arbitrary. The definition of T_r yields $T_r\left(M_\alpha\right) =$

$r^{\ell(\alpha)} M_\alpha$. Applying the map Δ to both sides of this identity, we obtain

$$\begin{aligned}
\Delta\left(T_r\left(M_\alpha\right)\right) &= \Delta\left(r^{\ell(\alpha)} M_\alpha\right) = r^{\ell(\alpha)} \Delta\left(M_\alpha\right) = r^{\ell(\alpha)} \sum_{\substack{\beta,\gamma\in\operatorname{Comp};\\ \alpha=\beta\gamma}} M_\beta \otimes M_\gamma \qquad (\text{by (31)}) \\
&= \sum_{\substack{\beta,\gamma\in\operatorname{Comp};\\ \alpha=\beta\gamma}} \underbrace{r^{\ell(\alpha)}}_{\substack{=r^{\ell(\beta\gamma)}\\ (\text{since } \alpha=\beta\gamma)}} M_\beta \otimes M_\gamma = \sum_{\substack{\beta,\gamma\in\operatorname{Comp};\\ \alpha=\beta\gamma}} \underbrace{r^{\ell(\beta\gamma)}}_{\substack{=r^{\ell(\beta)+\ell(\gamma)}\\ (\text{since the definition}\\ \text{of concatenation}\\ \text{yields } \ell(\beta\gamma)=\ell(\beta)+\ell(\gamma))}} M_\beta \otimes M_\gamma \\
&= \sum_{\substack{\beta,\gamma\in\operatorname{Comp};\\ \alpha=\beta\gamma}} \underbrace{r^{\ell(\beta)+\ell(\gamma)} M_\beta \otimes M_\gamma}_{=r^{\ell(\beta)} M_\beta \otimes r^{\ell(\gamma)} M_\gamma} = \sum_{\substack{\beta,\gamma\in\operatorname{Comp};\\ \alpha=\beta\gamma}} r^{\ell(\beta)} M_\beta \otimes r^{\ell(\gamma)} M_\gamma .
\end{aligned}$$

Comparing this with

$$\begin{aligned}
\left(T_r \otimes T_r\right)\left(\Delta\left(M_\alpha\right)\right) &= \left(T_r \otimes T_r\right)\left(\sum_{\substack{\beta,\gamma\in\operatorname{Comp};\\ \alpha=\beta\gamma}} M_\beta \otimes M_\gamma\right) \qquad (\text{by (31)}) \\
&= \sum_{\substack{\beta,\gamma\in\operatorname{Comp};\\ \alpha=\beta\gamma}} \underbrace{T_r\left(M_\beta\right)}_{\substack{=r^{\ell(\beta)} M_\beta\\ (\text{by the definition of } T_r)}} \otimes \underbrace{T_r\left(M_\gamma\right)}_{\substack{=r^{\ell(\gamma)} M_\gamma\\ (\text{by the definition of } T_r)}} \\
&= \sum_{\substack{\beta,\gamma\in\operatorname{Comp};\\ \alpha=\beta\gamma}} r^{\ell(\beta)} M_\beta \otimes r^{\ell(\gamma)} M_\gamma ,
\end{aligned}$$

we obtain

$$\Delta\left(T_r\left(M_\alpha\right)\right) = \left(T_r \otimes T_r\right)\left(\Delta\left(M_\alpha\right)\right) = \left(\left(T_r \otimes T_r\right) \circ \Delta\right)\left(M_\alpha\right).$$

Hence, $\left(\Delta \circ T_r\right)\left(M_\alpha\right) = \Delta\left(T_r\left(M_\alpha\right)\right) = \left(\left(T_r \otimes T_r\right) \circ \Delta\right)\left(M_\alpha\right)$.

Forget that we fixed α. We thus have proved that $\left(\Delta \circ T_r\right)\left(M_\alpha\right) = \left(\left(T_r \otimes T_r\right) \circ \Delta\right)\left(M_\alpha\right)$ for each $\alpha \in \operatorname{Comp}$. In other words, the two maps $\Delta \circ T_r$ and $\left(T_r \otimes T_r\right) \circ \Delta$ agree on every element of the basis $\left(M_\alpha\right)_{\alpha\in\operatorname{Comp}}$ of QSym. Since these two maps both are $\mathbf{k}$-linear, this entails that they are completely identical. In other words, we have $\Delta \circ T_r = \left(T_r \otimes T_r\right) \circ \Delta$. This proves Lemma 3.22. □

Proposition 3.23. The map $T_r : \operatorname{QSym} \to \operatorname{QSym}$ is a $\mathbf{k}$-coalgebra homomorphism.

Proof of Proposition 3.23. Let $\varepsilon : \operatorname{QSym} \to \mathbf{k}$ be the counit of the $\mathbf{k}$-coalgebra QSym. It is well-known (and easy to see) that each composition $\alpha \in \operatorname{Comp}$ satisfies $\varepsilon\left(M_\alpha\right) = [\alpha = \varnothing]$ (where we are using Convention 3.7 again). This, in turn, makes it straightforward to check that $\left(\varepsilon \circ T_r\right)\left(M_\alpha\right) = \varepsilon\left(M_\alpha\right)$ for each $\alpha \in \operatorname{Comp}$. In other words, the two maps $\varepsilon \circ T_r$ and ε agree on every element of the basis $\left(M_\alpha\right)_{\alpha\in\operatorname{Comp}}$ of QSym. Since these two maps both are $\mathbf{k}$-linear, this entails that they are completely identical. In other words, we have $\varepsilon \circ T_r = \varepsilon$. Combining this with the equality $\Delta \circ T_r = (T_r \otimes T_r) \circ \Delta$ from Lemma 3.22, we conclude that the linear map T_r is a $\mathbf{k}$-coalgebra homomorphism. This proves Proposition 3.23. $\square$

To us, the map T_r becomes useful thanks to the following slick expression for $\eta_\alpha^{(q)}$ that it allows:

Theorem 3.24. Let $S : \operatorname{QSym} \to \operatorname{QSym}$ be the antipode of the Hopf algebra QSym. Let $\alpha \in \operatorname{Comp}$. Then,

$$\eta_\alpha^{(q)} = (-1)^{\ell(\alpha)} T_r \left(S \left(M_{\operatorname{rev}\alpha} \right) \right).$$

Proof of Theorem 3.24. Write the composition α in the form $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$. Thus, the definition of $\ell(\alpha)$ yields $\ell(\alpha) = \ell$, whereas the definition of $\operatorname{rev}\alpha$ yields $\operatorname{rev}\alpha = (\alpha_\ell, \alpha_{\ell-1}, \ldots, \alpha_1)$. Moreover, a trivial fact ([GriVas23, Proposition 3.3]) yields $|\operatorname{rev}\alpha| = |\alpha|$.

Set $n = |\alpha|$. Thus, $\alpha \in \operatorname{Comp}_n$. Furthermore, $|\operatorname{rev}\alpha| = |\alpha| = n$, so that $\operatorname{rev}\alpha \in \operatorname{Comp}_n$. Also, as we know, we have $\operatorname{rev}\alpha = (\alpha_\ell, \alpha_{\ell-1}, \ldots, \alpha_1)$. Hence, (18) (applied to $\operatorname{rev}\alpha$ and $\alpha_{\ell+1-i}$ instead of α and α_i) yields

$$S\left(M_{\operatorname{rev}\alpha}\right) = (-1)^\ell \sum_{\substack{\gamma \in \operatorname{Comp}_n; \\ D(\gamma) \subseteq D(\alpha_1, \alpha_2, \ldots, \alpha_\ell)}} M_\gamma = (-1)^\ell \sum_{\substack{\gamma \in \operatorname{Comp}_n; \\ D(\gamma) \subseteq D(\alpha)}} M_\gamma$$

(since $(\alpha_1, \alpha_2, \ldots, \alpha_\ell) = \alpha$). Applying the map T_r to both sides of this equality, we obtain

$$\begin{aligned} T_r\left(S\left(M_{\operatorname{rev}\alpha}\right)\right) &= T_r\left((-1)^\ell \sum_{\substack{\gamma \in \operatorname{Comp}_n; \\ D(\gamma) \subseteq D(\alpha)}} M_\gamma \right) = (-1)^\ell \sum_{\substack{\gamma \in \operatorname{Comp}_n; \\ D(\gamma) \subseteq D(\alpha)}} \underbrace{T_r\left(M_\gamma\right)}_{\substack{= r^{\ell(\gamma)} M_\gamma \\ \text{(by the definition of } T_r)}} \\ &= (-1)^\ell \underbrace{\sum_{\substack{\gamma \in \operatorname{Comp}_n; \\ D(\gamma) \subseteq D(\alpha)}} r^{\ell(\gamma)} M_\gamma}_{\substack{= \sum_{\substack{\beta \in \operatorname{Comp}_n; \\ D(\beta) \subseteq D(\alpha)}} r^{\ell(\beta)} M_\beta \\ = \eta_\alpha^{(q)} \\ \text{(by (5))}}} = (-1)^\ell \eta_\alpha^{(q)}. \end{aligned}$$

Multiplying both sides of this equality by $(-1)^{\ell(\alpha)}$, we obtain

$$(-1)^{\ell(\alpha)} T_r\left(S\left(M_{\operatorname{rev}\alpha}\right)\right) = \underbrace{(-1)^{\ell(\alpha)}}_{\substack{= (-1)^\ell \\ \text{(since } \ell(\alpha) = \ell)}} (-1)^\ell \eta_\alpha^{(q)} = \underbrace{(-1)^\ell (-1)^\ell}_{\substack{= (-1)^{\ell+\ell} = 1 \\ \text{(since } \ell+\ell = 2\ell \text{ is even)}}} \eta_\alpha^{(q)} = \eta_\alpha^{(q)}.$$

This proves Theorem 3.24. $\square$

3.7. Another proof for $\Delta\left(\eta_\alpha^{(q)}\right)$

Let us now outline another proof of Theorem 3.18.

Second proof of Theorem 3.18 (sketched). For every $\alpha \in \operatorname{Comp}$, we have

$$\begin{aligned}
\Delta\left(M_{\operatorname{rev}\alpha}\right) &= \sum_{\substack{\beta,\gamma\in\operatorname{Comp};\\ \operatorname{rev}\alpha=\beta\gamma}} M_\beta \otimes M_\gamma \qquad \text{(by (31))} \\
&= \sum_{\substack{\beta,\gamma\in\operatorname{Comp};\\ \operatorname{rev}\alpha=(\operatorname{rev}\gamma)(\operatorname{rev}\beta)}} M_{\operatorname{rev}\gamma} \otimes M_{\operatorname{rev}\beta} \qquad \left(\begin{array}{c}\text{here, we have}\\ \text{substituted } \operatorname{rev}\gamma \text{ and } \operatorname{rev}\beta\\ \text{for } \beta \text{ and } \gamma \text{ in the sum}\end{array}\right) \\
&= \sum_{\substack{\beta,\gamma\in\operatorname{Comp};\\ \operatorname{rev}\alpha=\operatorname{rev}(\beta\gamma)}} M_{\operatorname{rev}\gamma} \otimes M_{\operatorname{rev}\beta} \qquad \left(\begin{array}{c}\text{since it is easy to see}\\ \text{that } (\operatorname{rev}\gamma)(\operatorname{rev}\beta) = \operatorname{rev}(\beta\gamma)\\ \text{for any } \beta,\gamma \in \operatorname{Comp}\end{array}\right) \\
&= \sum_{\substack{\beta,\gamma\in\operatorname{Comp};\\ \alpha=\beta\gamma}} M_{\operatorname{rev}\gamma} \otimes M_{\operatorname{rev}\beta} \qquad (34)
\end{aligned}$$

(since the condition "$\operatorname{rev}\alpha = \operatorname{rev}(\beta\gamma)$" is equivalent to "$\alpha = \beta\gamma$").

Let $S : \operatorname{QSym} \to \operatorname{QSym}$ be the antipode of the Hopf algebra QSym. A classical result (see, e.g., [GriRei20, Exercise 1.4.28]) says that the antipode of any Hopf algebra is a $\mathbf{k}$-coalgebra anti-endomorphism (see [GriRei20, Definition 1.4.8] for the definition of this concept). Thus, in particular, the antipode S of QSym is a $\mathbf{k}$-coalgebra anti-endomorphism. In particular, this says that

$$\Delta \circ S = T \circ (S \otimes S) \circ \Delta,$$

where $T : \operatorname{QSym} \otimes \operatorname{QSym} \to \operatorname{QSym} \otimes \operatorname{QSym}$ is the $\mathbf{k}$-linear map that sends each pure tensor $x \otimes y$ to $y \otimes x$. Applying both sides of this equality to $M_{\operatorname{rev}\alpha}$, we obtain

$$\begin{aligned}
(\Delta \circ S)\left(M_{\operatorname{rev}\alpha}\right) &= (T \circ (S \otimes S) \circ \Delta)\left(M_{\operatorname{rev}\alpha}\right) = T\left((S \otimes S)\left(\Delta\left(M_{\operatorname{rev}\alpha}\right)\right)\right) \\
&= T\left((S \otimes S)\left(\sum_{\substack{\beta,\gamma\in\operatorname{Comp};\\ \alpha=\beta\gamma}} M_{\operatorname{rev}\gamma} \otimes M_{\operatorname{rev}\beta}\right)\right) \qquad \text{(by (34))} \\
&= \sum_{\substack{\beta,\gamma\in\operatorname{Comp};\\ \alpha=\beta\gamma}} S\left(M_{\operatorname{rev}\beta}\right) \otimes S\left(M_{\operatorname{rev}\gamma}\right) \qquad (35)
\end{aligned}$$

(by the definitions of the maps T and $S \otimes S$).

However, Lemma 3.22 yields $\Delta \circ T_r = (T_r \otimes T_r) \circ \Delta$, so that

$$\begin{aligned}
(\Delta \circ T_r) (S (M_{\operatorname{rev}\alpha})) &= ((T_r \otimes T_r) \circ \Delta) (S (M_{\operatorname{rev}\alpha})) = (T_r \otimes T_r) ((\Delta \circ S) (M_{\operatorname{rev}\alpha})) \\
&= (T_r \otimes T_r) \left(\sum_{\substack{\beta,\gamma \in \operatorname{Comp}; \\ \alpha = \beta\gamma}} S \left(M_{\operatorname{rev}\beta} \right) \otimes S \left(M_{\operatorname{rev}\gamma} \right) \right) \qquad \text{(by (35))} \\
&= \sum_{\substack{\beta,\gamma \in \operatorname{Comp}; \\ \alpha = \beta\gamma}} \underbrace{T_r \left(S \left(M_{\operatorname{rev}\beta} \right) \right)}_{\substack{= \eta^{(q)}_\beta / (-1)^{\ell(\beta)} \\ \text{(since Theorem 3.24} \\ \text{yields } \eta^{(q)}_\beta = (-1)^{\ell(\beta)} T_r \left(S \left(M_{\operatorname{rev}\beta} \right) \right))}} \otimes \underbrace{T_r \left(S \left(M_{\operatorname{rev}\gamma} \right) \right)}_{\substack{= \eta^{(q)}_\gamma / (-1)^{\ell(\gamma)} \\ \text{(since Theorem 3.24} \\ \text{yields } \eta^{(q)}_\gamma = (-1)^{\ell(\gamma)} T_r (S (M_{\operatorname{rev}\gamma})))}} \\
&= \sum_{\substack{\beta,\gamma \in \operatorname{Comp}; \\ \alpha = \beta\gamma}} \left(\eta^{(q)}_\beta / (-1)^{\ell(\beta)} \right) \otimes \left(\eta^{(q)}_\gamma / (-1)^{\ell(\gamma)} \right) \\
&= \sum_{\substack{\beta,\gamma \in \operatorname{Comp}; \\ \alpha = \beta\gamma}} \left(\eta^{(q)}_\beta \otimes \eta^{(q)}_\gamma \right) / \underbrace{(-1)^{\ell(\beta) + \ell(\gamma)}}_{\substack{= (-1)^{\ell(\alpha)} \\ \text{(since it is easy to see that } \ell(\beta) + \ell(\gamma) = \ell(\beta\gamma), \\ \text{but using } \alpha = \beta\gamma \text{ we can rewrite} \\ \text{this as } \ell(\beta) + \ell(\gamma) = \ell(\alpha))}} \\
&= \sum_{\substack{\beta,\gamma \in \operatorname{Comp}; \\ \alpha = \beta\gamma}} \left(\eta^{(q)}_\beta \otimes \eta^{(q)}_\gamma \right) / (-1)^{\ell(\alpha)} .
\end{aligned}$$

Multiplying this equality by $(-1)^{\ell(\alpha)}$, we find

$$(-1)^{\ell(\alpha)} (\Delta \circ T_r) (S (M_{\operatorname{rev}\alpha})) = \sum_{\substack{\beta,\gamma \in \operatorname{Comp}; \\ \alpha = \beta\gamma}} \eta^{(q)}_\beta \otimes \eta^{(q)}_\gamma ,$$

so that

$$\begin{aligned}
\sum_{\substack{\beta,\gamma \in \operatorname{Comp}; \\ \alpha = \beta\gamma}} \eta^{(q)}_\beta \otimes \eta^{(q)}_\gamma &= (-1)^{\ell(\alpha)} (\Delta \circ T_r) (S (M_{\operatorname{rev}\alpha})) \\
&= (-1)^{\ell(\alpha)} \Delta (T_r (S (M_{\operatorname{rev}\alpha}))) \\
&= \Delta \left(\underbrace{(-1)^{\ell(\alpha)} T_r (S (M_{\operatorname{rev}\alpha}))}_{\substack{= \eta^{(q)}_\alpha \\ \text{(by Theorem 3.24)}}} \right) = \Delta \left(\eta^{(q)}_\alpha \right) .
\end{aligned}$$

This proves Theorem 3.18 again. □

4. The dual eta basis of NSym

4.1. NSym and the duality pairing

Let NSym denote the free $\mathbf{k}$-algebra with generators $H_1, H_2, H_3, \ldots$ (that is, the tensor algebra of the free $\mathbf{k}$-module with basis $(H_1, H_2, H_3, \ldots)$). This $\mathbf{k}$-algebra NSym is known as the *ring of noncommutative symmetric functions* over $\mathbf{k}$. We refer to [GriRei20, §5.4], [GKLLRT94] and [Meliot17, §6.1] for more about this $\mathbf{k}$-algebra[9]; we will only need a few basic properties.

We set $H_0 := 1 \in \mathrm{NSym}$. Thus, an element H_n of NSym is defined for each $n \in \mathbb{N}$. For any composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in \mathrm{Comp}$, we set

$$H_\alpha := H_{\alpha_1} H_{\alpha_2} \cdots H_{\alpha_k} \in \mathrm{NSym}.$$

The family $(H_\alpha)_{\alpha \in \mathrm{Comp}}$ is then a basis of the $\mathbf{k}$-module NSym. (Note that $H_{(n)} = H_n$ for each $n > 0$.)

The $\mathbf{k}$-algebra NSym is graded, with each generator H_n being homogeneous of degree n (and thus each basis element H_α being homogeneous of degree $|\alpha|$). It becomes a connected graded $\mathbf{k}$-bialgebra if we define its coproduct $\Delta : \mathrm{NSym} \to \mathrm{NSym} \otimes \mathrm{NSym}$ and its counit $\varepsilon : \mathrm{NSym} \to \mathbf{k}$ as follows:

- The coproduct $\Delta : \mathrm{NSym} \to \mathrm{NSym} \otimes \mathrm{NSym}$ is the $\mathbf{k}$-algebra homomorphism that sends each generator H_n to $\sum\limits_{i=0}^{n} H_i \otimes H_{n-i}$.
- The counit $\varepsilon : \mathrm{NSym} \to \mathbf{k}$ is the $\mathbf{k}$-algebra homomorphism that sends each generator H_n (with $n > 0$) to 0.

Therefore, NSym becomes a Hopf algebra (since any connected graded $\mathbf{k}$-bialgebra is a Hopf algebra). Its antipode S is described in [GriRei20, (5.4.12)].

Most importantly to us, the Hopf algebra NSym is isomorphic to the graded dual of QSym. Specifically, we can define a $\mathbf{k}$-bilinear form $\langle \cdot, \cdot \rangle : \mathrm{NSym} \times \mathrm{QSym} \to \mathbf{k}$ by requiring that

$$\left\langle H_\alpha, M_\beta \right\rangle = [\alpha = \beta] \tag{36}$$

for all $\alpha, \beta \in \mathrm{Comp}$ (where we are using Convention 3.7)[10]. It can be seen that this $\mathbf{k}$-bilinear form produces a canonical isomorphism

$$\begin{aligned} \mathrm{NSym} &\to \mathrm{QSym}^o, \\ f &\mapsto \langle f, \cdot \rangle \end{aligned}$$

[9] We note some notational differences:

- What we call H_α is called S_α in [GKLLRT94] and in [Meliot17].
- The algebra NSym is denoted by NCSym in [Meliot17] (unfortunately, since NCSym also has a different meaning).

[10] This bilinear form $\langle \cdot, \cdot \rangle$ is denoted by $(\cdot, \cdot)$ in [GriRei20, §5.4].

of graded Hopf algebras, where QSym^o is the graded dual of the Hopf algebra QSym. Thus, we can identify NSym with the graded dual of the Hopf algebra QSym. (In [GriRei20, §5.4], this is used as a definition of NSym, while the properties that we used to define NSym above are stated as [GriRei20, Theorem 5.4.2].)

4.2. The dual eta basis

We shall now construct a basis of NSym that is dual to the basis $\left(\eta_\alpha^{(q)}\right)_{\alpha\in\mathrm{Comp}}$ of QSym. This requires the assumption that r is invertible (since this assumption ensures that $\left(\eta_\alpha^{(q)}\right)_{\alpha\in\mathrm{Comp}}$ is a basis of QSym in the first place[11]). Thus, we make the following convention:

Convention 4.1. For the rest of Section 4, we assume that r is invertible in $\mathbf{k}$.

Definition 4.2. For each $n \in \mathbb{N}$ and each composition α of n, we define an element

$$\eta_\alpha^{*(q)} := \sum_{\substack{\beta\in\mathrm{Comp}_n; \\ D(\alpha)\subseteq D(\beta)}} \frac{1}{r^{\ell(\beta)}} (-1)^{\ell(\beta)-\ell(\alpha)} H_\beta \in \mathrm{NSym}.$$

Example 4.3. We have

$$\begin{aligned}
\eta_{()}^{*(q)} &= H_{()} = 1_{\mathrm{NSym}}; \\
\eta_{(1)}^{*(q)} &= \frac{1}{r} H_{(1)}; \\
\eta_{(2)}^{*(q)} &= \frac{1}{r} H_{(2)} - \frac{1}{r^2} H_{(1,1)}; \\
\eta_{(1,1)}^{*(q)} &= \frac{1}{r^2} H_{(1,1)}.
\end{aligned}$$

We now claim the following:

Proposition 4.4.

(a) The family $\left(\eta_\alpha^{*(q)}\right)_{\alpha\in\mathrm{Comp}}$ is the basis of NSym dual to the basis $\left(\eta_\alpha^{(q)}\right)_{\alpha\in\mathrm{Comp}}$ of QSym with respect to the bilinear form $\langle \cdot, \cdot \rangle$.

Here, the notion of a "dual basis" should be understood in the graded sense, as explained in [GriRei20, §1.6]. Concretely, our claim is saying that

[11] by Theorem 3.10

$\left(\eta_\alpha^{*(q)}\right)_{\alpha\in\mathrm{Comp}}$ is a graded basis of NSym and satisfies

$$\left\langle \eta_\alpha^{*(q)}, \eta_\beta^{(q)} \right\rangle = [\alpha = \beta] \tag{37}$$

for all $\alpha, \beta \in \mathrm{Comp}$.

(b) Let $n \in \mathbb{N}$. Consider the n-th graded components QSym_n and NSym_n of the graded $\mathbf{k}$-modules QSym and NSym. Then, the family $\left(\eta_\alpha^{*(q)}\right)_{\alpha\in\mathrm{Comp}_n}$ is the basis of NSym_n dual to the basis $\left(\eta_\alpha^{(q)}\right)_{\alpha\in\mathrm{Comp}_n}$ of QSym_n with respect to the bilinear form $\langle\cdot,\cdot\rangle$.

To prove this, we will use a general fact about dual bases of $\mathbf{k}$-modules that should be known from linear algebra, but is hard to find explicitly in the literature. This fact is a close relative of the classical linear-algebraic result that the transpose of a matrix represents the adjoint of its linear map:

Lemma 4.5. Let F and U be two free $\mathbf{k}$-modules, and let $\langle\cdot,\cdot\rangle : F \times U \to \mathbf{k}$ be a $\mathbf{k}$-bilinear form. Let A be a finite set. Let $(f_\alpha)_{\alpha\in A}$ be a basis of the $\mathbf{k}$-module F, and let $(g_\alpha)_{\alpha\in A}$ be a further family of elements of F. Let $(u_\alpha)_{\alpha\in A}$ and $(v_\alpha)_{\alpha\in A}$ be two bases of the $\mathbf{k}$-module U.

Assume that the basis $(u_\alpha)_{\alpha\in A}$ of U is dual to the basis $(f_\alpha)_{\alpha\in A}$ of F; in other words, assume that

$$\left\langle f_\beta, u_\alpha \right\rangle = [\beta = \alpha] \tag{38}$$

for all $\beta, \alpha \in A$.

Furthermore, let $c_{\alpha,\beta}$ be an element of $\mathbf{k}$ for each pair $(\alpha,\beta) \in A \times A$. Assume that

$$u_\beta = \sum_{\alpha\in A} c_{\alpha,\beta} v_\alpha \qquad \text{for each } \beta \in A. \tag{39}$$

Assume furthermore that

$$g_\alpha = \sum_{\beta\in A} c_{\alpha,\beta} f_\beta \qquad \text{for each } \alpha \in A. \tag{40}$$

Then, the families $(v_\alpha)_{\alpha\in A}$ and $(g_\alpha)_{\alpha\in A}$ are mutually dual bases of the $\mathbf{k}$-modules U and F, respectively.

Proof of Lemma 4.5. We shall prove several claims:

Claim 1: Each $w \in F$ satisfies

$$w = \sum_{\alpha\in A} \langle w, u_\alpha \rangle f_\alpha. \tag{41}$$

Proof of Claim 1. Let $w \in F$. The equality (41) is a well-known property of dual bases (since the basis $(u_\alpha)_{\alpha \in A}$ of U is dual to the basis $(f_\alpha)_{\alpha \in A}$ of F), but let us recall its proof for its sake of completeness.

The family $(f_\alpha)_{\alpha \in A}$ is a basis of F, thus spans F. Hence, $w \in F$ can be written as a $\mathbf{k}$-linear combination of this family. In other words, there exists a family $(d_\alpha)_{\alpha \in A} \in \mathbf{k}^A$ of coefficients such that

$$w = \sum_{\alpha \in A} d_\alpha f_\alpha. \tag{42}$$

Consider this family $(d_\alpha)_{\alpha \in A}$. Then,

$$w = \sum_{\alpha \in A} d_\alpha f_\alpha = \sum_{\beta \in A} d_\beta f_\beta$$

(here, we have renamed the summation index α as β). Hence, each $\alpha \in A$ satisfies

$$\langle w, u_\alpha \rangle = \left\langle \sum_{\beta \in A} d_\beta f_\beta, u_\alpha \right\rangle = \sum_{\beta \in A} d_\beta \underbrace{\langle f_\beta, u_\alpha \rangle}_{\substack{=[\beta=\alpha] \\ \text{(by (38))}}} = \sum_{\beta \in A} d_\beta [\beta = \alpha] = d_\alpha$$

(since the only nonzero addend in the sum $\sum_{\beta \in A} d_\beta [\beta = \alpha]$ is the addend for $\beta = \alpha$, and this latter addend equals d_α). Hence, we can rewrite (42) as

$$w = \sum_{\alpha \in A} \langle w, u_\alpha \rangle f_\alpha.$$

This proves (41). Thus, Claim 1 is proved. □

Claim 2: Each $w \in F$ satisfies

$$w = \sum_{\alpha \in A} \langle w, v_\alpha \rangle g_\alpha. \tag{43}$$

Proof of Claim 2. Let $w \in F$. Then,

$$\begin{aligned}
\sum_{\alpha \in A} \langle w, v_\alpha \rangle \underbrace{g_\alpha}_{\substack{= \sum_{\beta \in A} c_{\alpha,\beta} f_\beta \\ \text{(by (40))}}} &= \sum_{\alpha \in A} \langle w, v_\alpha \rangle \sum_{\beta \in A} c_{\alpha,\beta} f_\beta = \sum_{\beta \in A} \underbrace{\sum_{\alpha \in A} c_{\alpha,\beta} \langle w, v_\alpha \rangle}_{= \left\langle w, \sum_{\alpha \in A} c_{\alpha,\beta} v_\alpha \right\rangle} f_\beta \\
&= \sum_{\beta \in A} \left\langle w, \underbrace{\sum_{\alpha \in A} c_{\alpha,\beta} v_\alpha}_{\substack{= u_\beta \\ \text{(by (39))}}} \right\rangle f_\beta = \sum_{\beta \in A} \langle w, u_\beta \rangle f_\beta \\
&= \sum_{\alpha \in A} \langle w, u_\alpha \rangle f_\alpha = w \qquad \text{(by (41))}.
\end{aligned}$$

This proves Claim 2. □

Claim 3: The family $(g_\alpha)_{\alpha\in A}$ spans the $\mathbf{k}$-module F.

Proof of Claim 3. Let $w \in F$. Then, Claim 2 yields $w = \sum_{\alpha\in A} \langle w, v_\alpha\rangle g_\alpha$. Hence, w belongs to the span of the family $(g_\alpha)_{\alpha\in A}$.

Forget that we fixed w. We thus have proved that each $w \in F$ belongs to the span of the family $(g_\alpha)_{\alpha\in A}$. Hence, this family $(g_\alpha)_{\alpha\in A}$ spans the $\mathbf{k}$-module F. This proves Claim 3. □

Claim 4: The family $(g_\alpha)_{\alpha\in A}$ is $\mathbf{k}$-linearly independent.

Proof of Claim 4. Let $(m_\alpha)_{\alpha\in A} \in \mathbf{k}^A$ be a family of scalars such that $\sum_{\alpha\in A} m_\alpha g_\alpha = 0$. We shall show that $m_\alpha = 0$ for each $\alpha \in A$.

Recall that $(v_\alpha)_{\alpha\in A}$ is a basis of the $\mathbf{k}$-module U. Hence, we can define a $\mathbf{k}$-linear map $M : U \to \mathbf{k}$ by requiring that

$$M(v_\alpha) = m_\alpha \qquad \text{for each } \alpha \in A$$

(because a $\mathbf{k}$-linear map from a module can be uniquely defined by specifying its values on a given basis of this module). Consider this map M.

Now, for each $\beta \in A$, we can apply the map M to both sides of (39), and obtain

$$\begin{aligned} M(u_\beta) &= M\left(\sum_{\alpha\in A} c_{\alpha,\beta} v_\alpha\right) = \sum_{\alpha\in A} c_{\alpha,\beta} \underbrace{M(v_\alpha)}_{\substack{=m_\alpha \\ \text{(by the definition of } M)}} \\ &= \sum_{\alpha\in A} c_{\alpha,\beta} m_\alpha. \end{aligned} \tag{44}$$

From $\sum_{\alpha\in A} m_\alpha g_\alpha = 0$, we obtain

$$\begin{aligned} 0 &= \sum_{\alpha\in A} m_\alpha g_\alpha = \sum_{\alpha\in A} m_\alpha \sum_{\beta\in A} c_{\alpha,\beta} f_\beta \qquad \text{(by (40))} \\ &= \sum_{\beta\in A} \underbrace{\left(\sum_{\alpha\in A} c_{\alpha,\beta} m_\alpha\right)}_{\substack{=M(u_\beta) \\ \text{(by (44))}}} f_\beta = \sum_{\beta\in A} M(u_\beta) f_\beta. \end{aligned}$$

In other words,

$$\sum_{\beta\in A} M(u_\beta) f_\beta = 0. \tag{45}$$

However, the family $(f_\alpha)_{\alpha\in A}$ is a basis of the $\mathbf{k}$-module F. In other words, the family $(f_\beta)_{\beta\in A}$ is a basis of the $\mathbf{k}$-module F (here, we have renamed the index α as

β). Hence, this family is $\mathbf{k}$-linearly independent. In other words, if $\left(n_\beta\right)_{\beta\in A} \in \mathbf{k}^A$ is a family of scalars satisfying $\sum\limits_{\beta\in A} n_\beta f_\beta = 0$, then $n_\beta = 0$ for each $\beta \in A$.

We can apply this to $n_\beta = M\left(u_\beta\right)$ (since the family $\left(M\left(u_\beta\right)\right)_{\beta\in A} \in \mathbf{k}^A$ satisfies $\sum\limits_{\beta\in A} M\left(u_\beta\right) f_\beta = 0$ (by (45))). Thus, we conclude that

$$M\left(u_\beta\right) = 0 \qquad \text{for each } \beta \in A. \tag{46}$$

Recall that the family $(u_\alpha)_{\alpha\in A}$ is a basis of the $\mathbf{k}$-module U. In other words, the family $\left(u_\beta\right)_{\beta\in A}$ is a basis of the $\mathbf{k}$-module U (here, we have renamed the index α as β). Hence, this family spans U.

Now, recall a general fact from linear algebra: If a $\mathbf{k}$-linear map $f : V \to W$ sends a basis of its domain V to 0 (that is, if it satisfies $f(v) = 0$ for all vectors v in a certain basis of V), then $f = 0$ as a map. The equality (46) shows that the $\mathbf{k}$-linear map $M : U \to \mathbf{k}$ sends a basis of U to 0 (namely, the basis $\left(u_\beta\right)_{\beta\in A}$); therefore, the previous sentence shows that $M = 0$ as a map.

Now, let $\alpha \in A$ be arbitrary. Then, $M(v_\alpha) = 0$ (since we have just shown that $M = 0$). However, the definition of M yields $M(v_\alpha) = m_\alpha$. Thus, $m_\alpha = M(v_\alpha) = 0$.

Forget that we fixed α. We thus have shown that $m_\alpha = 0$ for each $\alpha \in A$.

Forget that we fixed $(m_\alpha)_{\alpha\in A}$. We thus have proved that if $(m_\alpha)_{\alpha\in A} \in \mathbf{k}^A$ is a family of scalars such that $\sum\limits_{\alpha\in A} m_\alpha g_\alpha = 0$, then $m_\alpha = 0$ for each $\alpha \in A$. In other words, the family $(g_\alpha)_{\alpha\in A}$ is $\mathbf{k}$-linearly independent. This proves Claim 4. $\square$

Now, we know that the family $(g_\alpha)_{\alpha\in A}$ spans the $\mathbf{k}$-module F (by Claim 3) and is $\mathbf{k}$-linearly independent (by Claim 4). In other words, this family is a basis of F.

Next, we claim the following:

Claim 5: We have $\left\langle g_\beta, v_\alpha\right\rangle = [\beta = \alpha]$ for all $\alpha, \beta \in A$.

Proof of Claim 5. Fix $\beta \in A$.

The sum $\sum\limits_{\alpha\in A} [\beta = \alpha]\, g_\alpha$ equals g_β, since all its addends except for the $\alpha = \beta$ addend are 0. Thus,

$$\sum_{\alpha\in A} [\beta = \alpha]\, g_\alpha = g_\beta = \sum_{\alpha\in A} \left\langle g_\beta, v_\alpha\right\rangle g_\alpha \tag{47}$$

(by Claim 2, applied to $w = g_\beta$).

Now, Claim 4 says that the family $(g_\alpha)_{\alpha\in A}$ is $\mathbf{k}$-linearly independent. In other words, if $(m_\alpha)_{\alpha\in A} \in \mathbf{k}^A$ and $(n_\alpha)_{\alpha\in A} \in \mathbf{k}^A$ are two families of scalars such that $\sum\limits_{\alpha\in A} m_\alpha g_\alpha = \sum\limits_{\alpha\in A} n_\alpha g_\alpha$, then $m_\alpha = n_\alpha$ for each $\alpha \in A$. Applying this to $m_\alpha = \left\langle g_\beta, v_\alpha\right\rangle$ and $n_\alpha = [\beta = \alpha]$, we conclude that

$$\left\langle g_\beta, v_\alpha\right\rangle = [\beta = \alpha] \qquad \text{for each } \alpha \in A$$

(since (47) yields $\sum\limits_{\alpha\in A} \left\langle g_\beta, v_\alpha\right\rangle g_\alpha = \sum\limits_{\alpha\in A} [\beta = \alpha]\, g_\alpha$). This proves Claim 5. $\square$

Now, recall that $(g_\alpha)_{\alpha \in A}$ is a basis of the $\mathbf{k}$-module F, whereas $(v_\alpha)_{\alpha \in A}$ is a basis of the $\mathbf{k}$-module U. Furthermore, Claim 5 shows that the bases $(v_\alpha)_{\alpha \in A}$ and $(g_\alpha)_{\alpha \in A}$ of U and F are mutually dual. This completes the proof of Lemma 4.5. $\square$

We are now ready to prove Proposition 4.4:

Proof of Proposition 4.4. **(b)** The scalar r is invertible (by Convention 4.1). Thus, we define an element

$$c_{\alpha,\beta} := \begin{cases} \dfrac{1}{r^{\ell(\beta)}} (-1)^{\ell(\beta)-\ell(\alpha)}, & \text{if } D(\alpha) \subseteq D(\beta); \\ 0, & \text{otherwise} \end{cases}$$

of $\mathbf{k}$ for every $\alpha, \beta \in \mathrm{Comp}_n$.

Now, let $\beta \in \mathrm{Comp}_n$. Then, Proposition 3.6 shows that

$$r^{\ell(\beta)} M_\beta = \sum_{\substack{\alpha \in \mathrm{Comp}_n; \\ D(\alpha) \subseteq D(\beta)}} (-1)^{\ell(\beta)-\ell(\alpha)} \eta_\alpha^{(q)}. \tag{48}$$

Dividing both sides of this equality by $r^{\ell(\beta)}$, we obtain

$$M_\beta = \sum_{\substack{\alpha \in \mathrm{Comp}_n; \\ D(\alpha) \subseteq D(\beta)}} \frac{1}{r^{\ell(\beta)}} (-1)^{\ell(\beta)-\ell(\alpha)} \eta_\alpha^{(q)} = \sum_{\alpha \in \mathrm{Comp}_n} c_{\alpha,\beta} \eta_\alpha^{(q)}$$

(since our definition of $c_{\alpha,\beta}$ ensures that all addends of the sum $\sum\limits_{\alpha \in \mathrm{Comp}_n} c_{\alpha,\beta} \eta_\alpha^{(q)}$ vanish except for the addends that satisfy $D(\alpha) \subseteq D(\beta)$, but these latter addends are precisely the addends of $\sum\limits_{\substack{\alpha \in \mathrm{Comp}_n; \\ D(\alpha) \subseteq D(\beta)}} \dfrac{1}{r^{\ell(\beta)}} (-1)^{\ell(\beta)-\ell(\alpha)} \eta_\alpha^{(q)}$).

Forget that we fixed β. We thus have proved that

$$M_\beta = \sum_{\alpha \in \mathrm{Comp}_n} c_{\alpha,\beta} \eta_\alpha^{(q)} \tag{49}$$

holds for each $\beta \in \mathrm{Comp}_n$.

Furthermore, for each $\alpha \in \mathrm{Comp}_n$, we have

$$\begin{aligned} \sum_{\beta \in \mathrm{Comp}_n} c_{\alpha,\beta} H_\beta &= \sum_{\substack{\beta \in \mathrm{Comp}_n; \\ D(\alpha) \subseteq D(\beta)}} \frac{1}{r^{\ell(\beta)}} (-1)^{\ell(\beta)-\ell(\alpha)} H_\beta && \left(\text{by the definition of the } c_{\alpha,\beta}\right) \\ &= \eta_\alpha^{*(q)} && (\text{by Definition 4.2}) \end{aligned}$$

and thus

$$\eta_\alpha^{*(q)} = \sum_{\beta \in \mathrm{Comp}_n} c_{\alpha,\beta} H_\beta. \tag{50}$$

Altogether, we now know the following:

- The $\mathbf{k}$-modules NSym_n and QSym_n are free, and $\langle\cdot,\cdot\rangle : \operatorname{NSym}_n \times \operatorname{QSym}_n \to \mathbf{k}$ is a $\mathbf{k}$-bilinear form.
- The set Comp_n is a finite set.
- The family $(H_\alpha)_{\alpha\in\operatorname{Comp}_n}$ is a basis of the $\mathbf{k}$-module NSym_n, and the family $\left(\eta_\alpha^{*(q)}\right)_{\alpha\in\operatorname{Comp}_n}$ is a further family of elements of NSym_n (since (50) readily yields that $\eta_\alpha^{*(q)} \in \operatorname{NSym}_n$ for each $\alpha \in \operatorname{Comp}_n$).
- The families $(M_\alpha)_{\alpha\in\operatorname{Comp}_n}$ and $\left(\eta_\alpha^{(q)}\right)_{\alpha\in\operatorname{Comp}_n}$ are two bases of the $\mathbf{k}$-module QSym_n (by Theorem 3.10 **(b)**).
- The basis $(M_\alpha)_{\alpha\in\operatorname{Comp}_n}$ of QSym_n is dual to the basis $(H_\alpha)_{\alpha\in\operatorname{Comp}_n}$ of NSym_n (because of (36)).
- The elements $c_{\alpha,\beta} \in \mathbf{k}$ are defined for all $(\alpha,\beta) \in \operatorname{Comp}_n \times \operatorname{Comp}_n$, and satisfy (49) for each $\beta \in \operatorname{Comp}_n$ and (50) for each $\alpha \in \operatorname{Comp}_n$.

Thus, Lemma 4.5 (applied to $F = \operatorname{NSym}_n$, $U = \operatorname{QSym}_n$, $A = \operatorname{Comp}_n$, $f_\alpha = H_\alpha$, $g_\alpha = \eta_\alpha^{*(q)}$, $u_\alpha = M_\alpha$ and $v_\alpha = \eta_\alpha^{(q)}$) shows that the families $\left(\eta_\alpha^{(q)}\right)_{\alpha\in\operatorname{Comp}_n}$ and $\left(\eta_\alpha^{*(q)}\right)_{\alpha\in\operatorname{Comp}_n}$ are mutually dual bases of QSym_n and NSym_n, respectively. In other words, the family $\left(\eta_\alpha^{*(q)}\right)_{\alpha\in\operatorname{Comp}_n}$ is the basis of NSym_n dual to the basis $\left(\eta_\alpha^{(q)}\right)_{\alpha\in\operatorname{Comp}_n}$ of QSym_n. This proves Proposition 4.4 **(b)**.

(a) This follows from part **(b)** by taking the direct sum over all n. □

4.3. The dual eta basis: product

We shall now study the multiplicative structure of the dual eta basis $\left(\eta_\alpha^{*(q)}\right)_{\alpha\in\operatorname{Comp}}$. First, we introduce a notation for the simplest entries of this basis:

Definition 4.6. For each positive integer n, we let

$$\eta_n^{*(q)} := \eta_{(n)}^{*(q)} = \sum_{\beta\in\operatorname{Comp}_n} \frac{1}{r^{\ell(\beta)}} (-1)^{\ell(\beta)-1} H_\beta \\ \in \operatorname{NSym}. \tag{51}$$

(The second equality sign here is easy to check.[12])

It turns out that we can easily express $\eta_\alpha^{*(q)}$ for any composition α using these $\eta_n^{*(q)}$:

Proposition 4.7. We have

$$\eta_\alpha^{*(q)} = \eta_{\alpha_1}^{*(q)} \eta_{\alpha_2}^{*(q)} \cdots \eta_{\alpha_k}^{*(q)} \qquad \text{for each composition } \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k).$$

The main idea of the proof of Proposition 4.7 is to recognize that if $n = |\alpha|$, then the compositions $\beta \in \operatorname{Comp}_n$ satisfying $D(\alpha) \subseteq D(\beta)$ are precisely the compositions obtained from α by breaking up each entry of α into pieces. A slicker way to formalize this proof proceeds using the notion of concatenation:

Definition 4.8. If $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_k)$ are two compositions, then the composition $\alpha\beta$ is defined by

$$\alpha\beta = (\alpha_1, \alpha_2, \ldots, \alpha_\ell, \beta_1, \beta_2, \ldots, \beta_k).$$

This composition $\alpha\beta$ is called the *concatenation* of α and β. The operation of concatenation (sending any two compositions α and β to $\alpha\beta$) is associative, and the empty composition $\varnothing$ is a neutral element for it; thus, the set of all compositions is a monoid under this operation.

The following proposition is saying (in the jargon of combinatorial Hopf algebras) that the basis $\left(\eta_\alpha^{*(q)}\right)_{\alpha \in \operatorname{Comp}}$ of NSym is multiplicative:

Proposition 4.9. Let α and β be two compositions. Then,

$$\eta_\alpha^{*(q)} \eta_\beta^{*(q)} = \eta_{\alpha\beta}^{*(q)}.$$

In order to prove this, we will use the comultiplication $\Delta : \operatorname{QSym} \to \operatorname{QSym} \otimes \operatorname{QSym}$ of the Hopf algebra QSym as well as the duality between NSym and QSym:

[12]Indeed, Definition 4.2 yields

$$\eta_{(n)}^{*(q)} = \sum_{\substack{\beta \in \operatorname{Comp}_n; \\ D((n)) \subseteq D(\beta)}} \frac{1}{r^{\ell(\beta)}} (-1)^{\ell(\beta) - \ell((n))} H_\beta.$$

However, every $\beta \in \operatorname{Comp}_n$ automatically satisfies $D((n)) \subseteq D(\beta)$ (because $D((n)) = \varnothing$). Hence, the condition "$D((n)) \subseteq D(\beta)$" under the summation sign is superfluous. Also, $\ell((n)) = 1$. Thus, the above equality simplifies to

$$\eta_{(n)}^{*(q)} = \sum_{\beta \in \operatorname{Comp}_n} \frac{1}{r^{\ell(\beta)}} (-1)^{\ell(\beta) - 1} H_\beta.$$

Lemma 4.10. Let $f, g \in \operatorname{NSym}$ and $h \in \operatorname{QSym}$ be arbitrary. Let the tensor $\Delta(h) \in \operatorname{QSym} \otimes \operatorname{QSym}$ be written in the form $\Delta(h) = \sum_{i \in I} s_i \otimes t_i$, where I is a finite set and where $s_i, t_i \in \operatorname{QSym}$ for each $i \in I$. Then,

$$\langle fg, h \rangle = \sum_{i \in I} \langle f, s_i \rangle \langle g, t_i \rangle .$$

Proof. Recall that the **k**-bilinear form $\langle \cdot, \cdot \rangle$ identifies NSym with the graded dual QSym^o as Hopf algebras. Thus, in particular, the multiplication of NSym and the comultiplication of QSym are mutually adjoint with respect to this form. In other words, if $f, g \in \operatorname{NSym}$ and $h \in \operatorname{QSym}$, then

$$\langle fg, h \rangle = \sum_{(h)} \left\langle f, h_{(1)} \right\rangle \left\langle g, h_{(2)} \right\rangle ,$$

where we are using the Sweedler notation $\sum_{(h)} h_{(1)} \otimes h_{(2)}$ for $\Delta(h)$ (see, e.g., [GriRei20, (1.2.3)]). Lemma 4.10 is just restating this fact without using the Sweedler notation. □

Proof of Proposition 4.9. This follows by dualization from Theorem 3.18. Here are the details:

Forget that we fixed α and β. Proposition 4.4 **(a)** shows that the families $\left(\eta_\alpha^{*(q)}\right)_{\alpha \in \operatorname{Comp}}$ and $\left(\eta_\alpha^{(q)}\right)_{\alpha \in \operatorname{Comp}}$ are mutually dual bases of NSym and QSym with respect to the bilinear form $\langle \cdot, \cdot \rangle$. This shows that

$$\left\langle \eta_\lambda^{*(q)}, \eta_\mu^{(q)} \right\rangle = [\lambda = \mu] \tag{52}$$

for all $\lambda, \mu \in \operatorname{Comp}$. But another consequence of this duality is that the bilinear form $\langle \cdot, \cdot \rangle$ is nondegenerate (since only nondegenerate forms have dual bases), and that the family $\left(\eta_\alpha^{(q)}\right)_{\alpha \in \operatorname{Comp}}$ is a basis of QSym. Hence, in order to prove that two elements $f, g \in \operatorname{NSym}$ are equal, it suffices to show that $\left\langle f, \eta_\gamma^{(q)} \right\rangle = \left\langle g, \eta_\gamma^{(q)} \right\rangle$ holds for each $\gamma \in \operatorname{Comp}$.

We shall use this strategy to prove $\eta_\alpha^{*(q)} \eta_\beta^{*(q)} = \eta_{\alpha\beta}^{*(q)}$ for all $\alpha, \beta \in \operatorname{Comp}$. Thus, we need to show that $\left\langle \eta_\alpha^{*(q)} \eta_\beta^{*(q)}, \eta_\gamma^{(q)} \right\rangle = \left\langle \eta_{\alpha\beta}^{*(q)}, \eta_\gamma^{(q)} \right\rangle$ holds for all $\alpha, \beta, \gamma \in \operatorname{Comp}$.

To show this, we fix $\alpha, \beta, \gamma \in \operatorname{Comp}$. Theorem 3.18 (with the letters α, β, γ replaced by γ, φ, ψ) says that

$$\Delta\left(\eta_\gamma^{(q)}\right) = \sum_{\substack{\varphi, \psi \in \operatorname{Comp}; \\ \gamma = \varphi\psi}} \eta_\varphi^{(q)} \otimes \eta_\psi^{(q)} .$$

Hence, Lemma 4.10 (applied to $f = \eta^{*(q)}_\alpha$ and $g = \eta^{*(q)}_\beta$ and $h = \eta^{(q)}_\gamma$ and

$$I = \{(\varphi, \psi) \in \text{Comp} \times \text{Comp} \mid \gamma = \varphi\psi\}$$

and $s_{(\varphi,\psi)} = \eta^{(q)}_\varphi$ and $t_{(\varphi,\psi)} = \eta^{(q)}_\psi$) yields

$$\begin{aligned}\left\langle \eta^{*(q)}_\alpha \eta^{*(q)}_\beta, \eta^{(q)}_\gamma \right\rangle &= \sum_{\substack{\varphi,\psi\in\text{Comp};\\ \gamma=\varphi\psi}} \underbrace{\left\langle \eta^{*(q)}_\alpha, \eta^{(q)}_\varphi \right\rangle}_{\substack{=[\alpha=\varphi]\\ \text{(by (52))}}} \underbrace{\left\langle \eta^{*(q)}_\beta, \eta^{(q)}_\psi \right\rangle}_{\substack{=[\beta=\psi]\\ \text{(by (52))}}} = \sum_{\substack{\varphi,\psi\in\text{Comp};\\ \gamma=\varphi\psi}} \underbrace{[\alpha=\varphi]\cdot[\beta=\psi]}_{=[\alpha=\varphi \text{ and } \beta=\psi]} \\ &= \sum_{\substack{\varphi,\psi\in\text{Comp};\\ \gamma=\varphi\psi}} [\alpha = \varphi \text{ and } \beta = \psi]. \qquad (53)\end{aligned}$$

The sum on the right hand side of this equality has at most one nonzero addend – namely the addend for $\varphi = \alpha$ and $\psi = \beta$, if this addend exists. Of course, this addend exists if and only if $\gamma = \alpha\beta$, and equals 1 in this case. Thus, the sum equals 1 if $\gamma = \alpha\beta$, and otherwise equals 0. In other words, this sum equals the truth value $[\gamma = \alpha\beta]$. Hence, we can rewrite (53) as

$$\left\langle \eta^{*(q)}_\alpha \eta^{*(q)}_\beta, \eta^{(q)}_\gamma \right\rangle = [\gamma = \alpha\beta].$$

Comparing this with

$$\begin{aligned}\left\langle \eta^{*(q)}_{\alpha\beta}, \eta^{(q)}_\gamma \right\rangle &= [\alpha\beta = \gamma] \qquad \text{(by (52))} \\ &= [\gamma = \alpha\beta],\end{aligned}$$

we obtain $\left\langle \eta^{*(q)}_\alpha \eta^{*(q)}_\beta, \eta^{(q)}_\gamma \right\rangle = \left\langle \eta^{*(q)}_{\alpha\beta}, \eta^{(q)}_\gamma \right\rangle$.

Forget that we fixed γ. We thus have shown that $\left\langle \eta^{*(q)}_\alpha \eta^{*(q)}_\beta, \eta^{(q)}_\gamma \right\rangle = \left\langle \eta^{*(q)}_{\alpha\beta}, \eta^{(q)}_\gamma \right\rangle$ for each $\gamma \in \text{Comp}$. Since $\left(\eta^{(q)}_\gamma\right)_{\gamma\in\text{Comp}}$ is a basis of the $\mathbf{k}$-module QSym, and since the bilinear form $\langle \cdot, \cdot \rangle$ is nondegenerate, we thus conclude that $\eta^{*(q)}_\alpha \eta^{*(q)}_\beta = \eta^{*(q)}_{\alpha\beta}$. This proves Proposition 4.9. $\square$

Corollary 4.11. Let $\beta_1, \beta_2, \ldots, \beta_k$ be finitely many compositions. Then,

$$\eta^{*(q)}_{\beta_1} \eta^{*(q)}_{\beta_2} \cdots \eta^{*(q)}_{\beta_k} = \eta^{*(q)}_{\beta_1\beta_2\cdots\beta_k}.$$

Proof. This follows by induction on k using Proposition 4.9. (The base case, $k = 0$, follows from $\eta^{*(q)}_{()} = 1$.) $\square$

Proof of Proposition 4.7. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ be a composition. Then, applying Corollary 4.11 to the 1-element compositions $\beta_i = (\alpha_i)$, we obtain

$$\eta^{*(q)}_{(\alpha_1)} \eta^{*(q)}_{(\alpha_2)} \cdots \eta^{*(q)}_{(\alpha_k)} = \eta^{*(q)}_{(\alpha_1)(\alpha_2)\cdots(\alpha_k)} = \eta^{*(q)}_{\alpha}$$

(since the concatenation $(\alpha_1)(\alpha_2)\cdots(\alpha_k)$ equals $(\alpha_1, \alpha_2, \ldots, \alpha_k) = \alpha$). Thus,

$$\eta^{*(q)}_{\alpha} = \eta^{*(q)}_{(\alpha_1)} \eta^{*(q)}_{(\alpha_2)} \cdots \eta^{*(q)}_{(\alpha_k)} = \eta^{*(q)}_{\alpha_1} \eta^{*(q)}_{\alpha_2} \cdots \eta^{*(q)}_{\alpha_k}$$

(since $\eta^{*(q)}_{(n)} = \eta^{*(q)}_{n}$ for each $n > 0$). This proves Proposition 4.7. □

4.4. The dual eta basis: generating function

We shall now work in the ring $\operatorname{NSym}[[t]]$ of formal power series in the indeterminate t over the ring NSym. This ring $\operatorname{NSym}[[t]]$ is noncommutative (since NSym is), but the indeterminate t commutes with all its elements.

We furthermore define two special series in this ring:

Definition 4.12. Define the formal power series

$$H(t) := \sum_{n \geq 0} H_n t^n \in \operatorname{NSym}[[t]]$$

and

$$G(t) := \sum_{n \geq 1} \eta^{*(q)}_{n} t^n \in \operatorname{NSym}[[t]].$$

Now, it is easy to see the following:

Proposition 4.13. We have

$$G(t) = 1 - \frac{1}{1 + \dfrac{H(t) - 1}{r}} = \frac{H(t) - 1}{H(t) + q}.$$

Proof. The power series $H(t)$ has constant term $H_0 = 1$. Thus, the power series $H(t) - 1$ has constant term 0. Hence, the power series $1 + \dfrac{H(t) - 1}{r}$ has constant term $1 + \dfrac{0}{r} = 1$, and thus is invertible (since every formal power series with constant term 1 is invertible). The fraction $\dfrac{1}{1 + \dfrac{H(t) - 1}{r}}$ is thus well-defined.

If $u \in \operatorname{NSym}[[t]]$ is a formal power series with constant term 0, then the geometric series formula yields

$$\frac{1}{1-u} = \sum_{k\geq 0} u^k = \underbrace{u^0}_{=1} + \sum_{k\geq 1} u^k = 1 + \sum_{k\geq 1} u^k,$$

so that

$$\sum_{k\geq 1} u^k = \frac{1}{1-u} - 1. \tag{54}$$

We shall use this result in a somewhat modified form: If $u \in \operatorname{NSym}[[t]]$ is a formal power series with constant term 0, then $\frac{-u}{r}$ is also a formal power series with constant term 0, and we have

$$\begin{aligned}
\sum_{k\geq 1} \frac{1}{r^k} \underbrace{(-1)^{k-1}}_{=-(-1)^k} u^k &= -\sum_{k\geq 1} \underbrace{\frac{1}{r^k} (-1)^k u^k}_{=\left(\frac{-u}{r}\right)^k} = - \underbrace{\sum_{k\geq 1} \left(\frac{-u}{r}\right)^k}_{\substack{=\frac{1}{1-\frac{-u}{r}}-1 \\ \text{(by (54),} \\ \text{applied to } \frac{-u}{r} \text{ instead of } u)}} \\
&= -\left(\frac{1}{1-\frac{-u}{r}} - 1\right) = 1 - \frac{1}{1-\frac{-u}{r}} \\
&= 1 - \frac{1}{1+\frac{u}{r}}.
\end{aligned} \tag{55}$$

We have $H(t) = \sum_{n\geq 0} H_n t^n = \underbrace{H_0}_{=1} \underbrace{t^0}_{=1} + \sum_{n\geq 1} H_n t^n = 1 + \sum_{n\geq 1} H_n t^n$, so that

$$\sum_{n\geq 1} H_n t^n = H(t) - 1.$$

The definition of $G(t)$ yields

$$\begin{aligned}
G(t) &= \sum_{n\geq 1} \eta_n^{*(q)} t^n \\
&= \sum_{n\geq 1} \left(\sum_{\beta\in\operatorname{Comp}_n} \frac{1}{r^{\ell(\beta)}} (-1)^{\ell(\beta)-1} H_\beta \right) t^n \qquad \text{(by (51))} \\
&= \sum_{n\geq 1} \sum_{\beta\in\operatorname{Comp}_n} \frac{1}{r^{\ell(\beta)}} (-1)^{\ell(\beta)-1} H_\beta \underbrace{t^n}_{\substack{=t^{|\beta|} \\ \text{(since } \beta\in\operatorname{Comp}_n\text{)}}} \\
&= \underbrace{\sum_{n\geq 1} \sum_{\beta\in\operatorname{Comp}_n}}_{\substack{=\sum_{\beta\in\operatorname{Comp};\ |\beta|\geq 1} \\ =\sum_{\beta\in\operatorname{Comp};\ \ell(\beta)\geq 1} \\ \text{(because for a composition } \beta, \\ \text{the condition ``} |\beta|\geq 1 \text{'' is} \\ \text{equivalent to ``} \ell(\beta)\geq 1 \text{'')}}} \frac{1}{r^{\ell(\beta)}} (-1)^{\ell(\beta)-1} H_\beta t^{|\beta|} \\
&= \underbrace{\sum_{\substack{\beta\in\operatorname{Comp}; \\ \ell(\beta)\geq 1}}}_{=\sum_{k\geq 1} \sum_{\beta\in\operatorname{Comp};\ \ell(\beta)=k}} \frac{1}{r^{\ell(\beta)}} (-1)^{\ell(\beta)-1} H_\beta t^{|\beta|} \\
&= \sum_{k\geq 1} \sum_{\substack{\beta\in\operatorname{Comp}; \\ \ell(\beta)=k}} \underbrace{\frac{1}{r^{\ell(\beta)}} (-1)^{\ell(\beta)-1}}_{\substack{=\frac{1}{r^k}(-1)^{k-1} \\ \text{(since } \ell(\beta)=k\text{)}}} H_\beta t^{|\beta|} \\
&= \sum_{k\geq 1} \frac{1}{r^k} (-1)^{k-1} \underbrace{\sum_{\substack{\beta\in\operatorname{Comp}; \\ \ell(\beta)=k}} H_\beta t^{|\beta|}}_{\substack{=\sum_{(n_1,n_2,\ldots,n_k)\in\operatorname{Comp}} H_{(n_1,n_2,\ldots,n_k)} t^{n_1+n_2+\cdots+n_k} \\ \text{(here, we have renamed the summation index } \beta \text{ as } (n_1,n_2,\ldots,n_k)\text{)}}} \\
&= \sum_{k\geq 1} \frac{1}{r^k} (-1)^{k-1} \underbrace{\sum_{(n_1,n_2,\ldots,n_k)\in\operatorname{Comp}}}_{=\sum_{n_1,n_2,\ldots,n_k\geq 1}} \underbrace{H_{(n_1,n_2,\ldots,n_k)} t^{n_1+n_2+\cdots+n_k}}_{\substack{=\left(H_{n_1}H_{n_2}\cdots H_{n_k}\right)\left(t^{n_1}t^{n_2}\cdots t^{n_k}\right) \\ =\left(H_{n_1}t^{n_1}\right)\left(H_{n_2}t^{n_2}\right)\cdots\left(H_{n_k}t^{n_k}\right)}}
\end{aligned}$$

$$= \sum_{k \geq 1} \frac{1}{r^k} (-1)^{k-1} \underbrace{\sum_{n_1, n_2, \ldots, n_k \geq 1} (H_{n_1} t^{n_1}) (H_{n_2} t^{n_2}) \cdots (H_{n_k} t^{n_k})}_{\substack{= \left(\sum_{n \geq 1} H_n t^n \right)^k \\ \text{(by the product rule)}}}$$

$$= \sum_{k \geq 1} \frac{1}{r^k} (-1)^{k-1} \left(\underbrace{\sum_{n \geq 1} H_n t^n}_{= H(t) - 1} \right)^k = \sum_{k \geq 1} \frac{1}{r^k} (-1)^{k-1} (H(t) - 1)^k$$

$$= 1 - \frac{1}{1 + \frac{H(t) - 1}{r}} \qquad \text{(by (55), applied to } u = H(t) - 1\text{)}$$

$$= 1 - \frac{r}{H(t) + r - 1} = \frac{H(t) - 1}{H(t) + r - 1} = \frac{H(t) - 1}{H(t) + q}$$

(since $r - 1 = q$ (because $r = q + 1$)). This proves Proposition 4.13. ☐

Proposition 4.14. Let $k \in \mathbb{N}$. Then,

$$G(t)^k = \sum_{\substack{\beta \in \mathrm{Comp}; \\ \ell(\beta) = k}} \eta_\beta^{*(q)} t^{|\beta|}. \tag{56}$$

Proof. From $G(t) = \sum_{n \geq 1} \eta_n^{*(q)} t^n$, we obtain

$$\begin{aligned}
G(t)^k &= \left(\sum_{n \geq 1} \eta_n^{*(q)} t^n \right)^k = \sum_{n_1, n_2, \ldots, n_k \geq 1} \underbrace{\left(\eta_{n_1}^{*(q)} t^{n_1} \right) \left(\eta_{n_2}^{*(q)} t^{n_2} \right) \cdots \left(\eta_{n_k}^{*(q)} t^{n_k} \right)}_{= \eta_{n_1}^{*(q)} \eta_{n_2}^{*(q)} \cdots \eta_{n_k}^{*(q)} t^{n_1 + n_2 + \cdots + n_k}} \\
&\qquad \text{(by the product rule)} \\
&= \sum_{n_1, n_2, \ldots, n_k \geq 1} \eta_{n_1}^{*(q)} \eta_{n_2}^{*(q)} \cdots \eta_{n_k}^{*(q)} t^{n_1 + n_2 + \cdots + n_k} \\
&= \sum_{\beta = (\beta_1, \beta_2, \ldots, \beta_k) \in \text{Comp}} \underbrace{\eta_{\beta_1}^{*(q)} \eta_{\beta_2}^{*(q)} \cdots \eta_{\beta_k}^{*(q)}}_{\substack{= \eta_\beta^{*(q)} \\ \text{(by Proposition 4.7)}}} \underbrace{t^{\beta_1 + \beta_2 + \cdots + \beta_k}}_{= t^{|\beta|}} \\
&\qquad \text{(here, we have renamed } n_1, n_2, \ldots, n_k \text{ as } \beta_1, \beta_2, \ldots, \beta_k) \\
&= \underbrace{\sum_{\beta = (\beta_1, \beta_2, \ldots, \beta_k) \in \text{Comp}}}_{= \sum_{\substack{\beta \in \text{Comp}; \\ \ell(\beta) = k}}} \eta_\beta^{*(q)} t^{|\beta|} = \sum_{\substack{\beta \in \text{Comp}; \\ \ell(\beta) = k}} \eta_\beta^{*(q)} t^{|\beta|}.
\end{aligned}$$

This proves Proposition 4.14. □

4.5. The dual eta basis: coproduct

Consider the comultiplication $\Delta : \text{NSym} \to \text{NSym} \otimes \text{NSym}$ of the Hopf algebra NSym. We again recall the Iverson bracket notation (Convention 3.7).

Theorem 4.15. For any positive integer n, we have

$$\Delta \left(\eta_n^{*(q)} \right) = \sum_{\substack{\beta, \gamma \in \text{Comp}; \\ |\beta| + |\gamma| = n; \\ |\ell(\beta) - \ell(\gamma)| \leq 1}} (-q)^{\max\{\ell(\beta), \ell(\gamma)\} - 1} (q-1)^{[\ell(\beta) = \ell(\gamma)]} \eta_\beta^{*(q)} \otimes \eta_\gamma^{*(q)}.$$

Example 4.16. For $n = 2$, there are exactly three pairs (β, γ) of compositions $\beta, \gamma \in \text{Comp}$ satisfying $|\beta| + |\gamma| = n$ and $|\ell(\beta) - \ell(\gamma)| \leq 1$: namely, the pairs

$(\varnothing, (2))$, $((1), (1))$ and $((2), \varnothing)$. Hence, Theorem 4.15 (applied to $n = 2$) yields

$$\begin{aligned}
\Delta\left(\eta_2^{*(q)}\right) &= (-q)^{1-1}(q-1)^0 \eta_{\varnothing}^{*(q)} \otimes \eta_{(2)}^{*(q)} + (-q)^{1-1}(q-1)^1 \eta_{(1)}^{*(q)} \otimes \eta_{(1)}^{*(q)} \\
&\qquad + (-q)^{1-1}(q-1)^0 \eta_{(2)}^{*(q)} \otimes \eta_{\varnothing}^{*(q)} \\
&= \eta_{\varnothing}^{*(q)} \otimes \eta_{(2)}^{*(q)} + (q-1)\eta_{(1)}^{*(q)} \otimes \eta_{(1)}^{*(q)} + \eta_{(2)}^{*(q)} \otimes \eta_{\varnothing}^{*(q)} \\
&= 1 \otimes \eta_2^{*(q)} + (q-1)\eta_1^{*(q)} \otimes \eta_1^{*(q)} + \eta_2^{*(q)} \otimes 1
\end{aligned}$$

(since $\eta_{(2)}^{*(q)} = \eta_2$ and $\eta_{(1)}^{*(q)} = \eta_1$ and $\eta_{\varnothing}^{*(q)} = 1$).

Similar computations show that

$$\Delta\left(\eta_1^{*(q)}\right) = 1 \otimes \eta_1^{*(q)} + \eta_1^{*(q)} \otimes 1$$

and

$$\begin{aligned}
\Delta\left(\eta_3^{*(q)}\right) &= 1 \otimes \eta_3^{*(q)} + (q-1)\eta_1^{*(q)} \otimes \eta_2^{*(q)} - q\eta_1^{*(q)} \otimes \left(\eta_1^{*(q)}\right)^2 \\
&\qquad - q\left(\eta_1^{*(q)}\right)^2 \otimes \eta_1^{*(q)} + (q-1)\eta_2^{*(q)} \otimes \eta_1^{*(q)} + \eta_3^{*(q)} \otimes 1
\end{aligned}$$

(since Proposition 4.7 yields $\eta_{(1,1)}^{*(q)} = \left(\eta_1^{*(q)}\right)^2$).

Our proof of Theorem 4.15 will use the following general fact from abstract algebra:

Lemma 4.17. Let A and B be any two $\mathbf{k}$-algebras. Then, there is a canonical $\mathbf{k}$-algebra homomorphism

$$\begin{aligned}
\iota : A[[t]] \otimes_{\mathbf{k}[[t]]} B[[t]] &\to (A \otimes B)[[t]], \\
\left(\sum_{i\in\mathbb{N}} a_i t^i\right) \otimes \left(\sum_{j\in\mathbb{N}} b_j t^j\right) &\mapsto \left(\sum_{i\in\mathbb{N}} (a_i \otimes 1) t^i\right)\left(\sum_{j\in\mathbb{N}} (1 \otimes b_j) t^j\right).
\end{aligned}$$

Proof of Lemma 4.17 (sketched). To construct the map ι, we need A and B only to be $\mathbf{k}$-modules, not $\mathbf{k}$-algebras. The well-definedness follows easily from the fact that $\sum_{k\in\mathbb{N}} (\lambda_k 1 \otimes 1) t^k = \sum_{k\in\mathbb{N}} (1 \otimes \lambda_k 1) t^k$ for any formal power series $\sum_{k\in\mathbb{N}} \lambda_k t^k \in \mathbf{k}[[t]]$. Finally, to prove that ι is a $\mathbf{k}$-algebra homomorphism, observe that any power series of the form $\sum_{i\in\mathbb{N}} (a_i \otimes 1) t^i \in (A \otimes B)[[t]]$ commutes with any power series of the form $\sum_{j\in\mathbb{N}} (1 \otimes b_j) t^j \in (A \otimes B)[[t]]$. The details are left to the reader. $\square$

Proof of Theorem 4.15. The comultiplication $\Delta : \mathrm{NSym} \to \mathrm{NSym} \otimes \mathrm{NSym}$ is a $\mathbf{k}$-algebra homomorphism (since NSym is a $\mathbf{k}$-bialgebra), and thus induces a $\mathbf{k}[[t]]$-

algebra homomorphism

$$\Delta_t : \operatorname{NSym}[[t]] \to (\operatorname{NSym} \otimes \operatorname{NSym})[[t]]$$

that sends each formal power series $\sum_{i\in\mathbb{N}} a_i t^i$ to $\sum_{i\in\mathbb{N}} \Delta(a_i) t^i$. This Δ_t is a $\mathbf{k}$-algebra homomorphism as well (since any $\mathbf{k}[[t]]$-algebra homomorphism is a $\mathbf{k}$-algebra homomorphism).

Furthermore, Lemma 4.17 shows that there is a canonical $\mathbf{k}$-algebra homomorphism

$$\iota : \operatorname{NSym}[[t]] \otimes_{\mathbf{k}[[t]]} \operatorname{NSym}[[t]] \to (\operatorname{NSym} \otimes \operatorname{NSym})[[t]],$$
$$\left(\sum_{i\in\mathbb{N}} a_i t^i\right) \otimes \left(\sum_{j\in\mathbb{N}} b_j t^j\right) \mapsto \left(\sum_{i\in\mathbb{N}} (a_i \otimes 1) t^i\right)\left(\sum_{j\in\mathbb{N}} (1 \otimes b_j) t^j\right).$$

Unlike some authors, we will not treat ι as an embedding, but we will often use the fact that ι is a $\mathbf{k}$-algebra homomorphism.

We recall that $\mathbf{k}$-algebra homomorphisms are always ring homomorphisms, and thus respect quotients. That is, if $f : A \to B$ is a $\mathbf{k}$-algebra homomorphism, and if a_1 and a_2 are two elements of A such that a_2 is invertible in A, then $f(a_2)$ is again invertible and we have $f\left(\dfrac{a_1}{a_2}\right) = \dfrac{f(a_1)}{f(a_2)}$. We will use this fact without saying a few times.

For the sake of brevity, we define the shorthands

$$\mathbf{G} := G(t) \qquad \text{and} \qquad \mathbf{H} := H(t).$$

Proposition 4.13 says that $G(t) = \dfrac{H(t) - 1}{H(t) + q}$. Using our abbreviations $\mathbf{G}$ and $\mathbf{H}$, we can rewrite this as

$$\mathbf{G} = \frac{\mathbf{H} - 1}{\mathbf{H} + q}. \tag{57}$$

Hence,

$$\Delta_t(\mathbf{G}) = \Delta_t\left(\frac{\mathbf{H} - 1}{\mathbf{H} + q}\right) = \frac{\Delta_t(\mathbf{H}) - 1}{\Delta_t(\mathbf{H}) + q} \tag{58}$$

(since Δ_t is a $\mathbf{k}$-algebra homomorphism).

Next, we observe the following:

Claim 1: We have

$$\Delta_t(\mathbf{H}) = \iota(\mathbf{H} \otimes \mathbf{H}). \tag{59}$$

[*Proof of Claim 1:* From $\mathbf{H} = H(t) = \sum_{n\in\mathbb{N}} H_n t^n = \sum_{i\in\mathbb{N}} H_i t^i$ and $\mathbf{H} = H(t) = \sum_{n\in\mathbb{N}} H_n t^n = \sum_{j\in\mathbb{N}} H_j t^j$, we obtain

$$\mathbf{H} \otimes \mathbf{H} = \left(\sum_{i\in\mathbb{N}} H_i t^i\right) \otimes \left(\sum_{j\in\mathbb{N}} H_j t^j\right).$$

By the definition of ι, this entails

$$\begin{aligned}
\iota\left(\mathbf{H}\otimes\mathbf{H}\right) &= \left(\sum_{i\in\mathbb{N}} \left(H_i\otimes 1\right) t^i\right) \left(\sum_{j\in\mathbb{N}} \left(1\otimes H_j\right) t^j\right) \\
&= \sum_{n\in\mathbb{N}} \left(\sum_{\substack{i,j\in\mathbb{N};\\ i+j=n}} \left(H_i\otimes 1\right)\left(1\otimes H_j\right)\right) t^n \qquad (60)
\end{aligned}$$

(by the definition of the product of two power series). On the other hand, $\mathbf{H} = \sum_{n\in\mathbb{N}} H_n t^n$. Thus,

$$\Delta_t\left(\mathbf{H}\right) = \Delta_t\left(\sum_{n\in\mathbb{N}} H_n t^n\right) = \sum_{n\in\mathbb{N}} \Delta\left(H_n\right) t^n \qquad (61)$$

(by the definition of Δ_t). However, for each $n\in\mathbb{N}$, we have

$$\begin{aligned}
\Delta\left(H_n\right) &= \sum_{\substack{i,j\in\mathbb{N};\\ i+j=n}} \underbrace{H_i\otimes H_j}_{=\left(H_i\otimes 1\right)\left(1\otimes H_j\right)} \qquad \text{(by [GriRei20, (5.4.2)])} \\
&= \sum_{\substack{i,j\in\mathbb{N};\\ i+j=n}} \left(H_i\otimes 1\right)\left(1\otimes H_j\right).
\end{aligned}$$

Hence, the right hand sides of the equalities (61) and (60) are equal. Therefore, so are their left hand sides. In other words, we have $\Delta_t\left(\mathbf{H}\right) = \iota\left(\mathbf{H}\otimes\mathbf{H}\right)$. This proves Claim 1.]

Define four elements h_1, h_2, g_1 and g_2 of $\left(\operatorname{NSym}\otimes\operatorname{NSym}\right)[[t]]$ by

$$\begin{aligned}
h_1 &= \iota\left(\mathbf{H}\otimes 1\right) \qquad \text{and} \qquad h_2 = \iota\left(1\otimes\mathbf{H}\right) \qquad \text{and} \\
g_1 &= \iota\left(\mathbf{G}\otimes 1\right) \qquad \text{and} \qquad g_2 = \iota\left(1\otimes\mathbf{G}\right).
\end{aligned}$$

The equality (59) becomes

$$\begin{aligned}
\Delta_t\left(\mathbf{H}\right) &= \iota\left(\underbrace{\mathbf{H}\otimes\mathbf{H}}_{=\left(\mathbf{H}\otimes 1\right)\left(1\otimes\mathbf{H}\right)}\right) = \iota\left(\left(\mathbf{H}\otimes 1\right)\left(1\otimes\mathbf{H}\right)\right) \\
&= \underbrace{\iota\left(\mathbf{H}\otimes 1\right)}_{=h_1}\cdot\underbrace{\iota\left(1\otimes\mathbf{H}\right)}_{=h_2} \qquad \text{(since } \iota \text{ is a ring homomorphism)} \\
&= h_1 h_2 \qquad (62)
\end{aligned}$$

and

$$\begin{aligned}\Delta_t(\mathbf{H}) &= \iota\left(\underbrace{\mathbf{H}\otimes\mathbf{H}}_{=(1\otimes\mathbf{H})(\mathbf{H}\otimes 1)}\right) = \iota\left((1\otimes\mathbf{H})(\mathbf{H}\otimes 1)\right) \\ &= \underbrace{\iota(1\otimes\mathbf{H})}_{=h_2}\cdot\underbrace{\iota(\mathbf{H}\otimes 1)}_{=h_1} \qquad \text{(since } \iota \text{ is a ring homomorphism)} \\ &= h_2h_1.\end{aligned}$$

Comparing these two equalities, we obtain $h_1h_2 = h_2h_1$. In other words, the elements h_1 and h_2 commute. The elements $\dfrac{1}{h_1+q}$, $\dfrac{1}{h_2+q}$ and $\dfrac{1}{h_1h_2+q}$ (which are easily seen to be well-defined[13]) are rational functions in these commuting elements h_1 and h_2, and therefore also commute with them (and with each other). Thus, the five elements h_1, h_2, $\dfrac{1}{h_1+q}$, $\dfrac{1}{h_2+q}$ and $\dfrac{1}{h_1h_2+q}$ generate a commutative $\mathbf{k}$-subalgebra of $(\operatorname{NSym}\otimes\operatorname{NSym})[[t]]$. Let us denote this commutative $\mathbf{k}$-subalgebra by $\mathcal{H}$. Clearly, the elements h_1+q, h_2+q and h_1h_2+q are invertible in $\mathcal{H}$. Also, the element $q+1=r$ is invertible in $\mathcal{H}$ (since it is invertible in $\mathbf{k}$ already).

From (57), we obtain

$$g_1 = \frac{h_1-1}{h_1+q} \qquad \text{and} \qquad g_2 = \frac{h_2-1}{h_2+q} \tag{63}$$

(since ι is a $\mathbf{k}$-algebra homomorphism[14]).

[13] *Proof.* The power series $h_1 = \iota(\mathbf{H}\otimes 1)$ has constant term 1 (since $\mathbf{H} = \sum_{n\in\mathbb{N}} H_n t^n$ entails $\iota(\mathbf{H}\otimes 1) = \sum_{n\in\mathbb{N}} (H_n\otimes 1)t^n$, and this latter series has constant term $\underbrace{H_0}_{=1}\otimes 1 = 1\otimes 1 = 1$). Thus, the power series h_1+q has constant term $1+q = q+1 = r$, which is invertible (by Convention 4.1). Thus, the power series h_1+q itself is invertible (since a formal power series whose constant term is invertible must itself be invertible). In other words, $\dfrac{1}{h_1+q}$ is well-defined. Similarly, $\dfrac{1}{h_2+q}$ and $\dfrac{1}{h_1h_2+q}$ are well-defined.

[14] Let us give some more details here: Let ι_1 be the map

$$\begin{aligned}\operatorname{NSym}[[t]] &\to \operatorname{NSym}[[t]]\otimes_{\mathbf{k}[[t]]}\operatorname{NSym}[[t]], \\ z &\mapsto z\otimes 1.\end{aligned}$$

Then, ι_1 is a $\mathbf{k}$-algebra homomorphism. Since ι is a $\mathbf{k}$-algebra homomorphism as well, we conclude that the composition $\iota\circ\iota_1$ is a $\mathbf{k}$-algebra homomorphism. However, the definition of ι_1 yields $\iota_1(\mathbf{G}) = \mathbf{G}\otimes 1$, so that

$$(\iota\circ\iota_1)(\mathbf{G}) = \iota\left(\underbrace{\iota_1(\mathbf{G})}_{=\mathbf{G}\otimes 1}\right) = \iota(\mathbf{G}\otimes 1) = g_1$$

Thus, the elements g_1 and g_2 also belong to the commutative $\mathbf{k}$-subalgebra $\mathcal{H}$ generated by h_1, h_2, $\dfrac{1}{h_1+q}$, $\dfrac{1}{h_2+q}$ and $\dfrac{1}{h_1h_2+q}$. Straightforward computations using (63) (and the commutativity of $\mathcal{H}$) show that

$$1+qg_1g_2 = \frac{(q+1)(h_1h_2+q)}{(h_1+q)(h_2+q)}.$$

Thus, $1+qg_1g_2$ is invertible in $\mathcal{H}$ (since $q+1$, h_1h_2+q, h_1+q and h_2+q are invertible in $\mathcal{H}$).

From (62), we obtain

$$\begin{aligned}\frac{\Delta_t(\mathbf{H})-1}{\Delta_t(\mathbf{H})+q} &= \frac{h_1h_2-1}{h_1h_2+q} \\ &= \frac{g_1+g_2+(q-1)g_1g_2}{1+qg_1g_2}.\end{aligned} \tag{64}$$

(Indeed, the last equality sign can easily be verified by straightforward computations in the commutative $\mathbf{k}$-algebra $\mathcal{H}$, using the equalities (63). For example, you can plug (63) into $\dfrac{g_1+g_2+(q-1)g_1g_2}{1+qg_1g_2}$ and simplify; the result will be $\dfrac{h_1h_2-1}{h_1h_2+q}$.)

From $g_1 = \iota(\mathbf{G}\otimes 1)$ and $g_2 = \iota(1\otimes\mathbf{G})$, we obtain

$$\begin{aligned} g_1g_2 &= \iota(\mathbf{G}\otimes 1)\cdot\iota(1\otimes\mathbf{G}) = \iota\left(\underbrace{(\mathbf{G}\otimes 1)\cdot(1\otimes\mathbf{G})}_{=\mathbf{G}\otimes\mathbf{G}}\right) \\ &\qquad (\text{since } \iota \text{ is a } \mathbf{k}\text{-algebra homomorphism}) \\ &= \iota(\mathbf{G}\otimes\mathbf{G}).\end{aligned} \tag{65}$$

Note that the formal power series qg_1g_2 has constant term 0 (since it is easy to see that both g_1 and g_2 have constant term 0).

(by the definition of g_1). Similarly, $(\iota\circ\iota_1)(\mathbf{H}) = h_1$. Now, applying the map $\iota\circ\iota_1$ to both sides of the equality (57), we obtain

$$(\iota\circ\iota_1)(\mathbf{G}) = (\iota\circ\iota_1)\left(\frac{\mathbf{H}-1}{\mathbf{H}+q}\right) = \frac{(\iota\circ\iota_1)(\mathbf{H})-1}{(\iota\circ\iota_1)(\mathbf{H})+q}$$

(since $\iota\circ\iota_1$ is a $\mathbf{k}$-algebra homomorphism). In view of $(\iota\circ\iota_1)(\mathbf{G}) = g_1$ and $(\iota\circ\iota_1)(\mathbf{H}) = h_1$, we can rewrite this as $g_1 = \dfrac{h_1-1}{h_1+q}$. Similarly, we can show that $g_2 = \dfrac{h_2-1}{h_2+q}$. Thus, (63) is proved.

Now, (58) becomes

$$\begin{aligned}
\Delta_t(\mathbf{G}) &= \frac{\Delta_t(\mathbf{H})-1}{\Delta_t(\mathbf{H})+q} = \frac{g_1+g_2+(q-1)g_1g_2}{1+qg_1g_2} \qquad \text{(by (64))} \\
&= \underbrace{\frac{1}{1+qg_1g_2}}_{\substack{=\sum_{i\in\mathbb{N}}(-qg_1g_2)^i \\ \text{(by the geometric series formula)}}} \cdot (g_1+g_2+(q-1)g_1g_2) \\
&= \sum_{i\in\mathbb{N}} \underbrace{(-qg_1g_2)^i}_{=(-q)^i(g_1g_2)^i} (g_1+g_2+(q-1)g_1g_2) \\
&= \sum_{i\in\mathbb{N}} (-q)^i \left(\underbrace{g_1g_2}_{\substack{=\iota(\mathbf{G}\otimes\mathbf{G}) \\ \text{(by (65))}}} \right)^i \left(\underbrace{g_1}_{=\iota(\mathbf{G}\otimes 1)} + \underbrace{g_2}_{=\iota(1\otimes\mathbf{G})} + (q-1) \underbrace{g_1g_2}_{\substack{=\iota(\mathbf{G}\otimes\mathbf{G}) \\ \text{(by (65))}}} \right) \\
&= \sum_{i\in\mathbb{N}} (-q)^i \underbrace{(\iota(\mathbf{G}\otimes\mathbf{G}))^i \cdot (\iota(\mathbf{G}\otimes 1)+\iota(1\otimes\mathbf{G})+(q-1)\iota(\mathbf{G}\otimes\mathbf{G}))}_{\substack{=\iota\left((\mathbf{G}\otimes\mathbf{G})^i(\mathbf{G}\otimes 1+1\otimes\mathbf{G}+(q-1)\mathbf{G}\otimes\mathbf{G})\right) \\ \text{(since } \iota \text{ is a } \mathbf{k}\text{-algebra homomorphism)}}} \\
&= \sum_{i\in\mathbb{N}} (-q)^i \iota\left((\mathbf{G}\otimes\mathbf{G})^i(\mathbf{G}\otimes 1+1\otimes\mathbf{G}+(q-1)\mathbf{G}\otimes\mathbf{G})\right).
\end{aligned}$$

In order to simplify the right hand side, we need two further claims:

Claim 2: Let $u, v \in \mathbb{N}$. Then,

$$\iota(\mathbf{G}^u \otimes \mathbf{G}^v) = \sum_{\substack{\beta,\gamma\in\mathrm{Comp}; \\ \ell(\beta)=u \text{ and } \ell(\gamma)=v}} \left(\eta_\beta^{*(q)} \otimes \eta_\gamma^{*(q)}\right) t^{|\beta|+|\gamma|}.$$

[*Proof of Claim 2:* From $\mathbf{G} = G(t)$, we obtain

$$\begin{aligned}
\mathbf{G}^u &= G(t)^u = \sum_{\substack{\beta\in\mathrm{Comp}; \\ \ell(\beta)=u}} \eta_\beta^{*(q)} t^{|\beta|} \qquad \text{(by (56), applied to } k=u) \\
&= \sum_{i\in\mathbb{N}} \sum_{\substack{\beta\in\mathrm{Comp}; \\ \ell(\beta)=u; \\ |\beta|=i}} \eta_\beta^{*(q)} \underbrace{t^{|\beta|}}_{\substack{=t^i \\ \text{(since } |\beta|=i)}} \\
&= \sum_{i\in\mathbb{N}} \left(\sum_{\substack{\beta\in\mathrm{Comp}; \\ \ell(\beta)=u; \\ |\beta|=i}} \eta_\beta^{*(q)} \right) t^i. \qquad (66)
\end{aligned}$$

The same argument (applied to v instead of u) shows that

$$\mathbf{G}^{v}=\sum_{i\in\mathbb{N}}\left(\sum_{\substack{\beta\in\operatorname{Comp};\\ \ell(\beta)=v;\\ |\beta|=i}}\eta_{\beta}^{*(q)}\right)t^{i}=\sum_{j\in\mathbb{N}}\left(\sum_{\substack{\gamma\in\operatorname{Comp};\\ \ell(\gamma)=v;\\ |\gamma|=j}}\eta_{\gamma}^{*(q)}\right)t^{j}$$

(here, we have renamed the summation indices i and β as j and γ). Substituting these two equalities into $\mathbf{G}^{u}\otimes\mathbf{G}^{v}$, we obtain

$$\mathbf{G}^{u}\otimes\mathbf{G}^{v}=\left(\sum_{i\in\mathbb{N}}\left(\sum_{\substack{\beta\in\operatorname{Comp};\\ \ell(\beta)=u;\\ |\beta|=i}}\eta_{\beta}^{*(q)}\right)t^{i}\right)\otimes\left(\sum_{j\in\mathbb{N}}\left(\sum_{\substack{\gamma\in\operatorname{Comp};\\ \ell(\gamma)=v;\\ |\gamma|=j}}\eta_{\gamma}^{*(q)}\right)t^{j}\right)$$

(note that the two outer sums here are infinite, so that we cannot simply expand them out using the bilinearity of the tensor product). Applying the map ι to both sides of this equality, we obtain

$$\begin{aligned}
\iota\left(\mathbf{G}^u \otimes \mathbf{G}^v\right) &= \iota\left(\left(\sum_{i\in\mathbb{N}}\left(\sum_{\substack{\beta\in\mathrm{Comp};\\ \ell(\beta)=u;\\ |\beta|=i}} \eta_\beta^{*(q)}\right) t^i\right) \otimes \left(\sum_{j\in\mathbb{N}}\left(\sum_{\substack{\gamma\in\mathrm{Comp};\\ \ell(\gamma)=v;\\ |\gamma|=j}} \eta_\gamma^{*(q)}\right) t^j\right)\right) \\
&= \left(\sum_{i\in\mathbb{N}}\left(\left(\sum_{\substack{\beta\in\mathrm{Comp};\\ \ell(\beta)=u;\\ |\beta|=i}} \eta_\beta^{*(q)}\right) \otimes 1\right) t^i\right)\left(\sum_{j\in\mathbb{N}}\left(1 \otimes \left(\sum_{\substack{\gamma\in\mathrm{Comp};\\ \ell(\gamma)=v;\\ |\gamma|=j}} \eta_\gamma^{*(q)}\right)\right) t^j\right) \\
&\qquad \text{(by the definition of } \iota\text{)} \\
&= \sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}} \underbrace{\left(\left(\sum_{\substack{\beta\in\mathrm{Comp};\\ \ell(\beta)=u;\\ |\beta|=i}} \eta_\beta^{*(q)}\right) \otimes 1\right)\left(1 \otimes \left(\sum_{\substack{\gamma\in\mathrm{Comp};\\ \ell(\gamma)=v;\\ |\gamma|=j}} \eta_\gamma^{*(q)}\right)\right)}_{\substack{=\left(\sum_{\substack{\beta\in\mathrm{Comp};\\ \ell(\beta)=u;\\ |\beta|=i}} \eta_\beta^{*(q)}\right)\otimes\left(\sum_{\substack{\gamma\in\mathrm{Comp};\\ \ell(\gamma)=v;\\ |\gamma|=j}} \eta_\gamma^{*(q)}\right) \\ =\sum_{\substack{\beta,\gamma\in\mathrm{Comp};\\ \ell(\beta)=u \text{ and } \ell(\gamma)=v;\\ |\beta|=i \text{ and } |\gamma|=j}} \eta_\beta^{*(q)}\otimes\eta_\gamma^{*(q)} \\ \text{(here, we expanded the tensor product,} \\ \text{since both sums involved are finite)}}} \underbrace{t^i t^j}_{=t^{i+j}} \\
&= \sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}} \sum_{\substack{\beta,\gamma\in\mathrm{Comp};\\ \ell(\beta)=u \text{ and } \ell(\gamma)=v;\\ |\beta|=i \text{ and } |\gamma|=j}} \left(\eta_\beta^{*(q)} \otimes \eta_\gamma^{*(q)}\right) \underbrace{t^{i+j}}_{\substack{=t^{|\beta|+|\gamma|} \\ \text{(since } i=|\beta| \text{ and } j=|\gamma|\text{)}}} \\
&= \underbrace{\sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}} \sum_{\substack{\beta,\gamma\in\mathrm{Comp};\\ \ell(\beta)=u \text{ and } \ell(\gamma)=v;\\ |\beta|=i \text{ and } |\gamma|=j}}}_{=\sum_{\substack{\beta,\gamma\in\mathrm{Comp};\\ \ell(\beta)=u \text{ and } \ell(\gamma)=v}}} \left(\eta_\beta^{*(q)} \otimes \eta_\gamma^{*(q)}\right) t^{|\beta|+|\gamma|} \\
&= \sum_{\substack{\beta,\gamma\in\mathrm{Comp};\\ \ell(\beta)=u \text{ and } \ell(\gamma)=v}} \left(\eta_\beta^{*(q)} \otimes \eta_\gamma^{*(q)}\right) t^{|\beta|+|\gamma|}.
\end{aligned}$$

This proves Claim 2.]

Claim 3: Let $i \in \mathbb{N}$. Then,

$$\begin{aligned}
&\iota\left(\left(\mathbf{G}\otimes\mathbf{G}\right)^{i}\left(\mathbf{G}\otimes 1+1\otimes\mathbf{G}+\left(q-1\right)\mathbf{G}\otimes\mathbf{G}\right)\right)\\
&=\sum_{\substack{\beta,\gamma\in\operatorname{Comp};\\ \left|\ell(\beta)-\ell(\gamma)\right|\leq 1;\\ \max\{\ell(\beta),\ell(\gamma)\}=i+1}}\left(q-1\right)^{[\ell(\beta)=\ell(\gamma)]}\left(\eta_{\beta}^{*(q)}\otimes\eta_{\gamma}^{*(q)}\right)t^{|\beta|+|\gamma|}.
\end{aligned}$$

[*Proof of Claim 3:* We have

$$\begin{aligned}
&\underbrace{\left(\mathbf{G}\otimes\mathbf{G}\right)^{i}}_{=\mathbf{G}^{i}\otimes\mathbf{G}^{i}}\left(\mathbf{G}\otimes 1+1\otimes\mathbf{G}+\left(q-1\right)\mathbf{G}\otimes\mathbf{G}\right)\\
&=\left(\mathbf{G}^{i}\otimes\mathbf{G}^{i}\right)\left(\mathbf{G}\otimes 1+1\otimes\mathbf{G}+\left(q-1\right)\mathbf{G}\otimes\mathbf{G}\right)\\
&=\mathbf{G}^{i+1}\otimes\mathbf{G}^{i}+\mathbf{G}^{i}\otimes\mathbf{G}^{i+1}+\left(q-1\right)\mathbf{G}^{i+1}\otimes\mathbf{G}^{i+1}.
\end{aligned}$$

Applying the map ι to both sides of this equality, we obtain

$$\begin{aligned}
&\iota\left(\left(\mathbf{G}\otimes\mathbf{G}\right)^{i}\left(\mathbf{G}\otimes 1+1\otimes\mathbf{G}+\left(q-1\right)\mathbf{G}\otimes\mathbf{G}\right)\right)\\
&=\iota\left(\mathbf{G}^{i+1}\otimes\mathbf{G}^{i}+\mathbf{G}^{i}\otimes\mathbf{G}^{i+1}+\left(q-1\right)\mathbf{G}^{i+1}\otimes\mathbf{G}^{i+1}\right)\\
&=\underbrace{\iota\left(\mathbf{G}^{i+1}\otimes\mathbf{G}^{i}\right)}_{\substack{=\sum_{\substack{\beta,\gamma\in\operatorname{Comp};\\ \ell(\beta)=i+1\text{ and }\ell(\gamma)=i}}\left(\eta_{\beta}^{*(q)}\otimes\eta_{\gamma}^{*(q)}\right)t^{|\beta|+|\gamma|}\\ \text{(by Claim 2)}}}+\underbrace{\iota\left(\mathbf{G}^{i}\otimes\mathbf{G}^{i+1}\right)}_{\substack{=\sum_{\substack{\beta,\gamma\in\operatorname{Comp};\\ \ell(\beta)=i\text{ and }\ell(\gamma)=i+1}}\left(\eta_{\beta}^{*(q)}\otimes\eta_{\gamma}^{*(q)}\right)t^{|\beta|+|\gamma|}\\ \text{(by Claim 2)}}}\\
&\qquad+\left(q-1\right)\underbrace{\iota\left(\mathbf{G}^{i+1}\otimes\mathbf{G}^{i+1}\right)}_{\substack{=\sum_{\substack{\beta,\gamma\in\operatorname{Comp};\\ \ell(\beta)=i+1\text{ and }\ell(\gamma)=i+1}}\left(\eta_{\beta}^{*(q)}\otimes\eta_{\gamma}^{*(q)}\right)t^{|\beta|+|\gamma|}\\ \text{(by Claim 2)}}}\\
&\qquad\qquad\left(\text{since }\iota\text{ is a }\mathbf{k}\text{-algebra homomorphism}\right)\\
&=\sum_{\substack{\beta,\gamma\in\operatorname{Comp};\\ \ell(\beta)=i+1\text{ and }\ell(\gamma)=i}}\left(\eta_{\beta}^{*(q)}\otimes\eta_{\gamma}^{*(q)}\right)t^{|\beta|+|\gamma|}+\sum_{\substack{\beta,\gamma\in\operatorname{Comp};\\ \ell(\beta)=i\text{ and }\ell(\gamma)=i+1}}\left(\eta_{\beta}^{*(q)}\otimes\eta_{\gamma}^{*(q)}\right)t^{|\beta|+|\gamma|}\\
&\qquad+\left(q-1\right)\sum_{\substack{\beta,\gamma\in\operatorname{Comp};\\ \ell(\beta)=i+1\text{ and }\ell(\gamma)=i+1}}\left(\eta_{\beta}^{*(q)}\otimes\eta_{\gamma}^{*(q)}\right)t^{|\beta|+|\gamma|}.
\end{aligned}\tag{67}$$

On the other hand, let us observe that two integers u and v satisfy the two conditions

$$\left|u-v\right|\leq 1\qquad\text{and}\qquad\max\left\{u,v\right\}=i+1$$

if and only if they satisfy one of the three mutually exclusive conditions

$$(u = i+1 \text{ and } v = i), \quad (u = i \text{ and } v = i+1) \quad \text{and} \quad (u = i+1 \text{ and } v = i+1).$$

Hence, two compositions $\beta, \gamma \in \operatorname{Comp}$ satisfy the two conditions

$$\left|\ell(\beta) - \ell(\gamma)\right| \leq 1 \qquad \text{and} \qquad \max\left\{\ell(\beta), \ell(\gamma)\right\} = i+1$$

if and only if they satisfy one of the three mutually exclusive conditions

$$(\ell(\beta) = i+1 \text{ and } \ell(\gamma) = i), \quad (\ell(\beta) = i \text{ and } \ell(\gamma) = i+1) \quad \text{and} \quad (\ell(\beta) = i+1 \text{ and } \ell(\gamma) = i+1).$$

Hence,

$$\begin{aligned}
&\sum_{\substack{\beta,\gamma\in\operatorname{Comp};\\ |\ell(\beta)-\ell(\gamma)|\leq 1;\\ \max\{\ell(\beta),\ell(\gamma)\}=i+1}} (q-1)^{[\ell(\beta)=\ell(\gamma)]} \left(\eta_\beta^{*(q)} \otimes \eta_\gamma^{*(q)}\right) t^{|\beta|+|\gamma|} \\
&= \sum_{\substack{\beta,\gamma\in\operatorname{Comp};\\ \ell(\beta)=i+1 \text{ and } \ell(\gamma)=i}} \underbrace{(q-1)^{[\ell(\beta)=\ell(\gamma)]}}_{\substack{=1\\ \text{(since } \ell(\beta)=i+1>i=\ell(\gamma)\\ \text{and thus } [\ell(\beta)=\ell(\gamma)]=0)}} \left(\eta_\beta^{*(q)} \otimes \eta_\gamma^{*(q)}\right) t^{|\beta|+|\gamma|} \\
&\qquad + \sum_{\substack{\beta,\gamma\in\operatorname{Comp};\\ \ell(\beta)=i \text{ and } \ell(\gamma)=i+1}} \underbrace{(q-1)^{[\ell(\beta)=\ell(\gamma)]}}_{\substack{=1\\ \text{(since } \ell(\beta)=i<i+1=\ell(\gamma)\\ \text{and thus } [\ell(\beta)=\ell(\gamma)]=0)}} \left(\eta_\beta^{*(q)} \otimes \eta_\gamma^{*(q)}\right) t^{|\beta|+|\gamma|} \\
&\qquad + \sum_{\substack{\beta,\gamma\in\operatorname{Comp};\\ \ell(\beta)=i+1 \text{ and } \ell(\gamma)=i+1}} \underbrace{(q-1)^{[\ell(\beta)=\ell(\gamma)]}}_{\substack{=q-1\\ \text{(since } \ell(\beta)=i+1=\ell(\gamma)\\ \text{and thus } [\ell(\beta)=\ell(\gamma)]=1)}} \left(\eta_\beta^{*(q)} \otimes \eta_\gamma^{*(q)}\right) t^{|\beta|+|\gamma|} \\
&= \sum_{\substack{\beta,\gamma\in\operatorname{Comp};\\ \ell(\beta)=i+1 \text{ and } \ell(\gamma)=i}} \left(\eta_\beta^{*(q)} \otimes \eta_\gamma^{*(q)}\right) t^{|\beta|+|\gamma|} + \sum_{\substack{\beta,\gamma\in\operatorname{Comp};\\ \ell(\beta)=i \text{ and } \ell(\gamma)=i+1}} \left(\eta_\beta^{*(q)} \otimes \eta_\gamma^{*(q)}\right) t^{|\beta|+|\gamma|} \\
&\qquad + (q-1) \sum_{\substack{\beta,\gamma\in\operatorname{Comp};\\ \ell(\beta)=i+1 \text{ and } \ell(\gamma)=i+1}} \left(\eta_\beta^{*(q)} \otimes \eta_\gamma^{*(q)}\right) t^{|\beta|+|\gamma|}.
\end{aligned}$$

Comparing this with (67), we obtain

$$\begin{aligned}
&\iota\left(\left(\mathbf{G} \otimes \mathbf{G}\right)^i \left(\mathbf{G} \otimes 1 + 1 \otimes \mathbf{G} + (q-1)\,\mathbf{G} \otimes \mathbf{G}\right)\right) \\
&= \sum_{\substack{\beta,\gamma\in\operatorname{Comp};\\ |\ell(\beta)-\ell(\gamma)|\leq 1;\\ \max\{\ell(\beta),\ell(\gamma)\}=i+1}} (q-1)^{[\ell(\beta)=\ell(\gamma)]} \left(\eta_\beta^{*(q)} \otimes \eta_\gamma^{*(q)}\right) t^{|\beta|+|\gamma|}.
\end{aligned}$$

This proves Claim 3.]

Now, we can finish our computation of $\Delta_t(\mathbf{G})$: As we know,

$$\begin{aligned}
&\Delta_t(\mathbf{G}) \\
&= \sum_{i\in\mathbb{N}} (-q)^i \underbrace{\iota\left((\mathbf{G}\otimes\mathbf{G})^i (\mathbf{G}\otimes 1 + 1\otimes\mathbf{G} + (q-1)\,\mathbf{G}\otimes\mathbf{G})\right)}_{\substack{= \sum\limits_{\substack{\beta,\gamma\in\mathrm{Comp};\\ |\ell(\beta)-\ell(\gamma)|\leq 1;\\ \max\{\ell(\beta),\ell(\gamma)\}=i+1}} (q-1)^{[\ell(\beta)=\ell(\gamma)]}\left(\eta_\beta^{*(q)}\otimes\eta_\gamma^{*(q)}\right) t^{|\beta|+|\gamma|} \\ \text{(by Claim 3)}}} \\
&= \sum_{i\in\mathbb{N}} (-q)^i \sum_{\substack{\beta,\gamma\in\mathrm{Comp};\\ |\ell(\beta)-\ell(\gamma)|\leq 1;\\ \max\{\ell(\beta),\ell(\gamma)\}=i+1}} (q-1)^{[\ell(\beta)=\ell(\gamma)]}\left(\eta_\beta^{*(q)}\otimes\eta_\gamma^{*(q)}\right) t^{|\beta|+|\gamma|} \\
&= \sum_{j>0} (-q)^{j-1} \sum_{\substack{\beta,\gamma\in\mathrm{Comp};\\ |\ell(\beta)-\ell(\gamma)|\leq 1;\\ \max\{\ell(\beta),\ell(\gamma)\}=j}} (q-1)^{[\ell(\beta)=\ell(\gamma)]}\left(\eta_\beta^{*(q)}\otimes\eta_\gamma^{*(q)}\right) t^{|\beta|+|\gamma|} \\
&\qquad\qquad \text{(here, we have substituted } j-1 \text{ for } i \text{ in the outer sum)} \\
&= \sum_{j>0} \sum_{\substack{\beta,\gamma\in\mathrm{Comp};\\ |\ell(\beta)-\ell(\gamma)|\leq 1;\\ \max\{\ell(\beta),\ell(\gamma)\}=j}} \underbrace{(-q)^{j-1}}_{\substack{=(-q)^{\max\{\ell(\beta),\ell(\gamma)\}-1} \\ (\text{since } j=\max\{\ell(\beta),\ell(\gamma)\})}} (q-1)^{[\ell(\beta)=\ell(\gamma)]}\left(\eta_\beta^{*(q)}\otimes\eta_\gamma^{*(q)}\right) t^{|\beta|+|\gamma|} \\
&= \underbrace{\sum_{j>0} \sum_{\substack{\beta,\gamma\in\mathrm{Comp};\\ |\ell(\beta)-\ell(\gamma)|\leq 1;\\ \max\{\ell(\beta),\ell(\gamma)\}=j}}}_{= \sum\limits_{\substack{\beta,\gamma\in\mathrm{Comp};\\ |\ell(\beta)-\ell(\gamma)|\leq 1;\\ \max\{\ell(\beta),\ell(\gamma)\}>0}}} (-q)^{\max\{\ell(\beta),\ell(\gamma)\}-1} (q-1)^{[\ell(\beta)=\ell(\gamma)]}\left(\eta_\beta^{*(q)}\otimes\eta_\gamma^{*(q)}\right) t^{|\beta|+|\gamma|} \\
&= \sum_{\substack{\beta,\gamma\in\mathrm{Comp};\\ |\ell(\beta)-\ell(\gamma)|\leq 1;\\ \max\{\ell(\beta),\ell(\gamma)\}>0}} (-q)^{\max\{\ell(\beta),\ell(\gamma)\}-1} (q-1)^{[\ell(\beta)=\ell(\gamma)]}\left(\eta_\beta^{*(q)}\otimes\eta_\gamma^{*(q)}\right) t^{|\beta|+|\gamma|}.
\end{aligned}$$

Comparing this with

$$\begin{aligned}
\Delta_t(\mathbf{G}) &= \Delta_t\left(\sum_{n\geq 1}\eta_n^{*(q)} t^n\right) && \left(\text{since } \mathbf{G} = G(t) = \sum_{n\geq 1}\eta_n^{*(q)} t^n\right) \\
&= \sum_{n\geq 1}\Delta\left(\eta_n^{*(q)}\right) t^n && (\text{by the definition of } \Delta_t),
\end{aligned}$$

we obtain

$$\sum_{n\geq 1}\Delta\left(\eta_n^{*(q)}\right)t^n$$
$$=\sum_{\substack{\beta,\gamma\in\operatorname{Comp};\\ |\ell(\beta)-\ell(\gamma)|\leq 1;\\ \max\{\ell(\beta),\ell(\gamma)\}>0}} (-q)^{\max\{\ell(\beta),\ell(\gamma)\}-1}(q-1)^{[\ell(\beta)=\ell(\gamma)]}\left(\eta_\beta^{*(q)}\otimes\eta_\gamma^{*(q)}\right)t^{|\beta|+|\gamma|}.$$

Comparing coefficients of t^n on both sides of this equality, we obtain

$$\Delta\left(\eta_n^{*(q)}\right)=\sum_{\substack{\beta,\gamma\in\operatorname{Comp};\\ |\ell(\beta)-\ell(\gamma)|\leq 1;\\ \max\{\ell(\beta),\ell(\gamma)\}>0;\\ |\beta|+|\gamma|=n}} (-q)^{\max\{\ell(\beta),\ell(\gamma)\}-1}(q-1)^{[\ell(\beta)=\ell(\gamma)]}\eta_\beta^{*(q)}\otimes\eta_\gamma^{*(q)} \qquad (68)$$

for each $n\geq 1$ (in fact, the condition "$|\beta|+|\gamma|=n$" under the summation sign ensures that the monomial $t^{|\beta|+|\gamma|}$ is t^n).

Now, fix a positive integer n. Then, any two compositions β and γ that satisfy $|\beta|+|\gamma|=n$ will automatically satisfy $\max\{\ell(\beta),\ell(\gamma)\}>0$ (since otherwise, they would satisfy $\max\{\ell(\beta),\ell(\gamma)\}=0$ and thus $\beta=\varnothing$ and $\gamma=\varnothing$, which would lead to $|\beta|+|\gamma|=|\varnothing|+|\varnothing|=0+0=0$, but this would contradict $|\beta|+|\gamma|=n>0$). Hence, in the summation sign on the right hand side of (68), the condition "$\max\{\ell(\beta),\ell(\gamma)\}>0$" is redundant. We can thus rewrite this summation sign as follows:

$$\sum_{\substack{\beta,\gamma\in\operatorname{Comp};\\ |\ell(\beta)-\ell(\gamma)|\leq 1;\\ \max\{\ell(\beta),\ell(\gamma)\}>0;\\ |\beta|+|\gamma|=n}} = \sum_{\substack{\beta,\gamma\in\operatorname{Comp};\\ |\ell(\beta)-\ell(\gamma)|\leq 1;\\ |\beta|+|\gamma|=n}} = \sum_{\substack{\beta,\gamma\in\operatorname{Comp};\\ |\beta|+|\gamma|=n;\\ |\ell(\beta)-\ell(\gamma)|\leq 1}}.$$

Hence, (68) rewrites as

$$\Delta\left(\eta_n^{*(q)}\right)=\sum_{\substack{\beta,\gamma\in\operatorname{Comp};\\ |\beta|+|\gamma|=n;\\ |\ell(\beta)-\ell(\gamma)|\leq 1}} (-q)^{\max\{\ell(\beta),\ell(\gamma)\}-1}(q-1)^{[\ell(\beta)=\ell(\gamma)]}\eta_\beta^{*(q)}\otimes\eta_\gamma^{*(q)}.$$

This proves Theorem 4.15. □

Using Theorem 4.15, we can easily compute the coproduct of any $\eta_\alpha^{*(q)}$:[15]

[15]The symbol "#" means "number". Thus, e.g., we have (# of odd numbers $i\in[2n]$) $=n$ for each $n\in\mathbb{N}$.

Corollary 4.18. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ be any composition. Then,

$$\Delta\left(\eta_\alpha^{*(q)}\right) = \sum_{\substack{\beta_1,\beta_2,\ldots,\beta_k \in \operatorname{Comp}; \\ \gamma_1,\gamma_2,\ldots,\gamma_k \in \operatorname{Comp}; \\ |\beta_s|+|\gamma_s| = \alpha_s \text{ for all } s; \\ |\ell(\beta_s)-\ell(\gamma_s)| \leq 1 \text{ for all } s}} (-q)^{\sum_{s=1}^k \max\{\ell(\beta_s),\ell(\gamma_s)\} - k}$$
$$\cdot (q-1)^{(\text{\# of all } s\in[k] \text{ such that } \ell(\beta_s)=\ell(\gamma_s))} \left(\eta^{*(q)}_{\beta_1\beta_2\cdots\beta_k} \otimes \eta^{*(q)}_{\gamma_1\gamma_2\cdots\gamma_k}\right).$$

Proof. We agree to understand an expression of the form $\prod_{s=1}^k u_s$ to mean $u_1u_2\cdots u_k$ whenever $u_1, u_2, \ldots, u_k$ are any k elements of any (not necessarily commutative) ring.

The comultiplication Δ of the $\mathbf{k}$-bialgebra NSym is a $\mathbf{k}$-algebra homomorphism (indeed, this is true for any $\mathbf{k}$-bialgebra), and thus respects products. However, Proposition 4.7 yields $\eta_\alpha^{*(q)} = \eta_{\alpha_1}^{*(q)} \eta_{\alpha_2}^{*(q)} \cdots \eta_{\alpha_k}^{*(q)}$. Hence,

$$\Delta\left(\eta_\alpha^{*(q)}\right) = \Delta\left(\eta_{\alpha_1}^{*(q)} \eta_{\alpha_2}^{*(q)} \cdots \eta_{\alpha_k}^{*(q)}\right) = \Delta\left(\eta_{\alpha_1}^{*(q)}\right) \Delta\left(\eta_{\alpha_2}^{*(q)}\right) \cdots \Delta\left(\eta_{\alpha_k}^{*(q)}\right)$$

(since Δ respects products). In other words (using the notation $\prod_{s=1}^k u_s$ as explained above),

$$\Delta\left(\eta_\alpha^{*(q)}\right) = \prod_{s=1}^k \Delta\left(\eta_{\alpha_s}^{*(q)}\right). \tag{69}$$

However, Theorem 4.15 shows that for each $s \in [k]$, we have

$$\Delta\left(\eta_{\alpha_s}^{*(q)}\right) = \sum_{\substack{\beta,\gamma \in \operatorname{Comp}; \\ |\beta|+|\gamma| = \alpha_s; \\ |\ell(\beta)-\ell(\gamma)| \leq 1}} (-q)^{\max\{\ell(\beta),\ell(\gamma)\}-1} (q-1)^{[\ell(\beta)=\ell(\gamma)]} \eta_\beta^{*(q)} \otimes \eta_\gamma^{*(q)}.$$

Multiplying these equalities for all $s \in [k]$, we obtain

$$\begin{aligned}
&\prod_{s=1}^{k} \Delta\left(\eta_{\alpha_s}^{*(q)}\right) \\
&= \prod_{s=1}^{k} \sum_{\substack{\beta,\gamma\in\operatorname{Comp};\\ |\beta|+|\gamma|=\alpha_s;\\ |\ell(\beta)-\ell(\gamma)|\leq 1}} (-q)^{\max\{\ell(\beta),\ell(\gamma)\}-1} (q-1)^{[\ell(\beta)=\ell(\gamma)]} \eta_{\beta}^{*(q)} \otimes \eta_{\gamma}^{*(q)} \\
&= \sum_{\substack{\beta_1,\beta_2,\ldots,\beta_k\in\operatorname{Comp};\\ \gamma_1,\gamma_2,\ldots,\gamma_k\in\operatorname{Comp};\\ |\beta_s|+|\gamma_s|=\alpha_s \text{ for all } s;\\ |\ell(\beta_s)-\ell(\gamma_s)|\leq 1 \text{ for all } s}} \prod_{s=1}^{k} \left((-q)^{\max\{\ell(\beta_s),\ell(\gamma_s)\}-1} (q-1)^{[\ell(\beta_s)=\ell(\gamma_s)]} \eta_{\beta_s}^{*(q)} \otimes \eta_{\gamma_s}^{*(q)}\right) \\
&\qquad \text{(by the product rule)} \\
&= \sum_{\substack{\beta_1,\beta_2,\ldots,\beta_k\in\operatorname{Comp};\\ \gamma_1,\gamma_2,\ldots,\gamma_k\in\operatorname{Comp};\\ |\beta_s|+|\gamma_s|=\alpha_s \text{ for all } s;\\ |\ell(\beta_s)-\ell(\gamma_s)|\leq 1 \text{ for all } s}} \underbrace{\left(\prod_{s=1}^{k} (-q)^{\max\{\ell(\beta_s),\ell(\gamma_s)\}-1}\right)}_{\substack{=(-q)^{\sum_{s=1}^{k}(\max\{\ell(\beta_s),\ell(\gamma_s)\}-1)}\\ =(-q)^{\sum_{s=1}^{k}\max\{\ell(\beta_s),\ell(\gamma_s)\}-k}}} \\
&\qquad \cdot \underbrace{\left(\prod_{s=1}^{k} (q-1)^{[\ell(\beta_s)=\ell(\gamma_s)]}\right)}_{\substack{=(q-1)^{\sum_{s=1}^{k}[\ell(\beta_s)=\ell(\gamma_s)]}\\ =(q-1)^{(\#\text{ of all } s\in[k] \text{ such that } \ell(\beta_s)=\ell(\gamma_s))}}} \underbrace{\left(\prod_{s=1}^{k}\left(\eta_{\beta_s}^{*(q)} \otimes \eta_{\gamma_s}^{*(q)}\right)\right)}_{\substack{=\eta_{\beta_1}^{*(q)}\eta_{\beta_2}^{*(q)}\cdots\eta_{\beta_k}^{*(q)}\otimes\eta_{\gamma_1}^{*(q)}\eta_{\gamma_2}^{*(q)}\cdots\eta_{\gamma_k}^{*(q)}\\ =\eta_{\beta_1\beta_2\cdots\beta_k}^{*(q)}\otimes\eta_{\gamma_1\gamma_2\cdots\gamma_k}^{*(q)}\\ \text{(since Corollary 4.11}\\ \text{yields } \eta_{\beta_1}^{*(q)}\eta_{\beta_2}^{*(q)}\cdots\eta_{\beta_k}^{*(q)}=\eta_{\beta_1\beta_2\cdots\beta_k}^{*(q)}\\ \text{and } \eta_{\gamma_1}^{*(q)}\eta_{\gamma_2}^{*(q)}\cdots\eta_{\gamma_k}^{*(q)}=\eta_{\gamma_1\gamma_2\cdots\gamma_k}^{*(q)})}} \\
&= \sum_{\substack{\beta_1,\beta_2,\ldots,\beta_k\in\operatorname{Comp};\\ \gamma_1,\gamma_2,\ldots,\gamma_k\in\operatorname{Comp};\\ |\beta_s|+|\gamma_s|=\alpha_s \text{ for all } s;\\ |\ell(\beta_s)-\ell(\gamma_s)|\leq 1 \text{ for all } s}} (-q)^{\sum_{s=1}^{k}\max\{\ell(\beta_s),\ell(\gamma_s)\}-k} \\
&\qquad \cdot (q-1)^{(\#\text{ of all } s\in[k] \text{ such that } \ell(\beta_s)=\ell(\gamma_s))} \left(\eta_{\beta_1\beta_2\cdots\beta_k}^{*(q)} \otimes \eta_{\gamma_1\gamma_2\cdots\gamma_k}^{*(q)}\right).
\end{aligned}$$

In view of (69), we can rewrite this as

$$\Delta\left(\eta_{\alpha}^{*(q)}\right) = \sum_{\substack{\beta_1,\beta_2,\ldots,\beta_k\in\operatorname{Comp};\\ \gamma_1,\gamma_2,\ldots,\gamma_k\in\operatorname{Comp};\\ |\beta_s|+|\gamma_s|=\alpha_s \text{ for all } s;\\ |\ell(\beta_s)-\ell(\gamma_s)|\leq 1 \text{ for all } s}} (-q)^{\sum_{s=1}^{k}\max\{\ell(\beta_s),\ell(\gamma_s)\}-k} \cdot (q-1)^{(\#\text{ of all } s\in[k] \text{ such that } \ell(\beta_s)=\ell(\gamma_s))} \left(\eta_{\beta_1\beta_2\cdots\beta_k}^{*(q)} \otimes \eta_{\gamma_1\gamma_2\cdots\gamma_k}^{*(q)}\right).$$

This proves Corollary 4.18. □

5. The product rule for $\eta_{\alpha}^{(q)}$

We now approach the most intricate of the rules for the $\eta_{\alpha}^{(q)}$ functions: the product rule, i.e., the expression of a product $\eta_{\delta}^{(q)}\eta_{\varepsilon}^{(q)}$ as a $\mathbb{Z}[q]$-linear combination of other $\eta_{\alpha}^{(q)}$'s. We shall give three different versions of this rule, all equivalent but using somewhat different indexing sets. Only the first version will be proved in detail, as it suffices for the applications we have in mind.

5.1. The product rule in terms of compositions

Our first version of the product rule is as follows:[16]

Theorem 5.1. Let δ and ε be two compositions. Then,

$$\eta_{\delta}^{(q)}\eta_{\varepsilon}^{(q)} = \sum_{\substack{k\in\mathbb{N};\\ \beta_1,\beta_2,\ldots,\beta_k\in\operatorname{Comp};\\ \gamma_1,\gamma_2,\ldots,\gamma_k\in\operatorname{Comp};\\ \beta_1\beta_2\cdots\beta_k=\delta;\\ \gamma_1\gamma_2\cdots\gamma_k=\varepsilon;\\ |\ell(\beta_s)-\ell(\gamma_s)|\leq 1 \text{ for all } s;\\ \ell(\beta_s)+\ell(\gamma_s)>0 \text{ for all } s}} (-q)^{\sum_{s=1}^{k}\max\{\ell(\beta_s),\ell(\gamma_s)\}-k} \cdot (q-1)^{(\#\text{ of all } s\in[k] \text{ such that } \ell(\beta_s)=\ell(\gamma_s))} \cdot \eta^{(q)}_{(|\beta_1|+|\gamma_1|,\ |\beta_2|+|\gamma_2|,\ \ldots,\ |\beta_k|+|\gamma_k|)}.$$

Remark 5.2. The compositions $\beta_1,\beta_2,\ldots,\beta_k$ and $\gamma_1,\gamma_2,\ldots,\gamma_k$ in the sum on the right hand side of Theorem 5.1 are allowed to be empty. Nevertheless, the sum

[16] The symbol "#" means "number" (so that, e.g., we have (# of odd numbers $i\in[2n]$) $= n$ for each $n\in\mathbb{N}$).

is finite. Indeed, if $k \in \mathbb{N}$ and $\beta_1, \beta_2, \ldots, \beta_k \in \operatorname{Comp}$ and $\gamma_1, \gamma_2, \ldots, \gamma_k \in \operatorname{Comp}$ satisfy

$$\beta_1 \beta_2 \cdots \beta_k = \delta \qquad \text{and} \qquad \gamma_1 \gamma_2 \cdots \gamma_k = \varepsilon \qquad \text{and}$$
$$|\ell(\beta_s) - \ell(\gamma_s)| \leq 1 \text{ for all } s \qquad \text{and} \qquad \ell(\beta_s) + \ell(\gamma_s) > 0 \text{ for all } s,$$

then $k \leq \ell(\delta) + \ell(\varepsilon)$, because

$$\begin{aligned}
&\underbrace{\ell(\delta)}_{\substack{=\ell(\beta_1)+\ell(\beta_2)+\cdots+\ell(\beta_k) \\ (\text{since } \delta=\beta_1\beta_2\cdots\beta_k)}} + \underbrace{\ell(\varepsilon)}_{\substack{=\ell(\gamma_1)+\ell(\gamma_2)+\cdots+\ell(\gamma_k) \\ (\text{since } \varepsilon=\gamma_1\gamma_2\cdots\gamma_k)}} \\
&= (\ell(\beta_1) + \ell(\beta_2) + \cdots + \ell(\beta_k)) + (\ell(\gamma_1) + \ell(\gamma_2) + \cdots + \ell(\gamma_k)) \\
&= \sum_{s=1}^k \ell(\beta_s) + \sum_{s=1}^k \ell(\gamma_s) = \sum_{s=1}^k \underbrace{(\ell(\beta_s) + \ell(\gamma_s))}_{\substack{\geq 1 \\ (\text{since our above assumptions} \\ \text{yield } \ell(\beta_s)+\ell(\gamma_s)>0, \\ \text{but } \ell(\beta_s)+\ell(\gamma_s) \text{ is an integer})}} \geq \sum_{s=1}^k 1 = k.
\end{aligned}$$

This narrows down the options for k to the finite set $\{0, 1, \ldots, \ell(\delta) + \ell(\varepsilon)\}$, and thus leaves only finitely many options for $\beta_1, \beta_2, \ldots, \beta_k$ (since there are only finitely many ways to decompose the composition δ as a concatenation $\delta = \beta_1 \beta_2 \cdots \beta_k$ when k is fixed) and for $\gamma_1, \gamma_2, \ldots, \gamma_k$ (similarly). Thus, the sum is finite.

Example 5.3. Let δ and ε be two compositions of the form $\delta = (a, b)$ and $\varepsilon = (c)$ for some positive integers a, b, c. Then, Theorem 5.1 expresses the product $\eta_\delta^{(q)} \eta_\varepsilon^{(q)} = \eta_{(a,b)}^{(q)} \eta_{(c)}^{(q)}$ as a sum over all choices of $k \in \mathbb{N}$ and of k compositions $\beta_1, \beta_2, \ldots, \beta_k \in \operatorname{Comp}$ and of k further compositions $\gamma_1, \gamma_2, \ldots, \gamma_k \in \operatorname{Comp}$ satisfying

$$\beta_1 \beta_2 \cdots \beta_k = \delta \qquad \text{and} \qquad \gamma_1 \gamma_2 \cdots \gamma_k = \varepsilon \qquad \text{and}$$
$$|\ell(\beta_s) - \ell(\gamma_s)| \leq 1 \text{ for all } s \qquad \text{and} \qquad \ell(\beta_s) + \ell(\gamma_s) > 0 \text{ for all } s.$$

These choices are

1. having $k = 1$ and $\beta_1 = \delta = (a, b)$ and $\gamma_1 = \varepsilon = (c)$;
2. having $k = 2$ and $\beta_1 = (a)$ and $\beta_2 = (b)$ and $\gamma_1 = \varnothing$ and $\gamma_2 = (c)$;
3. having $k = 2$ and $\beta_1 = (a)$ and $\beta_2 = (b)$ and $\gamma_1 = (c)$ and $\gamma_2 = \varnothing$;
4. having $k = 3$ and $\beta_1 = \varnothing$ and $\beta_2 = (a)$ and $\beta_3 = (b)$ and $\gamma_1 = (c)$ and $\gamma_2 = \varnothing$ and $\gamma_3 = \varnothing$;
5. having $k = 3$ and $\beta_1 = (a)$ and $\beta_2 = \varnothing$ and $\beta_3 = (b)$ and $\gamma_1 = \varnothing$ and $\gamma_2 = (c)$ and $\gamma_3 = \varnothing$;

6. having $k = 3$ and $\beta_1 = (a)$ and $\beta_2 = (b)$ and $\beta_3 = \varnothing$ and $\gamma_1 = \varnothing$ and $\gamma_2 = \varnothing$ and $\gamma_3 = (c)$.

Thus, Theorem 5.1 yields

$$\begin{aligned}
&\eta^{(q)}_{(a,b)}\eta^{(q)}_{(c)} \\
&= (-q)^{2-1}(q-1)^0 \eta^{(q)}_{(a+b+c)} + (-q)^{1+1-2}(q-1)^1 \eta^{(q)}_{(a,\ b+c)} \\
&\qquad + (-q)^{1+1-2}(q-1)^1 \eta^{(q)}_{(a+c,\ b)} + (-q)^{1+1+1-3}(q-1)^0 \eta^{(q)}_{(c,a,b)} \\
&\qquad + (-q)^{1+1+1-3}(q-1)^0 \eta^{(q)}_{(a,c,b)} + (-q)^{1+1+1-3}(q-1)^0 \eta^{(q)}_{(a,b,c)} \\
&= -q\eta^{(q)}_{(a+b+c)} + (q-1)\eta^{(q)}_{(a,\ b+c)} + (q-1)\eta^{(q)}_{(a+c,\ b)} + \eta^{(q)}_{(c,a,b)} + \eta^{(q)}_{(a,c,b)} + \eta^{(q)}_{(a,b,c)}.
\end{aligned}$$

Note that the last three addends $\eta^{(q)}_{(c,a,b)}, \eta^{(q)}_{(a,c,b)}, \eta^{(q)}_{(a,b,c)}$ here come from those choices in which $\min\{\ell(\beta_s), \ell(\gamma_s)\} = 0$ for each $s \in [k]$ (that is, for each $s \in [k]$, one of the two compositions β_s and γ_s is empty). In these choices, the two powers

$$(-q)^{\sum_{s=1}^{k} \max\{\ell(\beta_s),\ell(\gamma_s)\}-k} \qquad \text{and} \qquad (q-1)^{(\text{\# of all } s\in[k] \text{ such that } \ell(\beta_s)=\ell(\gamma_s))}$$

are equal to 1 (because the exponents are easily seen to be 0), whereas the composition $(|\beta_1| + |\gamma_1|, |\beta_2| + |\gamma_2|, \ldots, |\beta_k| + |\gamma_k|)$ is a shuffle of δ with ε. Thus, these choices contribute terms of the form $\eta^{(q)}_{\varphi}$, where φ is a shuffle of δ with ε, to the right hand side of Theorem 5.1, and these terms all have coefficient 1. These are the only choices of k, $\beta_1, \beta_2, \ldots, \beta_k$, $\gamma_1, \gamma_2, \ldots, \gamma_k$ that have $k = \ell(\delta) + \ell(\varepsilon)$. All other choices have $k < \ell(\delta) + \ell(\varepsilon)$, and these choices lead to addends that involve either a nontrivial power of $-q$ or a nontrivial power of $q-1$ (or both). In this sense, we can view Theorem 5.1 as a deformation of the overlapping shuffle product formula for $M_\delta M_\varepsilon$ (see, e.g., [GriRei20, Proposition 5.1.3]), although the concept of a "deformation" must be understood in a wide sense (we cannot obtain the latter just by specializing the former).

We will derive Theorem 5.1 from Corollary 4.18. For this, we will again use the duality between NSym and QSym:

Lemma 5.4. Let $f, g \in \mathrm{QSym}$ and $h \in \mathrm{NSym}$ be arbitrary. Let the tensor $\Delta(h) \in \mathrm{NSym} \otimes \mathrm{NSym}$ be written in the form $\Delta(h) = \sum_{i\in I} s_i \otimes t_i$, where I is a finite set and where $s_i, t_i \in \mathrm{NSym}$ for each $i \in I$. Then,

$$\langle h, fg \rangle = \sum_{i\in I} \langle s_i, f \rangle \langle t_i, g \rangle .$$

Proof. This is analogous to Lemma 4.10, except that the roles of QSym and NSym have now been switched. □

For the sake of convenience, let us extend Lemma 5.4 to infinite sums with only finitely many infinite addends:

Lemma 5.5. Let $f, g \in \operatorname{QSym}$ and $h \in \operatorname{NSym}$ be arbitrary. Let the tensor $\Delta(h) \in \operatorname{NSym} \otimes \operatorname{NSym}$ be written in the form $\Delta(h) = \sum\limits_{i \in I} s_i \otimes t_i$, where I is a set and where $s_i, t_i \in \operatorname{NSym}$ for each $i \in I$ are chosen such that only finitely many $i \in I$ satisfy $s_i \neq 0$. Then,

$$\langle h, fg \rangle = \sum_{i \in I} \langle s_i, f \rangle \langle t_i, g \rangle .$$

Proof. This is easily reduced to Lemma 5.4 (just replace the set I by its subset $I' := \{i \in I \mid s_i \neq 0\}$). □

Proof of Theorem 5.1. Forget that we fixed δ and ε. For any three compositions $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$, δ and ε, we define a polynomial

$$d^{\alpha}_{\delta,\varepsilon}(X) := \sum_{\substack{\beta_1,\beta_2,\ldots,\beta_k \in \operatorname{Comp}; \\ \gamma_1,\gamma_2,\ldots,\gamma_k \in \operatorname{Comp}; \\ \beta_1\beta_2\cdots\beta_k = \delta; \\ \gamma_1\gamma_2\cdots\gamma_k = \varepsilon; \\ |\ell(\beta_s) - \ell(\gamma_s)| \leq 1 \text{ for all } s; \\ |\beta_s| + |\gamma_s| = \alpha_s \text{ for all } s}} (-X)^{\sum\limits_{s=1}^{k} \max\{\ell(\beta_s),\ell(\gamma_s)\} - k} \cdot (X-1)^{(\text{\# of all } s \in [k] \text{ such that } \ell(\beta_s) = \ell(\gamma_s))} \in \mathbb{Z}[X] \tag{70}$$

(this really is a polynomial, since the exponent $\sum\limits_{s=1}^{k} \max\{\ell(\beta_s), \ell(\gamma_s)\} - k$ is easily seen to be a nonnegative integer). Thus, clearly, for any three compositions $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$, δ and ε, we have

$$d^{\alpha}_{\delta,\varepsilon}(q) = \sum_{\substack{\beta_1,\beta_2,\ldots,\beta_k \in \operatorname{Comp}; \\ \gamma_1,\gamma_2,\ldots,\gamma_k \in \operatorname{Comp}; \\ \beta_1\beta_2\cdots\beta_k = \delta; \\ \gamma_1\gamma_2\cdots\gamma_k = \varepsilon; \\ |\ell(\beta_s) - \ell(\gamma_s)| \leq 1 \text{ for all } s; \\ |\beta_s| + |\gamma_s| = \alpha_s \text{ for all } s}} (-q)^{\sum\limits_{s=1}^{k} \max\{\ell(\beta_s),\ell(\gamma_s)\} - k} \cdot (q-1)^{(\text{\# of all } s \in [k] \text{ such that } \ell(\beta_s) = \ell(\gamma_s))} \tag{71}$$

Note that the sums on the right hand sides of (70) and (71) are finite (because for a given $k \in \mathbb{N}$ and given compositions δ and ε, there are only finitely many ways to decompose δ as $\delta = \beta_1\beta_2 \cdots \beta_k$, and only finitely many ways to decompose ε as

$\varepsilon = \gamma_1 \gamma_2 \cdots \gamma_k$). All sums that will appear in this proof will be finite or essentially finite (i.e., have only finitely many nonzero addends). We note that **k**-linear maps always respect such sums.

Now, we shall proceed by proving several claims. Our first claim is a restatement of Corollary 4.18:

> *Claim 1:* Let α be any composition. Assume that r is invertible. Then,
> $$\Delta\left(\eta_\alpha^{*(q)}\right) = \sum_{\delta,\varepsilon \in \operatorname{Comp}} d_{\delta,\varepsilon}^{\alpha}(q)\, \eta_\delta^{*(q)} \otimes \eta_\varepsilon^{*(q)}. \qquad (72)$$

Proof of Claim 1. Write the composition α as $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$. Then, Corollary 4.18 yields

$$\begin{aligned}
&\Delta\left(\eta_\alpha^{*(q)}\right) \\
&= \underbrace{\sum_{\substack{\beta_1,\beta_2,\ldots,\beta_k \in \operatorname{Comp};\\ \gamma_1,\gamma_2,\ldots,\gamma_k \in \operatorname{Comp};\\ |\beta_s|+|\gamma_s|=\alpha_s \text{ for all } s;\\ |\ell(\beta_s)-\ell(\gamma_s)|\leq 1 \text{ for all } s}}}_{\substack{= \sum\limits_{\substack{\beta_1,\beta_2,\ldots,\beta_k \in \operatorname{Comp};\\ \gamma_1,\gamma_2,\ldots,\gamma_k \in \operatorname{Comp};\\ |\ell(\beta_s)-\ell(\gamma_s)|\leq 1 \text{ for all } s;\\ |\beta_s|+|\gamma_s|=\alpha_s \text{ for all } s}} \\ = \sum\limits_{\delta,\varepsilon\in\operatorname{Comp}} \sum\limits_{\substack{\beta_1,\beta_2,\ldots,\beta_k \in \operatorname{Comp};\\ \gamma_1,\gamma_2,\ldots,\gamma_k \in \operatorname{Comp};\\ \beta_1\beta_2\cdots\beta_k=\delta;\\ \gamma_1\gamma_2\cdots\gamma_k=\varepsilon;\\ |\ell(\beta_s)-\ell(\gamma_s)|\leq 1 \text{ for all } s;\\ |\beta_s|+|\gamma_s|=\alpha_s \text{ for all } s}} \\ \text{(here, we have split up the sum according} \\ \text{to the values of } \beta_1\beta_2\cdots\beta_k \text{ and } \gamma_1\gamma_2\cdots\gamma_k)}} (-q)^{\sum_{s=1}^{k} \max\{\ell(\beta_s),\ell(\gamma_s)\}-k} \\
&\qquad \cdot (q-1)^{(\#\text{ of all } s\in[k] \text{ such that } \ell(\beta_s)=\ell(\gamma_s))} \left(\eta_{\beta_1\beta_2\cdots\beta_k}^{*(q)} \otimes \eta_{\gamma_1\gamma_2\cdots\gamma_k}^{*(q)}\right) \\
&= \sum_{\delta,\varepsilon\in\operatorname{Comp}} \sum_{\substack{\beta_1,\beta_2,\ldots,\beta_k \in \operatorname{Comp};\\ \gamma_1,\gamma_2,\ldots,\gamma_k \in \operatorname{Comp};\\ \beta_1\beta_2\cdots\beta_k=\delta;\\ \gamma_1\gamma_2\cdots\gamma_k=\varepsilon;\\ |\ell(\beta_s)-\ell(\gamma_s)|\leq 1 \text{ for all } s;\\ |\beta_s|+|\gamma_s|=\alpha_s \text{ for all } s}} (-q)^{\sum_{s=1}^{k} \max\{\ell(\beta_s),\ell(\gamma_s)\}-k} \\
&\qquad \cdot (q-1)^{(\#\text{ of all } s\in[k] \text{ such that } \ell(\beta_s)=\ell(\gamma_s))} \underbrace{\left(\eta_{\beta_1\beta_2\cdots\beta_k}^{*(q)} \otimes \eta_{\gamma_1\gamma_2\cdots\gamma_k}^{*(q)}\right)}_{\substack{=\eta_\delta^{*(q)}\otimes\eta_\varepsilon^{*(q)} \\ \text{(since } \beta_1\beta_2\cdots\beta_k=\delta \\ \text{and } \gamma_1\gamma_2\cdots\gamma_k=\varepsilon)}}
\end{aligned}$$

$$= \sum_{\delta,\varepsilon\in\operatorname{Comp}} \sum_{\substack{\beta_1,\beta_2,\ldots,\beta_k\in\operatorname{Comp};\\ \gamma_1,\gamma_2,\ldots,\gamma_k\in\operatorname{Comp};\\ \beta_1\beta_2\cdots\beta_k=\delta;\\ \gamma_1\gamma_2\cdots\gamma_k=\varepsilon;\\ |\ell(\beta_s)-\ell(\gamma_s)|\leq 1 \text{ for all } s;\\ |\beta_s|+|\gamma_s|=\alpha_s \text{ for all } s}} (-q)^{\sum_{s=1}^{k}\max\{\ell(\beta_s),\ell(\gamma_s)\}-k} \cdot (q-1)^{(\# \text{ of all } s\in[k] \text{ such that } \ell(\beta_s)=\ell(\gamma_s))} \eta_\delta^{*(q)} \otimes \eta_\varepsilon^{*(q)}$$

$$= \sum_{\delta,\varepsilon\in\operatorname{Comp}} \underbrace{\left(\sum_{\substack{\beta_1,\beta_2,\ldots,\beta_k\in\operatorname{Comp};\\ \gamma_1,\gamma_2,\ldots,\gamma_k\in\operatorname{Comp};\\ \beta_1\beta_2\cdots\beta_k=\delta;\\ \gamma_1\gamma_2\cdots\gamma_k=\varepsilon;\\ |\ell(\beta_s)-\ell(\gamma_s)|\leq 1 \text{ for all } s;\\ |\beta_s|+|\gamma_s|=\alpha_s \text{ for all } s}} (-q)^{\sum_{s=1}^{k}\max\{\ell(\beta_s),\ell(\gamma_s)\}-k} (q-1)^{(\# \text{ of all } s\in[k] \text{ such that } \ell(\beta_s)=\ell(\gamma_s))} \right)}_{\substack{=d^\alpha_{\delta,\varepsilon}(q)\\ \text{(by (71))}}} \eta_\delta^{*(q)} \otimes \eta_\varepsilon^{*(q)}$$

$$= \sum_{\delta,\varepsilon\in\operatorname{Comp}} d^\alpha_{\delta,\varepsilon}(q)\, \eta_\delta^{*(q)} \otimes \eta_\varepsilon^{*(q)}.$$

Thus, Claim 1 is proved. □

Claim 2: Let δ and ε be two compositions. If r is invertible, then

$$\eta_\delta^{(q)} \eta_\varepsilon^{(q)} = \sum_{\alpha\in\operatorname{Comp}} d^\alpha_{\delta,\varepsilon}(q)\, \eta_\alpha^{(q)}.$$

Proof of Claim 2. Essentially, this follows by duality (Lemma 5.4) from Claim 1. Here are the details:

Assume that r is invertible. For any composition α, we have

$$\Delta\left(\eta_\alpha^{*(q)}\right) = \sum_{\lambda,\mu\in\operatorname{Comp}} d^\alpha_{\lambda,\mu}(q)\, \eta_\lambda^{*(q)} \otimes \eta_\mu^{*(q)} \tag{73}$$

(by Claim 1, with the letters δ and ε renamed as λ and μ).

Let I be the set $\operatorname{Comp}\times\operatorname{Comp}$. Then, we can rewrite (73) as follows: For any composition α, we have

$$\Delta\left(\eta_\alpha^{*(q)}\right) = \sum_{(\lambda,\mu)\in I} d^\alpha_{\lambda,\mu}(q)\, \eta_\lambda^{*(q)} \otimes \eta_\mu^{*(q)}. \tag{74}$$

Then, Theorem 3.10 **(a)** shows that the family $\left(\eta_\alpha^{(q)}\right)_{\alpha\in\operatorname{Comp}}$ is a basis of the $\mathbf{k}$-module QSym. In other words, the family $\left(\eta_\beta^{(q)}\right)_{\beta\in\operatorname{Comp}}$ is a basis of the $\mathbf{k}$-module QSym. Hence, we can write the quasisymmetric function $\eta_\delta^{(q)}\eta_\varepsilon^{(q)} \in \operatorname{QSym}$ as

$$\eta_\delta^{(q)}\eta_\varepsilon^{(q)} = \sum_{\beta\in\operatorname{Comp}} c_\beta \eta_\beta^{(q)}, \tag{75}$$

where $\left(c_\beta\right)_{\beta\in\operatorname{Comp}} \in \mathbf{k}^{\operatorname{Comp}}$ is a family of coefficients (with $c_\beta = 0$ for all but finitely many $\beta \in \operatorname{Comp}$). Consider this family.

For every $\alpha \in \operatorname{Comp}$, we have

$$\begin{aligned}
\left\langle \eta_\alpha^{*(q)},\ \eta_\delta^{(q)}\eta_\varepsilon^{(q)} \right\rangle &= \left\langle \eta_\alpha^{*(q)},\ \sum_{\beta\in\operatorname{Comp}} c_\beta \eta_\beta^{(q)} \right\rangle \qquad \text{(by (75))} \\
&= \sum_{\beta\in\operatorname{Comp}} c_\beta \underbrace{\left\langle \eta_\alpha^{*(q)}, \eta_\beta^{(q)} \right\rangle}_{\substack{=[\alpha=\beta] \\ \text{(by (37))}}} = \sum_{\beta\in\operatorname{Comp}} c_\beta \left[\alpha=\beta\right] = c_\alpha
\end{aligned}$$

(since all addends of the sum $\sum\limits_{\beta\in\operatorname{Comp}} c_\beta \left[\alpha=\beta\right]$ except for the $\beta = \alpha$ addend are 0) and therefore

$$\begin{aligned}
c_\alpha &= \left\langle \eta_\alpha^{*(q)},\ \eta_\delta^{(q)}\eta_\varepsilon^{(q)} \right\rangle \\
&= \underbrace{\sum_{(\lambda,\mu)\in I}}_{=\sum\limits_{\lambda,\mu\in\operatorname{Comp}}} \underbrace{\left\langle d_{\lambda,\mu}^{\alpha}(q)\,\eta_\lambda^{*(q)},\ \eta_\delta^{(q)} \right\rangle}_{=d_{\lambda,\mu}^{\alpha}(q)\left\langle \eta_\lambda^{*(q)},\ \eta_\delta^{(q)} \right\rangle} \left\langle \eta_\mu^{*(q)},\ \eta_\varepsilon^{(q)} \right\rangle \\
&\qquad \left(\begin{array}{c}
\text{by Lemma 5.5, applied to } f = \eta_\delta^{(q)} \text{ and } g = \eta_\varepsilon^{(q)} \\
\text{and } h = \eta_\alpha^{*(q)} \text{ and } s_{(\lambda,\mu)} = d_{\lambda,\mu}^{\alpha}(q)\,\eta_\lambda^{*(q)} \text{ and } t_{(\lambda,\mu)} = \eta_\mu^{*(q)} \\
\text{(since (74) yields } \Delta\left(\eta_\alpha^{*(q)}\right) = \sum\limits_{(\lambda,\mu)\in I} d_{\lambda,\mu}^{\alpha}(q)\,\eta_\lambda^{*(q)} \otimes \eta_\mu^{*(q)}\text{)}
\end{array}\right) \\
&= \sum_{\lambda,\mu\in\operatorname{Comp}} d_{\lambda,\mu}^{\alpha}(q) \underbrace{\left\langle \eta_\lambda^{*(q)},\ \eta_\delta^{(q)} \right\rangle}_{\substack{=[\lambda=\delta] \\ \text{(by (37))}}} \underbrace{\left\langle \eta_\mu^{*(q)},\ \eta_\varepsilon^{(q)} \right\rangle}_{\substack{=[\mu=\varepsilon] \\ \text{(by (37))}}} \\
&= \sum_{\lambda,\mu\in\operatorname{Comp}} d_{\lambda,\mu}^{\alpha}(q) \underbrace{\left[\lambda=\delta\right]\cdot\left[\mu=\varepsilon\right]}_{\substack{=[\lambda=\delta \text{ and } \mu=\varepsilon] \\ =[(\lambda,\mu)=(\delta,\varepsilon)]}} = \sum_{\lambda,\mu\in\operatorname{Comp}} d_{\lambda,\mu}^{\alpha}(q) \left[(\lambda,\mu)=(\delta,\varepsilon)\right] \\
&= d_{\delta,\varepsilon}^{\alpha}(q)
\end{aligned} \tag{76}$$

(since all addends of the sum $\sum\limits_{\lambda,\mu\in\operatorname{Comp}} d_{\lambda,\mu}^{\alpha}(q)\left[(\lambda,\mu)=(\delta,\varepsilon)\right]$ except for the $(\lambda,\mu) = (\delta,\varepsilon)$ addend are 0).

Now, (75) becomes

$$\eta_{\delta}^{(q)}\eta_{\varepsilon}^{(q)} = \sum_{\beta\in\mathrm{Comp}} c_{\beta}\eta_{\beta}^{(q)} = \sum_{\alpha\in\mathrm{Comp}} \underbrace{c_{\alpha}}_{\substack{=d_{\delta,\varepsilon}^{\alpha}(q)\\ \text{(by (76))}}} \eta_{\alpha}^{(q)} = \sum_{\alpha\in\mathrm{Comp}} d_{\delta,\varepsilon}^{\alpha}\left(q\right)\eta_{\alpha}^{(q)}.$$

This proves Claim 2. $\square$

In the rest of this proof, we will use several different base rings. Thus, we shall use the notation $\mathrm{QSym}_{\mathbf{k}}$ for what we have previously been calling QSym (that is, the ring of quasisymmetric functions over the ring $\mathbf{k}$). Clearly, any ring homomorphism $f : \mathbf{k} \to \mathbf{l}$ between two commutative rings $\mathbf{k}$ and $\mathbf{l}$ canonically induces a ring homomorphism $\mathrm{QSym}_{\mathbf{k}} \to \mathrm{QSym}_{\mathbf{l}}$, which we denote by QSym_f. Moreover, if $\mathbf{k}$ is a subring of a commutative ring $\mathbf{l}$, then $\mathrm{QSym}_{\mathbf{k}}$ canonically becomes a subring of $\mathrm{QSym}_{\mathbf{l}}$.

We note that the definition of the power series $\eta_{\alpha}^{(q)}$ does not depend on the base ring. Thus, if $f : \mathbf{k} \to \mathbf{l}$ is a ring homomorphism between two commutative rings $\mathbf{k}$ and $\mathbf{l}$, then any $\alpha \in \mathrm{Comp}$ and any $q \in \mathbf{k}$ satisfy

$$\mathrm{QSym}_f\left(\eta_{\alpha}^{(q)}\right) = \eta_{\alpha}^{(f(q))} \tag{77}$$

(where the $\eta_{\alpha}^{(q)}$ on the left hand side is defined in $\mathrm{QSym}_{\mathbf{k}}$, whereas the $\eta_{\alpha}^{(f(q))}$ on the right hand side is defined in $\mathrm{QSym}_{\mathbf{l}}$). Likewise, if $\mathbf{k}$ is a subring of $\mathbf{l}$, then the $\eta_{\alpha}^{(q)}$ in $\mathrm{QSym}_{\mathbf{k}}$ equals the $\eta_{\alpha}^{(q)}$ in $\mathrm{QSym}_{\mathbf{l}}$. We shall use this tacitly soon.

Claim 3: Let δ and ε be two compositions. If $\mathbf{k}$ is the polynomial ring $\mathbb{Z}\left[X\right]$, and if q is the indeterminate X in this ring, then

$$\eta_{\delta}^{(q)}\eta_{\varepsilon}^{(q)} = \sum_{\alpha\in\mathrm{Comp}} d_{\delta,\varepsilon}^{\alpha}\left(q\right)\eta_{\alpha}^{(q)}.$$

In other words, we have

$$\eta_{\delta}^{(X)}\eta_{\varepsilon}^{(X)} = \sum_{\alpha\in\mathrm{Comp}} d_{\delta,\varepsilon}^{\alpha}\left(X\right)\eta_{\alpha}^{(X)} \qquad \text{in } \mathrm{QSym}_{\mathbb{Z}[X]}. \tag{78}$$

Proof of Claim 3. Consider the field $\mathbb{Q}\left(X\right)$ of rational functions in X over $\mathbb{Q}$. Clearly, $\mathbb{Z}\left[X\right]$ is a subring of $\mathbb{Q}\left(X\right)$. Thus, $\mathrm{QSym}_{\mathbb{Z}[X]}$ becomes a subring of $\mathrm{QSym}_{\mathbb{Q}(X)}$.

In the ring $\mathbb{Q}\left(X\right)$, the polynomial $X+1$ is invertible. Thus, Claim 2 (applied to $\mathbb{Q}\left(X\right)$, X and $X+1$ instead of $\mathbf{k}$, q and r) yields that

$$\eta_{\delta}^{(X)}\eta_{\varepsilon}^{(X)} = \sum_{\alpha\in\mathrm{Comp}} d_{\delta,\varepsilon}^{\alpha}\left(X\right)\eta_{\alpha}^{(X)} \qquad \text{in } \mathrm{QSym}_{\mathbb{Q}(X)}. \tag{79}$$

But $\operatorname{QSym}_{\mathbb{Z}[X]}$ is a subring of $\operatorname{QSym}_{\mathbb{Q}(X)}$, and both sides of the equality (79) belong to $\operatorname{QSym}_{\mathbb{Z}[X]}$ (since $d^{\alpha}_{\delta,\varepsilon}(X) \in \mathbb{Z}[X]$ and $\eta^{(X)}_{\alpha} \in \operatorname{QSym}_{\mathbb{Z}[X]}$ for all $\alpha \in \operatorname{Comp}$), and do not depend on the base ring[17]. Hence, the equality (79) holds in $\operatorname{QSym}_{\mathbb{Z}[X]}$ as well. In other words, we have

$$\eta^{(X)}_{\delta}\eta^{(X)}_{\varepsilon} = \sum_{\alpha \in \operatorname{Comp}} d^{\alpha}_{\delta,\varepsilon}(X)\,\eta^{(X)}_{\alpha} \qquad \text{in } \operatorname{QSym}_{\mathbb{Z}[X]}.$$

In other words, (78) holds. This proves Claim 3. $\square$

Claim 4: Let δ and ε be two compositions. Then,

$$\eta^{(q)}_{\delta}\eta^{(q)}_{\varepsilon} = \sum_{\alpha \in \operatorname{Comp}} d^{\alpha}_{\delta,\varepsilon}(q)\,\eta^{(q)}_{\alpha}.$$

Proof of Claim 4. Consider the polynomial ring $\mathbb{Z}[X]$. By the universal property of a polynomial ring, there exists a unique $\mathbb{Z}$-algebra homomorphism $f : \mathbb{Z}[X] \to \mathbf{k}$ that sends X to q. Consider this f. Explicitly, f is given by

$$f(u(X)) = u(q) \qquad \text{for any polynomial } u(X) \in \mathbb{Z}[X]. \tag{80}$$

The map f is a $\mathbb{Z}$-algebra homomorphism, thus a ring homomorphism, and therefore induces a ring homomorphism $\operatorname{QSym}_f : \operatorname{QSym}_{\mathbb{Z}[X]} \to \operatorname{QSym}_{\mathbf{k}}$. Applying this ring homomorphism QSym_f to both sides of (78), we obtain

$$\operatorname{QSym}_f\left(\eta^{(X)}_{\delta}\right)\cdot\operatorname{QSym}_f\left(\eta^{(X)}_{\varepsilon}\right) = \sum_{\alpha \in \operatorname{Comp}} \operatorname{QSym}_f\left(d^{\alpha}_{\delta,\varepsilon}(X)\right)\cdot\operatorname{QSym}_f\left(\eta^{(X)}_{\alpha}\right) \qquad \text{in } \operatorname{QSym}_{\mathbf{k}}.$$

Since every composition $\alpha \in \operatorname{Comp}$ satisfies

$$\begin{aligned}\operatorname{QSym}_f\left(\eta^{(X)}_{\alpha}\right) &= \eta^{(f(X))}_{\alpha} && \text{(by (77))}\\ &= \eta^{(q)}_{\alpha} && \text{(since } f(X) = q\text{)},\end{aligned}$$

we can rewrite this as

$$\begin{aligned}\eta^{(q)}_{\delta}\eta^{(q)}_{\varepsilon} &= \sum_{\alpha \in \operatorname{Comp}} \underbrace{\operatorname{QSym}_f\left(d^{\alpha}_{\delta,\varepsilon}(X)\right)}_{\substack{=f\left(d^{\alpha}_{\delta,\varepsilon}(X)\right)\\ \text{(since the homomorphism } \operatorname{QSym}_f\\ \text{acts as } f \text{ on } \mathbb{Z}[X])}} \eta^{(q)}_{\alpha}\\ &= \sum_{\alpha \in \operatorname{Comp}} \underbrace{f\left(d^{\alpha}_{\delta,\varepsilon}(X)\right)}_{\substack{=d^{\alpha}_{\delta,\varepsilon}(q)\\ \text{(by (80))}}} \eta^{(q)}_{\alpha} = \sum_{\alpha \in \operatorname{Comp}} d^{\alpha}_{\delta,\varepsilon}(q)\,\eta^{(q)}_{\alpha}.\end{aligned}$$

This proves Claim 4. $\square$

[17] Indeed, the power series $\eta^{(X)}_{\alpha}$ defined over $\mathbb{Z}[X]$ equals the power series $\eta^{(X)}_{\alpha}$ defined over $\mathbb{Q}(X)$.

Claim 5: Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$, δ and ε be three compositions. Then,

$$d^{\alpha}_{\delta,\varepsilon}(q) = \sum_{\substack{\beta_1,\beta_2,\ldots,\beta_k \in \operatorname{Comp}; \\ \gamma_1,\gamma_2,\ldots,\gamma_k \in \operatorname{Comp}; \\ \beta_1\beta_2\cdots\beta_k = \delta; \\ \gamma_1\gamma_2\cdots\gamma_k = \varepsilon; \\ |\ell(\beta_s) - \ell(\gamma_s)| \leq 1 \text{ for all } s; \\ \alpha = (|\beta_1|+|\gamma_1|,\ |\beta_2|+|\gamma_2|,\ \ldots,\ |\beta_k|+|\gamma_k|)}} (-q)^{\sum\limits_{s=1}^{k} \max\{\ell(\beta_s),\ell(\gamma_s)\} - k} \cdot (q-1)^{(\# \text{ of all } s \in [k] \text{ such that } \ell(\beta_s) = \ell(\gamma_s))}. \tag{81}$$

Proof of Claim 5. The condition "$|\beta_s| + |\gamma_s| = \alpha_s$ for all s" under the summation sign in (71) is equivalent to the condition "$\alpha = (|\beta_1| + |\gamma_1|,\ |\beta_2| + |\gamma_2|,\ \ldots,\ |\beta_k| + |\gamma_k|)$" (since $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$). Thus, we can replace the former condition in (71) by the latter. The result of this replacement is precisely the equality (81). Hence, Claim 5 is proved. $\square$

Now, Theorem 5.1 is just a restatement of Claim 4. Indeed, let δ and ε be two compositions. Then,

$$\begin{aligned}
&\eta_{\delta}^{(q)}\eta_{\varepsilon}^{(q)} \\
&= \underbrace{\sum_{\alpha\in\mathrm{Comp}}}_{\substack{=\sum\limits_{k\in\mathbb{N}}\ \sum\limits_{\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_k)\in\mathrm{Comp}} \\ \text{(since any composition} \\ \text{has a unique length)}}} d_{\delta,\varepsilon}^{\alpha}(q)\,\eta_{\alpha}^{(q)} \qquad \text{(by Claim 4)} \\
&= \sum_{k\in\mathbb{N}}\ \sum_{\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_k)\in\mathrm{Comp}} d_{\delta,\varepsilon}^{\alpha}(q)\,\eta_{\alpha}^{(q)} \\
&= \sum_{k\in\mathbb{N}}\ \sum_{\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_k)\in\mathrm{Comp}} \left(\sum_{\substack{\beta_1,\beta_2,\ldots,\beta_k\in\mathrm{Comp}; \\ \gamma_1,\gamma_2,\ldots,\gamma_k\in\mathrm{Comp}; \\ \beta_1\beta_2\cdots\beta_k=\delta; \\ \gamma_1\gamma_2\cdots\gamma_k=\varepsilon; \\ \left|\ell(\beta_s)-\ell(\gamma_s)\right|\leq 1 \text{ for all } s; \\ \alpha=(|\beta_1|+|\gamma_1|,\ |\beta_2|+|\gamma_2|,\ \ldots,\ |\beta_k|+|\gamma_k|)}} (-q)^{\sum\limits_{s=1}^{k}\max\{\ell(\beta_s),\ell(\gamma_s)\}-k} \right. \\
&\qquad \left. \cdot (q-1)^{(\#\text{ of all } s\in[k] \text{ such that } \ell(\beta_s)=\ell(\gamma_s))} \right) \cdot \eta_{\alpha}^{(q)} \\
&\qquad\qquad \text{(by (81))} \\
&= \sum_{k\in\mathbb{N}}\ \sum_{\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_k)\in\mathrm{Comp}}\ \sum_{\substack{\beta_1,\beta_2,\ldots,\beta_k\in\mathrm{Comp}; \\ \gamma_1,\gamma_2,\ldots,\gamma_k\in\mathrm{Comp}; \\ \beta_1\beta_2\cdots\beta_k=\delta; \\ \gamma_1\gamma_2\cdots\gamma_k=\varepsilon; \\ \left|\ell(\beta_s)-\ell(\gamma_s)\right|\leq 1 \text{ for all } s; \\ \alpha=(|\beta_1|+|\gamma_1|,\ |\beta_2|+|\gamma_2|,\ \ldots,\ |\beta_k|+|\gamma_k|)}} (-q)^{\sum\limits_{s=1}^{k}\max\{\ell(\beta_s),\ell(\gamma_s)\}-k} \\
&\qquad \cdot (q-1)^{(\#\text{ of all } s\in[k] \text{ such that } \ell(\beta_s)=\ell(\gamma_s))} \\
&\qquad \cdot \underbrace{\eta_{\alpha}^{(q)}}_{\substack{=\eta^{(q)}_{(|\beta_1|+|\gamma_1|,\ |\beta_2|+|\gamma_2|,\ \ldots,\ |\beta_k|+|\gamma_k|)} \\ \text{(since } \alpha=(|\beta_1|+|\gamma_1|,\ |\beta_2|+|\gamma_2|,\ \ldots,\ |\beta_k|+|\gamma_k|))}}
\end{aligned}$$

$$= \sum_{k\in\mathbb{N}} \sum_{\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_k)\in\mathrm{Comp}} \sum_{\substack{\beta_1,\beta_2,\ldots,\beta_k\in\mathrm{Comp};\\ \gamma_1,\gamma_2,\ldots,\gamma_k\in\mathrm{Comp};\\ \beta_1\beta_2\cdots\beta_k=\delta;\\ \gamma_1\gamma_2\cdots\gamma_k=\varepsilon;\\ |\ell(\beta_s)-\ell(\gamma_s)|\leq 1 \text{ for all } s;\\ \alpha=(|\beta_1|+|\gamma_1|,\ |\beta_2|+|\gamma_2|,\ \ldots,\ |\beta_k|+|\gamma_k|)}} (-q)^{\sum_{s=1}^{k}\max\{\ell(\beta_s),\ell(\gamma_s)\}-k}$$
$$\cdot (q-1)^{(\#\text{ of all } s\in[k] \text{ such that } \ell(\beta_s)=\ell(\gamma_s))}$$
$$\cdot \eta^{(q)}_{(|\beta_1|+|\gamma_1|,\ |\beta_2|+|\gamma_2|,\ \ldots,\ |\beta_k|+|\gamma_k|)}. \tag{82}$$

However, for each $k \in \mathbb{N}$, we have the following equality of summation signs:

$$\sum_{\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_k)\in\mathrm{Comp}} \sum_{\substack{\beta_1,\beta_2,\ldots,\beta_k\in\mathrm{Comp};\\ \gamma_1,\gamma_2,\ldots,\gamma_k\in\mathrm{Comp};\\ \beta_1\beta_2\cdots\beta_k=\delta;\\ \gamma_1\gamma_2\cdots\gamma_k=\varepsilon;\\ |\ell(\beta_s)-\ell(\gamma_s)|\leq 1 \text{ for all } s;\\ \alpha=(|\beta_1|+|\gamma_1|,\ |\beta_2|+|\gamma_2|,\ \ldots,\ |\beta_k|+|\gamma_k|)}}$$
$$= \sum_{\substack{\beta_1,\beta_2,\ldots,\beta_k\in\mathrm{Comp};\\ \gamma_1,\gamma_2,\ldots,\gamma_k\in\mathrm{Comp};\\ \beta_1\beta_2\cdots\beta_k=\delta;\\ \gamma_1\gamma_2\cdots\gamma_k=\varepsilon;\\ |\ell(\beta_s)-\ell(\gamma_s)|\leq 1 \text{ for all } s;\\ (|\beta_1|+|\gamma_1|,\ |\beta_2|+|\gamma_2|,\ \ldots,\ |\beta_k|+|\gamma_k|)\in\mathrm{Comp}}} \qquad \left(\begin{array}{c}\text{since the condition ``}\alpha = (|\beta_1|+|\gamma_1|,\ |\beta_2|+|\gamma_2|,\ \ldots,\ |\beta_k|+|\gamma_k|)\text{''}\\ \text{under the second summation sign uniquely determines } \alpha\end{array}\right)$$
$$= \sum_{\substack{\beta_1,\beta_2,\ldots,\beta_k\in\mathrm{Comp};\\ \gamma_1,\gamma_2,\ldots,\gamma_k\in\mathrm{Comp};\\ \beta_1\beta_2\cdots\beta_k=\delta;\\ \gamma_1\gamma_2\cdots\gamma_k=\varepsilon;\\ |\ell(\beta_s)-\ell(\gamma_s)|\leq 1 \text{ for all } s;\\ |\beta_s|+|\gamma_s|>0 \text{ for all } s}} \qquad \left(\begin{array}{c}\text{since the condition}\\ \text{``}(|\beta_1|+|\gamma_1|,\ |\beta_2|+|\gamma_2|,\ \ldots,\ |\beta_k|+|\gamma_k|)\in\mathrm{Comp}\text{''}\\ \text{is equivalent to ``}|\beta_s|+|\gamma_s|>0 \text{ for all } s\text{''}\end{array}\right)$$
$$= \sum_{\substack{\beta_1,\beta_2,\ldots,\beta_k\in\mathrm{Comp};\\ \gamma_1,\gamma_2,\ldots,\gamma_k\in\mathrm{Comp};\\ \beta_1\beta_2\cdots\beta_k=\delta;\\ \gamma_1\gamma_2\cdots\gamma_k=\varepsilon;\\ |\ell(\beta_s)-\ell(\gamma_s)|\leq 1 \text{ for all } s;\\ \ell(\beta_s)+\ell(\gamma_s)>0 \text{ for all } s}} \qquad \left(\begin{array}{c}\text{since the condition ``}|\beta_s|+|\gamma_s|>0\text{''}\\ \text{on two compositions } \beta_s \text{ and } \gamma_s\\ \text{is equivalent to ``}\ell(\beta_s)+\ell(\gamma_s)>0\text{''}\\ \text{(indeed, both conditions are}\\ \text{equivalent to } (\beta_s,\gamma_s)\neq(\varnothing,\varnothing)\text{)}\end{array}\right).$$

Hence, we can rewrite (82) as

$$\eta_{\delta}^{(q)}\eta_{\varepsilon}^{(q)} = \sum_{k\in\mathbb{N}} \sum_{\substack{\beta_1,\beta_2,\ldots,\beta_k\in\operatorname{Comp};\\ \gamma_1,\gamma_2,\ldots,\gamma_k\in\operatorname{Comp};\\ \beta_1\beta_2\cdots\beta_k=\delta;\\ \gamma_1\gamma_2\cdots\gamma_k=\varepsilon;\\ |\ell(\beta_s)-\ell(\gamma_s)|\leq 1 \text{ for all } s;\\ \ell(\beta_s)+\ell(\gamma_s)>0 \text{ for all } s}} (-q)^{\sum_{s=1}^{k}\max\{\ell(\beta_s),\ell(\gamma_s)\}-k} \cdot (q-1)^{(\#\text{ of all } s\in[k] \text{ such that } \ell(\beta_s)=\ell(\gamma_s))} \cdot \eta^{(q)}_{(|\beta_1|+|\gamma_1|,\ |\beta_2|+|\gamma_2|,\ \ldots,\ |\beta_k|+|\gamma_k|)}.$$

Combining the two summation signs here into a single sum, we can rewrite this as

$$\eta_{\delta}^{(q)}\eta_{\varepsilon}^{(q)} = \sum_{\substack{k\in\mathbb{N};\\ \beta_1,\beta_2,\ldots,\beta_k\in\operatorname{Comp};\\ \gamma_1,\gamma_2,\ldots,\gamma_k\in\operatorname{Comp};\\ \beta_1\beta_2\cdots\beta_k=\delta;\\ \gamma_1\gamma_2\cdots\gamma_k=\varepsilon;\\ |\ell(\beta_s)-\ell(\gamma_s)|\leq 1 \text{ for all } s;\\ \ell(\beta_s)+\ell(\gamma_s)>0 \text{ for all } s}} (-q)^{\sum_{s=1}^{k}\max\{\ell(\beta_s),\ell(\gamma_s)\}-k} \cdot (q-1)^{(\#\text{ of all } s\in[k] \text{ such that } \ell(\beta_s)=\ell(\gamma_s))} \cdot \eta^{(q)}_{(|\beta_1|+|\gamma_1|,\ |\beta_2|+|\gamma_2|,\ \ldots,\ |\beta_k|+|\gamma_k|)}.$$

Thus, Theorem 5.1 is proved. □

5.2. The product rule in terms of stufflers

We will next rewrite Theorem 5.1 in a somewhat different language, using certain surjective maps instead of factorizations of compositions. First, we introduce several pieces of notation:

Definition 5.6. Let i and j be two integers. Then, we write $i \approx j$ (and say that i is *nearly equal* to j) if and only if $|i-j| \leq 1$.

(Of course, $\approx$ is not an equivalence relation.)

Definition 5.7. Let $\delta = (\delta_1, \delta_2, \ldots, \delta_\ell)$ and $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m)$ be two compositions.

Fix two chains (i.e., totally ordered sets) $P = \{p_1 < p_2 < \cdots < p_\ell\}$ and $Q = \{q_1 < q_2 < \cdots < q_m\}$, and let

$$U = P \sqcup Q$$

be their disjoint union. This U is a poset with $\ell + m$ elements $p_1, p_2, \ldots, p_\ell, q_1, q_2, \ldots, q_m$, whose relations are given by $p_1 < p_2 < \cdots < p_\ell$ and $q_1 < q_2 < \cdots < q_m$ (while each p_i is incomparable to each q_j).

If $f : U \to X$ is a map from U to any set X, and if $s \in X$ is any element, then we define the two sets

$$f_P^{-1}(s) := \{u \in [\ell] \mid f(p_u) = s\} \qquad \text{and}$$
$$f_Q^{-1}(s) := \{v \in [m] \mid f(q_v) = s\}.$$

(Essentially, $f_P^{-1}(s)$ and $f_Q^{-1}(s)$ are the sets of the preimages of s in P and Q, respectively, except that they consist of numbers instead of actual elements of P and Q.)

A *stufufuffler* for δ and ε shall mean a surjective and weakly order-preserving map

$$f : U \to \{1 < 2 < \cdots < k\} \qquad \text{for some } k \in \mathbb{N}$$

with the property that each $s \in \{1 < 2 < \cdots < k\}$ satisfies

$$\left|f_P^{-1}(s)\right| \approx \left|f_Q^{-1}(s)\right|. \tag{83}$$

("Weakly order-preserving" means that if u and v are two elements of the poset U satisfying $u < v$, then $f(u) \leq f(v)$.)

If $f : U \to \{1 < 2 < \cdots < k\}$ is a stufufuffler for δ and ε, then we define three further concepts:

- We define the *weight* $\operatorname{wt}(f)$ of f to be the composition $(\operatorname{wt}_1(f), \operatorname{wt}_2(f), \ldots, \operatorname{wt}_k(f))$, where

$$\begin{aligned}\operatorname{wt}_s(f) &= \sum_{u \in f_P^{-1}(s)} \delta_u + \sum_{v \in f_Q^{-1}(s)} \varepsilon_v \\ &= \sum_{\substack{u \in [\ell];\\ f(p_u) = s}} \delta_u + \sum_{\substack{v \in [m];\\ f(q_v) = s}} \varepsilon_v \qquad \text{for each } s \in [k].\end{aligned}$$

  (Note that (83) ensures that the two sums on the right hand side here have nearly equal numbers of addends. Moreover, the surjectivity of f ensures that at least one of these two sums has at least one addend, and thus $\operatorname{wt}_s(f)$ is a positive integer; therefore, $\operatorname{wt}(f)$ is a composition.)

- We define the *loss* of f to be the nonnegative integer

$$\operatorname{loss}(f) := \sum_{s=1}^{k} \max\left\{\left|f_P^{-1}(s)\right|, \ \left|f_Q^{-1}(s)\right|\right\} - k.$$

  (This really is a nonnegative integer, since the surjectivity of f yields that $\max\left\{\left|f_P^{-1}(s)\right|, \ \left|f_Q^{-1}(s)\right|\right\} \geq 1$ for each $s \in [k]$, and thus $\operatorname{loss}(f) = \sum_{s=1}^{k} \underbrace{\max\left\{\left|f_P^{-1}(s)\right|, \ \left|f_Q^{-1}(s)\right|\right\}}_{\geq 1} - k \geq \underbrace{\sum_{s=1}^{k} 1}_{=k} - k = 0$.)

- We define the *poise* of f to be the nonnegative integer

$$\operatorname{poise}(f) := \left(\text{\# of all } s \in [k] \text{ such that } \left|f_P^{-1}(s)\right| = \left|f_Q^{-1}(s)\right|\right).$$

Example 5.8. Let $\delta = (a, b)$ and $\varepsilon = (c, d, e)$ be two compositions. Then, the poset U in Definition 5.7 is $U = \{p_1 < p_2\} \sqcup \{q_1 < q_2 < q_3\}$. The following maps (written in two-line notation) are stufufufflers for δ and ε:

$$\begin{pmatrix} p_1 & p_2 & q_1 & q_2 & q_3 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}, \qquad \begin{pmatrix} p_1 & p_2 & q_1 & q_2 & q_3 \\ 2 & 5 & 1 & 3 & 4 \end{pmatrix},$$
$$\begin{pmatrix} p_1 & p_2 & q_1 & q_2 & q_3 \\ 1 & 1 & 1 & 2 & 3 \end{pmatrix}, \qquad \begin{pmatrix} p_1 & p_2 & q_1 & q_2 & q_3 \\ 1 & 2 & 2 & 2 & 3 \end{pmatrix},$$
$$\begin{pmatrix} p_1 & p_2 & q_1 & q_2 & q_3 \\ 2 & 2 & 1 & 2 & 3 \end{pmatrix}, \qquad \begin{pmatrix} p_1 & p_2 & q_1 & q_2 & q_3 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix},$$
$$\begin{pmatrix} p_1 & p_2 & q_1 & q_2 & q_3 \\ 1 & 1 & 1 & 1 & 2 \end{pmatrix}, \qquad \begin{pmatrix} p_1 & p_2 & q_1 & q_2 & q_3 \\ 1 & 2 & 1 & 1 & 2 \end{pmatrix}.$$

(The list is not exhaustive – there are many more stufufufflers for δ and ε.)

On the other hand, here are some maps (in two-line notation) that are not stufufufflers for δ and ε:

- The map $\begin{pmatrix} p_1 & p_2 & q_1 & q_2 & q_3 \\ 1 & 2 & 1 & 1 & 1 \end{pmatrix}$ is not a stufufuffler, since it violates (83) for $s = 1$.
- The map $\begin{pmatrix} p_1 & p_2 & q_1 & q_2 & q_3 \\ 1 & 2 & 2 & 1 & 2 \end{pmatrix}$ is not a stufufuffler, since it is not weakly increasing ($f(q_1) > f(q_2)$).
- The map $\begin{pmatrix} p_1 & p_2 & q_1 & q_2 & q_3 \\ 2 & 2 & 2 & 2 & 2 \end{pmatrix}$ is not a stufufuffler, since it fails to be surjective onto $\{1 < 2 < \cdots < k\}$ whatever k is.

Here are the weights of the eight stufufufflers listed above:

$$\operatorname{wt}\begin{pmatrix} p_1 & p_2 & q_1 & q_2 & q_3 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} = (a,b,c,d,e),$$
$$\operatorname{wt}\begin{pmatrix} p_1 & p_2 & q_1 & q_2 & q_3 \\ 2 & 5 & 1 & 3 & 4 \end{pmatrix} = (c,a,d,e,b),$$
$$\operatorname{wt}\begin{pmatrix} p_1 & p_2 & q_1 & q_2 & q_3 \\ 1 & 1 & 1 & 2 & 3 \end{pmatrix} = (a+b+c,\ d,\ e),$$
$$\operatorname{wt}\begin{pmatrix} p_1 & p_2 & q_1 & q_2 & q_3 \\ 1 & 2 & 2 & 2 & 3 \end{pmatrix} = (a,\ b+c+d,\ e),$$
$$\operatorname{wt}\begin{pmatrix} p_1 & p_2 & q_1 & q_2 & q_3 \\ 2 & 2 & 1 & 2 & 3 \end{pmatrix} = (c,\ a+b+d,\ e),$$
$$\operatorname{wt}\begin{pmatrix} p_1 & p_2 & q_1 & q_2 & q_3 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} = (a+b+c+d+e),$$
$$\operatorname{wt}\begin{pmatrix} p_1 & p_2 & q_1 & q_2 & q_3 \\ 1 & 1 & 1 & 1 & 2 \end{pmatrix} = (a+b+c+d,\ e),$$
$$\operatorname{wt}\begin{pmatrix} p_1 & p_2 & q_1 & q_2 & q_3 \\ 1 & 2 & 1 & 1 & 2 \end{pmatrix} = (a+c+d,\ b+e).$$

The losses of these stufufufflers are 0, 0, 1, 1, 1, 2, 1 and 1, respectively. Their poises are 0, 0, 0, 0, 0, 0, 1 and 1, respectively.

Intuitively, the composition $\operatorname{wt}(f)$ in Definition 5.7 can be thought of as a variant of a stuffle[18] of δ with ε, but instead of adding an entry of δ with an entry of ε, it allows adding i consecutive entries of δ and j consecutive entries of ε whenever i and j are integers satisfying $i \approx j$. (Such a sum can be obtained by starting with 0 and taking turns at adding the next available entry from δ or from ε; thus the name "stufufuffle".) The poise statistic $\operatorname{poise}(f)$ tells us how often this $i \approx j$ relation becomes an equality. The loss statistic $\operatorname{loss}(f)$ tells how much is being added, i.e., how far this "stufufuffle" deviates from a stuffle.

Now we can restate the multiplication rule for $\eta_\delta^{(q)} \eta_\varepsilon^{(q)}$ in terms of stufufufflers:

Theorem 5.9. Let δ and ε be two compositions. Then,

$$\eta_\delta^{(q)} \eta_\varepsilon^{(q)} = \sum_{\substack{f \text{ is a stufufuffler} \\ \text{for } \delta \text{ and } \varepsilon}} (-q)^{\operatorname{loss}(f)} (q-1)^{\operatorname{poise}(f)} \eta_{\operatorname{wt}(f)}^{(q)}.$$

[18]"Stuffles" are also known as "overlapping shuffles"; see [GriRei20, Proposition 5.1.3 and Example 5.1.4] for the meaning of this concept (and [DEMT17] for more).

Example 5.10. Let $\delta = (a, b)$ and $\varepsilon = (c, d)$ be two compositions of length 2. Let us compute $\eta^{(q)}_{(a,b)} \eta^{(q)}_{(c,d)}$ using Theorem 5.9. The stufufufflers for δ and ε are the maps (written here in two-line notation)

$$\begin{array}{ccc}
\begin{pmatrix} p_1 & p_2 & q_1 & q_2 \\ 1 & 2 & 3 & 4 \end{pmatrix}, & \begin{pmatrix} p_1 & p_2 & q_1 & q_2 \\ 1 & 3 & 2 & 4 \end{pmatrix}, & \begin{pmatrix} p_1 & p_2 & q_1 & q_2 \\ 1 & 4 & 2 & 3 \end{pmatrix}, \\
\begin{pmatrix} p_1 & p_2 & q_1 & q_2 \\ 2 & 3 & 1 & 4 \end{pmatrix}, & \begin{pmatrix} p_1 & p_2 & q_1 & q_2 \\ 2 & 4 & 1 & 3 \end{pmatrix}, & \begin{pmatrix} p_1 & p_2 & q_1 & q_2 \\ 3 & 4 & 1 & 2 \end{pmatrix}, \\
\begin{pmatrix} p_1 & p_2 & q_1 & q_2 \\ 1 & 2 & 2 & 2 \end{pmatrix}, & \begin{pmatrix} p_1 & p_2 & q_1 & q_2 \\ 2 & 2 & 1 & 2 \end{pmatrix}, & \begin{pmatrix} p_1 & p_2 & q_1 & q_2 \\ 1 & 1 & 1 & 2 \end{pmatrix}, \\
\begin{pmatrix} p_1 & p_2 & q_1 & q_2 \\ 1 & 2 & 1 & 1 \end{pmatrix}, & \begin{pmatrix} p_1 & p_2 & q_1 & q_2 \\ 1 & 2 & 1 & 2 \end{pmatrix}, & \begin{pmatrix} p_1 & p_2 & q_1 & q_2 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \\
\begin{pmatrix} p_1 & p_2 & q_1 & q_2 \\ 1 & 2 & 1 & 3 \end{pmatrix}, & \begin{pmatrix} p_1 & p_2 & q_1 & q_2 \\ 1 & 3 & 1 & 2 \end{pmatrix}, & \begin{pmatrix} p_1 & p_2 & q_1 & q_2 \\ 1 & 3 & 2 & 3 \end{pmatrix}, \\
\begin{pmatrix} p_1 & p_2 & q_1 & q_2 \\ 2 & 3 & 1 & 3 \end{pmatrix}, & \begin{pmatrix} p_1 & p_2 & q_1 & q_2 \\ 1 & 2 & 2 & 3 \end{pmatrix}, & \begin{pmatrix} p_1 & p_2 & q_1 & q_2 \\ 2 & 3 & 1 & 2 \end{pmatrix}.
\end{array}$$

Their respective weights are

$$\begin{array}{ccc}
(a, b, c, d), & (a, c, b, d), & (a, c, d, b), \\
(c, a, b, d), & (c, a, d, b), & (c, d, a, b), \\
(a,\ b + c + d), & (c,\ a + b + d), & (a + b + c,\ d), \\
(a + c + d,\ b), & (a + c,\ b + d), & (a + b + c + d), \\
(a + c,\ b,\ d), & (a + c,\ d,\ b), & (a,\ c,\ b + d), \\
(c,\ a,\ b + d), & (a,\ b + c,\ d), & (c,\ a + d,\ b);
\end{array}$$

their respective losses are

$$\begin{array}{c}
0, 0, 0, \\
0, 0, 0, \\
1, 1, 1, \\
1, 0, 1, \\
0, 0, 0, \\
0, 0, 0,
\end{array}$$

whereas their respective poises are

$$\begin{array}{c}
0, 0, 0, \\
0, 0, 0, \\
0, 0, 0, \\
0, 2, 1, \\
1, 1, 1, \\
1, 1, 1.
\end{array}$$

Thus, Theorem 5.9 yields

$$\begin{aligned}
\eta^{(q)}_{(a,b)}\eta^{(q)}_{(c,d)} = \eta^{(q)}_{(a,b,c,d)} &+ \eta^{(q)}_{(a,c,b,d)} + \eta^{(q)}_{(a,c,d,b)} \\
&+ \eta^{(q)}_{(c,a,b,d)} + \eta^{(q)}_{(c,a,d,b)} + \eta^{(q)}_{(c,d,a,b)} \\
&- q\eta^{(q)}_{(a,\ b+c+d)} - q\eta^{(q)}_{(c,\ a+b+d)} - q\eta^{(q)}_{(a+b+c,\ d)} \\
&- q\eta^{(q)}_{(a+c+d,\ b)} + (q-1)^2 \eta^{(q)}_{(a+c,\ b+d)} - q(q-1)\eta^{(q)}_{(a+b+c+d)} \\
&+ (q-1)\eta^{(q)}_{(a+c,\ b,\ d)} + (q-1)\eta^{(q)}_{(a+c,\ d,\ b)} + (q-1)\eta^{(q)}_{(a,\ c,\ b+d)} \\
&+ (q-1)\eta^{(q)}_{(c,\ a,\ b+d)} + (q-1)\eta^{(q)}_{(a,\ b+c,\ d)} + (q-1)\eta^{(q)}_{(c,\ a+d,\ b)}.
\end{aligned}$$

Let us now outline how Theorem 5.9 can be derived from Theorem 5.1.

Proof of Theorem 5.9 (sketched). Let us define the polynomials $d^{\alpha}_{\delta,\varepsilon}(X) \in \mathbb{Z}[X]$ as in the proof of Theorem 5.1. Then, Claim 4 in said proof shows that

$$\eta^{(q)}_{\delta}\eta^{(q)}_{\varepsilon} = \sum_{\alpha\in\mathrm{Comp}} d^{\alpha}_{\delta,\varepsilon}(q)\,\eta^{(q)}_{\alpha}. \tag{84}$$

Now, fix a composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$. Let $\mathbf{P}$ be the set of all pairs

$$((\beta_1, \beta_2, \ldots, \beta_k),\ (\gamma_1, \gamma_2, \ldots, \gamma_k))$$

satisfying the six conditions

$$\begin{aligned}
&\beta_1, \beta_2, \ldots, \beta_k \in \mathrm{Comp}; \qquad \gamma_1, \gamma_2, \ldots, \gamma_k \in \mathrm{Comp}; \\
&\beta_1\beta_2\cdots\beta_k = \delta; \qquad \gamma_1\gamma_2\cdots\gamma_k = \varepsilon; \\
&|\ell(\beta_s) - \ell(\gamma_s)| \leq 1 \text{ for each } s; \\
&|\beta_s| + |\gamma_s| = \alpha_s \text{ for each } s.
\end{aligned}$$

Then, the equality (71) rewrites as

$$\begin{aligned}
d^{\alpha}_{\delta,\varepsilon}(q) = \sum_{((\beta_1,\beta_2,\ldots,\beta_k),\ (\gamma_1,\gamma_2,\ldots,\gamma_k))\in\mathbf{P}} &(-q)^{\sum\limits_{s=1}^{k}\max\{\ell(\beta_s),\ell(\gamma_s)\}-k} \\
&\cdot (q-1)^{(\#\text{ of all } s\in[k] \text{ such that } \ell(\beta_s)=\ell(\gamma_s))}.
\end{aligned} \tag{85}$$

On the other hand, let $\mathbf{S}$ be the set of all stufufufflers f for δ and ε satisfying $\mathrm{wt}(f) = \alpha$.

We shall construct a bijection Φ from $\mathbf{S}$ to $\mathbf{P}$. Namely, Φ shall send any stufufuffler $f \in \mathbf{S}$ to the pair

$$((\beta_1, \beta_2, \ldots, \beta_k),\ (\gamma_1, \gamma_2, \ldots, \gamma_k)),$$

where

$$\beta_s = \Big(\text{the composition consisting of the } \delta_u \text{ for all } u \in f_P^{-1}(s)$$
$$\text{(in the order of increasing } u)\Big) \qquad \text{and}$$
$$\gamma_s = \Big(\text{the composition consisting of the } \varepsilon_v \text{ for all } v \in f_Q^{-1}(s)$$
$$\text{(in the order of increasing } v)\Big) \qquad \text{for all } s \in [k].$$

(We are here using the fact that our stufufuffler f must necessarily be a map from U to $\{1 < 2 < \cdots < k\}$, because its weight $\operatorname{wt}(f) = \alpha$ is a composition of length k.) It is easy to see that this pair really belongs to $\mathbf{P}$, and that Φ is indeed a bijection[19].

This bijection Φ has a further useful property: If Φ sends a stufufuffler f to a pair $((\beta_1, \beta_2, \ldots, \beta_k), (\gamma_1, \gamma_2, \ldots, \gamma_k))$, then

$$\sum_{s=1}^{k} \max\{\ell(\beta_s), \ell(\gamma_s)\} - k = \operatorname{loss}(f) \qquad \text{and}$$
$$(\# \text{ of all } s \in [k] \text{ such that } \ell(\beta_s) = \ell(\gamma_s)) = \operatorname{poise}(f).$$

(This is easily seen from the definitions of Φ, of the loss and of the poise.)

Thus, we can use the bijection Φ to rewrite (85) as

$$\begin{aligned} d^{\alpha}_{\delta,\varepsilon}(q) &= \sum_{f \in \mathbf{S}} (-q)^{\operatorname{loss}(f)} (q-1)^{\operatorname{poise}(f)} \\ &= \sum_{\substack{f \text{ is a stufufuffler} \\ \text{for } \delta \text{ and } \varepsilon; \\ \operatorname{wt}(f) = \alpha}} (-q)^{\operatorname{loss}(f)} (q-1)^{\operatorname{poise}(f)} \end{aligned} \qquad (86)$$

(by the definition of $\mathbf{S}$).

Forget that we fixed α. We thus have proved (86) for each composition $\alpha \in \operatorname{Comp}$.

[19] Its inverse map Φ^{-1} can easily be constructed: It sends each pair $((\beta_1, \beta_2, \ldots, \beta_k), (\gamma_1, \gamma_2, \ldots, \gamma_k)) \in \mathbf{P}$ to the map $f : U \to [k]$ that is given by

$$f(p_u) = \min\{s \in [k] \mid \ell(\beta_1 \beta_2 \cdots \beta_s) \geq u\} \qquad \text{for all } u \in [\ell]$$

and

$$f(q_v) = \min\{s \in [k] \mid \ell(\gamma_1 \gamma_2 \cdots \gamma_s) \geq v\} \qquad \text{for all } v \in [m].$$

The idea behind this is that $f(p_u)$ is the number s such that the u-th entry of the concatenated composition $\beta_1 \beta_2 \cdots \beta_k$ is taken from its s-th factor β_s (and similarly $f(q_v)$).

Hence, we can rewrite (84) as

$$\begin{aligned}
\eta_\delta^{(q)} \eta_\varepsilon^{(q)} &= \sum_{\alpha \in \operatorname{Comp}} \left(\sum_{\substack{f \text{ is a stufufuffler} \\ \text{for } \delta \text{ and } \varepsilon; \\ \operatorname{wt}(f) = \alpha}} (-q)^{\operatorname{loss}(f)} (q-1)^{\operatorname{poise}(f)} \right) \eta_\alpha^{(q)} \\
&= \sum_{\alpha \in \operatorname{Comp}} \sum_{\substack{f \text{ is a stufufuffler} \\ \text{for } \delta \text{ and } \varepsilon; \\ \operatorname{wt}(f) = \alpha}} (-q)^{\operatorname{loss}(f)} (q-1)^{\operatorname{poise}(f)} \underbrace{\eta_\alpha^{(q)}}_{\substack{= \eta_{\operatorname{wt}(f)}^{(q)} \\ (\text{since } \alpha = \operatorname{wt}(f))}} \\
&= \underbrace{\sum_{\alpha \in \operatorname{Comp}} \sum_{\substack{f \text{ is a stufufuffler} \\ \text{for } \delta \text{ and } \varepsilon; \\ \operatorname{wt}(f) = \alpha}}}_{= \sum_{\substack{f \text{ is a stufufuffler} \\ \text{for } \delta \text{ and } \varepsilon}}} (-q)^{\operatorname{loss}(f)} (q-1)^{\operatorname{poise}(f)} \eta_{\operatorname{wt}(f)}^{(q)} \\
&= \sum_{\substack{f \text{ is a stufufuffler} \\ \text{for } \delta \text{ and } \varepsilon}} (-q)^{\operatorname{loss}(f)} (q-1)^{\operatorname{poise}(f)} \eta_{\operatorname{wt}(f)}^{(q)}.
\end{aligned}$$

This proves Theorem 5.9. □

5.3. The product rule in terms of subsets

Finally, let us state the product rule for the $\eta_\alpha^{(q)}$ (Theorem 5.1) in yet another form, using classical shuffles ([GriVas22, Corollary 1]):

Definition 5.11. If T is any set of integers, then $T-1$ shall denote the set $\{t-1 \mid t \in T\}$.

Definition 5.12. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ be a composition with n entries. For any $i \in [n-1]$, we let $\alpha^{\downarrow i}$ denote the following composition with $n-1$ entries:

$$\alpha^{\downarrow i} := (\alpha_1, \ldots, \alpha_{i-1}, \alpha_i + \alpha_{i+1}, \alpha_{i+2}, \ldots, \alpha_n).$$

Furthermore, for any subset $I \subseteq [n-1]$, we set

$$\alpha^{\downarrow I} := \left(\left(\cdots \left(\alpha^{\downarrow i_k} \right) \cdots \right)^{\downarrow i_2} \right)^{\downarrow i_1},$$

where $i_1, i_2, \ldots, i_k$ are the elements of I in increasing order.

Finally, if I and J are two subsets of $[n-1]$, then we set

$$\alpha^{\downarrow I \downarrow\downarrow J} := \alpha^{\downarrow K}, \qquad \text{where } K = I \cup J \cup (J-1).$$

Example 5.13. Let $\alpha = (a, b, c, d, e, f, g)$ be a composition with 7 entries. Then,

$$\alpha^{\downarrow 2} = (a,\ b+c,\ d,\ e,\ f,\ g)\,;$$
$$\alpha^{\downarrow \{2,4,5\}} = (a,\ b+c,\ d+e+f,\ g)\,;$$
$$\alpha^{\downarrow \{2\} \downarrow\downarrow \{6\}} = \alpha^{\downarrow \{2,5,6\}} = (a,\ b+c,\ d,\ e+f+g)\,.$$

Theorem 5.14. Let $\delta = (\delta_1, \delta_2, \ldots, \delta_n)$ and $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m)$ be two compositions.

If T is any m-element subset of $[n+m]$, then we define the *T-shuffle* of δ with ε to be the composition

$$\delta \lfloor T \rfloor \varepsilon := (\gamma_1, \gamma_2, \ldots, \gamma_{n+m})\,,$$

where

$$\gamma_k := \begin{cases} \delta_i, & \text{if } k \text{ is the } i\text{-th smallest element of } [n+m] \setminus T; \\ \varepsilon_j, & \text{if } k \text{ is the } j\text{-th smallest element of } T. \end{cases}$$

Furthermore, if T is any subset of $[n+m]$, then we define a further subset

$$T' := (T \setminus (T-1)) \setminus \{n+m\}\,.$$

Then,

$$\eta_\delta^{(q)} \eta_\varepsilon^{(q)} = \sum_{\substack{\text{triples } (T,I,J);\\ T \subseteq [n+m];\\ |T|=m;\\ I \subseteq T';\\ J \subseteq T' \setminus \{1\};\\ I \cap J = \varnothing}} (-q)^{|J|} (q-1)^{|I|}\, \eta^{(q)}_{(\delta \lfloor T \rfloor \varepsilon)^{\downarrow I \downarrow\downarrow J}}.$$

Example 5.15. Let $\delta = (a)$ and $\varepsilon = (b, c)$ be two compositions. Then, applying Theorem 5.14 (with $n = 1$ and $m = 2$), we see that $\eta_\delta^{(q)} \eta_\varepsilon^{(q)} = \eta_{(a)}^{(q)} \eta_{(b,c)}^{(q)}$ is a sum over all triples (T, I, J) satisfying

$$T \subseteq [3]\,, \qquad |T| = 2, \qquad I \subseteq T', \qquad J \subseteq T' \setminus \{1\}\,, \qquad I \cap J = \varnothing.$$

There are exactly six such triples (T, I, J), namely

$$(\{1,2\}\,,\ \varnothing,\ \varnothing)\,,\ (\{1,2\}\,,\ \varnothing,\ \{2\})\,,\ (\{1,2\}\,,\ \{2\}\,,\ \varnothing)\,,$$
$$(\{1,3\}\,,\ \varnothing,\ \varnothing)\,,\ (\{1,3\}\,,\ \{1\}\,,\ \varnothing)\,,\ (\{2,3\}\,,\ \varnothing,\ \varnothing)\,.$$

Thus, the claim of Theorem 5.14 becomes

$$\eta_{(a)}^{(q)} \eta_{(b,c)}^{(q)} = \eta_{(b,c,a)}^{(q)} - q\eta_{(a+b+c)}^{(q)} + (q-1)\,\eta_{(b,\ a+c)}^{(q)} + \eta_{(b,a,c)}^{(q)} + (q-1)\,\eta_{(a+b,\ c)}^{(q)} + \eta_{(a,b,c)}^{(q)}$$

(here, we have listed the addends in the same order in which the corresponding triples were listed above).

Theorem 5.14 can be derived from Theorem 5.9 by constructing a bijection between the stufufufflers of δ and ε and the triples (T, I, J) from Theorem 5.14. The details of this bijection are somewhat bothersome, so we shall omit them, not least because Theorem 5.14 can also be proved in a different way (using enriched P-partitions). The latter proof has been outlined in [GriVas22, Corollary 1] and will be elaborated upon in forthcoming work.

6. Applications

We shall now discuss some applications of the basis $\left(\eta_\alpha^{(q)}\right)_{\alpha\in\mathrm{Comp}}$ and its features.

6.1. Hopf subalgebras of QSym

The $q = 1$ case in particular is useful for constructing Hopf subalgebras of QSym, such as the peak subalgebra Π introduced by Stembridge [Stembr97, §3] and later studied by various authors ([AgBeSo14, §6, particularly Proposition 6.5], [BMSW99], [BMSW00, §5], [Hsiao07] etc.). We shall now briefly survey some Hopf subalgebras that can be obtained in this way.

Convention 6.1. For the rest of Subsection 6.1, we fix a set T of compositions (i.e., a subset T of Comp).

We let $\mathrm{QSym}_T^{(q)}$ be the $\mathbf{k}$-submodule of QSym spanned by the family $\left(\eta_\alpha^{(q)}\right)_{\alpha\in T}$.

When is this $\mathbf{k}$-submodule $\mathrm{QSym}_T^{(q)}$ a subcoalgebra of QSym? The answer is simple:[20]

Proposition 6.2. For any subset Y of $\{1, 2, 3, \ldots\}$, we let

$$\begin{aligned} Y^* &:= \{\text{all compositions whose entries all belong to } Y\} \\ &= \{(\alpha_1, \alpha_2, \ldots, \alpha_k) \in \mathrm{Comp} \mid \alpha_i \in Y \text{ for each } i\}. \end{aligned}$$

(a) If $T = Y^*$ for some subset Y of $\{1, 2, 3, \ldots\}$, then $\mathrm{QSym}_T^{(q)}$ is a subcoalgebra of QSym.

(b) If $\mathbf{k}$ is a field and $r \neq 0$, then the converse holds as well: If $\mathrm{QSym}_T^{(q)}$ is a subcoalgebra of QSym, then $T = Y^*$ for some subset Y of $\{1, 2, 3, \ldots\}$.

[20] We are being sloppy: For us here, a "subcoalgebra" of a coalgebra C means a $\mathbf{k}$-submodule D of C that satisfies

$$\Delta(D) \subseteq (\text{image of the canonical map } D \otimes D \to C \otimes C).$$

This is **not** the algebraically literate definition of a "subcoalgebra", as it does not imply that D itself becomes a $\mathbf{k}$-coalgebra (after all, the canonical map $D \otimes D \to C \otimes C$ might fail to be injective, and then it is not clear how to "restrict" Δ to a map $D \to D \otimes D$). Fortunately, the two definitions are equivalent when $\mathbf{k}$ is a field (or when D is a direct addend of C as a $\mathbf{k}$-module).

Proof sketch. **(a)** This follows from Theorem 3.18.

(b) Use the graded dual NSym of QSym and Proposition 4.9. (The orthogonal complement of a subcoalgebra is an ideal.) $\square$

Proposition 6.2 allows us to restrict ourselves to sets T of the form Y^* for $Y \subseteq \{1,2,3,\ldots\}$ if we want $\mathrm{QSym}_T^{(q)}$ to be a Hopf subalgebra of QSym. However, not every set T of this form Y^* results in a Hopf subalgebra. For generic q, this happens fairly rarely:

Proposition 6.3. Let Y be a subset of $\{1,2,3,\ldots\}$ that is closed under addition (i.e., satisfies $y+z \in Y$ for every $y,z \in Y$). Let $T := Y^*$. Then, $\mathrm{QSym}_T^{(q)}$ is a Hopf subalgebra of QSym.

Proof sketch. Clearly, $1 = \eta_{\varnothing}^{(q)} \in \mathrm{QSym}_T^{(q)}$, and Proposition 6.2 **(a)** shows that $\mathrm{QSym}_T^{(q)}$ is a subcoalgebra of QSym. Next, we will show that $\mathrm{QSym}_T^{(q)}$ is closed under multiplication. In view of Theorem 5.1, this will follow once we can show the following claim:

Claim 1: Let $k \in \mathbb{N}$. Let $\delta \in Y^*$ and $\varepsilon \in Y^*$ be two compositions all of whose entries are $\in Y$. Let $\beta_1, \beta_2, \ldots, \beta_k \in \mathrm{Comp}$ and $\gamma_1, \gamma_2, \ldots, \gamma_k \in \mathrm{Comp}$ be $2k$ compositions satisfying

$$\beta_1\beta_2\cdots\beta_k = \delta \qquad \text{and} \qquad \gamma_1\gamma_2\cdots\gamma_k = \varepsilon$$
$$\text{and} \qquad \ell(\beta_s) + \ell(\gamma_s) > 0 \text{ for all } s.$$

Then,
$$(|\beta_1| + |\gamma_1|,\ |\beta_2| + |\gamma_2|,\ \ldots,\ |\beta_k| + |\gamma_k|) \in Y^*.$$

Proof of Claim 1. We need to show that $|\beta_s| + |\gamma_s| \in Y$ for each $s \in [k]$. To do so, we fix $s \in [k]$. Then, $\ell(\beta_s) + \ell(\gamma_s) > 0$ (by assumption). In other words, at least one of the compositions β_s and γ_s is nonempty.

However, all entries of the composition β_s are entries of the composition $\beta_1\beta_2\cdots\beta_k = \delta$, and thus belong to Y (since $\delta \in Y^*$). Thus, the sum of all entries of β_s either equals 0 or belongs to Y (since Y is closed under addition). In other words, the size $|\beta_s|$ either equals 0 or belongs to Y. Similarly, $|\gamma_s|$ either equals 0 or belongs to Y. Hence, the sum $|\beta_s| + |\gamma_s|$ either equals 0 or belongs to Y as well (since Y is closed under addition). Since $|\beta_s| + |\gamma_s|$ cannot equal 0 (because at least one of the compositions β_s and γ_s is nonempty), we thus conclude that $|\beta_s| + |\gamma_s|$ belongs to Y. In other words, $|\beta_s| + |\gamma_s| \in Y$. As we said, this completes the proof of Claim 1. $\square$

Now, Claim 1 (together with $1 \in \mathrm{QSym}_T^{(q)}$) shows that $\mathrm{QSym}_T^{(q)}$ is a $\mathbf{k}$-subalgebra of QSym. As we saw above, $\mathrm{QSym}_T^{(q)}$ is a $\mathbf{k}$-subcoalgebra of QSym as well, and thus is a $\mathbf{k}$-subbialgebra of QSym. This bialgebra $\mathrm{QSym}_T^{(q)}$ is connected graded, and therefore a Hopf algebra (by Takeuchi's famous result [GriRei20, Proposition 1.4.16]). The inclusion map $\operatorname{QSym}_T^{(q)} \to \operatorname{QSym}$ is a bialgebra morphism between two Hopf algebras, and thus automatically a Hopf algebra morphism (by another well-known result: [GriRei20, Corollary 1.4.27]). Hence, $\operatorname{QSym}_T^{(q)}$ is a Hopf subalgebra of QSym. This proves Proposition 6.3. $\square$

Example 6.4. The subset $\{2,4,6,8,\ldots\}$ of $\{1,2,3,\ldots\}$ is closed under addition. Thus, Proposition 6.3 shows that $\operatorname{QSym}_T^{(q)}$ is a Hopf subalgebra of QSym for $Y := \{2,4,6,8,\ldots\}$ and $T := Y^*$. This Hopf subalgebra can be viewed as a copy of QSym in the indeterminates $x_1^2, x_2^2, x_3^2, \ldots$, and thus is isomorphic to QSym.

Example 6.5. The subset $\{2,3,4,5,\ldots\}$ of $\{1,2,3,\ldots\}$ is closed under addition. Thus, Proposition 6.3 shows that $\operatorname{QSym}_T^{(q)}$ is a Hopf subalgebra of QSym for $Y := \{2,3,4,5,\ldots\}$ and $T := Y^*$.

Proposition 6.3 is not very surprising. In fact, (5) shows that (under the assumptions of Proposition 6.3) the space $\operatorname{QSym}_T^{(q)}$ is just the $\mathbf{k}$-linear span of the functions $r^{\ell(\alpha)} M_\alpha$ with $\alpha \in Y^*$; but the latter span is easily seen to be a Hopf subalgebra (using [GriRei20, Proposition 5.1.3] and (31)).

If $q \neq 1$ and if r is invertible, then Proposition 6.3 has a converse (i.e., $\operatorname{QSym}_T^{(q)}$ is a Hopf subalgebra of QSym only when Y is closed under addition), since it is easy to see that

$$\eta_{(a)}^{(q)} \eta_{(b)}^{(q)} = (q-1)\, \eta_{(a+b)}^{(q)} + \eta_{(a,b)}^{(q)} + \eta_{(b,a)}^{(q)} \qquad \text{for any } a, b \geq 1.$$

However, more interesting behavior emerges when $q = 1$:

Proposition 6.6. Let Y be a subset of $\{1,2,3,\ldots\}$ that is closed under ternary addition (i.e., satisfies $y+z+w \in Y$ for every $y,z,w \in Y$). Let $T := Y^*$. Then, $\operatorname{QSym}_T^{(1)}$ is a Hopf subalgebra of QSym.

Proof sketch. This is similar to Proposition 6.3, but now we set $q = 1$ and observe that all addends on the right hand side of Theorem 5.1 that satisfy

$$\ell(\beta_s) = \ell(\gamma_s) \qquad \text{for at least one } s \in [k]$$

are 0 (because they include the factor $(1-1)^{\text{a positive integer}}$, which vanishes), and all the remaining addends have the property that $|\ell(\beta_s) - \ell(\gamma_s)| = 1$ for all s (since $|\ell(\beta_s) - \ell(\gamma_s)| \leq 1$ and $\ell(\beta_s) \neq \ell(\gamma_s)$). Hence, the following claim now replaces Claim 1:

Claim 1′: Let $k \in \mathbb{N}$. Let $\delta \in Y^*$ and $\varepsilon \in Y^*$ be two compositions all of whose entries are $\in Y$. Let $\beta_1, \beta_2, \ldots, \beta_k \in \operatorname{Comp}$ and $\gamma_1, \gamma_2, \ldots, \gamma_k \in$

Comp be $2k$ compositions satisfying

$$\beta_1\beta_2\cdots\beta_k=\delta \qquad \text{and} \qquad \gamma_1\gamma_2\cdots\gamma_k=\varepsilon$$
$$\text{and} \qquad \left|\ell\left(\beta_s\right)-\ell\left(\gamma_s\right)\right|=1 \text{ for all } s.$$

Then,

$$\left(\left|\beta_1\right|+\left|\gamma_1\right|,\ \left|\beta_2\right|+\left|\gamma_2\right|,\ \ldots,\ \left|\beta_k\right|+\left|\gamma_k\right|\right)\in Y^*.$$

Proof of Claim 1′. We need to show that $|\beta_s|+|\gamma_s|\in Y$ for each $s\in[k]$. To do so, we fix $s\in[k]$. Then, $\left|\ell\left(\beta_s\right)-\ell\left(\gamma_s\right)\right|=1$ (by assumption), and thus $\ell\left(\beta_s\right)+\ell\left(\gamma_s\right)$ is odd. Hence, $|\beta_s|+|\gamma_s|$ is a sum of an odd number of entries of δ and ε, and therefore a sum of an odd number of elements of Y (since δ and ε belong to Y^*). But Y is closed under ternary addition, and therefore any sum of an odd number of elements of Y must belong to Y (easy induction exercise). Hence, $|\beta_s|+|\gamma_s|\in Y$, and thus Claim 1′ is proved. □

The rest of the proof proceeds as for Proposition 6.3. □

Example 6.7. The subset $\{1,3,5,7,\ldots\}$ of $\{1,2,3,\ldots\}$ is closed under ternary addition. Thus, Proposition 6.6 shows that $\operatorname{QSym}_T^{(1)}$ is a Hopf subalgebra of QSym for $Y:=\{1,3,5,7,\ldots\}$ and $T:=Y^*$. This Hopf subalgebra is precisely the peak algebra Π of [Stembr97, §3], [AgBeSo14, §6, particularly Proposition 6.5], [BMSW99], [BMSW00, §5] and [Hsiao07] (since [Hsiao07, (2.1) and (2.2)] shows that the θ_α for α odd have the same span as the η_α for α odd, but [Hsiao07, Proposition 2.1] shows that the latter η_α are precisely our $\eta_\alpha^{(1)}$ up to sign).

Example 6.8. The subset $\{\text{positive integers } \neq 2\}=\{1,3,4,5,\ldots\}$ of $\{1,2,3,\ldots\}$ is closed under ternary addition. Thus, Proposition 6.6 shows that $\operatorname{QSym}_T^{(1)}$ is a Hopf subalgebra of QSym for $Y:=\{\text{positive integers } \neq 2\}$ and $T:=Y^*$. This Hopf subalgebra is the Hopf subalgebra Ξ constructed in [BMSW00, Theorem 5.7]. (Indeed, both Hopf subalgebras have the same orthogonal complement: the principal ideal of NSym generated by $\eta_2^*=\dfrac{1}{4}X_2=\dfrac{1}{4}\left(2H_2-H_1H_1\right)$.)

Example 6.9. More generally, if we pick a positive integer k and set

$$Y:=\{\text{odd positive integers}\}\cup\{k,k+1,k+2,\ldots\}$$

and $T:=Y^*$, then Proposition 6.6 shows that $\operatorname{QSym}_T^{(1)}$ is a Hopf subalgebra of QSym (since Y is closed under ternary addition).

The reader can find more examples without trouble. When $\mathbf{k}$ is nontrivial and 2 is invertible in $\mathbf{k}$, Proposition 6.6 is easily seen to have a converse (using Example 5.3).

6.2. A new shuffle algebra

Next, we shall use the enriched q-monomial quasisymmetric functions to realize a certain deformed version of the shuffle product, which has appeared in recent work of [BoNoTh22] by Bouillot, Novelli and Thibon (generalizing the "block shuffle product" of Hirose and Sato [HirSat22, ◊]).

Shuffle products are a broad and deep subject with a long history and many applications (e.g., to multiple zeta values, algebraic topology and stochastic differential equations). An overview of known variants (such as the stuffles, the "muffles", the infiltrations and many more) can be found in [DEMT17, Table 1]. In the following, we shall discuss a variant that does not directly fit into the framework of [DEMT17], but is sufficiently similar to enjoy some of the same behavior. To our knowledge, this variant has first appeared in [BoNoTh22]. We will use the letters a and b for what was called α and β in [BoNoTh22], as we prefer to use Greek letters for compositions.

Let $\mathcal{F}$ be the free $\mathbf{k}$-algebra with generators $x_1, x_2, x_3, \ldots$. It has a basis consisting of all words over the alphabet $\{x_1, x_2, x_3, \ldots\}$; these words are in bijection with the compositions. In fact, let us set

$$x_\gamma := x_{\gamma_1} x_{\gamma_2} \cdots x_{\gamma_k} \tag{87}$$

for every composition $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_k)$. Then, the bijection sends each composition γ to the word x_γ.

For any $k \in \mathbb{N}$, we let $\zeta_k : \mathcal{F} \to \mathcal{F}$ be the $\mathbf{k}$-linear operator defined by

$$\begin{aligned} \zeta_k(1) &= 0; \\ \zeta_k(x_i w) &= x_{i+k} w \qquad \text{for each } i \geq 1 \text{ and any word } w. \end{aligned}$$

(Thus, explicitly, the map ζ_k sends 1 to 0, and transforms any nonempty word by adding k to the subscript of its first letter. For example, $\zeta_k(x_u x_v x_w) = x_{u+k} x_v x_w$ for any $u, v, w \geq 1$.)

Fix two elements a and b of the base ring $\mathbf{k}$.

Let $\# : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$ be the $\mathbf{k}$-bilinear map on $\mathcal{F}$ defined recursively by the requirements

$$\begin{aligned} 1 \# w &= w \qquad \text{for any word } w; \\ w \# 1 &= w \qquad \text{for any word } w; \\ (x_i u) \# (x_j v) &= x_i (u \# (x_j v)) + x_j ((x_i u) \# v) + a x_{i+j} (u \# v) + b \zeta_{i+j} (u \# v) \\ &\qquad \text{for any } i, j \geq 1 \text{ and any words } u \text{ and } v. \end{aligned}$$

We call this bilinear map # the *stufufuffle*[21]. Explicitly, we can compute this operation as follows:

[21]This is a riff on the notion of "stuffle" (which is recovered when $a = 1$ and $b = 0$) and the fact that multiple letters of both words u and v can get combined into one in $u \# v$.

Proposition 6.10. Let $\delta = (\delta_1, \delta_2, \ldots, \delta_\ell)$ and $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m)$ be two compositions. Then, using the notation of (87), we have

$$x_\delta \# x_\varepsilon = \sum_{\substack{f \text{ is a stufufuffler} \\ \text{for } \delta \text{ and } \varepsilon}} b^{\operatorname{loss}(f)} a^{\operatorname{poise}(f)} x_{\operatorname{wt}(f)}.$$

Proof sketch. Use strong induction on $\ell + m$.

Induction step: If $\delta = \varnothing$ or $\varepsilon = \varnothing$, then the claim is easy to check. Thus, assume WLOG that neither δ nor ε is $\varnothing$. Let $i = \delta_1$ and $j = \varepsilon_1$ and $\overline{\delta} = (\delta_2, \delta_3, \ldots, \delta_\ell)$ and $\overline{\varepsilon} = (\varepsilon_2, \varepsilon_3, \ldots, \varepsilon_m)$. Hence, $x_\delta = x_i x_{\overline{\delta}}$ and $x_\varepsilon = x_j x_{\overline{\varepsilon}}$, so that

$$\begin{aligned} x_\delta \# x_\varepsilon &= \left(x_i x_{\overline{\delta}}\right) \# \left(x_j x_{\overline{\varepsilon}}\right) \\ &= x_i \left(x_{\overline{\delta}} \# \underbrace{\left(x_j x_{\overline{\varepsilon}}\right)}_{= x_\varepsilon} \right) + x_j \left(\underbrace{\left(x_i x_{\overline{\delta}}\right)}_{= x_\delta} \# x_{\overline{\varepsilon}} \right) + a x_{i+j} \left(x_{\overline{\delta}} \# x_{\overline{\varepsilon}}\right) + b \zeta_{i+j} \left(x_{\overline{\delta}} \# x_{\overline{\varepsilon}}\right) \\ &\qquad \text{(by the recursive definition of \#)} \\ &= x_i \left(x_{\overline{\delta}} \# x_\varepsilon\right) + x_j \left(x_\delta \# x_{\overline{\varepsilon}}\right) + a x_{i+j} \left(x_{\overline{\delta}} \# x_{\overline{\varepsilon}}\right) + b \zeta_{i+j} \left(x_{\overline{\delta}} \# x_{\overline{\varepsilon}}\right). \end{aligned} \tag{88}$$

On the other hand, the stufufufflers f for δ and ε can be classified into four types:

1. *Type 1* consists of those stufufufflers f that satisfy $\left|f_P^{-1}(1)\right| = 1$ and $\left|f_Q^{-1}(1)\right| = 0$ (so that the composition $\operatorname{wt}(f)$ begins with the entry $\delta_1 = i$).

2. *Type 2* consists of those stufufufflers f that satisfy $\left|f_P^{-1}(1)\right| = 0$ and $\left|f_Q^{-1}(1)\right| = 1$ (so that the composition $\operatorname{wt}(f)$ begins with the entry $\varepsilon_1 = j$).

3. *Type 3* consists of those stufufufflers f that satisfy $\left|f_P^{-1}(1)\right| = 1$ and $\left|f_Q^{-1}(1)\right| = 1$ (so that the composition $\operatorname{wt}(f)$ begins with the entry $\delta_1 + \varepsilon_1 = i + j$).

4. *Type 4* consists of those stufufufflers f that satisfy $\left|f_P^{-1}(1)\right| + \left|f_Q^{-1}(1)\right| > 2$ (so that both numbers $\left|f_P^{-1}(1)\right|$ and $\left|f_Q^{-1}(1)\right|$ are positive[22], and one of them is at least 2, and therefore the composition $\operatorname{wt}(f)$ begins with the entry $\delta_1 + \varepsilon_1 + (\text{some further numbers})$).

A type-1 stufufuffler f for δ and ε becomes a stufufuffler for $\overline{\delta}$ and ε if we decrease all its values by 1 and remove p_1 from P. This is furthermore a bijection from $\{\text{type-1 stufufufflers for } \delta \text{ and } \varepsilon\}$ to $\left\{\text{stufufufflers for } \overline{\delta} \text{ and } \varepsilon\right\}$, and this bijection preserves both loss and poise while removing the first entry from the weight.

[22] by (83), applied to $s = 1$

Hence, we obtain

$$\begin{aligned}
&\sum_{\substack{f \text{ is a type-1 stufufuffler} \\ \text{for } \delta \text{ and } \varepsilon}} b^{\operatorname{loss}(f)} a^{\operatorname{poise}(f)} x_{\operatorname{wt}(f)} \\
&= \sum_{\substack{f \text{ is a stufufuffler} \\ \text{for } \overline{\delta} \text{ and } \varepsilon}} b^{\operatorname{loss}(f)} a^{\operatorname{poise}(f)} x_i x_{\operatorname{wt}(f)} \\
&= x_i \cdot \underbrace{\sum_{\substack{f \text{ is a stufufuffler} \\ \text{for } \overline{\delta} \text{ and } \varepsilon}} b^{\operatorname{loss}(f)} a^{\operatorname{poise}(f)} x_{\operatorname{wt}(f)}}_{\substack{= x_{\overline{\delta}} \# x_\varepsilon \\ \text{(by the induction hypothesis,} \\ \text{since } \overline{\delta} \text{ has length } \ell - 1 < \ell)}} \\
&= x_i \left(x_{\overline{\delta}} \# x_\varepsilon \right).
\end{aligned}$$

Similar reasoning leads to

$$\begin{aligned}
\sum_{\substack{f \text{ is a type-2 stufufuffler} \\ \text{for } \delta \text{ and } \varepsilon}} b^{\operatorname{loss}(f)} a^{\operatorname{poise}(f)} x_{\operatorname{wt}(f)} &= x_j \left(x_\delta \# x_{\overline{\varepsilon}} \right); \\
\sum_{\substack{f \text{ is a type-3 stufufuffler} \\ \text{for } \delta \text{ and } \varepsilon}} b^{\operatorname{loss}(f)} a^{\operatorname{poise}(f)} x_{\operatorname{wt}(f)} &= a x_{i+j} \left(x_{\overline{\delta}} \# x_{\overline{\varepsilon}} \right); \\
\sum_{\substack{f \text{ is a type-4 stufufuffler} \\ \text{for } \delta \text{ and } \varepsilon}} b^{\operatorname{loss}(f)} a^{\operatorname{poise}(f)} x_{\operatorname{wt}(f)} &= b \zeta_{i+j} \left(x_{\overline{\delta}} \# x_{\overline{\varepsilon}} \right).
\end{aligned}$$

Adding these four equalities together (and recalling that each stufufuffler for δ and ε belongs to exactly one of the four types 1, 2, 3 and 4), we obtain

$$\begin{aligned}
&\sum_{\substack{f \text{ is a stufufuffler} \\ \text{for } \delta \text{ and } \varepsilon}} b^{\operatorname{loss}(f)} a^{\operatorname{poise}(f)} x_{\operatorname{wt}(f)} \\
&= x_i \left(x_{\overline{\delta}} \# x_\varepsilon \right) + x_j \left(x_\delta \# x_{\overline{\varepsilon}} \right) + a x_{i+j} \left(x_{\overline{\delta}} \# x_{\overline{\varepsilon}} \right) + b \zeta_{i+j} \left(x_{\overline{\delta}} \# x_{\overline{\varepsilon}} \right).
\end{aligned}$$

Comparing this with (88), we obtain

$$x_\delta \# x_\varepsilon = \sum_{\substack{f \text{ is a stufufuffler} \\ \text{for } \delta \text{ and } \varepsilon}} b^{\operatorname{loss}(f)} a^{\operatorname{poise}(f)} x_{\operatorname{wt}(f)}.$$

This completes the induction step, and thus Proposition 6.10 is proved. □

Theorem 6.11. The bilinear map # is commutative and associative, and the element $1 \in \mathcal{F}$ is a neutral element for it. Thus, the $\mathbf{k}$-module $\mathcal{F}$, equipped with the operation # (as multiplication), becomes a commutative $\mathbf{k}$-algebra with unity 1.

It appears possible to prove Theorem 6.11 by induction, but the most convenient method at this point is to deduce this from the properties of the enriched q-monomial basis of QSym. To wit, the following proposition connects the map # to the latter basis:

Proposition 6.12. Let q and u be two elements of $\mathbf{k}$ such that $a = (q-1)\,u$ and $b = -qu^2$. (Such q and u do not always exist, of course.)

Let $\operatorname{eta} : \mathcal{F} \to \operatorname{QSym}$ be the $\mathbf{k}$-linear map that sends the word $x_\alpha = x_{\alpha_1} x_{\alpha_2} \cdots x_{\alpha_k} \in \mathcal{F}$ to $u^{\ell(\alpha)} \eta_\alpha^{(q)} \in \operatorname{QSym}$ for each composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$. Then, $\operatorname{eta}(g \# h) = (\operatorname{eta} g) \cdot (\operatorname{eta} h)$ for any $g, h \in \mathcal{F}$.

Proof sketch. Let $g, h \in \mathcal{F}$. We WLOG assume that $g = x_\delta$ and $h = x_\varepsilon$ for two compositions δ and ε. Consider these δ and ε. Thus,

$$\operatorname{eta} g = \operatorname{eta} x_\delta = u^{\ell(\delta)} \eta_\delta^{(q)} \qquad \text{(by the definition of eta)}$$

and similarly $\operatorname{eta} h = u^{\ell(\varepsilon)} \eta_\varepsilon^{(q)}$. Multiplying these two equalities, we find

$$\begin{aligned} &(\operatorname{eta} g) \cdot (\operatorname{eta} h) \\ &= u^{\ell(\delta)} \eta_\delta^{(q)} \cdot u^{\ell(\varepsilon)} \eta_\varepsilon^{(q)} = u^{\ell(\delta)+\ell(\varepsilon)} \eta_\delta^{(q)} \eta_\varepsilon^{(q)} \\ &= u^{\ell(\delta)+\ell(\varepsilon)} \sum_{\substack{f \text{ is a stufufuffler} \\ \text{for } \delta \text{ and } \varepsilon}} (-q)^{\operatorname{loss}(f)} (q-1)^{\operatorname{poise}(f)} \eta_{\operatorname{wt}(f)}^{(q)} \end{aligned} \tag{89}$$

(by Theorem 5.9).

On the other hand, from $g = x_\delta$ and $h = x_\varepsilon$, we obtain

$$g \# h = x_\delta \# x_\varepsilon = \sum_{\substack{f \text{ is a stufufuffler} \\ \text{for } \delta \text{ and } \varepsilon}} b^{\operatorname{loss}(f)} a^{\operatorname{poise}(f)} x_{\operatorname{wt}(f)}$$

(by Proposition 6.10). Hence, by the definition of eta, we obtain

$$\operatorname{eta}(g \# h) = \sum_{\substack{f \text{ is a stufufuffler} \\ \text{for } \delta \text{ and } \varepsilon}} b^{\operatorname{loss}(f)} a^{\operatorname{poise}(f)} u^{\ell(\operatorname{wt}(f))} \eta_{\operatorname{wt}(f)}^{(q)}. \tag{90}$$

We must prove that the left hand sides of (90) and (89) are equal. Of course, it suffices to show that the right hand sides are equal. For that purpose, it suffices to show that

$$b^{\operatorname{loss}(f)} a^{\operatorname{poise}(f)} u^{\ell(\operatorname{wt}(f))} = u^{\ell(\delta)+\ell(\varepsilon)} (-q)^{\operatorname{loss}(f)} (q-1)^{\operatorname{poise}(f)}$$

whenever f is a stufufuffler for δ and ε. Recalling that $a = (q-1)\,u$ and $b = -qu^2$, we can easily boil this down to the fact that every stufufuffler f for δ and ε satisfies

$$2 \operatorname{loss}(f) + \operatorname{poise}(f) + \ell(\operatorname{wt}(f)) = \ell(\delta) + \ell(\varepsilon);$$

but this fact is easily verified combinatorially. $\square$

Proof of Theorem 6.11 (sketched). All claims of this theorem boil down to polynomial identities in a and b. For example, associativity of # is saying that the elements $(u\#v)\#w$ and $u\#(v\#w)$ of $\mathcal{F}$ have the same t-coefficient whenever u, v, w, t are four words; but this is easily revealed (upon expanding all products) to be an equality between two polynomials in a and b (when u, v, w, t are fixed). Note that all relevant polynomials have integer coefficients.

Thus, in order to prove Theorem 6.11, we can WLOG assume that a and b are two distinct indeterminates in a polynomial ring over $\mathbb{Z}$ (for example, $a = X$ and $b = Y$ in the polynomial ring $\mathbb{Z}[X, Y]$). Even better, we can WLOG assume that a and b are two algebraically independent elements of a $\mathbb{Z}$-algebra.

However, in the ring $\mathbb{Z}[X, Y]$, the two elements $X + Y$ and XY are algebraically independent (since they are the elementary symmetric polynomials in the indeterminates X and Y). Thus, we can WLOG assume that $\mathbf{k} = \mathbb{Z}[X, Y]$ and that $a = X + Y$ and $b = XY$. Moreover, we can extend the base ring $\mathbf{k}$ to its quotient field $\mathbb{Q}(X, Y)$. So we assume that $\mathbf{k} = \mathbb{Q}(X, Y)$ and $a = X + Y$ and $b = XY$.

Set $q := -XY^{-1}$ and $u := -Y$ in $\mathbf{k}$. Then, simple computations confirm that $a = (q-1)u$ and $b = -qu^2$. Hence, the map eta : $\mathcal{F} \to \operatorname{QSym}$ constructed in Proposition 6.12 satisfies

$$\operatorname{eta}(g\#h) = (\operatorname{eta} g) \cdot (\operatorname{eta} h) \qquad \text{for any } g, h \in \mathcal{F} \tag{91}$$

(by Proposition 6.12). Moreover, the element $u = -Y \in \mathbf{k}$ is invertible (since $\mathbf{k}$ is a field), and so is the element $r := q + 1 = -XY^{-1} + 1 \in \mathbf{k}$ (for the same reason, since $r \neq 0$). Thus, the family $\left(u^{\ell(\alpha)} \eta_\alpha^{(q)}\right)_{\alpha \in \operatorname{Comp}}$ is a basis of QSym (by Theorem 3.10 **(a)**). Hence, the map eta is a $\mathbf{k}$-module isomorphism (since it sends the basis $(x_\alpha)_{\alpha \in \operatorname{Comp}}$ of $\mathcal{F}$ to the basis $\left(u^{\ell(\alpha)} \eta_\alpha^{(q)}\right)_{\alpha \in \operatorname{Comp}}$ of QSym). The equality (91) shows that this isomorphism eta transfers the multiplication of QSym to the binary operation # on $\mathcal{F}$. Since the former multiplication is associative, we thus conclude that the latter operation # is associative as well. Similarly, we can see that # is commutative. Finally, it is clear that 1 is a neutral element for #. Thus, Theorem 6.11 is proved. □

In view of Theorem 6.11, we can restate Proposition 6.12 as follows:

Theorem 6.13. Let q and u be two elements of $\mathbf{k}$ such that $a = (q-1)u$ and $b = -qu^2$.

Let eta : $\mathcal{F} \to \operatorname{QSym}$ be the $\mathbf{k}$-linear map that sends the word $x_\alpha = x_{\alpha_1} x_{\alpha_2} \cdots x_{\alpha_k} \in \mathcal{F}$ to $u^{\ell(\alpha)} \eta_\alpha^{(q)} \in \operatorname{QSym}$ for each composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$. Then, eta is a $\mathbf{k}$-algebra homomorphism from the $\mathbf{k}$-algebra $(\mathcal{F}, \#)$ to the $\mathbf{k}$-algebra QSym.

We can also turn $\mathcal{F}$ into a coalgebra. In fact, let $\Delta : \mathcal{F} \to \mathcal{F} \otimes \mathcal{F}$ be the $\mathbf{k}$-linear map that sends each word $w_1 w_2 \cdots w_n$ to $\sum\limits_{i=0}^{n} w_1 w_2 \cdots w_i \otimes w_{i+1} w_{i+2} \cdots w_n$. This map Δ is called the *deconcatenation coproduct* (or the *cut coproduct*). This coproduct turns $\mathcal{F}$ into a coalgebra (with counit $\varepsilon : \mathcal{F} \to \mathbf{k}$ sending each word $w_1 w_2 \cdots w_n$ to $\begin{cases} 1, & \text{if } n = 0; \\ 0, & \text{if } n > 0 \end{cases}$). The map $\operatorname{eta} : \mathcal{F} \to \operatorname{QSym}$ from Theorem 6.13 is then easily seen to be a $\mathbf{k}$-coalgebra homomorphism (by Theorem 3.18).

The stufufuffle product # on $\mathcal{F}$ respects the deconcatenation coproduct Δ of $\mathcal{F}$, in the following sense:

Theorem 6.14. The $\mathbf{k}$-algebra $(\mathcal{F}, \#)$, equipped with the coproduct Δ and the counit ε constructed above, is a commutative connected graded Hopf algebra.

Theorem 6.15. Let q and u be two elements of $\mathbf{k}$ such that $a = (q-1)\,u$ and $b = -qu^2$.

Let $\operatorname{eta} : \mathcal{F} \to \operatorname{QSym}$ be the $\mathbf{k}$-linear map from Theorem 6.13. Then, eta is a Hopf algebra homomorphism from the Hopf algebra $(\mathcal{F}, \#, \Delta, \varepsilon)$ to the Hopf algebra QSym.

We leave the proofs of these two theorems to the reader. (They follow the same mold as our above proof of Theorem 6.11.)

Likewise, using Theorem 3.16 and the proof method of Theorem 6.11 above, we can prove the following:

Theorem 6.16. Let S be the antipode of the Hopf algebra $(\mathcal{F}, \#)$ constructed in Theorem 6.14. Let $n \in \mathbb{N}$ and $\alpha \in \operatorname{Comp}_n$. Then, in $\mathcal{F}$, we have

$$S(x_\alpha) = (-1)^{\ell(\alpha)} \sum_{\substack{\beta \in \operatorname{Comp}_n; \\ D(\beta) \subseteq D(\operatorname{rev} \alpha)}} a^{\ell(\alpha) - \ell(\beta)} x_\beta.$$

The recent work [BoNoTh22, Theorem 5.2] constructs another basis (X_I) of QSym (indexed by subsets I of $[n-1]$ instead of compositions α, but this difference is insubstantial) that multiplies according to the stufufuffle product (thus obtaining another $\mathbf{k}$-algebra homomorphism from $\mathcal{F}$ to QSym, and with it another proof of Theorem 6.11). While similar to ours, it uses the alphabet-transformed functions $H_k((s-t)A)$ instead of the plain H_k, which lead to a basis of QSym that does not appear to have a simple combinatorial formula like our $\eta_\alpha^{(q)}$.

Remark 6.17. Assume that $\mathbf{k}$ is a field of characteristic 0. Then, Leray's theorem ([GriRei20, Theorem 1.7.29(f)]) shows that any commutative connected graded $\mathbf{k}$-bialgebra A is isomorphic as a graded $\mathbf{k}$-algebra to the symmetric algebra of a certain graded $\mathbf{k}$-module (namely, of $(\ker \varepsilon) / (\ker \varepsilon)^2$, where ε is the counit of A). In other words, any such A is isomorphic as a graded $\mathbf{k}$-algebra to a polynomial ring whose generators are homogeneous of various positive degrees, with exactly

$\dim\left(\left(\ker\varepsilon\right) / \left(\ker\varepsilon\right)^2\right)_i$ many generators of degree i. This applies, in particular, to our connected graded $\mathbf{k}$-bialgebra $(\mathcal{F}, \#)$. Moreover, using standard Hilbert-series arguments, it is easy to see that the number of generators of each given degree does not depend on the parameters a and b. Hence, as a graded $\mathbf{k}$-algebra, our $(\mathcal{F}, \#)$ is isomorphic to the usual shuffle algebra (which is obtained for $a = 0$ and $b = 0$).

However, this is not a $\mathbf{k}$-coalgebra isomorphism; nor is it canonical (although we suspect that a canonical $\mathbf{k}$-algebra isomorphism may exist); nor does it extend to fields of positive characteristic.

7. Appendix: The map R_q

We finish by stating yet another formula for $\eta_\alpha^{(q)}$, which may eventually prove useful in understanding these functions. This formula relies on some more notations. We first define a simple combinatorial operation on compositions:

Definition 7.1. Let $\alpha \in \operatorname{Comp}$, and let $n = |\alpha|$. Then, $\overline{\alpha}$ shall denote the unique composition γ of n such that $D(\gamma) = [n-1] \setminus D(\alpha)$. (This γ is indeed unique, since the map D is a bijection.) This composition $\overline{\alpha}$ is called the *complement* of α.

For example, $\overline{(2,5,1,1)} = (1,2,1,1,1,3)$. We observe some simple properties of complements of compositions:

Proposition 7.2.

- **(a)** Every composition α satisfies $\overline{\overline{\alpha}} = \alpha$.
- **(b)** For each $n \in \mathbb{N}$, the map $\operatorname{Comp}_n \to \operatorname{Comp}_n,\ \beta \mapsto \overline{\beta}$ is a bijection.
- **(c)** If α and β are two compositions of n for some $n \in \mathbb{N}$, then the statements "$D(\beta) \subseteq D(\alpha)$" and "$D\left(\overline{\beta}\right) \supseteq D(\overline{\alpha})$" are equivalent.
- **(d)** If α is a composition of a positive integer n, then $\ell(\overline{\alpha}) + \ell(\alpha) = n + 1$.

Proof. **(a)** Let α be a composition. Let $n = |\alpha|$. Thus, $D(\alpha) \subseteq [n-1]$.

The definition of $\overline{\alpha}$ yields that $\overline{\alpha}$ is a composition of n and satisfies $D(\overline{\alpha}) = [n-1] \setminus D(\alpha)$. Hence, the definition of $\overline{\overline{\alpha}}$ yields that $\overline{\overline{\alpha}}$ is a composition of n and satisfies

$$D\left(\overline{\overline{\alpha}}\right) = [n-1] \setminus \underbrace{D(\overline{\alpha})}_{=[n-1]\setminus D(\alpha)} = [n-1] \setminus \left([n-1] \setminus D(\alpha)\right) = D(\alpha)$$

(since $D(\alpha) \subseteq [n-1]$). Since the map $D : \operatorname{Comp}_n \to \mathcal{P}([n-1])$ is a bijection (and since both $\overline{\overline{\alpha}}$ and α belong to Comp_n), we thus conclude that $\overline{\overline{\alpha}} = \alpha$. This proves Proposition 7.2 **(a)**.

(b) Let $n \in \mathbb{N}$. Then, $\overline{\beta} \in \operatorname{Comp}_n$ for each $\beta \in \operatorname{Comp}_n$ (by the definition of $\overline{\beta}$). Hence, the map $\operatorname{Comp}_n \to \operatorname{Comp}_n,\ \beta \mapsto \overline{\beta}$ is well-defined. This map is furthermore its own inverse (by Proposition 7.2 **(a)**). Thus, it is invertible, i.e., a bijection. This proves Proposition 7.2 **(b)**.

(c) Let α and β be two compositions of n for some $n \in \mathbb{N}$. Then, $D(\alpha)$ and $D(\beta)$ are two subsets of $[n-1]$. Meanwhile, $D(\overline{\alpha})$ and $D(\overline{\beta})$ are the complements of these two subsets (i.e., we have $D(\overline{\alpha}) = [n-1] \setminus D(\alpha)$ and $D(\overline{\beta}) = [n-1] \setminus D(\beta)$), by the definition of $\overline{\alpha}$ and $\overline{\beta}$. However, it is well-known from set theory that complementation reverses inclusion of subsets: That is, if X and Y are two subsets of a given set Z, then the statement "$X \subseteq Y$" is equivalent to the statement "$Z \setminus X \supseteq Z \setminus Y$". Applying this to $X = D(\beta)$ and $Y = D(\alpha)$ and $Z = [n-1]$, we conclude that the statement "$D(\beta) \subseteq D(\alpha)$" is equivalent to the statement "$[n-1] \setminus D(\beta) \supseteq [n-1] \setminus D(\alpha)$". In other words, the statement "$D(\beta) \subseteq D(\alpha)$" is equivalent to the statement "$D(\overline{\beta}) \supseteq D(\overline{\alpha})$" (since $D(\overline{\alpha}) = [n-1] \setminus D(\alpha)$ and $D(\overline{\beta}) = [n-1] \setminus D(\beta)$). This proves Proposition 7.2 **(c)**.

(d) Let α be a composition of a positive integer n. Then, $[n \neq 0] = 1$ (since n is positive), so that Lemma 3.9 **(a)** yields $\ell(\alpha) = |D(\alpha)| + \underbrace{[n \neq 0]}_{=1} = |D(\alpha)| + 1$. Similarly, $\ell(\overline{\alpha}) = |D(\overline{\alpha})| + 1$ (since $\overline{\alpha}$ is a composition of n as well). But the definition of $\overline{\alpha}$ yields $D(\overline{\alpha}) = [n-1] \setminus D(\alpha)$, so that

$$|D(\overline{\alpha})| = |[n-1] \setminus D(\alpha)| = |[n-1]| - |D(\alpha)|$$

(since $D(\alpha) \subseteq [n-1]$). Therefore, $|D(\overline{\alpha})| + |D(\alpha)| = |[n-1]| = n-1$ (since $n \geq 1$). Now,

$$\begin{aligned} \underbrace{\ell(\overline{\alpha})}_{=|D(\overline{\alpha})|+1} + \underbrace{\ell(\alpha)}_{=|D(\alpha)|+1} &= |D(\overline{\alpha})| + 1 + |D(\alpha)| + 1 = \underbrace{|D(\overline{\alpha})| + |D(\alpha)|}_{=n-1} + 2 \\ &= n - 1 + 2 = n + 1. \end{aligned}$$

This proves Proposition 7.2 **(d)**. □

We now define a linear endomorphism of QSym:

Definition 7.3. We let R_q be the $\mathbf{k}$-linear map from QSym to QSym that sends each M_α (with $\alpha \in \operatorname{Comp}$) to $r^{\ell(\overline{\alpha})} M_{\overline{\alpha}}$. (This is well-defined, since $(M_\alpha)_{\alpha \in \operatorname{Comp}}$ is a basis of QSym.)

This map R_q is neither an algebra endomorphism nor a coalgebra endomorphism of QSym (not even when $r = 1$), but it is exactly what we need for our formula. First, let us observe that the map R_q is "close to an involution" in the following sense:

Proposition 7.4. Let n be a positive integer. Let $f \in \operatorname{QSym}$ be homogeneous of degree n. Then, $\left(R_q \circ R_q\right)(f) = r^{n+1} f$.

Proof. Both sides of this equality are $\mathbf{k}$-linear in f. Thus, it suffices to prove this equality for $f = M_\alpha$ for all compositions $\alpha \in \operatorname{Comp}_n$ (since the family $(M_\alpha)_{\alpha \in \operatorname{Comp}_n}$ is a basis of the n-th graded component of QSym). In other words, it suffices to show that $\left(R_q \circ R_q\right)(M_\alpha) = r^{n+1} M_\alpha$ for every $\alpha \in \operatorname{Comp}_n$. However, this is easy:

Let $\alpha \in \operatorname{Comp}_n$. Then, the definition of R_q yields $R_q(M_\alpha) = r^{\ell(\overline{\alpha})} M_{\overline{\alpha}}$. Applying the map R_q to both sides of this equality, we find

$$R_q\left(R_q\left(M_\alpha\right)\right) = R_q\left(r^{\ell(\overline{\alpha})} M_{\overline{\alpha}}\right) = r^{\ell(\overline{\alpha})} R_q\left(M_{\overline{\alpha}}\right). \tag{92}$$

But the definition of R_q yields $R_q(M_{\overline{\alpha}}) = r^{\ell(\overline{\overline{\alpha}})} M_{\overline{\overline{\alpha}}} = r^{\ell(\alpha)} M_\alpha$ (since Proposition 7.2 **(a)** yields $\overline{\overline{\alpha}} = \alpha$). Thus, (92) becomes

$$R_q\left(R_q\left(M_\alpha\right)\right) = r^{\ell(\overline{\alpha})} \underbrace{R_q\left(M_{\overline{\alpha}}\right)}_{= r^{\ell(\alpha)} M_\alpha} = r^{\ell(\overline{\alpha})} r^{\ell(\alpha)} M_\alpha = r^{\ell(\overline{\alpha}) + \ell(\alpha)} M_\alpha = r^{n+1} M_\alpha$$

(since Proposition 7.2 **(d)** yields $\ell(\overline{\alpha}) + \ell(\alpha) = n+1$). Hence, $\left(R_q \circ R_q\right)(M_\alpha) = R_q\left(R_q\left(M_\alpha\right)\right) = r^{n+1} M_\alpha$. As explained above, this completes the proof of Proposition 7.4. $\square$

Now we can state our final formula for $\eta_\alpha^{(q)}$:

Theorem 7.5. Let $\alpha \in \operatorname{Comp}$. Then,

$$\eta_\alpha^{(q)} = R_q\left(L_{\overline{\alpha}}\right).$$

Proof. Let $n = |\alpha|$, so that $\alpha \in \operatorname{Comp}_n$. Therefore, $\overline{\alpha} \in \operatorname{Comp}_n$ as well (by the definition of $\overline{\alpha}$), so that $n = |\overline{\alpha}|$. Hence, applying (3) to $\overline{\alpha}$ instead of α, we obtain

$$L_{\overline{\alpha}} = \sum_{\substack{\beta \in \operatorname{Comp}_n; \\ D(\beta) \supseteq D(\overline{\alpha})}} M_\beta = \sum_{\substack{\beta \in \operatorname{Comp}_n; \\ D(\overline{\beta}) \supseteq D(\overline{\alpha})}} M_{\overline{\beta}} \tag{93}$$

(here, we have substituted $\overline{\beta}$ for β in the sum, since Proposition 7.2 **(b)** shows that the map $\operatorname{Comp}_n \to \operatorname{Comp}_n,\ \beta \mapsto \overline{\beta}$ is a bijection). However, the condition "$D\left(\overline{\beta}\right) \supseteq D(\overline{\alpha})$" under the summation sign on the right hand side of (93) is equivalent to "$D(\beta) \subseteq D(\alpha)$" (by Proposition 7.2 **(c)**). Hence, we can rewrite (93) as

$$L_{\overline{\alpha}} = \sum_{\substack{\beta \in \operatorname{Comp}_n; \\ D(\beta) \subseteq D(\alpha)}} M_{\overline{\beta}}.$$

Applying the map R_q to both sides of this equality, we find

$$R_q\left(L_{\overline{\alpha}}\right) = R_q\left(\sum_{\substack{\beta\in\operatorname{Comp}_n;\\ D(\beta)\subseteq D(\alpha)}} M_{\overline{\beta}}\right) = \sum_{\substack{\beta\in\operatorname{Comp}_n;\\ D(\beta)\subseteq D(\alpha)}} \underbrace{R_q\left(M_{\overline{\beta}}\right)}_{\substack{=r^{\ell\left(\overline{\overline{\beta}}\right)}M_{\overline{\overline{\beta}}} \\ \text{(by the definition of } R_q\text{)}}}$$

$$= \sum_{\substack{\beta\in\operatorname{Comp}_n;\\ D(\beta)\subseteq D(\alpha)}} \underbrace{r^{\ell\left(\overline{\overline{\beta}}\right)}M_{\overline{\overline{\beta}}}}_{\substack{=r^{\ell(\beta)}M_{\beta} \\ \text{(since Proposition 7.2 \textbf{(a)}} \\ \text{yields } \overline{\overline{\beta}}=\beta)}} = \sum_{\substack{\beta\in\operatorname{Comp}_n;\\ D(\beta)\subseteq D(\alpha)}} r^{\ell(\beta)}M_\beta = \eta_\alpha^{(q)} \qquad \text{(by (5))}.$$

This proves Theorem 7.5. □

Remark 7.6. Let n be a positive integer, and let $\alpha \in \operatorname{Comp}_n$. Combining Theorem 7.5 with Proposition 7.4, we can easily see that $R_q\left(\eta_\alpha^{(q)}\right) = r^{n+1}L_{\overline{\alpha}}$. Contrasting this equality with Theorem 7.5 reveals a symmetry of sorts between the $\eta_\alpha^{(q)}$ and $L_{\overline{\alpha}}$. This symmetry explains the similarity between Proposition 3.11 and Proposition 3.12 (and allows one to derive one of these propositions from the other with a bit of work).

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